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Nonlinear Images of Sets.
I: Strong and Weak Convexity

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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Developing the Polyak convexity principle, we show that strong convexity of a set in a Hilbert space is preserved under a $C^{1,1}$ covering mapping provided that the parameters of the mapping and the set are related properly. We also prove that weak convexity of a set is preserved under $C^{1,1}$ bi-Lipschitz mapping and obtain exact estimates of the convexity parameters of the images of the Cartesian product of two sets depending on the parameters of the sets and the mapping.

Keywords: $C^{1,1}$ mapping, covering mapping, bi-Lipschitz mapping, strong convexity, weak convexity, the Polyak convexity principle, Minkowski sum.

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1. Introduction

Convexity is a very important and quite often critical property in solving optimization and optimal control problems. The fact that some modifications of convex optimization methods can be applied to problems possessing weak convexity stimulated investigations of weakly convex sets, which were considered in literature under different names. In Federer [11] this class of sets was considered in $\mathbb{R}^n$ under the name "sets with positive reach". Further properties of weakly convex sets in $\mathbb{R}^n$ were investigated by Vial [30]. Clarke, Stern and Wolenski [8] introduced and studied the class of proximally smooth sets in a Hilbert space, which coincides with the class of weakly convex sets. Poliquin, Rockafellar and Thibault [24] showed that this class in a Hilbert space is the same as the class of prox-regular sets introduced by Poliquin and Rockafellar [23]. In Banach spaces the prox-regular sets were investigated by Bernard, Thibault and Zlateva [5, 6].

Classes of weakly convex objects (functions and sets) are rather wide and include both convex and smooth (in the proper sense) objects. On the other hand, strong convexity sometimes gives fundamentally new results in comparison with classical

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convexity, in particular, in dealing with convergence rates of optimization algorithms [22, 25]. In some cases unconvexity of some weakly convex terms of the problem can be compensated by strong convexity of other terms of the problem, in such cases the relation between parameters of convexity is critical [3, 21]. In practice a term of an optimization (or some other) problem may be an image of a set through some smooth mapping. In this connection the following question arises: "Which properties of the mapping f and the set S are sufficient for the convexity (in some sense) of the image $f(S)$?"

Polyak in [26] showed that the image of a ball with sufficiently small radius through $C^{1,1}$ mapping between Hilbert spaces is convex provided that the center of the ball is a regular point of the mapping. This Polyak convexity principle and its applications in optimization have been developed in [7, 27, 29]. Polyak in [26] noted that the ball in the convexity principle can be replaced by a strongly convex set.

We develop this idea and obtain sufficient conditions of strong and weak convexity of the image $f(S)$ in terms of convexity parameters of the set S, wherein instead of the assumption about the regular point of the mapping f we assume that the mapping is covering (open at a linear rate) in case of strong convexity or co-Lipschitz in case of weak convexity. We obtain exact estimates of the convexity parameters of the image and show that in general the covering condition can not be substituted by the co-Lipschitz condition and vice versa, though these conditions are equivalent for a bijective mapping. Note that our results are nonlocal unlike the Polyak convexity principle. The covering property is widely used in nonlinear analysis, optimization, differential equations etc. (see e.g. [1, 2, 10, 13, 14, 15, 16]). As known, see e.g. Ioffe [13, 14], the covering property is equivalent to the metric regularity. Our result about weak convexity of the image $f(S)$ correlates with the results of Thibault [28, Theorems 9.3, 9.5] about preservation of subsmoothness for inverse images provided that the mapping is metrically subregular.

Further we obtain sufficient conditions for strong and weak convexity of the image $f(S, U)$ of the Cartesian product $S \times U$ of two sets S and U and provide exact estimates of the convexity parameters of the image in terms of convexity parameters of S, U and parameters of the mapping f. These results can be considered as nonlinear versions of known results about convexity parameters of the Minkowski sum $S + U$. On the other hand, our previously mentioned results about convexity parameters of the image $f(S)$ can be treated as degenerate cases of results about convexity parameters of $f(S, U)$. In the companion paper [19] we apply these results for nonlinear differential games and obtain sufficient conditions for the game reachable sets to be smooth.

The rest of the paper is organized as follows. In Section 2 we recall auxiliary notions and results concerning Lipschitz, co-Lipschitz, covering, $C^{1,1}$ mappings and also the signed distance. Section 3 is devoted to some known results on strongly and weakly convex sets in a Hilbert space, which are used in next sections. In Sections 4 and 5 we obtain correspondingly strong and weak convexity of the images of $C^{1,1}$ mappings.

2. Auxiliary notions and results

We use $B_r(c)$ to denote the *closed ball* in a metric space (X, ϱ_X) with center $c \in X$ and radius $r \geq 0$:

$$B_r(c) = \{x \in X : \ \varrho_X(x, c) \leq r\}.$$

Let (X, ϱ_X) and (Y, ϱ_Y) be metric spaces. For a mapping $f\colon X \to Y$ and a set $S \subset X$ we use $\operatorname{lip}(f, S)$ to denote the *Lipschitz constant* of f on S, i.e. the infimum of $L > 0$ such that

$$\varrho_Y(f(x_1), f(x_2)) \leq L\varrho_X(x_1, x_2) \qquad \forall x_1, x_2 \in S.$$

As usual, the infimum of the empty set is defined as $+\infty$.

A mapping $f\colon X \to Y$ is called *co-Lipschitz* on $S \subset X$ if there exists $\beta > 0$ such that

$$\varrho_Y(f(x_1), f(x_2)) \geq \beta\varrho_X(x_1, x_2) \qquad \forall x_1, x_2 \in S. \tag{1}$$

Denote by $\operatorname{colip}(f, S)$ the supremum of such $\beta > 0$. If no such $\beta > 0$ exists, set $\operatorname{colip}(f, S) = 0$.

If a mapping $f\colon X \to Y$ is Lipschitz continuous and co-Lipschitz on $S \subset X$, it is called *bi-Lipschitz* on S.

A mapping $f\colon X \to Y$ is called *covering (open at a linear rate)* on $S \subset X$ if there exists $\sigma > 0$ such that for every $x \in S$ and every $t > 0$, with $B_t(x) \subset S$, it holds

$$B_{\sigma t}(f(x)) \subset f(B_t(x)). \tag{2}$$

Denote by $\operatorname{sur}(f, S)$ the supremum of such $\sigma > 0$. If no such $\sigma > 0$ exists, set $\operatorname{sur}(f, S) = 0$.

Recall that $f\colon X \to Y$ is called *injection* if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ for all $x_1, x_2 \in X$ and $f\colon X \to Y$ is called *surjection* if $f(X) = Y$. If f is an injection and a surjection, it is called *bijection*.

Remark 2.1. If $f\colon X \to Y$ is co-Lipschitz on X, it is an injection. If $f\colon X \to Y$ is covering on X, it is a surjection.

For the sake of completeness of the paper, we provide the following rather obvious proposition with its proof.

Proposition 2.2. *Consider two metric spaces* (X, ϱ_X) *and* (Y, ϱ_Y) *and a function* $f\colon X \to Y$*.*

(i) *If f is an injection, then* $\operatorname{sur}(f, X) \leq \operatorname{colip}(f, X)$*.*

(ii) *If f is a surjection, then* $\operatorname{colip}(f, X) \leq \operatorname{sur}(f, X)$*.*

(iii) *If f is a bijection, then* $\operatorname{colip}(f, X) = \operatorname{sur}(f, X)$*.*

Proof. (i): If $\operatorname{sur}(f, X) = 0$, then $\operatorname{sur}(f, X) \leq \operatorname{colip}(f, X)$ holds trivially. Assume that $\operatorname{sur}(f, X) > 0$ and f is an injection. Fix some $\sigma \in (0, \operatorname{sur}(f, X))$. Then for every $x \in X$ and every $t > 0$ one has (2). Fix any $x_1, x_2 \in X$ and use (2) for $x = x_1$ and $t = \varrho_Y(f(x_1), f(x_2))/\sigma$. We have $f(x_2) \in B_{\sigma t}(f(x_1)) \subset f(B_t(x_1))$.

Hence there exists $x \in B_t(x_1)$ such that $f(x_2) = f(x)$. Since f is an injection it follows that $x_2 = x$. So, $\varrho_X(x_1, x_2) \le t = \varrho_Y(f(x_1), f(x_2))/\sigma$ and (1) holds for $\beta = \sigma$. So, $\operatorname{colip}(f, X) \ge \sigma$ for any $\sigma \in (0, \operatorname{sur}(f, X))$ and, consequently, $\operatorname{sur}(f, X) \le \operatorname{colip}(f, X)$.

(ii): If $\operatorname{colip}(f, X) = 0$, then $\operatorname{colip}(f, X) \le \operatorname{sur}(f, X)$ holds trivially. Assume that $\operatorname{colip}(f, X) > 0$ and f is a surjection. Fix any $\beta \in (0, \operatorname{colip}(f, X))$. Then (1) holds. Fix any $x \in X$, $t > 0$ and $y_1 \in B_{\beta t}(f(x))$. By surjectivety of f there exists $x_1 \in X$ such that $y_1 = f(x_1)$. Using (1) for $x_2 = x$, we obtain $\beta\varrho_X(x_1, x) \le \varrho_Y(f(x_1), f(x)) \le \beta t$. Thus, $x_1 \in B_t(x)$ and hence $y_1 \in f(B_t(x))$. This yields (2) for $\sigma = \beta$. So, $\operatorname{sur}(f, X) \ge \beta$ for any $\beta \in (0, \operatorname{colip}(f, X))$ and, consequently, $\operatorname{colip}(f, X) \le \operatorname{sur}(f, X)$.

Assertion (iii) immediately follows from (i) and (ii). □

Remark 2.3. In general $\operatorname{colip}(f, S) \ne \operatorname{sur}(f, S)$. For example, if $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $f(x_1, x_2) = x_1$ for all $(x_1, x_2) \in \mathbb{R}^2$, then $\operatorname{colip}(f, X) = 0 < 1 = \operatorname{sur}(f, X)$. On the other hand, if $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $f(x) = (x, 0)$ for all $x \in \mathbb{R}$, then $\operatorname{sur}(f, X) = 0 < 1 = \operatorname{colip}(f, X)$.

Let X and Y be Banach spaces. Denote $\mathcal{L}(X, Y)$ the Banach space of all linear continuous operators from X to Y equipped with the operator norm. Given an open set $V \subset X$, a function $f\colon V \to Y$ and a point $x_0 \in V$, we denote by $\mathcal{D}f(x_0) \in \mathcal{L}(X, Y)$ the *Fréchet derivative* of f at x_0. Let $S \subset X$. A mapping $f\colon S \to Y$ is called *Fréchet differentiable* if it can be extended on an open subset $V \subset X$ with $S \subset V$ such that the extension $\tilde{f}\colon V \to Y$ is Fréchet differentiable on S. We use $\mathcal{D}f(x_0)$ to define the Fréchet derivative $\mathcal{D}\tilde{f}(x_0)$ of the extension $\tilde{f}\colon V \to Y$ at the point $x_0 \in S$. Denote $C^{1,1}(S, Y)$ the class of Fréchet differentiable functions $f\colon S \to Y$ such that the Fréchet derivative $\mathcal{D}f\colon S \to \mathcal{L}(X, Y)$ satisfies the Lipschitz condition

$$\|\mathcal{D}f(x_1) - \mathcal{D}f(x_2)\| \le \gamma\|x_1 - x_2\| \quad \forall x_1, x_2 \in S.$$

Denote by $\operatorname{lip}(\mathcal{D}f, S)$ the infimum of such γ.

Proposition 2.4. [29, Lemma 2.7]. *Let $S \subset X$ be a convex set and consider $f \in C^{1,1}(S, Y)$. Then*

$$\left\| \frac{f(x_1) + f(x_2)}{2} - f\left(\frac{x_1 + x_2}{2}\right) \right\| \le \frac{\gamma}{8}\|x_1 - x_2\|^2 \quad \forall x_1, x_2 \in S.$$

The *distance* from $x \in X$ to $S \subset X$ is $d(x, S) = \inf_{y \in S} \|x - y\|$.

The *excess* of $A \subset X$ over $C \subset X$ and the *Hausdorff distance* between A and C are defined as

$$ex(A, C) = \sup_{a \in A} d(a, C), \quad \text{and} \quad h(A, C) = \max\{ex(A, C), ex(C, A)\}.$$

Following [9] we define the *signed (oriented) distance* from $x \in X$ to $S \subset X$ as

$$sd(x,S) = \begin{cases} \inf\limits_{y\in S} \|x-y\|, & x \in X \setminus S, \\ -\inf\limits_{y\in X\setminus S} \|x-y\|, & x \in S. \end{cases}$$

Remark 2.5. If $x \in S$, then $sd(x,S) = -\sup\{r \geq 0 : \; B_r(x) \subset S\}$.

It follows from the definitions that the signed distance function is Lipschitz continuous with constant 1 relative to the point:

$$|sd(x_1,S) - sd(x_2,S)| \leq \|x_1 - x_2\| \quad \forall x_1, x_2 \in X \quad \forall S \subset X. \tag{3}$$

The next proposition states that the signed distance function is Lipschitz continuous with constant 1 relative to the set with respect to the Hausdorff distance provided that the set is convex.

Proposition 2.6. ([20, Theorem 5].) *Let $S_1 \subset X$ be convex and $S_2 \subset X$. Then*

$$sd(x,S_1) - sd(x,S_2) \leq ex(S_2,S_1).$$

3. Strongly and weakly convex sets

Let H be a Hilbert space. We identify H with its dual and use $\langle x,y\rangle$ to denote the inner product of $x,y \in H$. As usual, we consider H as a Banach space, a metric space and a topological space with the norm, the metric and the topology induced by the inner product. For a set $S \subset H$ by $\operatorname{int} S$, $\overline{S}$ and ∂S we denote the interior, the closure and the boundary of S respectively.

Let $r > 0$. A set $S \subset H$ is called *strongly convex* with radius r if it can be represented as an intersection of closed balls with radius r, i.e. there exists $C \subset H$ such that

$$S = \bigcap_{c\in C} B_r(c).$$

Denote by $\mathcal{R}^+(S)$ the infimum of such $r > 0$.

Remark 3.1. If $S \subset H$ is strongly convex with radius r, it is closed, convex and strongly convex with any radius $r' > r$.

We define
$$\eta(t) = 1 - \sqrt{1-t^2}, \quad t \in [0,1]. \tag{4}$$

Note that $\eta(\frac{t}{2}) = 1 - \sqrt{1-\frac{t^2}{4}}$ is the modulus of convexity of a Hilbert space. It follows directly from (4) that the function $\eta\colon [0,1] \to \mathbb{R}$ is convex, increasing and

$$\eta(Lt) \geq L^2\eta(t) \qquad \forall L \geq 1, \quad \forall t \in \left[0; \frac{1}{L}\right], \tag{5}$$

and
$$\frac{t^2}{2} \leq \eta(t) \qquad \forall t \in [0;1]. \tag{6}$$

Recall that the *Fréchet normal cone* to a set $S \subset H$ at a point $x_0 \in S$ is defined as

$$N(x_0, S) = \{x^* \in H: \ \forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in S \cap B_\delta(x_0) \quad \langle x^*, x - x_0 \rangle \leq \varepsilon \|x - x_0\| \}.$$

Proposition 3.2. *Given a closed set $S \subset H$ and $r > 0$, the following assertions are equivalent:*

(i) *S is strongly convex with radius r.*

(ii) *$r \geq \mathcal{R}^+(S)$.*

(iii) *For any $x_1, x_2 \in S$ one has $\|x_1 - x_2\| \leq 2r$ and*

$$sd\left(\frac{x_1 + x_2}{2}, S\right) \leq -r\eta\left(\frac{\|x_1 - x_2\|}{2r}\right).$$

(iv) *For any $x_0 \in S$ and $p \in N(x_0, S)$ with $\|p\| = 1$ one has $S \subset B_r(x_0 - rp)$.*

A closed set $S \subset H$ is called *weakly convex* with radius $r > 0$ if for any $x_0 \in S$ and $p \in N(x_0, S)$ with $\|p\| = 1$ one has $S \cap \operatorname{int} B_r(x_0 + rp) = \emptyset$. Denote by $\mathcal{R}^-(S)$ the supremum of such $r > 0$. If no such $r > 0$ exists, set $\mathcal{R}^-(S) = 0$.

Remark 3.3. If $S \subset H$ is convex and closed, then it is weakly convex with any radius $r > 0$. If $S \subset H$ is closed and weakly convex with radius $r > 0$, it is weakly convex with any radius $r' \in (0, r)$.

Proposition 3.4. *Given a closed set $S \subset H$ and $r > 0$, the following assertions are equivalent:*

(i) *S is weakly convex with radius r.*

(ii) *$r \leq \mathcal{R}^-(S)$.*

(iii) *For any $x_1, x_2 \in S$ such that $\|x_1 - x_2\| < 2r$ one has*

$$sd\left(\frac{x_1 + x_2}{2}, S\right) \leq r\eta\left(\frac{\|x_1 - x_2\|}{2r}\right).$$

Propositions 3.2 and 3.4 are proved in [12]. Note that in Banach spaces Propositions 3.2 and 3.4 may be false. Properties of strongly and weakly convex sets in Banach spaces were studied in [4, 5, 6, 18].

The *Minkowski operations (sum and difference)* for $A, C \subset H$ are

$$A + C = \{a + c : \ a \in A, \ c \in C\}, \quad \text{and} \quad A \overset{*}{-} C = \{x \in H : \ x + C \subset A\}.$$

Proposition 3.5. ([17, Section 1.12]).

(i) *Let $A \subset H$ and $C \subset H$ be strongly convex sets. Then $A + C$ is strongly convex and*

$$\mathcal{R}^+(A + C) \leq \mathcal{R}^+(A) + \mathcal{R}^+(C).$$

(ii) *Let $A \subset H$ be weakly convex and $C \subset H$ be strongly convex, $\mathcal{R}^+(C) < \mathcal{R}^-(A)$. Then $A + C$ is closed and weakly convex,*

$$\mathcal{R}^-(A + C) \geq \mathcal{R}^-(A) - \mathcal{R}^+(C).$$

4. Strong convexity of the image

Lemma 4.1. *Let X and Y be Hilbert spaces and let $S \subset X$ be a strongly convex set. Assume that a mapping $f \in C^{1,1}(S, Y)$ is covering on S. Let*

$$r = \mathcal{R}^+(S) > 0, \quad \gamma = \operatorname{lip}(\mathcal{D}f, S), \quad \sigma = \operatorname{sur}(f, S), \quad r\gamma < \sigma.$$

Then, for all $x_1, x_2 \in S$,

$$sd\left(\frac{f(x_1) + f(x_2)}{2}, f(S)\right) \leq -r(\sigma - r\gamma)\eta\left(\frac{\|x_1 - x_2\|}{2r}\right).$$

Proof. Fix any $x_1, x_2 \in S$. Denote $\overline{f} = (f(x_1) + f(x_2))/2$, $\overline{x} = (x_1 + x_2)/2$. Proposition 3.2 implies that $\|x_1 - x_2\| \leq 2r$ and $sd(\overline{x}, S) \leq -r\eta\left(\|x_1 - x_2\|/2r\right)$. Using closedness of S we get $B_t(\overline{x}) \subset S$ with $t = r\eta\left(\|x_1 - x_2\|/2r\right)$. Since $\sigma = \operatorname{sur}(f, S)$ we get $B_{\sigma' t}(f(\overline{x})) \subset f\left(B_t(\overline{x})\right) \subset f(S)$ for all $\sigma' \in (0, \sigma)$ and hence

$$sd(f(\overline{x}), f(S)) \leq -\sigma t.$$

According to Proposition 2.4 and by (6) we have

$$\|f(\overline{x}) - \overline{f}\| \leq \frac{\gamma}{8}\|x_1 - x_2\|^2 \leq r^2\gamma\eta\left(\frac{\|x_1 - x_2\|}{2r}\right)$$

and hence by (3)

$$sd(\overline{f}, f(S)) \leq sd(f(\overline{x}), f(S)) + \|f(\overline{x}) - \overline{f}\| \leq -(\sigma r - r^2\gamma)\eta\left(\frac{\|x_1 - x_2\|}{2r}\right). \qquad \square$$

Theorem 4.2. *Let X and Y be Hilbert spaces and let $S \subset X$ be a strongly convex set. Assume that a mapping $f \in C^{1,1}(S, Y)$ is covering on S. Let $\mathcal{R}^+(S) > 0$ and $\mathcal{R}^+(S) \cdot \operatorname{lip}(\mathcal{D}f, S) < \operatorname{sur}(f, S)$. Then $f(S)$ is strongly convex and*

$$\mathcal{R}^+(f(S)) \leq \frac{\mathcal{R}^+(S) \cdot \operatorname{lip}^2(f, S)}{\operatorname{sur}(f, S) - \mathcal{R}^+(S) \cdot \operatorname{lip}(\mathcal{D}f, S)}.$$

Proof. Denote $r = \mathcal{R}^+(S)$, $\gamma = \operatorname{lip}(\mathcal{D}f, S)$, $L = \operatorname{lip}(f, S)$, and $\sigma = \operatorname{sur}(f, S)$.

To prove that $f(S)$ is closed assume the contrary: there exists a sequence $\{f_k\}$ in $f(S)$ such that $\lim_{k\to\infty} f_k = f_0 \notin f(S)$. Since $f_k \in f(S)$ for any $k \in \mathbb{N}$ one can find $x_k \in S$ such that $f_k = f(x_k)$. According to Lemma 4.1

$$sd\left(\frac{f_k + f_n}{2}, f(S)\right) \leq -r(\sigma - r\gamma)\eta\left(\frac{\|x_k - x_n\|}{2r}\right) \qquad \forall k, n \in \mathbb{N}.$$

Using (3) we get $sd\left(\frac{f_k + f_n}{2}, f(S)\right) \overset{k,n\to\infty}{\longrightarrow} sd(f_0, f(S)) \geq 0$. So,

$$\eta\left(\frac{\|x_k - x_n\|}{2r}\right) \overset{k,n\to\infty}{\longrightarrow} 0$$

and hence $\{x_k\}$ is a Cauchy sequence. Consequently, $\{x_k\}$ converges to some $x_0 \in S$ and $f_0 = \lim\limits_{k\to\infty} f(x_k) = f(x_0) \in f(S)$. The contradiction shows that $f(S)$ is closed.

Fix any $f_1, f_2 \in f(S)$. One can find $x_1, x_2 \in S$ such that $f_k = f(x_k)$, $k = 1, 2$. Lemma 4.1 implies that

$$sd\left(\frac{f_1+f_2}{2}, f(S)\right) \le -r(\sigma - r\gamma)\eta\left(\frac{\|x_1-x_2\|}{2r}\right).$$

Denote $\varrho = \frac{rL^2}{\sigma - r\gamma}$. Since $L = \mathrm{lip}(f,S)$ we have $\|f_1 - f_2\| \le L\|x_1 - x_2\| \le 2rL \le 2\varrho$ and by (5) we get

$$sd\left(\frac{f_1+f_2}{2}, f(S)\right) \le -r(\sigma - r\gamma)\eta\left(\frac{\|f_1-f_2\|}{2rL}\right) \le -\varrho\eta\left(\frac{\|f_1-f_2\|}{2\varrho}\right).$$

Thus, Proposition 3.2 implies that $\mathcal{R}^+(f(S)) \le \varrho$. $\square$

Example 4.3. Fix any positive reals r, γ, any $\sigma > r\gamma$, and $L > \sigma$.

Let $X = Y = \mathbb{R}^2$ and the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$f(x_1, x_2) = \left(Lx_1,\ \sigma x_2 + \frac{\gamma}{2}x_1^2\right), \quad (x_1, x_2) \in \mathbb{R}^2.$$

For any $\varepsilon \in (0, r)$ consider

$$S_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 :\ x_1^2 + x_2^2 \le r^2,\ x_1^2 + (x_2 - r)^2 \le \varepsilon^2\}.$$

One can see that

$$\mathcal{R}^+(S_\varepsilon) = r, \quad \mathrm{lip}(\mathcal{D}f, S_\varepsilon) = \gamma, \quad \lim_{\varepsilon\to+0} \mathrm{lip}(f, S_\varepsilon) = L, \quad \lim_{\varepsilon\to+0} \mathrm{sur}(f, S_\varepsilon) = \sigma.$$

Around the point $(0, \sigma r)$ the condition $(y_1, y_2) \in f(S_\varepsilon)$ is equivalent to the inequality

$$y_2 - \sigma r \le -\frac{\sigma - r\gamma}{2rL^2}y_1^2 + o(y_1^2),\ y_1 \to 0.$$

Hence, $\lim\limits_{\varepsilon\to+0} \mathcal{R}^+(f(S_\varepsilon)) \ge \dfrac{rL^2}{\sigma - r\gamma} = \lim\limits_{\varepsilon\to+0} \dfrac{\mathcal{R}^+(S_\varepsilon)\cdot \mathrm{lip}^2(f, S_\varepsilon)}{\mathrm{sur}(f, S_\varepsilon) - \mathcal{R}^+(S_\varepsilon)\cdot \mathrm{lip}(\mathcal{D}f, S_\varepsilon)}$.

So, the estimate obtained in Theorem 4.2 is exact.

Example 4.4. Let the mapping $f\colon \mathbb{R} \to \mathbb{R}^2$ be defined as

$$f(x) = \left(x, \frac{x^2}{2}\right), \qquad x \in \mathbb{R}.$$

For any $r > 0$ consider $S = [-r, r] \subset \mathbb{R}$. We see that $\mathcal{R}^+(S) = r$, $\mathrm{lip}(\mathcal{D}f, S) = 1$, $\mathrm{lip}(f, S) = \sqrt{1 + r^2}$, $\mathrm{sur}(f, S) = 0$, $\mathrm{colip}(f, S) = 1$, and $f(S)$ is not convex for any $r > 0$. So, the covering condition in Theorem 4.2 can not be substituted by the co-Lipschitz condition.

Define the scalar product in the Cartesian product $X \times Y$ in a standard way:

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle \quad \forall (x_1, y_1) \in X \times Y, \quad \forall (x_2, y_2) \in X \times Y.$$

Therefore, the norm in the Hilbert space $X \times Y$ will be expressed through norms in X and Y as

$$\|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2} \qquad \forall (x, y) \in X \times Y. \tag{7}$$

Lemma 4.5. *Let X, Y and Z be Hilbert spaces; let $S \subset X$ and $U \subset Y$ be strongly convex sets. Assume that $f \in C^{1,1}(S \times U, Z)$ and for all $x \in S$ and $u \in U$ the mappings $f(x, \cdot)$ and $f(\cdot, u)$ are covering on U and S correspondingly. Let*

$$r_S = \mathcal{R}^+(S) > 0, \quad r_U = \mathcal{R}^+(U) > 0, \quad \gamma = \operatorname{lip}(\mathcal{D}f, S \times U),$$

$$L_x = \sup_{u \in U} \operatorname{lip}(f(\cdot, u), S), \quad L_u = \sup_{x \in S} \operatorname{lip}(f(x, \cdot), U), \quad \sigma_x = \inf_{u \in U} \operatorname{sur}(f(\cdot, u), S),$$

$$\sigma_u = \inf_{x \in S} \operatorname{sur}(f(x, \cdot), U), \quad r_S\gamma < \sigma_x, \quad r_U\gamma < \sigma_u,$$

$$\varrho_x = \frac{r_S L_x^2}{\sigma_x - r_S\gamma}, \qquad \varrho_u = \frac{r_U L_u^2}{\sigma_u - r_U\gamma}, \qquad \varrho = \varrho_x + \varrho_u.$$

Then, for all $x_1, x_2 \in S$, $u_1, u_2 \in U$, one has $L_x\|x_1 - x_2\| + L_u\|u_1 - u_2\| \le 2\varrho$ and

$$sd\left(\frac{f(x_1, u_1) + f(x_2, u_2)}{2}, f(S, U)\right) \le -\varrho\eta\left(\frac{L_x\|x_1 - x_2\| + L_u\|u_1 - u_2\|}{2\varrho}\right).$$

Proof. Fix any $x_1, x_2 \in S$ and $u_1, u_2 \in U$. Denote

$$\overline{f} = \frac{f(x_1, u_1) + f(x_2, u_2)}{2}, \quad \overline{u} = \frac{u_1 + u_2}{2}, \quad \overline{x} = \frac{x_1 + x_2}{2}.$$

Proposition 3.2 implies that $\|x_1 - x_2\| \le 2r_S$, $\|u_1 - u_2\| \le 2r_U$ and $sd(\overline{x}, S) \le -t_x$, $sd(\overline{u}, S) \le -t_u$ with

$$t_x = r_S\eta\left(\frac{\|x_1 - x_2\|}{2r_S}\right), \quad t_u = r_U\eta\left(\frac{\|u_1 - u_2\|}{2r_U}\right).$$

Hence, $L_x\|x_1 - x_2\| + L_u\|u_1 - u_2\| \le 2L_x r_S + 2L_u r_U \le 2(\varrho_x + \varrho_u) = 2\varrho$.
Using the closedness of S and U we get $B_{t_x}(\overline{x}) \subset S$ and $B_{t_u}(\overline{u}) \subset U$.
Since $\sigma_x = \inf_{u \in U} \operatorname{sur}(f(\cdot, u), S)$ it follows that

$$B_{\sigma'_x t_x}(f(\overline{x}, u)) \subset f\left(B_{t_x}(\overline{x}), u\right) \subset f(S, u) \qquad \forall u \in U, \quad \forall \sigma'_x \in (0, \sigma_x). \tag{8}$$

Similarly,

$$B_{\sigma'_u t_u}(f(x, \overline{u})) \subset f\left(x, B_{t_u}(\overline{u})\right) \subset f(x, U) \qquad \forall x \in S, \quad \forall \sigma'_u \in (0, \sigma_u).$$

Hence, for each $\sigma'_u \in (0, \sigma_u)$,

$$\bigcup_{x\in S} B_{\sigma'_u t_u}(f(x,\overline{u})) \subset \bigcup_{x\in S} f(x,U) = f(S,U). \tag{9}$$

Since $B_{\sigma'_u t_u}(f(x,\overline{u})) = \bigcup\limits_{z\in B_{\sigma'_u t_u}(0)} (f(x,\overline{u}) + z)$ it follows that

$$\begin{aligned}\bigcup_{x\in S} B_{\sigma'_u t_u}(f(x,\overline{u})) &= \bigcup_{z\in B_{\sigma'_u t_u}(0)} \bigcup_{x\in S} (f(x,\overline{u}) + z)\\ &= \bigcup_{z\in B_{\sigma'_u t_u}(0)} (f(S,\overline{u}) + z) = f(S,\overline{u}) + B_{\sigma'_u t_u}(0).\end{aligned}$$

Thus, (9) means that

$$f(S,\overline{u}) + B_{\sigma'_u t_u}(0) \subset f(S,U) \quad \forall \sigma'_u \in (0,\sigma_u).$$

Using (8), we get for all $\sigma'_x \in (0,\sigma_x)$ and $\sigma'_u \in (0,\sigma_u)$

$$B_{\sigma'_x t_x + \sigma'_u t_u}(f(\overline{x},\overline{u})) = B_{\sigma'_x t_x}(f(\overline{x},\overline{u})) + B_{\sigma'_u t_u}(0) \subset f(S,\overline{u}) + B_{\sigma'_u t_u}(0) \subset f(S,U).$$

This means that $sd(f(\overline{x},\overline{u}), f(S,U)) \le -(\sigma_x t_x + \sigma_u t_u)$.

According to Proposition 2.4 and by (7), (6) we have

$$\begin{aligned}\|f(\overline{x},\overline{u}) - \overline{f}\| &\le \frac{\gamma}{8}(\|x_1 - x_2\|^2 + \|u_1 - u_2\|^2)\\ &\le r_S^2\gamma\eta\left(\frac{\|x_1-x_2\|}{2r_S}\right) + r_U^2\gamma\eta\left(\frac{\|u_1-u_2\|}{2r_U}\right)\end{aligned}$$

and hence by (3)

$$\begin{aligned}sd(\overline{f}, f(S,U)) &\le sd(f(\overline{x},\overline{u}), f(S,U)) + \|f(\overline{x},\overline{u}) - \overline{f}\|\\ &\le -r_S(\sigma_x - r_S\gamma)\eta\left(\frac{\|x_1-x_2\|}{2r_S}\right) - r_U(\sigma_u - r_U\gamma)\eta\left(\frac{\|u_1-u_2\|}{2r_U}\right).\end{aligned}$$

Using (5) and convexity of $\eta(\cdot)$ we get

$$\begin{aligned}sd(\overline{f}, f(S,U)) &\le -\frac{r_S^2 L_x^2}{\varrho_x}\eta\left(\frac{\|x_1-x_2\|}{2r_S}\right) - \frac{r_U^2 L_u^2}{\varrho_u}\eta\left(\frac{\|u_1-u_2\|}{2r_U}\right)\\ &\le -\varrho_x\eta\left(\frac{L_x\|x_1-x_2\|}{2\varrho_x}\right) - \varrho_u\eta\left(\frac{L_u\|u_1-u_2\|}{2\varrho_u}\right)\\ &\le -\varrho\eta\left(\frac{L_x\|x_1-x_2\| + L_u\|u_1-u_2\|}{2\varrho}\right). \qquad \square\end{aligned}$$

Theorem 4.6. *Let X, Y and Z be Hilbert spaces; let $S \subset X$ and $U \subset Y$ be strongly convex sets. Assume that $f \in C^{1,1}(S\times U, Z)$ and for all $x \in S$ and $u \in U$ the mappings $f(x,\cdot)$ and $f(\cdot,u)$ are covering on U and S correspondingly. Assume*

$$\mathcal{R}^+(S) > 0, \quad \mathcal{R}^+(S)\cdot \operatorname{lip}(\mathcal{D}f, S\times U) < \inf_{u\in U} \operatorname{sur}(f(\cdot,u),S),$$

$$\mathcal{R}^+(U) > 0, \quad \mathcal{R}^+(U)\cdot \operatorname{lip}(\mathcal{D}f, S\times U) < \inf_{x\in S} \operatorname{sur}(f(x,\cdot),U).$$

Then $f(S,U)$ is strongly convex and

$$\mathcal{R}^+(f(S,U)) \;\le\; \frac{\mathcal{R}^+(S)\cdot\sup\limits_{u\in U}\operatorname{lip}^2(f(\cdot,u),S)}{\inf\limits_{u\in U}\operatorname{sur}(f(\cdot,u),S)-\mathcal{R}^+(S)\cdot\operatorname{lip}(\mathcal{D}f,S\times U)} + \frac{\mathcal{R}^+(U)\cdot\sup\limits_{x\in S}\operatorname{lip}^2(f(x,\cdot),U)}{\inf\limits_{x\in S}\operatorname{sur}(f(x,\cdot),U)-\mathcal{R}^+(U)\cdot\operatorname{lip}(\mathcal{D}f,S\times U)}.$$

Proof. Denote $r_S=\mathcal{R}^+(S),\ r_U=\mathcal{R}^+(U),\ \gamma=\operatorname{lip}(\mathcal{D}f,S\times U)$,

$$L_x=\sup_{u\in U}\operatorname{lip}(f(\cdot,u),S),\quad L_u=\sup_{x\in S}\operatorname{lip}(f(x,\cdot),U),$$

$$\sigma_x=\inf_{u\in U}\operatorname{sur}(f(\cdot,u),S),\quad \sigma_u=\inf_{x\in S}\operatorname{sur}(f(x,\cdot),U),$$

$$\varrho_x=\frac{r_SL_x^2}{\sigma_x-r_S\gamma},\qquad \varrho_u=\frac{r_UL_u^2}{\sigma_u-r_U\gamma},\qquad \varrho=\varrho_x+\varrho_u.$$

We prove the closedness of $f(S,U)$ by contradiction. Assume that there exists a sequence $\{f_k\}$ in $f(S,U)$ such that $\lim\limits_{k\to\infty}f_k=f_0\notin f(S,U)$. Since $f_k\in f(S,U)$ for any $k\in\mathbb{N}$ one can find $x_k\in S$ and $u_k\in U$ such that $f_k=f(x_k,u_k)$. According to Lemma 4.5

$$sd\left(\frac{f_k+f_n}{2},f(S,U)\right)\le-\varrho\eta\left(\frac{L_x\|x_k-x_n\|+L_u\|u_k-u_n\|}{2\varrho}\right)\qquad\forall k,n\in\mathbb{N}.$$

Hence, in view of (3) we have

$$\begin{aligned}\limsup_{k,n\to\infty}\varrho\eta\left(\frac{L_x\|x_k-x_n\|+L_u\|u_k-u_n\|}{2\varrho}\right) &\le -\limsup_{k,n\to\infty}sd\left(\frac{f_k+f_n}{2},f(S,U)\right)\\ &= -sd(f_0,f(S,U))\le 0.\end{aligned}$$

So, $\{x_k\}$ and $\{u_k\}$ are Cauchy sequences. Consequently, $\{x_k\}$ converges to some $x_0\in S$, $\{u_k\}$ converges to some $u_0\in U$ and thus

$$f_0=\lim_{k\to\infty}f(x_k,u_k)=f(x_0,u_0)\in f(S,U).$$

The contradiction shows that $f(S,U)$ is closed.

Fix any $f_1,f_2\in f(S,U)$. Then one can find $x_1,x_2\in S$ and $u_1,u_2\in U$ such that $f_k=f(x_k,u_k)$, $k=1,2$. Lemma 4.5 implies that $L_x\|x_1-x_2\|+L_u\|u_1-u_2\|\le2\varrho$ and

$$sd\left(\frac{f_1+f_2}{2},f(S,U)\right)\le-\varrho\eta\left(\frac{L_x\|x_1-x_2\|+L_u\|u_1-u_2\|}{2\varrho}\right).$$

Since $L_x = \sup_{u\in U} \operatorname{lip}(f(\cdot,u),S)$ and $L_u = \sup_{x\in S} \operatorname{lip}(f(x,\cdot),U)$ it follows immediately that $\|f_1 - f_2\| \le L_x\|x_1 - x_2\| + L_u\|u_1 - u_2\|$. Hence, $\|f_1 - f_2\| \le 2\varrho$ and

$$sd\left(\frac{f_1+f_2}{2}, f(S,U)\right) \le -\varrho\eta\left(\frac{\|f_1-f_2\|}{2\varrho}\right).$$

Applying Proposition 3.2 yields $\mathcal{R}^+(f(S,U)) \le \varrho$. □

Remark 4.7. Theorem 4.2 can be considered as a degenerate case of Theorem 4.6 with $U = \{0\}$, $\mathcal{R}^+(U) = 0$.

Remark 4.8. Since the Cartesian product $S \times U$ of two strongly convex sets $S \subset X$ and $U \subset Y$ in general is not strongly convex in $X \times Y$ Theorem 4.6 does not follow from Theorem 4.2.

Remark 4.9. Theorem 4.6 is a generalization (a nonlinear version) of Proposition 3.5(i). Indeed, given $A \subset H$ and $C \subset H$ as in Proposition 3.5(i), applying Theorem 4.6 for $X = Y = H$, $S = A$, $U = C$ and $f(x,u) = x + u$, we obtain the assertion of Proposition 3.5(i).

Example 4.10. Fix any positive reals r_S, r_U, γ, $\sigma_x > r_S\gamma$, $\sigma_u > r_U\gamma$, $L_x > \sigma_x$, $L_u > \sigma_u$. Consider $X = Y = Z = \mathbb{R}^2$ and the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x_1,x_2,u_1,u_2) = \big(L_x x_1 + L_u u_1,\ \sigma_x x_2 + \sigma_u u_2 + \frac{\gamma}{2}(x_1^2 + u_1^2)\big),$$

for $(x_1,x_2) \in \mathbb{R}^2$ and $(u_1,u_2) \in \mathbb{R}^2$. For any $\varepsilon \in (0, \min\{r_S, r_U\})$ consider

$$S_\varepsilon = \{(x_1,x_2) \in \mathbb{R}^2 :\ x_1^2 + x_2^2 \le r_S^2,\ x_1^2 + (x_2 - r_S)^2 \le \varepsilon^2\},$$
$$U_\varepsilon = \{(u_1,u_2) \in \mathbb{R}^2 :\ u_1^2 + u_2^2 \le r_U^2,\ u_1^2 + (u_2 - r_U)^2 \le \varepsilon^2\}.$$

One can see that

$$\mathcal{R}^+(S_\varepsilon) = r_S, \quad \mathcal{R}^+(U_\varepsilon) = r_U, \quad \operatorname{lip}(\mathcal{D}f, S_\varepsilon \times U_\varepsilon) = \gamma,$$
$$\lim_{\varepsilon\to+0} \sup_{u\in U_\varepsilon} \operatorname{lip}(f(\cdot,u),S_\varepsilon) = L_x, \quad \lim_{\varepsilon\to+0} \sup_{x\in S_\varepsilon} \operatorname{lip}(f(x,\cdot),U_\varepsilon) = L_u,$$
$$\lim_{\varepsilon\to+0} \inf_{u\in U_\varepsilon} \operatorname{sur}(f(\cdot,u),S_\varepsilon) = \sigma_x, \quad \lim_{\varepsilon\to+0} \inf_{x\in S_\varepsilon} \operatorname{sur}(f(x,\cdot),U_\varepsilon) = \sigma_u.$$

Consider the function $h(y_1) = \sup\{y_2 :\ (y_1,y_2) \in f(S_\varepsilon,U_\varepsilon)\}$.

It follows from the definition of f, S_ε and U_ε that

$$\begin{aligned} h(y_1) &= \sup_{\substack{(x_1,y_1)\in S_\varepsilon,\,(x_2,y_2)\in U_\varepsilon:\\ L_x x_1 + L_u u_1 = y_1}} \left\{\sigma_x x_2 + \sigma_u u_2 + \frac{\gamma}{2}(x_1^2+u_1^2)\right\} \\ &= \sup_{\substack{x_1,u_1:\\ L_x x_1 + L_u u_1 = y_1}} \left\{\sigma_x\left(r_S - \frac{x_1^2}{2r_S}\right) + \sigma_u\left(r_U - \frac{u_1^2}{2r_U}\right) + \frac{\gamma}{2}(x_1^2+u_1^2)\right\} + o(y_1^2) \\ &= \sigma_x r_S + \sigma_u r_U - \frac{y_1^2}{2\varrho} + o(y_1^2), \quad y_1 \to 0, \end{aligned}$$

with $\varrho = \dfrac{r_S L_x^2}{\sigma_x - r_S\gamma} + \dfrac{r_U L_u^2}{\sigma_u - r_U\gamma}$. Consequently, $\lim_{\varepsilon\to+0}\ \mathcal{R}^+(f(S_\varepsilon, U_\varepsilon)) \geq \varrho$.

This example shows that the estimate of the radius of strong convexity of $f(S,U)$ obtained in Theorem 4.6 is exact.

5. Weak convexity of the image

We use $\operatorname{conv} S$ to define the convex hull of $S \subset X$.

Theorem 5.1. *Let X and Y be Hilbert spaces and $S \subset X$ be a closed and weakly convex set. Assume that a mapping $f \in C^{1,1}(\operatorname{conv} S, Y)$ is bi-Lipschitz on* $\operatorname{conv} S$. *Then $f(S)$ is closed and weakly convex, wherein*

$$\mathcal{R}^-(f(S)) \geq \frac{\mathcal{R}^-(S)\cdot \operatorname{colip}^2(f,S)}{\operatorname{lip}(f,\operatorname{conv} S) + \mathcal{R}^-(S)\cdot \operatorname{lip}(\mathcal{D}f, \operatorname{conv} S)}.$$

Proof. Since f is bi-Lipschitz on a closed set S, it follows that $f(S)$ is closed as well. Denote $r = \mathcal{R}^-(S)$, $L = \operatorname{lip}(f,\operatorname{conv} S)$, $\sigma = \operatorname{colip}(f,S)$, $\gamma = \operatorname{lip}(\mathcal{D}f,\operatorname{conv} S)$, and $\varrho = r\sigma^2/(L + r\gamma)$.

Fix $f_1, f_2 \in f(S)$ with $\|f_1 - f_2\| < 2\varrho$. One can find $x_1, x_2 \in S$ with $f_k = f(x_k)$, $k = 1,2$. Denote $\overline{f} = (f_1 + f_2)/2$, $\overline{x} = (x_1 + x_2)/2$. Since $\sigma = \operatorname{colip}(f,S)$, it follows that $\|x_1 - x_2\| \leq \|f_1 - f_2\|/\sigma < 2\varrho/\sigma \leq 2r$. According to Proposition 3.4 we have $sd(\overline{x}, S) \leq r\eta(\|x_1 - x_2\|/2r)$. Fix any $\varepsilon > 0$. One can find $\hat{x} \in S$ such that $\|\hat{x} - \overline{x}\| \leq r\eta(\|x_1 - x_2\|/2r) + \varepsilon$. In view of $L = \operatorname{lip}(f,\operatorname{conv} S)$ we get

$$sd(f(\overline{x}), f(S)) \leq \|f(\overline{x}) - f(\hat{x})\| \leq L\|\overline{x} - \hat{x}\| \leq Lr\eta\left(\frac{\|x_1 - x_2\|}{2r}\right) + L\varepsilon.$$

Passing to the limit as $\varepsilon \to +0$ we obtain $sd(f(\overline{x}), f(S)) \leq Lr\eta(\|x_1 - x_2\|/2r)$. Since $\gamma = \operatorname{lip}(\mathcal{D}f, S)$ according to Proposition 2.4 and in view of (6) we obtain $\|f(\overline{x}) - \overline{f}\| \leq \frac{\gamma}{8}\|x_1 - x_2\|^2 \leq r^2\gamma\,\eta(\|x_1 - x_2\|/2r)$. Hence by (3)

$$sd(\overline{f}, f(S)) \leq sd(f(\overline{x}), f(S)) + \|f(\overline{x}) - \overline{f}\| \leq (L + r\gamma)r\eta\left(\frac{\|x_1 - x_2\|}{2r}\right).$$

In view of $\|f_1 - f_2\| \geq \sigma\|x_1 - x_2\|$ by (5) we get

$$sd(\overline{f}, f(S)) \leq (L + r\gamma)r\eta\left(\frac{\|f_1 - f_2\|}{2r\sigma}\right) \leq \varrho\eta\left(\frac{\|f_1 - f_2\|}{2\varrho}\right).$$

Using Proposition 3.4 we complete the proof. □

Example 5.2. Fix any positive reals r, γ, σ and any $L > \sigma$.
Consider $X = Y = \mathbb{R}^2$ and the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x_1, x_2) = \left(\sigma x_1,\ Lx_2 - \frac{\gamma}{2}x_1^2\right), \quad (x_1, x_2) \in \mathbb{R}^2.$$

For any $\varepsilon \in (0, r)$ consider

$$S_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 : \; x_1^2 + x_2^2 \geq r^2, \; x_1^2 + (x_2 - r)^2 \leq \varepsilon^2\}.$$

Similarly to Example 4.3 one can see that $\mathcal{R}^-(S_\varepsilon) = r$, $\operatorname{lip}(\mathcal{D}f, \operatorname{conv} S_\varepsilon) = \gamma$, $\lim_{\varepsilon\to+0} \operatorname{lip}(f, \operatorname{conv} S_\varepsilon) = L$, $\lim_{\varepsilon\to+0} \operatorname{colip}(f, S_\varepsilon) = \sigma$, and

$$\lim_{\varepsilon\to+0} \mathcal{R}^-(f(S_\varepsilon)) \leq \frac{r\sigma^2}{L + r\gamma} = \lim_{\varepsilon\to+0} \frac{\mathcal{R}^-(S_\varepsilon) \cdot \operatorname{colip}^2(f, S_\varepsilon)}{\operatorname{lip}(f, \operatorname{conv} S_\varepsilon) + \mathcal{R}^-(S_\varepsilon) \cdot \operatorname{lip}(\mathcal{D}f, \operatorname{conv} S_\varepsilon)}.$$

So, Theorem 5.1 provides an exact estimate.

Example 5.3. Consider the covering mapping $f\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$f(x_1, x_2, x_3) = (x_1, x_3), \qquad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

One can see that the image of the weakly convex set $S = \{(\cos t, \sin t, \sin 2t) : t \in [0, 2\pi]\}$ is not weakly convex. This example shows that the co-Lipschitz condition in Theorem 5.1 can not be substituted by the covering condition.

Lemma 5.4. *Let X, Y and Z be Hilbert spaces; let $S \subset X$ be a weakly convex set and $U \subset Y$ be a strongly convex set. Assume that $f \in C^{1,1}(\operatorname{conv} S \times U, Z)$ is such that for all $u \in U$ the mapping $f(\cdot, u)$ is bi-Lipschitz on* $\operatorname{conv} S$ *and for all $x \in S$ the mapping $f(x, \cdot)$ is covering on U. Let*

$$r_S = \mathcal{R}^-(S) > 0, \quad r_U = \mathcal{R}^+(U) > 0, \quad \gamma = \operatorname{lip}(\mathcal{D}f, \operatorname{conv} S \times U),$$
$$L_x = \sup_{u\in U} \operatorname{lip}(f(\cdot, u), \operatorname{conv} S), \quad L_u = \sup_{x\in S} \operatorname{lip}(f(x, \cdot), U),$$
$$\sigma_x = \inf_{u\in U} \operatorname{colip}(f(\cdot, u), S), \quad \sigma_u = \inf_{x\in\operatorname{conv} S} \operatorname{sur}(f(x, \cdot), U),$$
$$r_U\gamma < \sigma_u, \;\; \varrho_x = \frac{r_S\sigma_x^2}{L_x + r_S\gamma}, \;\; \varrho_u = \frac{r_U L_u^2}{\sigma_u - r_U\gamma}, \;\; \varrho = \varrho_x - \varrho_u > 0, \;\; \varepsilon \in (0, \varrho).$$

Then, for any $x_1, x_2 \in S$, $u_1, u_2 \in U$ such that $\|f(x_1, u_1) - f(x_2, u_2)\| < 2\varrho$ one has

$$sd\left(\frac{f(x_1, u_1) + f(x_2, u_2)}{2}, f(S, U)\right) + \frac{\varepsilon L_u^2 r_U^2}{\varrho_u(\varrho_u + \varepsilon)}\eta\left(\frac{\|u_1 - u_2\|}{2r_U}\right)$$
$$\leq (\varrho - \varepsilon)\eta\left(\frac{\|f(x_1, u_1) - f(x_2, u_2)\|}{2(\varrho - \varepsilon)}\right).$$

Proof. Fix some $x_1, x_2 \in S$, $u_1, u_2 \in U$ such that $\|f(x_1, u_1) - f(x_2, u_2)\| < 2\varrho$. Denote $f_k = f(x_k, u_k)$, $k = 1, 2$, $\overline{f} = (f_1 + f_2)/2$, $\overline{x} = (x_1 + x_2)/2$, $\overline{u} = (u_1 + u_2)/2$, $t = r_U\eta(\|u_1 - u_2\|/2r_U)$. Since $r_U = \mathcal{R}^+(U)$ it follows by Proposition 3.2 that $\|u_1 - u_2\| \leq 2r_U$ and $B_t(\overline{u}) \subset U$. In view of $\sigma_u = \inf_{x\in\operatorname{conv} S} \operatorname{sur}(f(x, \cdot), U)$ we see that $B_{\sigma_u' t}(f(\overline{x}, \overline{u})) \subset f(\overline{x}, B_t(\overline{u})) \subset f(\overline{x}, U)$ for all $\sigma_u' \in (0, \sigma_u)$ and hence

$$sd(f(\overline{x}, \overline{u}), f(\overline{x}, U)) \leq -\sigma_u t = -\sigma_u r_U \eta\left(\frac{\|u_1 - u_2\|}{2r_U}\right).$$

According to Proposition 2.4 and in view of (6) we have

$$\begin{aligned}\|f(\overline{x},\overline{u})-\overline{f}\| &\le \frac{\gamma}{8}(\|x_1-x_2\|^2+\|u_1-u_2\|^2)\\ &\le r_S^2\gamma\eta\left(\frac{\|x_1-x_2\|}{2r_S}\right)+r_U^2\gamma\eta\left(\frac{\|u_1-u_2\|}{2r_U}\right).\end{aligned}$$

and hence by (3)

$$sd(\overline{f},f(\overline{x},U))\le -r_U(\sigma_u-r_U\gamma)\eta\left(\frac{\|u_1-u_2\|}{2r_U}\right)+r_S^2\gamma\eta\left(\frac{\|x_1-x_2\|}{2r_S}\right). \tag{10}$$

In view of definition of σ_x we have

$$\sigma_x\|x_1-x_2\|\le\|f(x_1,\overline{u})-f(x_2,\overline{u})\|\le\|f_1-f_2\|+\|f(x_1,\overline{u})-f_1\|+\|f(x_2,\overline{u})-f_2\|$$

So, by definition of L_u

$$\|x_1-x_2\|\le\frac{\|f_1-f_2\|+L_u\|u_1-u_2\|}{\sigma_x}. \tag{11}$$

Consequently, $$\|x_1-x_2\|<\frac{2\varrho+2L_ur_U}{\sigma_x}\le\frac{2\varrho_x}{\sigma_x}\le 2r_S.$$

Fix any $\delta>0$. Since S is r_S-weakly convex by Proposition 3.4 there exists $\hat{x}\in S$ such that $\|\overline{x}-\hat{x}\|\le r_S\eta(\|x_1-x_2\|/2r_S)+\delta$. According to Theorem 4.2 the set $f(\hat{x},U)$ is convex. In view of Proposition 2.6 we get

$$sd(\overline{f},f(\hat{x},U))-sd(\overline{f},f(\overline{x},U))\le ex\Big(f(\overline{x},U),f(\hat{x},U)\Big)\le L_x\|\overline{x}-\hat{x}\|,$$

where the latter inequality follows from the definition of L_x. This and (10) imply

$$sd(\overline{f},f(S,U))\le sd(\overline{f},f(\hat{x},U))\le sd(\overline{f},f(\overline{x},U))+L_x\|\overline{x}-\hat{x}\|$$

$$\le -r_U(\sigma_u-r_U\gamma)\eta\left(\frac{\|u_1-u_2\|}{2r_U}\right)+r_S(L_x+r_S\gamma)\eta\left(\frac{\|x_1-x_2\|}{2r_S}\right)+L_x\delta.$$

Passing to the limit as $\delta\to+0$, we obtain

$$sd(\overline{f},f(S,U))\le -r_U(\sigma_u-r_U\gamma)\eta\left(\frac{\|u_1-u_2\|}{2r_U}\right)+r_S(L_x+r_S\gamma)\eta\left(\frac{\|x_1-x_2\|}{2r_S}\right),$$

that is

$$sd(\overline{f},f(S,U))\le -\frac{r_U^2L_u^2}{\varrho_u}\eta\left(\frac{\|u_1-u_2\|}{2r_U}\right)+\frac{r_S^2\sigma_x^2}{\varrho_x}\eta\left(\frac{\|x_1-x_2\|}{2r_S}\right). \tag{12}$$

Denote $$\lambda=\frac{\varrho_u+\varepsilon}{\varrho_x}. \tag{13}$$

Since $0<\varepsilon<\varrho=\varrho_x-\varrho_u$ we have $\lambda\in(0,1)$. By convexity of $\eta(\cdot)$ it follows from (11) that

$$\eta\left(\frac{\|x_1-x_2\|}{2r_S}\right)\le\eta\left(\frac{\|f_1-f_2\|+L_u\|u_1-u_2\|}{2r_S\sigma_x}\right)$$
$$\le(1-\lambda)\eta\left(\frac{\|f_1-f_2\|}{2r_S\sigma_x(1-\lambda)}\right)+\lambda\eta\left(\frac{L_u\|u_1-u_2\|}{2r_S\sigma_x\lambda}\right).$$

In view of the inequalities $\varrho_x\le r_S\sigma_x$, $\frac{r_UL_u}{r_S\sigma_x}\le\frac{\varrho_u}{\varrho_x}<\lambda$ by virtue of (5) we get

$$\eta\left(\frac{\|f_1-f_2\|}{2r_S\sigma_x(1-\lambda)}\right)\le\left(\frac{\varrho_x}{r_S\sigma_x}\right)^2\eta\left(\frac{\|f_1-f_2\|}{2(1-\lambda)\varrho_x}\right),\quad\text{and}$$

$$\eta\left(\frac{L_u\|u_1-u_2\|}{2r_S\sigma_x\lambda}\right)\le\left(\frac{r_UL_u}{r_S\sigma_x\lambda}\right)^2\eta\left(\frac{\|u_1-u_2\|}{2r_U}\right).$$

Hence

$$\frac{r_S^2\sigma_x^2}{\varrho_x}\eta\left(\frac{\|x_1-x_2\|}{2r_S}\right)\le(1-\lambda)\varrho_x\eta\left(\frac{\|f_1-f_2\|}{2(1-\lambda)\varrho_x}\right)+\frac{r_U^2L_u^2}{\lambda\varrho_x}\eta\left(\frac{\|u_1-u_2\|}{2r_U}\right).$$

In view of (12), (13) this completes the proof. □

Theorem 5.5. *Let X, Y and Z be Hilbert spaces; let $S\subset X$ be a weakly convex set and $U\subset Y$ be a strongly convex set. Assume that $f\in C^{1,1}(\operatorname{conv}S\times U,Z)$ is such that for all $u\in U$ the mapping $f(\cdot,u)$ is bi-Lipschitz on $\operatorname{conv}S$ and for all $x\in S$ the mapping $f(x,\cdot)$ is covering on U. Let*

$$\mathcal{R}^-(S)>0,\ \mathcal{R}^+(U)>0,\ \inf_{x\in S}\operatorname{sur}(f(x,\cdot),U)>\mathcal{R}^+(U)\cdot\operatorname{lip}(\mathcal{D}f,\operatorname{conv}S\times U),$$

$$\varrho_x=\frac{\mathcal{R}^-(S)\cdot\inf_{u\in U}\operatorname{colip}^2(f(\cdot,u),S)}{\sup_{u\in U}\operatorname{lip}(f(\cdot,u),\operatorname{conv}S)+\mathcal{R}^-(S)\cdot\operatorname{lip}(\mathcal{D}f,\operatorname{conv}S\times U)},$$

$$\varrho_u=\frac{\mathcal{R}^+(U)\cdot\sup_{x\in S}\operatorname{lip}^2(f(x,\cdot),U)}{\inf_{x\in\operatorname{conv}S}\operatorname{sur}(f(x,\cdot),U)-\mathcal{R}^+(U)\cdot\operatorname{lip}(\mathcal{D}f,\operatorname{conv}S\times U)}.$$

$$\textit{and}\quad\varrho=\varrho_x-\varrho_u>0.$$

Then $f(S,U)$ is closed and weakly convex with $\mathcal{R}^-(f(S,U))\ge\varrho$.

Proof. Denote $r_S=\mathcal{R}^-(S)$, $r_U=\mathcal{R}^+(U)$, $\gamma=\operatorname{lip}(\mathcal{D}f,\operatorname{conv}S\times U)$,

$$L_x=\sup_{u\in U}\operatorname{lip}(f(\cdot,u),\operatorname{conv}S),\qquad L_u=\sup_{x\in S}\operatorname{lip}(f(x,\cdot),U),$$

$$\sigma_x=\inf_{u\in U}\operatorname{colip}(f(\cdot,u),S),\qquad\sigma_u=\inf_{x\in\operatorname{conv}S}\operatorname{sur}(f(x,\cdot),U).$$

Suppose that $f(S,U)$ is not closed. Then there exists a sequence $\{f_k\}$ in $f(S,U)$ such that $\lim_{k\to\infty} f_k = f_0 \notin f(S,U)$. Without loss of generality we can assume that $\|f_k - f_0\| < \varrho$ for all $k \in \mathbb{N}$. Since $f_k \in f(S,U)$ for any $k \in \mathbb{N}$ one can find $x_k \in S$, $u_k \in U$ such that $f_k = f(x_k, u_k)$. Fix any $\varepsilon \in (0,\varrho)$. According to Lemma 5.4 for any $n, k \in \mathbb{N}$

$$sd\left(\frac{f_k + f_n}{2}, f(S,U)\right) + \frac{\varepsilon L_u^2 r_U^2}{\varrho_u(\varrho_u + \varepsilon)}\eta\left(\frac{\|u_k - u_n\|}{2r_U}\right)$$
$$\le (\varrho - \varepsilon)\eta\left(\frac{\|f_k - f_n\|}{2(\varrho - \varepsilon)}\right) \overset{k,n\to\infty}{\longrightarrow} 0.$$

Using (3) we get $\quad sd\left(\dfrac{f_k + f_n}{2}, f(S,U)\right) \overset{k,n\to\infty}{\longrightarrow} sd(f_0, f(S,U)) \ge 0.$

So,
$$\eta\left(\frac{\|u_k - u_n\|}{2r_U}\right) \overset{k,n\to\infty}{\longrightarrow} 0$$

and hence $\{u_k\}$ is a Cauchy sequence. Consequently, $\{u_k\}$ converges to some $u_0 \in U$. Using the definitions of σ_x and L_u we get

$$\sigma_x\|x_k - x_n\| \le \|f(x_k,u_0) - f(x_n,u_0)\|$$
$$\le \|f_k - f_n\| + L_u(\|u_k - u_0\| + \|u_n - u_0\|) \overset{k,n\to\infty}{\longrightarrow} 0.$$

Thus, $\{x_k\}$ is a Cauchy sequence and consequently converges to some $x_0 \in S$. The relations

$$f_0 = \lim_{k\to\infty} f_k = \lim_{k\to\infty} f(x_k,u_k) = f(x_0,u_0) \in f(S,U)$$

contradict the assumption that $f_0 \notin f(S,U)$. This proves that $f(S,U)$ is closed.

Now fix any $f_1, f_2 \in f(S,U)$ such that $\|f_1 - f_2\| < 2\varrho$. One can find $u_1, u_2 \in U$ and $x_1, x_2 \in S$ such that $f_k = f(x_k,u_k)$, $k = 1,2$. According to Lemma 5.4 for any $\varepsilon \in (0,\varrho)$ we have

$$sd\left(\frac{f_1 + f_2}{2}, f(S,U)\right) \le (\varrho - \varepsilon)\eta\left(\frac{\|f_1 - f_2\|}{2(\varrho - \varepsilon)}\right).$$

Passing to the limit as $\varepsilon \to +0$, we obtain

$$sd\left(\frac{f_1 + f_2}{2}, f(S,U)\right) \le \varrho\eta\left(\frac{\|f_1 - f_2\|}{2\varrho}\right).$$

By Proposition 3.4 we get that $f(S,U)$ is weakly convex and $\mathcal{R}^-(f(S,U)) \ge \varrho$. $\square$

Remark 5.6. Theorem 4.2 can be considered as a degenerate case of Theorem 5.5 with $S = \{0\}$ and S in place of U, Theorem 5.1 is a degenerate case of the same theorem with $U = \{0\}$.

Remark 5.7. Since the mapping $f\colon \operatorname{conv} S \times U \to Z$ in Theorem 5.5 is not co-Lipschitz, Theorem 5.5 does not follow from Theorem 5.1.

Remark 5.8. Theorem 5.5 is a generalization (a nonlinear version) of Proposition 3.5(ii). Indeed, given $A \subset H$ and $C \subset H$ as in Proposition 3.5(ii), applying Theorem 5.5 for $X = Y = H$, $S = A$, $U = C$ and $f(x, u) = x + u$, we obtain the assertion of Proposition 3.5(ii).

Remark 5.9. Fix any positive reals r_S, r_U, γ, σ_x, σ_u, $L_x > \sigma_x$, $L_u > \sigma_u$ such that $r_U\gamma < \sigma_u$ and

$$\frac{r_S\sigma_x^2}{L_x + r_S\gamma} = \varrho_x > \varrho_u = \frac{r_U L_u^2}{\sigma_u - r_U\gamma}.$$

Consider $X = Y = Z = \mathbb{R}^2$ and the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x_1, x_2, u_1, u_2) = \Big(\sigma_x x_1 + L_u u_1,\ L_x x_2 - \sigma_u u_2 - \frac{\gamma}{2}(x_1^2 + u_1^2)\Big),$$

for $(x_1, x_2) \in \mathbb{R}^2$ and $(u_1, u_2) \in \mathbb{R}^2$. For any $\varepsilon \in (0, \min\{r_S, r_U\})$ consider

$$S_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 :\ x_1^2 + x_2^2 \geq r_S^2,\ x_1^2 + (x_2 - r_S)^2 \leq \varepsilon^2\},$$

$$U_\varepsilon = \{(u_1, u_2) \in \mathbb{R}^2 :\ u_1^2 + u_2^2 \leq r_U^2,\ u_1^2 + (u_2 - r_U)^2 \leq \varepsilon^2\}.$$

Similarly to Example 4.10 one can see that

$$\mathcal{R}^-(S_\varepsilon) = r_S, \quad \mathcal{R}^+(U_\varepsilon) = r_U, \quad \operatorname{lip}(\mathcal{D}f, \operatorname{conv} S_\varepsilon \times U_\varepsilon) = \gamma,$$

$$\lim_{\varepsilon\to+0} \sup_{u\in U_\varepsilon} \operatorname{lip}(f(\cdot, u), \operatorname{conv} S_\varepsilon) = L_x, \quad \lim_{\varepsilon\to+0} \sup_{x\in S_\varepsilon} \operatorname{lip}(f(x, \cdot), U_\varepsilon) = L_u,$$

$$\lim_{\varepsilon\to+0} \inf_{u\in U_\varepsilon} \operatorname{colip}(f(\cdot, u), S_\varepsilon) = \sigma_x, \quad \lim_{\varepsilon\to+0} \inf_{x\in\operatorname{conv} S_\varepsilon} \operatorname{sur}(f(x, \cdot), U_\varepsilon) = \sigma_u,$$

$$\lim_{\varepsilon\to+0} \mathcal{R}^-(f(S_\varepsilon, U_\varepsilon)) \leq \varrho_x - \varrho_u.$$

This example shows that the estimate of the radius of strong convexity of $f(S, U)$ obtained in Theorem 5.5 is exact.

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