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Subdifferentiation of the Infimal Convolution and Minimal Time Problems*

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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We investigate in the Banach setting (not necessarily reflexive) first-order variations of the infimal convolution of fairly general functions. We characterize different subdifferentials and differentiability concepts of this infimal convolution by means of the corresponding subdifferentials and differentiability concepts, respectively, of data functions, at points where the infimal convolution is attained, well-posed, or strongly attained. Next, we apply these results to study the (sub)differentiability of minimal time functions associated with constant dynamics satisfying appropriate interiority conditions.

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1. Introduction

Cibulka and Fabian [4] have shown that the Clarke generalized subdifferential of the infimal convolution of a bounded-below proper function $f\colon X \to \mathbb{R} \cup \{+\infty\}$, defined on a Banach space X, and the corresponding square-norm function,

$$(f\Box\|\cdot\|^2)(x) := \inf_{z\in X}\{f(z) + \|x-z\|^2\}$$

is contained in the subdifferential of the square-norm function at an appropriate point, which is related to the points where the infimal is attained. This result requires that the infimal convolution is strongly attained at the reference point. In addition, if the norm is Gâteaux (Fréchet, respectively) differentiable, then the

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infimal convolution is Gâteaux (Fréchet, respectively) differentiable too. We also refer for related results to [1, 2, 5, 10, 15, 19], among others.

In this paper, we give new formulas for different kinds of generalized subdifferentials of the infimal convolution above under less restrictive conditions on the function f than those used in [4]. We show that the Clarke generalized subdifferential of the (Lipschitzian) infimal convolution is included into the Clarke subdifferential of each one of these functions, at the corresponding points where the infimal convolution is attained. This result is achieved provided that the infimal convolution is well-posed at the point in consideration. The well-posedness condition is slightly weaker than the strongly attainability property used in [4]. If, in addition, one of these functions is strictly Hadamard (Fréchet, respectively) differentiable, then the infimal convolution is strictly Hadamard (Fréchet, respectively) differentiable, and its strict derivative is equal to the strict derivative of the corresponding function. We also provide a characterization of the Clarke directional derivative when the infimal convolution is not necessary attained but one of the function is uniformly Gâteaux-differentiable.

The above (sub)differential properties of the infimal convolution will be next applied to investigate (sub)differential properties of the minimal time function

$$T(x) := \inf\{t \geq 0 : \Omega \cap (x + tF) \neq \emptyset\},$$

associated to the following minimal time problem

$$\text{minimize } t \geq 0 \;\text{ such that } \Omega \cap (x + tF) \neq \emptyset, \tag{1}$$

where Ω is a closed target set, and $F \subset X$. In case F is convex, the optimal value in problem (1) is the minimal time needed to attain the target set Ω from x by trajectories of the differential inclusion $\dot{x}(t) \in F$. The proximal and Fréchet subdifferentials of the minimal time function, under the standard interiority condition $0 \in \operatorname{int} F$, were first computed in [7, 8] within the framework of Hilbert spaces. In general Banach spaces, formulas for evaluating various subdifferentials of the minimal time function were obtained in [12, 13, 11, 17], while in Asplund spaces it was shown in [6] that the well-posedness of the minimal time function implies its Fréchet-differentiability, provided that the associated Minkowski functional is Fréchet-differentiable.

Here, we give new formulas for the Clarke subdifferential and study different concepts of differentiability of the above minimal time function. We also approach the case when $0 \notin \operatorname{int} F$. We conclude the paper with an application to a generalized location problem, deriving necessary optimality conditions via Clarke normal cones. All these results are obtained in the Banach setting.

This paper is organized as follows. After introducing the necessary notations and preliminary tools in Section 2, we give our main results in Section 3, stated in Theorems 3.2 and 3.3. Section 4 is dedicated to the minimal time problem. In the final section 5, we give an application to a location problem.

2. Notations and preliminary results

In the sequel, we consider a Banach space X endowed with a norm $\|\cdot\|$. We denote by $\mathbb{B}$ the *unit ball* in X, and by $\langle\ ,\ \rangle$ the *duality product* in $X \times X^*$. Given two proper functions $f, g\colon X \to \mathbb{R} \cup \{+\infty\}$, the *infimal convolution* of f and g is the function $f \Box g$ defined on X as

$$(f \Box g)(x) := \inf_{z \in X} \{f(z) + g(x - z)\}.$$

A sequence $(z_n)_n$ in X is called *minimizing* for $(f\Box g)(x)$, $x \in X$, if

$$\lim_{n\to\infty} \big(f(z_n) + g(x - z_n)\big) = (f\Box g)(x).$$

Definition 2.1. Fix $x \in \operatorname{dom}(f\Box g)$. We say that:

(i) $(f\Box g)(x)$ *is attained at* $z \in X$ if

$$(f\Box g)(x) = f(z) + g(x - z); \tag{2}$$

(ii) $(f\Box g)(x)$ is strongly attained at $z \in X$, if every minimizing sequence of $(f\Box g)(x)$ converges to z;

(iii) $f\Box g$ is well-posed at x if every minimizing sequence of $(f\Box g)(x)$ possesses a convergent subsequence.

Let $T := f\Box g$. Although the attainability of $T(x) = (f\Box g)(x)$ depends not only on the function T and the point x but also on the functions f and g, for the sake of brevity we shall say that $T(x)$ is *strongly attained* at z if $(f\Box g)(x)$ is strongly attained at z. Similarly, we shall say that T is *well-posed* at x if $f\Box g$ is well-posed at x. It is worth mentioning that known results about attainability of the infimal convolution are given in [12, 13, Theorems 3.5 and 4.1].

When the two functions f and g are lower-semicontinuous, (lsc, for short), then (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

For $\varepsilon \geq 0$, we define

$$\Pi_\varepsilon(x) := \{z \in X : f(z) + g(x - z) \leq (f\Box g)(x) + \varepsilon\};$$

when $\varepsilon = 0$ we simply write $\Pi(x) := \Pi_0(x)$.

When $g := \frac{1}{2\lambda}\|\ \|^2$, the function $f\Box\frac{1}{\lambda}\|\ \|^2$ is called the *Moreau envelope* of f of index λ and is denoted as $e_\lambda f$.

We shall use the following Lemma.

Lemma 2.2. *Let $f, g\colon X \to \mathbb{R}\cup\{+\infty\}$ be two proper lsc functions such that $f\Box g$ is locally Lipschitz at $x \in X$. Assume that $(f\Box g)(x)$ is strongly attained at z.*

Then
$$\limsup_{\substack{x'\to x,\\ \varepsilon\downarrow 0}} \sup_{z' \in \Pi_\varepsilon(x')} \|z' - z\| = 0.$$

Proof. If not, then there would exist $\alpha > 0$ and sequences $(x_n)_n$ and $(\varepsilon_n)_n$ converging to x and 0, respectively, together with $(z_n)_n$ such that $\|z_n - z\| \geq \alpha$ and

$$f(z_n) + g(x_n - z_n) \leq (f \square g)(x_n) + \varepsilon_n.$$

As $(f \square g)(x_n) \to (f \square g)(x)$, the strong attainment property yields the contradiction $z_n \to z$. $\square$

Let us recall the following well-known classes of subdifferentials.

(i) The *Gâteaux subdifferential* of a function $\varphi \colon X \to \mathbb{R} \cup \{+\infty\}$ at x, written $\partial_G \varphi(x)$, is the set of vectors $x^* \in X^*$ such that

$$\liminf_{t \to 0} \frac{\varphi(x + th) - \varphi(x)}{t} \geq \langle x^*, h \rangle \quad \text{for all} \quad h \in X.$$

Recall that function φ is Gâteaux-differentiable at x if there exists an element $\varphi'_G(x) \in X^*$, called the Gâteaux-derivative of φ at x, such that

$$\lim_{t \to 0} \frac{\varphi(x + th) - \varphi(x)}{t} = \langle \varphi'_G(x), h \rangle \quad \text{for all} \quad h \in X.$$

If φ is Fréchet-differentiable at x, then φ is Gâteaux-differentiable at x and $\varphi'_G(x) = \varphi'_F(x)$, the Fréchet–derivative of φ at x.

(ii) The *Fréchet subdifferential* of φ at x, written $\partial_F \varphi(x)$, is the set of vectors $x^* \in X^*$ such that

$$\liminf_{h \to 0} \frac{\varphi(x + h) - \varphi(x) - \langle x^*, h \rangle}{\|h\|} \geq 0.$$

(iii) Assume that the function φ is lsc at x. The *Clarke subdifferential* of φ at x, written $\partial_C \varphi(x)$, is the set of vectors $x^* \in X^*$ such that

$$\langle x^*, h \rangle \leq \varphi^{\uparrow}(x; h) \quad \text{for every } h \in X,$$

where $\varphi^{\uparrow}(x; \cdot)$ is the Clarke-Rockafellar generalized directional derivative given by

$$\varphi^{\uparrow}(x; h) := \sup_{\varepsilon > 0} \; \limsup_{t \downarrow 0,\, y \to_{\varphi} x} \; \inf_{\|h - v\| \leq \varepsilon} \frac{1}{t} \big(\varphi(y + tv) - \varphi(y) \big);$$

here, $y \to_{\varphi} x$ means that $y \to x$ and $\varphi(y) \to \varphi(x)$.

If φ is locally Lipschitz at x, then $\partial_C \varphi(x)$ coincides with the *Clarke generalized subdifferential*,

$$\partial_C \varphi(x) = \big\{ x^* \in X^* : \langle x^*, h \rangle \leq \varphi^{\circ}(x; h) \quad \text{for every } h \in X \big\},$$

where $\varphi^{\circ}(x; h) := \limsup_{y \to x,\ t \downarrow 0} \frac{1}{t} \big(\varphi(y + th) - \varphi(y) \big)$ is the generalized directional derivative of φ at x in the direction $h \in X$.

We also use the Michel-Penot directional derivative of φ at x, introduced in [16],

$$\varphi^{\diamond}(x;h) := \sup_{z\in X} \limsup_{t\downarrow 0} \frac{1}{t}\big(\varphi(x+tz+th)-\varphi(x+tz)\big),$$

and the corresponding *Michel-Penot subdifferential* given by

$$\partial_{MP}\varphi(x) := \big\{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^{\diamond}(x;h) \ \text{ for every } h\in X\big\}.$$

Observe that $\partial_{MP}\varphi(x)$ may be strictly contained in $\partial_C\varphi(x)$ (see, e.g., [16]); for instance, $\partial_{MP}\varphi(x)$ reduces to a singleton if and only if the function φ is Gâteaux-differentiable at x.

If φ is convex, then all the subdifferentials above coincide with the *Fenchel subdifferential*, that we simply denote by $\partial\varphi(x)$,

$$\partial\varphi(x) := \{x^*\in X^* : \varphi(x+h)-\varphi(x) \geq \langle x^*, h\rangle \ \text{ for every } h\in X\}.$$

Recall that a function $\varphi\colon X\to\mathbb{R}$ is said to be *strictly Hadamard-differentiable* at $x\in X$ (see [5, Chapter 2]) if there exists some $x^*\in X^*$ such that

$$\lim_{x'\to x,\ t\downarrow 0} \frac{\varphi(x'+th)-\varphi(x')}{t} = \langle x^*, h\rangle,$$

uniformly on compact subsets of X. The vector x^* above is called the strict (Hadamard) derivative of φ at x and is denoted by $\varphi'_{sH}(x)$. By [5, Proposition 2.2.4], a Lipschitz function φ is strictly Hadamard-differentiable at $x\in X$ with $\varphi'_{sH}(x) = x^*$ if and only if $\partial_C\varphi(x) = x^*$.

A function $\varphi\colon X\to\mathbb{R}$ is said to be *strictly Fréchet-differentiable* at $x\in X$ if there exists $x^*\in X^*$ such that

$$\lim_{x'\to x,\ h\downarrow 0} \frac{\varphi(x'+h)-\varphi(x')-\langle x^*, h\rangle}{\|h\|} = 0.$$

The vector x^* above is called the *strict Fréchet derivative* of φ at x and is denoted by $\varphi'_{sF}(x)$. It is easily seen that if φ is convex (or concave) continuous and Fréchet-differentiable at $x\in X$, then it is strictly Fréchet-differentiable at x. Hence, so is every difference of Fréchet-differentiable (at x) convex continuous functions. It is also known ([21, Theorem 10.33]) that lower-$\mathcal{C}^1$ functions defined on $\mathbb{R}^n$ are strictly Fréchet-differentiable at Fréchet-differentiability points.

We recall that a locally Lipschitz function $\varphi\colon X\to\mathbb{R}$ is said to be *regular* at x (see, e.g., [5, Definition 2.3.4]), if for all $h\in X$

$$\varphi^{\circ}(x;h) = \lim_{t\downarrow 0} \frac{1}{t}\big(\varphi(x+th)-\varphi(x)\big) \in \mathbb{R}.$$

Examples of regular functions are convex continuous functions and strictly Hadamard-differentiable functions. Notice that for this class of functions we have $\partial_G\varphi(x) = \partial_C\varphi(x)$.

The *Clarke tangent* and *normal cones* of a closed set $\Omega \subset X$ are defined by

$$T(\Omega, x) := \liminf_{\substack{x' \xrightarrow{\Omega} x, \\ \xi \downarrow 0}} \frac{\Omega - x'}{\xi}, \quad N_\Omega^C(x) := \{x^* \in H : \langle x^*, h\rangle \leq 0 \ \ \forall h \in T(\Omega, x)\}.$$

3. (Sub)differentiability of the infimal convolution

We start with the following result.

Proposition 3.1. *Let $f, g\colon X \to \mathbb{R} \cup \{+\infty\}$ be two proper functions. Given $x \in \operatorname{dom}(f \square g)$, we assume that f is Lipschitz continuous on $\cup_{x' \in B(x,\varepsilon)} \Pi_\varepsilon(x') + \varepsilon\mathbb{B}$, for some $\varepsilon > 0$. Then $f \square g$ is locally Lipschitz around x.*

Proof. Let us first check that $B(x, \varepsilon) \subset \operatorname{dom}(f \square g)$. Take $x' \in B(x, \varepsilon)$ and choose $z \in \Pi_\varepsilon(x)$ (such z exists because $x \in \operatorname{dom}(f \square g)$). Since $z + (x' - x) \in \cup_{x' \in B(x,\varepsilon)} \Pi_\varepsilon(x') + \varepsilon\mathbb{B}$ and f is Lipschitz continuous on this last set, we obtain

$$\begin{aligned}(f \square g)(x') &\leq f(z + (x' - x)) + g(x - z) \\ &\leq f(z) + L\|x' - x\| + g(x - z) \leq (f \square g)(x) + \varepsilon + L\|x' - x\| < \infty,\end{aligned}$$

where L is the Lipschitz constant of f. Now we take $x', x'' \in B(x, \frac{\varepsilon}{2})$. Then for any $0 < \eta < \varepsilon$ there exists $z'' \in \Pi_\eta(x'')$ such that

$$\begin{aligned}(f \square g)(x') - (f \square g)(x'') &\leq f(x' - x'' + z'') + g(x'' - z'') - f(z'') - g(x'' - z'') + \eta \\ &= f(x' - x'' + z'') - f(z'') + \eta \leq L\|x' - x''\| + \eta.\end{aligned}$$

This implies that $f \square g$ is locally Lipschitz at x. $\square$

The following result extends [4, Proposition 2]. The proof is an adaptation of the techniques used in that work.

Theorem 3.2. *Assume that $f, g\colon X \to \mathbb{R} \cup \{+\infty\}$ are two proper functions and fix $x \in \operatorname{dom}(f \square g)$. If $(f \square g)(x)$ is attained at z, then*

$$\partial_G(f \square g)(x) \subset \partial_G f(z) \cap \partial_G g(x - z).$$

If, in addition, g is Gâteaux (Fréchet, resp.) differentiable at the point $x - z$ and $\partial_G(f \square g)(x) \neq \emptyset$ ($\partial_F(f \square g)(x) \neq \emptyset$, resp.), then $f \square g$ is Gâteaux (Fréchet, resp.) differentiable at x and the corresponding derivative is $(f \square g)'_G(x) = g'_G(x - z)$ ($(f \square g)'_F(x) = g'_F(x - z)$, resp.).

Proof. Fix $x^* \in \partial_G(f \square g)(x)$. Then for every $h \in X$ and every $t \in (0, 1)$ we have

$$\begin{aligned}\langle x^*, h\rangle &\leq \liminf_{t \to 0} \frac{1}{t}\big((f \square g)(x + th) - (f \square g)(x)\big) \\ &\leq \liminf_{t \to 0} \frac{1}{t}\big(f(z) + g(x + th - z) - (f(z) + g(x - z))\big) \\ &\leq \liminf_{t \to 0} \frac{1}{t}\big(g(x + th - z) - g(x - z)\big),\end{aligned}$$

and therefore $x^* \in \partial_G g(x - z)$. In a similar way we show that $x^* \in \partial_G f(z)$.

Now, we assume that g is Gâteaux-differentiable at $x-z$, with $g'_G(x-z)=\zeta$, and that $\partial_G(f\square g)(x)\neq\emptyset$. Thus, $\partial_G(f\square g)(x)=\{\zeta\}$ and we have that, arguing as above,

$$\begin{aligned}\limsup_{t\to 0}\frac{1}{t}\big((f\square g)(x+th)-(f\square g)(x)\big) &\leq \limsup_{t\to 0}\frac{1}{t}\big(g(x+th-z)-g(x-z)\big)\\ &\leq \langle\zeta,h\rangle.\end{aligned}$$

Consequently, $f\square g$ is Gâteaux-differentiable at x with $(f\square g)'_G(x)=\zeta$.

The proof of the Fréchet-differentiability is similar. $\square$

We will say that $\varphi\colon X\to\mathbb{R}\cup\{+\infty\}$ is uniformly Gâteaux-differentiable on a subset U of X if φ is Gâteaux differentiable at any point $x\in U$ and for each $h\in X$ one has

$$\lim_{t\downarrow 0}\ \sup_{x\in U}\left|\frac{\varphi(x+th)-\varphi(x)}{t}-\langle\varphi'_G(x),h\rangle\right|=0.$$

Theorem 3.3. *Let $f,g\colon X\to\mathbb{R}\cup\{+\infty\}$ be two proper lsc functions. Given $x\in\operatorname{dom}(f\square g)$, we assume that f is Lipschitz continuous on $\cup_{x'\in B(x,\varepsilon)}\Pi_\varepsilon(x')+\varepsilon\mathbb{B}$, for some $\varepsilon>0$. Then the following assertions hold:*

(i) *If $(f\square g)(x)$ is attained at $z\in X$, then*

$$\partial_{MP}(f\square g)(x)\cap\partial_C f(z)\neq\emptyset.$$

(ii) *If $(f\square g)(x)$ is attained at z and $f\square g$ is regular at x, then*

$$\partial_C(f\square g)(x)\subset\partial_G f(z)\cap\partial_G g(z-x).$$

(iii) *If $f\square g$ is well-posed at x, then for some $z\in\Pi(x)$ we have that*

$$\partial_C(f\square g)(x)\subset\partial_C f(z)\cap\partial_C g(x-z).$$

(iv) *If $(f\square g)(x)$ is strongly attained at z and f is strictly Fréchet-differentiable at z, then $f\square g$ is strictly Fréchet differentiable at x and $(f\square g)'_{sF}(x)=f'_{sF}(z)$.*

(v) *If $f\square g$ is well-posed at x and f is strictly Hadamard differentiable at each point of $\Pi(x)$, then $f\square g$ is strictly Hadamard differentiable at x and $(f\square g)'_{sH}(x)=f'_{sH}(z)$, for some $z\in\Pi(x)$.*

(vi) *If $f\square g$ is well-posed at x and $\partial_C g(x-z)$ is a singleton for every $z\in\Pi(x)$, then $f\square g$ is strictly Hadamard differentiable at x and $(f\square g)'_{sH}(x)=g^{\uparrow}(x-z,\cdot)$, for some $z\in\Pi(x)$.*

(vii) *If f is uniformly Gâteaux differentiable on $\cup_{x'\in B(x,\varepsilon)}\Pi_\varepsilon(x')$, then for any $x,h\in X$*

$$(f\square g)^\circ(x;h)=\limsup_{\substack{x'\to x,\ t\downarrow 0,\\ z\in\Pi_t(x')}}\langle f'_G(z),h\rangle.$$

Proof. To show (i) we suppose that $z \in \Pi(x)$. According to Proposition 3.1, the function $f\Box g$ is Lipschitz continuous around x. Then for every $h \in X$

$$\begin{aligned}(f\Box g)^{\diamond}(x;h) &\geq \limsup_{t\to 0^+} \frac{1}{t}\big((f\Box g)(x)-(f\Box g)(x-th)\big)\\ &\geq \limsup_{t\to 0^+} \frac{1}{t}\big(f(z)+g(x-z)-f(z-th)-g(x-z)\big)\\ &= -\liminf_{t\to 0^+} \frac{1}{t}\big(f(z-th)-f(z)\big)\\ &\geq -f^{\circ}(z;-h).\end{aligned}$$

Hence, since the function $(f\Box g)^{\diamond}(x;\cdot)$ is convex, and $f^{\circ}(z;-\cdot)$ is convex and continuous, by Sandwich's Theorem (see, e.g., [3]) there is $\zeta \in X^*$ such that

$$-f^{\circ}(z;-h) \leq \langle \zeta, h\rangle \leq (f\Box g)^{\diamond}(x;h) \quad \text{for all} \quad h \in X.$$

Therefore, $\zeta \in \partial_{MP}(f\Box g)(x) \cap \partial_C f(z)$, and assertion (i) follows.

Statement (ii). From the Lipschitz continuity and the regularity of $f\Box g$ at x it follows that $\partial_G(f\Box g)(x) = \partial_C(f\Box g)(x)$. Then (ii) follows by Theorem 3.2.

To prove assertion (iii) we assume that $(f\Box g)(x)$ is well-posed at x. Take $x^* \in \partial_C(f\Box g)(x)$. Then for any fixed $\tau > 0$ and $y \in X$ one could choose $\theta = \theta(\tau)$ and sequences $(x_n)_n \subset X$ and $(t_n)_n$ converging to x and 0, respectively, such that $(f\Box g)(x_n) \to (f\Box g)(x)$ (recall that $f\Box g$ is Lipschitz) and

$$\langle x^*, y-x\rangle \leq \lim_{n\to\infty} \inf_{\|v-(y-x)\|\leq\theta} \frac{1}{t_n}\big((f\Box g)(x_n+t_n v)-(f\Box g)(x_n)\big)+\tau. \tag{3}$$

For $n \in \mathbb{N}$, let $z_n \in X$ be such that

$$(f\Box g)(x_n) \leq f(z_n)+g(x_n-z_n) < (f\Box g)(x_n)+t_n^2;$$

hence, by the current assumption we may assume (without loss of generality) that $\lim_{n\to\infty} z_n = z \in \Pi(x)$. Since f is Lipschitz continuous at z, the last inequality also ensures that $\lim_{n\to\infty} g(x_n-z_n) = g(x-z)$.

We are going to show that $x^* \in \partial_C g(z-x)$. Using (3), we write

$$\begin{aligned}&\langle x^*, y-x\rangle \leq \lim_{n\to\infty} \inf_{\|v-(y-x)\|\leq\theta} \frac{1}{t_n}\big((f\Box g)(x_n+t_n v)-(f\Box g)(x_n)\big)+\tau\\ &\leq \lim_{n\to\infty} \inf_{\|v-(y-x)\|\leq\theta} \frac{1}{t_n}\big(f(z_n)+g(x_n+t_n v-z_n)-f(z_n)-g(x_n-z_n)+t_n^2\big)+\tau\\ &= \lim_{n\to\infty} \inf_{\|v-(y-x)\|\leq\theta} \frac{1}{t_n}\big(g(x_n+t_n v-z_n)-g(x_n-z_n)\big)+\tau\\ &\leq \limsup_{t\downarrow 0, z'\to_g x-z} \inf_{\|v-(y-x)\|\leq\theta} \frac{1}{t_n}\big(g(z'+t_n v)-g(z')\big)+\tau \leq g^{\uparrow}(x-z;y-x)+\tau.\end{aligned}$$

As τ goes to 0, by the arbitrariness of y we get $x^* \in \partial_C g(x-z)$, which shows that $\partial_C(f\square g)(x) \subset \partial_C g(x-z)$. This establishes assertion (ii) because the relation $x^* \in \partial_C f(z)$ follows in a similar way.

To prove assertion (iv), suppose that $(f\square g)(x)$ is strongly attained at z and f is strictly Fréchet differentiable at z with $f'_{sF}(z) = \zeta$; that is, for every $\varepsilon > 0$ there is $\delta > 0$ such that, for all $h \in B(0,\delta)$ and $y \in B(z,\delta)$,

$$-\varepsilon\|h\| \leq f(y+h) - f(y) - \langle \zeta, h\rangle \leq \varepsilon\|h\|. \tag{4}$$

Since $f\square g$ is Lipschitz continuous around x, by Lemma 2.2 we find $0 < \eta < \delta$ such that, for all $x' \in B(x,\eta)$ and $h \in B(0,\eta)$,

$$\Pi_{\varepsilon\|h\|}(x') \subset B(z,\delta).$$

Given $x' \in B(x,\eta)$ and $h \in B(0,\eta)$, we choose $z' \in X$ such that

$$f(z') + g(x'-z') \leq (f\square g)(x') + \varepsilon\|h\|;$$

hence, $z' \in B(z,\delta)$. Then

$$\begin{aligned}(f\square g)(x'+h) - (f\square g)(x') &\leq f(z'+h) + g(x'-z') - (f(z') + g(x'-z')) + \varepsilon\|h\| \\ &= f(z'+h) - f(z') + \varepsilon\|h\|,\end{aligned}$$

and, so, by (4), $(f\square g)(x'+h) - (f\square g)(x') - \langle \zeta, h\rangle \leq 2\varepsilon\|h\|$.

Similarly, we prove that

$$(f\square g)(x'+h) - (f\square g)(x') - \langle \zeta, h\rangle \geq 2\varepsilon\|h\|,$$

for all x' close to x and h close to 0. This yields the desired strict Fréchet differentiability.

To prove assertion (v), observe that since $\partial_C(f\square g)(x) \neq \emptyset$ ($f\square g$ is Lipschitz), from (ii) we get

$$\partial_C(f\square g)(x) = \{f'_{sF}(z)\}.$$

The conclusion of (v) follows then from [5, Proposition 2.2.4].

The proof of (vi) follows the same route as the one of (v).

Finally, to prove (vii) we take an arbitrary $h \in X$. By definition we have

$$\begin{aligned}(f\square g)^\circ(x;h) &= \limsup_{x'\to x,\ t\downarrow 0} \frac{1}{t}\big((f\square g)(x'+th) - (f\square g)(x')\big) \\ &\leq \limsup_{\substack{x'\to x,\ t\downarrow 0,\\ z\in\Pi_{t^2}(x')}} \frac{1}{t}\big(f(z+th) + g(x'-z) - f(z) - g(x'-z) + t^2\big) \\ &= \limsup_{\substack{x'\to x,\ t\downarrow 0,\\ z\in\Pi_{t^2}(x')}} \frac{1}{t}\big(f(z+th) - f(z)\big) = \limsup_{\substack{x'\to x,\ t\downarrow 0,\\ z\in\Pi_{t}(x')}} \langle f'_G(z), h\rangle.\end{aligned}$$

In the same way one can see that

$$(f\Box g)^\circ(x;h) \geq \limsup_{\substack{x'\to x,\ t\downarrow 0,\\ z\in\Pi_{t^2}(x'+th)}} \langle f'_G(z),h\rangle.$$

Since $t\downarrow 0$, the condition $x'\to x$ is equivalent to $x''\to x$ with $x''=x+th$, which proves that the above two limits are equal. Thus, we obtain that

$$(f\Box g)^\circ(x;h) = \limsup_{\substack{x'\to x,\ t\downarrow 0,\\ z\in\Pi_t(x')}} \langle f'_G(z),h\rangle.$$

The proof is complete. $\square$

Remark 3.4. A formula similar to that of Theorem 3.3(ii) for the limiting subdifferential of $f\Box g$ at $x\in X$ has been given in [19] when (i) f is lsc, and Lipschitz continuous on $\operatorname{dom} f$ with constant $\ell\geq 0$, (ii) function g is lsc, subadditive, continuous at 0 with $g(0)=0$, and coercive in the following sense

$$m\|x\|\leq g(x),$$

where $m>\ell$, and (iii) x satisfies $(f\Box g)(x)=f(x)+g(0)$.

Corollary 3.5. *Let $f,g\colon X\to\mathbb{R}\cup\{+\infty\}$ be two proper lsc functions and fix $x\in\operatorname{dom}(f\Box g)$. We assume that f is locally Lipschitz at a given $z\in\Pi(x)$. Then the two assertions are equivalent*:

(i) $\partial_C(f\Box g)(x)\subset\partial_C f(z)\cap\partial_C g(x-z)$,

(ii) $(f\Box g)^\circ(x;h)\leq (f^\circ(z;\cdot)\Box g^\uparrow(x-z;\cdot))(h)$ *for all* $h\in X$.

Consequently, provided that f is Lipschitz continuous, if $f\Box g$ is well-posed at x, then there exists $z\in\Pi(x)$ such that

$$(f\Box g)^\circ(x;h)\leq (f^\circ(z;\cdot)\Box g^\uparrow(x-z;\cdot))(h) \text{ for all } h\in X.$$

Proof. We denote by $\sigma_E\colon X\to\mathbb{R}\cup\{+\infty\}$, $E\subset X^*$, the support function of E defined by

$$\sigma_E(h):=\sup_{x^*\in E}\langle h,x^*\rangle.$$

We observe that $f\Box g$ is locally Lipschitz at x, according to Proposition 3.1. Then (i) holds iff for all $h\in X$ one has that

$$(f\Box g)^\circ(x;h)=\sigma_{\partial_C(f\Box g)(x)}(h)\leq\sigma_{\partial_C f(z)\cap\partial_C g(x-z)}(h)=\operatorname{cl}(\sigma_{\partial_C f(z)}\Box\sigma_{\partial_C g(x-z)})(h),$$

where cl refers to the closure envelope. Thus, since $(f\Box g)^\circ(x;\cdot)$ is lsc, (i) holds iff for all $h\in X$

$$(f\Box g)^\circ(x;h)\leq(\sigma_{\partial_C f(z)}\Box\sigma_{\partial_C g(x-z)})(h)=(f^\circ(z;\cdot)\Box g^\uparrow(x-z;\cdot)(h),$$

which is condition (ii). The last assertion of the theorem follows, consequently, by Theorem 3.3(iii) and the Lipschitz continuity of the functions f and $f\Box g$. $\square$

Proposition 3.2 of [12], given in the following corollary, deals with the infimal convolution of an extended real-valued function f defined on X and the *Minkowski functional* ρ_G of a non-empty closed and bounded convex set $G \subset X$ satisfying $0 \in \operatorname{int} G$, defined as

$$\rho_G(u) := \inf\{t > 0 : t^{-1}u \in G\}, \quad u \in X.$$

The current assumptions on the set G ensure the Lipchitz continuity of the function ρ_G, as well as the existence of positive constants $\alpha, \beta > 0$ such that $\frac{1}{\beta}\mathbb{B} \subset G \subset \frac{1}{\alpha}\mathbb{B}$ (see [12]); hence,

$$\alpha\|x\| \leq \rho_G(x) \leq \beta\|x\| \quad \text{for all } x \in X. \tag{5}$$

Corollary 3.6. *Let f, G and α be as in the last paragraph above, and let $x \in X$ and $0 \leq \alpha' < \alpha$ be such that $f(x) = \inf(f + \alpha'\|x - \cdot\|)$.*
Assume that f is lsc. Then we have that

$$(f \square \rho_G)^\circ(x; h) \leq (f^\uparrow(x; \cdot) \square \rho_G)(h) \text{ for all } h \in X.$$

Proof. Under our assumptions we have

$$f(x) = \inf(f + \alpha'\|x - \cdot\|) \leq \inf(f + \alpha\|x - \cdot\|) \leq \inf(f + \rho_G(x - \cdot)) \leq f(x),$$

which implies that $\inf_y\{f(x) + \rho_G(x - y)\} = (f \square \rho_G)(x) = f(x)$; that is, $x \in \Pi(x)$. Moreover, $(f \square \rho_G)(x)$ is strongly attained at x and, particularly, $\Pi(x) = \{x\}$. Indeed, let $(y_n)_n \subset X$ be a minimizing sequence for $(f \square \rho_G)(x)$; that is,

$$f(y_n) + \rho_G(x - y_n) \to (f \square \rho_G)(x) = f(x). \tag{6}$$

Then, by the current assumption together with (5),

$$\begin{aligned} f(x) &\leq f(x) + (\alpha - \alpha')\|x - y_n\| \\ &= \inf(f + \alpha'\|x - \cdot\|) + (\alpha - \alpha')\|x - y_n\| \\ &\leq f(y_n) + \alpha\|x - y_n\| \leq f(y_n) + \rho_G(x - y_n), \end{aligned}$$

and so (6) ensures that $(\alpha - \alpha')\|x - y_n\| \to 0$; that is, $(y_n)_n$ converges to x and $(f \square \rho_G)(x)$ is strongly attained at x. Finally, to conclude it suffices to apply Corollary 3.6 (with ρ_G playing the role of the first function in that corollary), for all $h \in X$

$$(f \square \rho_G)^\circ(x; h) \leq (f^\uparrow(x; \cdot) \square \rho_G^\circ(0; \cdot))(h) = (f^\uparrow(x; \cdot) \square \rho_G)(h),$$

where we used the fact that $\rho_G^\circ(0; \cdot) = \rho_G$, since this last function is Lipschitz continuous and positively homogeneous. $\square$

4. Minimal time problems for constant dynamics

In this section, we consider the following optimization problem

$$\text{minimize}\,\{t \geq 0 : \Omega \cap (x + tF) \neq \emptyset\}, \tag{7}$$

where Ω is a closed *target* set, and $F \subset X$ is a closed set with $\operatorname{int} F \neq \emptyset$. In case F is convex, the optimal value in problem (7) is the minimal time needed to attain the target set Ω from x by trajectories of the differential inclusion $\dot{x}(t) \in F$. Associated to problem (7) we consider the *minimal value/time function* defined by

$$T_\Omega^F(x) := T(x) := \inf\{t > 0 : \Omega \cap (x + tF) \neq \emptyset\} = \inf_{y \in \Omega} \rho_F(y - x),$$

where ρ_F is the Minkowski functional,

$$\rho_F(u) := \inf\{t > 0 : t^{-1}u \in F\}, \quad u \in X.$$

Notice that $T_\Omega^F(x)$ coincides with the distance function $d_\Omega(x) := \inf_{y\in\Omega} \|y - x\|$ when $F = \mathbb{B}$.

The minimal time function can be seen as an infimal convolution,

$$T(x) = \inf_{y \in \Omega} \rho_F(y - x) = \inf_{y \in X} \left\{\delta_\Omega(y) + \rho_F\big(-(x - y)\big)\right\} = (\delta_\Omega \square g)(x), \tag{8}$$

where δ_Ω is the indicator function of Ω, and g is the function given by $g(v) := \rho_F(-v)$. From the definition it follows that ρ_F is positively homogeneous. A sequence $(y_n)_n$ in Ω is said to be minimizing for $T(x)$, $x \in X$, if

$$\lim_{n\to\infty} \rho_F(y_n - x) = T(x).$$

The *minimal time projection* is defined by

$$\Pi_\Omega^F(x) := \Pi(x) := \{y \in \Omega : \rho_F(y - x) = T(x)\}.$$

From Theorem 3.2 we directly get the following result.

Theorem 4.1. *Let F and Ω be nonempty closed subsets of X. If $T(x)$ is attained at $z \in X$, then $\partial_G T(x) \subset \partial_G \rho_F(x - z)$.*

If, in addition, ρ_F is Gâteaux (Fréchet, resp.) differentiable at the point $x - z$ and $\partial_G T(x) \neq \emptyset$ ($\partial_F T(x) \neq \emptyset$, resp.), then T is Gâteaux (Fréchet, resp.) differentiable at x and the corresponding derivative is $T'_G(x) = \rho'_G(x - z)$ ($T'_F(x) = \rho'_F(x - z)$, resp.).

To proceed we will consider two cases $0 \in \operatorname{int} F$ and $0 \notin \operatorname{int} F$.

The case when $0 \in \operatorname{int} F$.

To apply the main result obtained in the previous section we need the Lipschitz continuity of the function ρ_F. In general, we are interested in the Lipschitz continuity of homogeneous functions. Notice that if F is convex and $0 \in \operatorname{int} F$, then ρ_F is Lipschitz continuous on X with constant $\big(\inf_{x\in\operatorname{bd} F} \|x\|\big)^{-1}$ (see, e.g., Proposition 2.1 in [11]).

Lemma 4.2. *Let $g\colon X \to \mathbb{R}$ be a positively homogeneous function such that $|g(x)-g(y)| \leq k\|x-y\|$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$. Then g is Lipschitz continuous on X.*

Proof. Fix some $\bar{z} \in X$, $\|\bar{z}\| = 1$ and denote $q := g(\bar{z}) + 2k$. Then, by assumption, $g(z) \leq q$ for all $\|z\| = 1$. For arbitrary $x, y \in X$ we want to show that

$$g(x) - g(y) \leq (2k+q)\|x-y\|.$$

If $x = y = 0$ then the proof is obvious. If $x = 0$ and $y \neq 0$ then

$$g(y) - g(0) = \|y\| g(\frac{y}{\|y\|}) \leq q\|y\|.$$

Now let $x \neq 0$ and $y \neq 0$. Then one has

$$\begin{aligned} g(x) - g(y) &= \|x\| g(\frac{x}{\|x\|}) - \|y\| g(\frac{y}{\|y\|}) \\ &= \|x\| g(\frac{x}{\|x\|}) - \|x\| g(\frac{y}{\|y\|}) + \|x\| g(\frac{y}{\|y\|}) - \|y\| g(\frac{y}{\|y\|}) \\ &\leq k\|x\| \Big\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \Big\| + g(\frac{y}{\|y\|}) \|x-y\| \\ &\leq k\|x\| \Big\| \frac{x}{\|x\|} - \frac{y}{\|x\|} + \frac{y}{\|x\|} - \frac{y}{\|y\|} \Big\| + q\|x-y\| \\ &\leq (2k+q)\|x-y\|. \end{aligned}$$

□

The following two propositions give sufficient and necessary conditions for the function ρ_F to be Lipschitz continuous.

Proposition 4.3. *Let F be a closed subset of X. The following two assertions are equivalent:*

(i) *F is bounded, $0 \in \operatorname{int} F$ and ρ_F is Lipschitz continuous on X,*

(ii) *there is $\alpha > 0$ such that the function $x \to \frac{\rho_F(x)}{\|x\|}$ is Lipschitz continuous on the set $\{x \in X : \|x\| \geq 1\}$ and $\rho_F(x) \geq \alpha\|x\|$ for all $\|x\| \geq 1$.*

Proof. (i) ⇒ (ii). Assume that ρ_F is Lipschitz continuous with constant k. Since F is bounded and $0 \in \operatorname{int} F$, we have that

$$\frac{\|x\|}{|F|} \leq \rho_F(x) \leq \frac{1}{|F|_*}\|x\|,$$

where $|F| := \max\{\|x\| : x \in F\}$ and $|F|_* := \max\{r > 0 : r\mathbb{B} \subseteq F\}$. Thus $h(x) := \frac{\rho_F(x)}{\|x\|} \geq \frac{1}{|F|} > 0$ for all $\|x\| \geq 1$. Let $x_1, x_2 \in X$ be such that $\|x_1\|, \|x_2\| \geq 1$. Then

$$|h(x_1) - h(x_2)| \leq \Big| \frac{\rho_F(x_1)}{\|x_1\|} - \frac{\rho_F(x_2)}{\|x_2\|} \Big| \leq \frac{1}{\|x_1\|\|x_2\|} \Big| \|x_2\|\rho_F(x_1) - \|x_1\|\rho_F(x_2) \Big|$$

$$\begin{aligned}
&\leq \frac{1}{\|x_1\|\|x_2\|}\Big(\|x_2\|(\rho_F(x_1)-\rho_F(x_2))+\rho_F(x_2)(\|x_2\|-\|x_1\|)\Big)\\
&\leq \frac{1}{\|x_1\|\|x_2\|}\big(k\|x_2\|+\rho_F(x_2)\big)\|x_1-x_2\|\\
&\leq \frac{1}{\|x_1\|}\Big(k+\frac{1}{|F|_*}\Big)\|x_1-x_2\| \leq \Big(k+\frac{1}{|F|_*}\Big)\|x_1-x_2\|,
\end{aligned}$$

and so the function h is Lipschitz continuous on the set $\{x \in X : \|x\| \geq 1\}$.

(ii) $\Rightarrow$ (i). Since h is Lipschitz continuous, we put $\varepsilon := \sup\{h(z) : \|z\| = 1\} < \infty$. From the definition of ρ_F we have

$$F = \{x \in X : \rho_F \leq 1\} = \{x \in X : \|x\|h(x) \leq 1\},$$

so that $\frac{1}{\varepsilon}\mathbb{B} \subseteq F \subseteq \frac{1}{\alpha}\mathbb{B}$. On the other hand, we have $\rho_F(x) = \|x\|h(x)$ for all $x \in X\backslash\{0\}$. By Lemma 4.2 we derive that ρ_F is Lipschitz continuous on X. $\square$

Remark 4.4. The above proposition shows that a Lipschitz continuous and positively homogeneous function $g\colon X \to \mathbb{R}$ has bounded sublevel sets if and only if there exists $f\colon X \to \mathbb{R}$ a Lipschitz continuous function on X such that $f(x) \geq \alpha > 0$ for all $\|x\| \geq 1$, and

$$g(x) = \|x\|f\Big(\frac{x}{\|x\|}\Big) \text{ for all } x \in X\backslash\{0\}.$$

Proposition 4.5. *Let F and Ω be nonempty closed subsets of X with $0 \in \operatorname{int} F$. Then each one of the following conditions implies that ρ_F is Lipschitz continuous:*

(i) *$F := \{x \in X : p_2(x) - p_1(x) \leq 1\}$, where $p_1, p_2\colon X \to \mathbb{R}$ are given continuous sublinear functions such that $p_1(x) \leq p_2(x)$ for all $x \in X$.*

(ii) *$F := \operatorname{cl}\{\bigcup_{i\in I} F_i\}$, where $\{F_i\}_{i\in I}$ is a given collection of closed convex subsets with $0 \in \operatorname{int}\{\bigcap_{i\in I} F_i\}$.*

Proof. (i). From the definition of F it follows that $\rho_F(x) = p_2(x) - p_1(x)$. The continuous sublinear functions p_2 and p_1 are known to be Lipschitz continuous on X (see [20]), and thus ρ_F is Lipschitz continuous on X as well.

(ii). Assume that $\delta\mathbb{B} \subset \bigcap_{i\in I} F_i$. From the definition we know that $F_i = \{y \in X : \rho_{F_i}(y) \leq 1\}$. Then $F = \operatorname{cl}\{\bigcup_{i\in I}\{y \in X : \rho_{F_i}(y) \leq 1\}\}$. We aim to prove that

$$F = \{y \in X : \inf_{i\in I} \rho_{F_i}(y) \leq 1\}. \tag{9}$$

Let $x \in F = \operatorname{cl}\{\bigcup_{i\in I}\{y \in X : \rho_{F_i}(y) \leq 1\}\}$. Then there is a sequence $(x_n)_n$ converging to x and a sequence $(j(n))_n \subset I$ such that $\rho_{F_{j(n)}}(x_n) \leq 1$ for all $n \in \mathbb{N}$. Since ρ_{F_i} is sublinear we get

$$\rho_{F_i}(x) - \rho_{F_i}(y) \leq \rho_{F_i}(x-y) \leq \frac{1}{|F_i|_*}\|x-y\| \leq \frac{1}{\delta}\|x-y\|, \tag{10}$$

where as before $|F_i|_* = \max\{r > 0 : r\mathbb{B} \subseteq F_i\}$. Therefore

$$\inf_{i\in I}\rho_{F_i}(x) \le \inf_{n\in\mathbb{N}}\rho_{F_{j(n)}}(x) \le \inf_{n\in\mathbb{N}}\Big(\rho_{F_{j(n)}}(x_n) + \frac{1}{|F_{j(n)}|_*}\|x-x_n\|\Big)$$
$$\le \inf_{n\in\mathbb{N}}\Big(1+\frac{1}{\delta}\|x-x_n\|\Big) \le \liminf_{n\to\infty}\Big(1+\frac{1}{\delta}\|x-x_n\|\Big) \le 1.$$

For the opposite side we assume that $x \in \{y \in X : \inf_{i\in I}\rho_{F_i}(y) \le 1\}$. Then there exists a sequenece $(j(n))_n \subset I$ such that $\rho_{F_{j(n)}}(x) \le 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$, and thus

$$\rho_{F_{j(n)}}(x_n) \le 1, \quad \text{where } x_n := \frac{x}{1+\frac{1}{n}}.$$

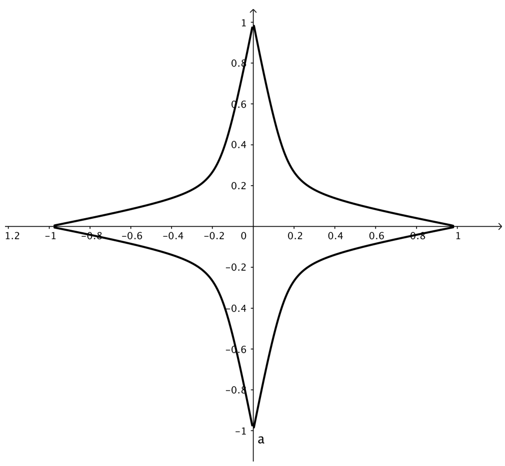
Since $x_n \to x$, we obtain that $x \in F$. This concludes the proof of (9). From this we derive that

$$\begin{aligned}\rho_F(x) &= \inf\{t>0 : x \in tF\} = \inf\{t>0 : \inf_{i\in I}\rho_{F_i}\big(\frac{x}{t}\big) \le 1\}\\ &= \inf\{t>0 : \inf_{i\in I}\rho_{F_i}(x) \le t\} = \inf_{i\in I}\rho_{F_i}(x).\end{aligned}$$

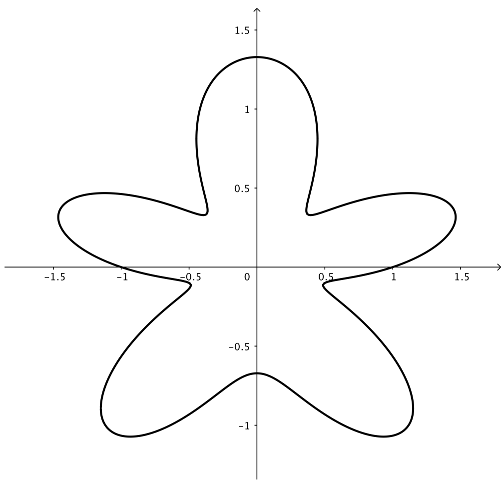
By (10), ρ_{F_i} is Lipschitz continuous with a constant $\frac{1}{\delta}$, and so ρ_F is Lipschitz continuous with a constant $\frac{1}{\delta}$ as well. □

We give two examples in $\mathbb{R}^2$ with non convex set F.

Example 4.6. In Figure 1 the set F_1 is represented as the interior of the parametric curve $F_1 := \{(x,y) \in \mathbb{R}^2 : 5|x| + 5|y| - 4\sqrt{x^2+y^2} \le 1\}$. In this case, $\rho_{F_1}(x,y) := 5|x| + 5|y| - 4\sqrt{x^2+y^2}$ and the Lipschitz continuity of ρ_{F_1} follows from Proposition 4.5 (i).



(a) Figure 1



(b) Figure 2

In Figure 2, the set F_2 is represented as the interior of the parametric curve

$$F_2 := \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \le 1 + \frac{1}{2}\sin\frac{7y}{\sqrt{x^2+y^2}}\} \cup \{(0,0)\}.$$

So, one has the following representation of ρ_{F_2}:

$$\rho_{F_2}(x,y) := \begin{cases} \sqrt{x^2+y^2}/(1+\frac{1}{2}\sin\frac{7y}{\sqrt{x^2+y^2}}) & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

We define the function h on $\mathbb{A} := \{(x,y) : x^2+y^2 \geq 1\}$ as

$$h(x,y) := \frac{\rho_{F_2}(x,y)}{\sqrt{x^2+y^2}} = 1/(1+\frac{1}{2}\sin\frac{7y}{\sqrt{x^2+y^2}}).$$

Then $h(x,y) \geq \frac{1}{2}$, and we can show that h is Lipschitz continuous.
Let $(x_1,y_1),(x_2,y_2) \in \mathbb{A}$ then

$$\begin{aligned}
\big|h(x_1,y_1)-h(x_2,y_2)\big| &\leq \Bigg|\frac{1}{1+\frac{1}{2}\sin\frac{7y_1}{\sqrt{x_1^2+y_1^2}}} - \frac{1}{1+\frac{1}{2}\sin\frac{7y_2}{\sqrt{x_2^2+y_2^2}}}\Bigg| \\
&\leq \frac{1}{2}\frac{\Big|\sin\frac{7y_2}{\sqrt{x_2^2+y_2^2}} - \sin\frac{7y_1}{\sqrt{x_1^2+y_1^2}}\Big|}{\big(1+\frac{1}{2}\sin\frac{7y_1}{\sqrt{x_1^2+y_1^2}}\big)\big(1+\frac{1}{2}\sin\frac{7y_2}{\sqrt{x_2^2+y_2^2}}\big)} \\
&\leq 2\Bigg|\frac{7y_2}{\sqrt{x_2^2+y_2^2}} - \frac{7y_1}{\sqrt{x_1^2+y_1^2}}\Bigg| \\
&\leq 2\frac{7\sqrt{x_1^2+y_1^2}|y_2-y_1| + 7|y_1|\big|\sqrt{x_1^2+y_1^2}-\sqrt{x_2^2+y_2^2}\big|}{\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}} \\
&\leq 14|y_2-y_1| + 14\big|\sqrt{x_1^2+y_1^2}-\sqrt{x_2^2+y_2^2}\big| \\
&\leq 14|y_2-y_1| + 14\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\
&\leq 28\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.
\end{aligned}$$

By Proposition 4.3 we obtain that $\rho_{F_2}(x,y)$ is Lipschitz continuous. □

Theorem 4.7. *Let Ω and F be closed subsets of X and assume that ρ_F is Lipschitz continuous on X. Then the following assertions hold true for every $x \in \operatorname{dom} T$:*

(i) *If $\Pi(x) \neq \emptyset$, then for every $y \in \Pi(x)$*

$$\partial_{MP}T(x) \cap (-\partial_C\rho_F(y-x)) \neq \emptyset.$$

(ii) *If $T(x)$ is attained at y and T is regular at x, then*

$$\partial_C T(x) \subset \partial_G\delta_\Omega(y) \cap (-\partial_G\rho_F(y-x)).$$

(iii) *If T is well-posed at x, then for some $y \in \Pi(x)$ we have that*

$$\partial_C T(x) \subset N_\Omega^C(y) \cap (-\partial_C\rho_F(y-x)). \tag{11}$$

(iv) *If $T(x)$ is strongly attained at y and ρ_F is strictly Fréchet-differentiable at $y-x$, then T is strictly Fréchet-differentiable at x and* $T'_{sF}(x) = -(\rho_F)'_{sF}(y-x)$.

(v) *If T is well-posed at x and ρ_F is strictly Hadamard-differentiable at each point of $\Pi(x)-x$, then T is strictly Hadamard-differentiable at x and* $T'_{sH}(x) = -(\rho_F)'_{sH}(y-x)$ *for some* $y \in \Pi(x)$.

(vi) *If T is well-posed at x and $N^C_\Omega(y)$ is a singleton for every $y \in \Pi(x)$, then T is strictly Hadamard-differentiable at x and $T'_{sH}(x) = \delta^\uparrow_\Omega(x-y;\cdot)$ for some $y \in \Pi(x)$.*

Proof. The proof is a direct consequence of Theorem 3.3 and Proposition 4.3. □

Remark 4.8. The last result gives an extension of Theorem 5.6 in [6] to arbitrary Banach spaces. Similar results were first established in [8] for the case of Hilbert spaces. Many other results for evaluating various subdifferentials of T in Hilbert and Banach spaces under different assumptions exist in the literature; see, e.g., [6, 11, 8]. Namely, similar formulas to the one in Theorem 4.7(ii) has been established for the limiting subdifferential of T in [17, Theorem 3.8], but using a slightly stronger assumption than the well-posedness of T at x.

Theorem 4.9. *Let F and Ω be nonempty closed subsets of X such that $0 \in \operatorname{int} F$. Assume that F satisfies (i) or (ii) of Proposition 4.5. Then assertions (i)-(vi) of Theorem 4.7 hold.*

Corollary 4.10. *Let $F \subset X$ be a closed bounded convex set with $0 \in \operatorname{int} F$. Then the following assertions hold true:*

(i) *For every $x \in \Omega$ we have $\partial_C T(x) \subset N^C_\Omega(x) \cap (-F^\circ)$, where $F^\circ := \{\xi \in X^* \mid \langle \xi, u\rangle \leq 1 \,\forall u \in F\}$ refers to the polar set of F.*

(ii) *Let $\operatorname{bd}(F^\circ)$ denote the boundary of F°. Then, for every $x \notin \Omega$ such that T is well-posed at x, there is some $y \in \Pi(x)$ such that*

$$\partial_C T(x) \subset N^C_\Omega(y) \cap (-\operatorname{bd}(F^\circ)).$$

Proof. For $x \in \Omega$ we have $\Pi(x) = x$ and $T(x) = 0$, which implies that T is well-posed at x. Indeed, if $(z_n)_n$ is a minimizing sequence for $T(x)$ then

$$0 = T(x) = \lim_{n\to\infty} \rho_F(z_n - x) \geq \lim_{n\to\infty} \frac{1}{|F|}\|z_n - x\|$$

and, consequently, $z_n \to x$. Thus, (i) follows directly from Theorem 4.7(iii) and the fact that $\partial\rho_F(0) = F^\circ$. For assertion (ii) we apply Theorem 4.7(iii) and [7, Corollary 2.3]:

$$\begin{aligned}\partial_C T(x) &\subset N^C_\Omega(y) \cap (-\partial\rho_F(y-x)) \subset N^C_\Omega(y) \cap \{\xi \in X^* : \rho_{F^\circ}(-\xi) = 1\} \\ &= N^C_\Omega(y) \cap (-\operatorname{bd}(F^\circ)).\end{aligned}$$

In the literature (e.g., [11]) one also can find other estimates for different subdifferentials of the minimal time function T by means of the corresponding normal cone to the sublevel set $\Omega_r := \{y \in X : T(y) \le r\}$, where $r := T(x),\ x \in X$. Using Theorem 4.7 we give a short proof of this result.

Corollary 4.11. ([11, Theorem 3.2]) *Let $F \subset X$ be a closed bounded convex set with $0 \in \operatorname{int} F$. Assume that T is attained and regular at $x \in X \setminus \Omega$. Then*

$$\partial_C T(x) = N^C_{\Omega_r}(x) \cap (-\mathrm{bd}(F^\circ)).$$

Proof. By Theorem 4.7 we have

$$\partial_C T(x) \subset -\partial \rho_F(y-x) \subset \{\xi \in X^* : \rho_{F^\circ}(-\xi) = 1\};$$

hence, in particular, $0 \notin \partial_C T(x)$. As T is regular at x, we know that $N^C_{\Omega_r}(x) = \cup_{\lambda \ge 0} \lambda \partial_C T(x)$ (see [5, Theorem 2.4.7]), and the opposite inclusion follows:

$$N^C_{\Omega_r}(x) \cap (-\mathrm{bd}(F^\circ)) \subset \cup_{\lambda \ge 0} \lambda \partial_C T(x) \subset \partial_C T(x). \qquad \square$$

Recall that the support function of F, namely $\sigma_F : X^* \to \mathbb{R} \cup \{+\infty\}$, is defined by $\sigma_F(x^*) := \sup_{x \in F} \langle x^*, x \rangle$. Applying [6, Theorem 4.1] we get the following result:

Corollary 4.12. *Let X be reflexive, $F \subset X$ be a closed bounded convex set with $0 \in \operatorname{int} F$ and let $x \notin \Omega$ be such that $\partial_F T(x) \neq \emptyset$. Assume that the support function of F is Fréchet differentiable at $-x^*$ for some $x^* \in \partial_F T(x)$. Then $T(x)$ is strongly attained at $y := \Pi(x)$ and*

$$\partial_C T(x) \subset N^C_\Omega(y) \cap (-\partial_C \rho_F(y-x)).$$

In particular, if ρ_F is strictly Fréchet-differentiable at $y-x$, then T is also strictly Fréchet-differentiable at x and $T'_{sF}(x) = -(\rho_F)'_{sF}(y-x)$.

Proof. We can show under the current assumptions that $T(x)$ is strongly attained at y (see Theorem 4.1 of [6], proving that the minimal time projection is a singleton, but the same proof is adapted to prove the strong attainability at y). Then the conclusion follows from Theorem 4.7(iii)–(iv). $\square$

The case when $0 \notin \operatorname{int} F$.

For nonempty subsets Ω and F we define $K := \mathbb{R}_+ F$ and

$$\mathscr{G}^F_\Omega(x) := \Omega \cap (x + K), \quad \mathscr{A}_F(\Omega) := \{x \in X : \Omega \cap (x + \operatorname{int} K) \neq \emptyset\}.$$

From these definitions we have the following properties.

Lemma 4.13. *Let F and Ω be nonempty subsets of X such that* $\operatorname{int} F \neq \emptyset$. *Then*

(i) $\mathscr{A}_F(\Omega)$ *is open,* (ii) $\mathscr{G}^F_\Omega(x) - x \subset \operatorname{dom} \rho_F$, (iii) $\mathscr{A}_F(\Omega) \subset \operatorname{int}(\operatorname{dom} T)$.

Proof. Statement (i) is direct; indeed, let $x \in \mathscr{A}_F(\Omega)$ and $y \in \Omega$ be such that $y - x \in \operatorname{int} K$. Then there is $\delta > 0$ and $v \in \operatorname{int} K$ such that $B(y - x, \delta) \subset \operatorname{int} K$, and thus we have $y - B(x, \delta) \subset \operatorname{int} K$. Consequently, $B(x, \delta) \subset \mathscr{A}_F(\Omega)$. Statement (ii) follows from the definition of ρ_F. To prove assertion (iii) we take $x \in \mathscr{A}_F(\Omega)$. Then there exists $y \in \Omega$ such that $y - x \in \operatorname{int} K$. Thus, there is a neighborhood U of x such that $y - U \subset \operatorname{int} \mathbb{R}_+ F$, and this implies that $U \subset \operatorname{dom} T$. □

Define $\Pi_\varepsilon(x) := \{y \in \Omega : \rho_F(y - x) \leq T(x) + \varepsilon\}$ and

$$\mathscr{L}_F(\Omega) := \left\{ x \in \mathscr{A}_F(\Omega) \;\middle|\; \begin{array}{l} \text{there exist } \delta_x, k_x > 0 \text{ such that for all } x_1, x_2 \in B(x, \delta_x) \\ \text{and } \varepsilon > 0,\ \Pi_\varepsilon(x_1) \cap (\mathscr{G}^F_\Omega(x_2) + k_x \|x_1 - x_2\| \mathbb{B}) \neq \emptyset \end{array} \right\}.$$

Lemma 4.14. *Let F and Ω be nonempty closed subsets of X such that* $\operatorname{int} F \neq \emptyset$ *and Ω is bounded. Then $\mathscr{L}_F(\Omega)$ is nonempty.*

Proof. Let $R > 0$ be such that $\Omega \subset R\mathbb{B}$. Let $x \in \operatorname{int} F$ and $\delta > 0$ be such that $B(x, \delta) \subset \operatorname{int} F$. Then we have

$$\Big(\Omega + B(\frac{2R}{\delta} x, R)\Big) \subset \Big(R\mathbb{B} + B(\frac{2R}{\delta} x, R)\Big) = \frac{2R}{\delta} B(x, \delta) \subset \frac{2R}{\delta} \operatorname{int} F \subset \operatorname{int} K.$$

This implies that $B(\frac{2R}{\delta} x, R) \subset \mathscr{A}_F(\Omega)$ and $\mathscr{G}^F_\Omega(x') = \Omega$ for all $x' \in B(\frac{2R}{\delta} x, R)$. We conclude that $\frac{2R}{\delta} x \in \mathscr{L}_F(\Omega)$. □

Remark 4.15. We have introduced $\mathscr{L}_F(\Omega)$ to study the Lipschitz continuity of T on $\mathscr{L}_F(\Omega)$. It is known that when F is convex with $0 \in \operatorname{int} F$ and $\Omega \neq \emptyset$, then T is Lipschitz continuous on X. As matter of fact, when $0 \in \operatorname{int} F$ we have that $K = X$ and, consequently, $\mathscr{G}^F_\Omega(x) = \Omega$, $\mathscr{A}_F(\Omega) = X$ and $\mathscr{L}_F(\Omega) = X$.

Proposition 4.16. *Let F and Ω be nonempty closed subsets of X such that* $\operatorname{int} F \neq \emptyset$ *and ρ_F is Lipschitz on its domain. Then T is locally Lipschitz at any point of $\mathscr{L}_F(\Omega)$.*

Proof. Let $x \in \mathscr{L}_F(\Omega)$. Assume that $\delta, k > 0$ are chosen as in the definition of $\mathscr{L}_F(\Omega)$. Let $x_1, x_2 \in B(x, \delta)$. Then for all $\varepsilon > 0$ there exist y_ε and z_ε that satisfy

$$y_\varepsilon \in \Pi_\varepsilon(x_1),\ z_\varepsilon \in \mathscr{G}^F_\Omega(x_2) \ \text{ with } \ \|y_\varepsilon - z_\varepsilon\| \leq k\|x_1 - x_2\|.$$

It follows from Lemma 4.13(ii) that $z_\varepsilon - x_2 \in \operatorname{dom} \rho_F$ and thus

$$\begin{aligned} T(x_2) - T(x_1) &\leq \rho_F(z_\varepsilon - x_2) - \rho_F(y_\varepsilon - x_1) + \varepsilon \leq l\|z_\varepsilon - x_2 - y_\varepsilon + x_1\| + \varepsilon \\ &\leq l\|z_\varepsilon - y_\varepsilon\| + l\|x_2 - x_1\| + \varepsilon \leq (lk + l)\|x_2 - x_1\| + \varepsilon, \end{aligned}$$

where l is the Lipschitz constant of ρ_F. Since ε is arbitrary we obtain that T is Lipschitz continuous at x. □

Theorem 4.17. *Let F and Ω be nonempty closed subsets of X such that* $\operatorname{int} F \neq \emptyset$ *and ρ_F is Lipschitz on its domain. At any point of $\mathscr{L}_F(\Omega)$, assertions* (i)–(vi) *of Theorem* 4.7 *hold.*

Proof. Let $x \in \mathscr{L}_F(\Omega)$. Then by Proposition 4.16 we know that T is Lipschitz continuous at x. Thus we can follow the proof of Theorem 3.3. □

5. Applications to location problems

We give an application of our previous results to a generalized version of the Fermat-Torricelli problem (see, e.g., [18] for other generalizations). Given a finite number of nonempty closed sets $\Omega_i \subset X$, $i \in \{0, \dots, n\}$, and nonempty bounded closed convex sets $F_i \subset X$, $i \in \{1, \dots, n\}$, such that $0 \in \cap_{i\in 1,n} \mathrm{int}\, F_i$, one is interested in finding and characterizing a point $\bar{x} \in \Omega_0$ such that the total times for the sets $\bar{x} + tF_i$ to reach the target set Ω_i is minimal. In other words, $\bar{x}$ is an optimal solution of the following optimization problem, stated in the Banach space X,

$$\min_{x\in\Omega_0} \sum_{i\in\overline{1,n}} T_i(x), \tag{12}$$

where $T_i(x) := T_{\Omega_i}^{F_i}(x) = \inf\{t \geq 0 : (x + tF_i) \cap \Omega_i \neq \emptyset\}$ is the minimal time function corresponding to the target set Ω_i and the velocity set F_i. We also denote $\Pi_i := \Pi_{\Omega_i}^{F_i}$.

Theorem 5.1. *Let $\bar{x} \in \Omega_0$ be an optimal solution of* (12) *and assume that T_i is well-posed at $\bar{x}$ for each $i \notin I(\bar{x}) := \{i \in \{1, \dots, n\} \mid \bar{x} \in \Omega_i\}$. Then there are $\bar{y}_i \in \Pi_i(\bar{x})$ $(i \in \{1, \dots, n\})$ such that*

$$0 \in N_{\Omega_0}^C(\bar{x}) + \sum_{i\in I(\bar{x})} N_{\Omega_i}^C(\bar{x}) \cap (-F_i^\circ) + \sum_{i\notin I(\bar{x})} N_{\Omega_i}^C(\bar{y}_i) \cap \{\xi \in X^* \mid \rho_{F_i^\circ}(-\xi) = 1\}.$$

Proof. As the functions T_i are Lipschitz, by [5, Corollary on p. 52 and Propositions 2.3.2 and 2.3.3] we have that

$$0 \in \partial_C\Big(\sum_{i\in\overline{1,n}} T_i + \delta_{\Omega_0}\Big)(\bar{x}) \subset \sum_{i\in\overline{1,n}} \partial_C T_i(\bar{x}) + N_{\Omega_0}^C(\bar{x}).$$

Thus, the conclusion follows by applying Corollary 4.10. □

The well-posedness condition used in the theorem above is a minimal hypothesis; for instance, it always holds when the underlying space X is finite-dimensional.

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