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On Characterizations of Metric Regularity of Multi-Valued Maps

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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We provide a new proof along the lines of the recent book of A. Ioffe [*Variational Analysis of Regular Mappings: Theory and Applications*, Springer Monographs in Mathematics (2017)] of a result of H. Frankowska [*Some inverse mapping theorems*, Ann. Inst. H. Poincaré, Anal. Non Linéaire 7 (1990) 183–234] showing that metric regularity of a multi-valued map can be characterized by regularity of its contingent variation – a notion extending contingent derivative.

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1. Introduction

Metric regularity, as well as, the equivalent to it linear openness and pseudo-Lipschitz property of the inverse, are very important concepts in Variational Analysis. They have been intensively studied as it can be seen in a number of recent monographs, e.g. [1, 3, 7, 8] and the references therein. A very rich and instructive survey on metric regularity is the book of A. Ioffe [6].

It may be noted in Chapter V of [6] that the modulus of regularity of a multi-valued map between Banach spaces is estimated in terms of the tangential cones to its graph. The estimates are precise, but they are not characteristic. This is because in infinite dimensions a map may well be regular and the tangential cones to its graph be insufficiently informative, for details see [5].

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In [4] H. Frankowska introduced the notion of contingent variation of a multi-valued map which extends Bouligand tangential cone. This notion precisely characterizes metric regularity.

Let (X, d) and (Y, d) be metric spaces and let $F\colon X \rightrightarrows Y$ be a multi-valued map. For $V \subset Y$ the restriction F^V is defined by

$$F^V(x) := F(x) \cap V, \quad \forall x \in X,$$

see [6, p.54]. The properties related to the so restricted map are called *restricted*.

For example, the multi-valued map $F\colon X \rightrightarrows Y$ is called *restrictedly Milyutin regular* on (U, V), where $U \subset X$ and $V \subset Y$, if there exists a number $r > 0$ such that

$$B(v, rt) \cap V \subset F(B(x, t))$$

whenever $(x, v) \in \operatorname{Gr} F \cap (U \times V)$ and $t < d(x, X \setminus U)$, where $B(x, t)$ is the closed ball with center x and radius t: $B(x, t) := \{u \in X : \ d(u, x) \le t\}$, and $\operatorname{Gr} F = \{(x, v) : \ v \in F(x)\}$.

The supremum of all such r is called *modulus of surjection*, denoted by

$$\operatorname{sur}_m F^V(U|V).$$

By convention, $\operatorname{sur}_m F^V(U|V) = 0$ means that F is not restrictedly Milyutin regular on (U, V).

This notion taken from [6] is explained in great detail in Section 2 below.

In the literature, e.g. [6, Section 5.2], there are various estimates of $\operatorname{sur}_m F^V(U|V)$ and related moduli in terms of derivative-like objects. Unlike the so called *co-derivative criterion*, see [6, Section 5.2.3], most of the *primal* estimates are not characteristic in general. Here we re-establish one primal criterion which complements [6, Section 5.2.2] and is, moreover, characteristic. It is essentially done by H. Frankowska in [4], see also [5]. There a new derivative-like object is defined as follows.

Let (X, d) be a metric space, $(Y, \|\cdot\|)$ be a Banach space, $F\colon X \rightrightarrows Y$ be a multi-valued map. For $(x, y) \in \operatorname{Gr} F$ the *contingent variation* of F at (x, y) is the closed set

$$F^{(1)}(x, y) := \limsup_{t \downarrow 0} \frac{F(B(x, t)) - y}{t},$$

where $\limsup$ stands for the Kuratowski limit superior of sets.

Equivalently, $v \in F^{(1)}(x, y)$ exactly when there exist a sequence of reals $t_n \downarrow 0$ and a sequence $(x_n, y_n) \in \operatorname{Gr} F$ such that $d(x, x_n) \le t_n$ and

$$\left\| v - \frac{y_n - y}{t_n} \right\| \to 0, \text{ when } n \to \infty.$$

This notion extends the so-called contingent, or graphical, derivative usually denoted by $DF(x, y)$, e.g. [6, pp.163, 202].

Our main result can now be stated. As usual, B_Y denotes the closed unit ball of the Banach space $(Y, \|\cdot\|)$: $B_Y := \{y \in Y : \|y\| \leq 1\}$.

Theorem 1.1. *Let (X, d) be a metric space and $(Y, \|\cdot\|)$ be a Banach space, let $U \subset X$ and $V \subset Y$ be non-empty open sets.*

Let $F\colon X \rightrightarrows Y$ be a multi-valued map with complete graph. F is restrictedly Milyutin regular on (U, V) with $\mathrm{sur}_m F^V(U|V) \geq r > 0$ if and only if

$$F^{(1)}(x, v) \supset rB_Y \quad \textit{for all } (x, v) \in \mathrm{Gr}\, F \cap (U \times V). \tag{1}$$

This result is essentially established by H. Frankowska in [4, Theorem 6.1 and Corollary 6.2]. However, there it is presented as a characterization of local modulus of regularity in terms of the local variant of the condition (1). Here we render the characterization global. The technique in [4] is different, but it also depends on Ekeland Variational Principle.

The rest of the article is organized as follows. In Section 2 we provide for reader's convenience the relevant material from [6]. We also present in another form the first criterion for Milyutin regularity from [6]. In Section 3 we prove Theorem 1.1.

2. Milyutin regularity

Let (X, d) and (Y, d) be metric spaces. Let $U \subset X$ and $V \subset Y$, let $F\colon X \rightrightarrows Y$ be a multi-valued map and let $\gamma(\cdot)$ be an extended real-valued function on X assuming positive values (possibly infinite) on U.

Definition 2.1. (Linear openness, [6, Definition 2.21]) F is said to be *γ-open at linear rate* on (U, V) if there is an $r > 0$ such that

$$B(F(x), rt) \cap V \subset F(B(x, t)),$$

if $x \in U$ and $t < \gamma(x)$, i.e.

$$B(v, rt) \cap V \subset F(B(x, t)),$$

whenever $(x, v) \in \mathrm{Gr}\, F$, $x \in U$ and $t < \gamma(x)$.

Denote by $\mathrm{sur}_\gamma F(U|V)$ the upper bound of all such $r > 0$ and call it *modulus of γ-surjection* of F on (U, V). If no such r exists, set $\mathrm{sur}_\gamma F(U|V)$=0.

Definition 2.2. (Metric regularity, [6, Definition 2.22]) F is said to be *γ-metrically regular* on (U, V) if there is $\kappa > 0$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)),$$

provided $x \in U$, $y \in V$ and $\kappa d(y, F(x)) < \gamma(x)$.

Denote by $\mathrm{reg}_\gamma F(U|V)$ the lower bound of all such $\kappa > 0$ and call it *modulus of γ-metric regularity* of F on (U, V). If no such κ exists, set $\mathrm{reg}_\gamma F(U|V) = \infty$.

Theorem 2.3. (Equivalence Theorem, [6, Theorem 2.25]) *The following are equivalent for any metric spaces X, Y, any $F\colon X \rightrightarrows Y$, any $U \subset X$, $V \subset Y$ and any extended real-valued function $\gamma(\cdot)$ which is positive on U:*

(a) *F is γ-open at linear rate on (U, V);*

(b) *F is γ-metrically regular on (U, V).*

Moreover (under the convention $0.\infty = 1$), $\operatorname{sur}_\gamma F(U|V).\operatorname{reg}_\gamma F(U|V) = 1.$

Definition 2.4. (Regularity, [6, Definition 2.26]) We say that $F\colon X \rightrightarrows Y$ is γ-*regular* on (U, V) if the equivalent properties of Theorem 2.3 are satisfied.

Definition 2.5. (Milyutin regularity, [6, Definition 2.28]) Set

$$m_U(x) := d(x, X \setminus U),$$

where $d(x, X \setminus U) := \inf\{d(x, y) : y \in X \setminus U\}$. We say that F is *Milyutin regular* on (U, V) if it is γ-regular on (U, V) with $\gamma(x) = m_U(x)$.

We need also *Ekeland Variational Principle* (see [9, p.45]): Let (M, d) be a complete metric space, and $f\colon M \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and bounded from below function. Assume that $f(\overline{x}) \le \inf f + \lambda\varepsilon$ for some $\overline{x} \in M$ and $\lambda\varepsilon > 0$. Then there is $\overline{y} \in M$ such that

(i) $f(\overline{y}) \le f(\overline{x}) - \lambda d(\overline{x}, \overline{y})$;

(ii) $d(\overline{x}, \overline{y}) \le \varepsilon$;

(iii) $f(x) + \lambda d(x, \overline{y}) \ge f(\overline{y})$, for all $x \in M$.

The following characterization of Milyutin regularity is very similar in form (in fact equivalent) to the so called *first criterion for Milyutin regularity*, see [6, Theorem 2.47]. It is also similar to [2, Proposition 2.2], but there it is stated in local form. We present here a proof for reader's convenience.

Following [6, p.35] for $\xi > 0$ we denote by d_ξ the product metric

$$d_\xi((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), \xi d(y_1, y_2)\}, \tag{2}$$

where $x_i \in X$, $y_i \in Y$, $i = 1, 2$, and (X, d) and (Y, d) are metric spaces.

Theorem 2.6. *Let (X, d), (Y, d) be metric spaces. Let $F\colon X \rightrightarrows Y$ be a multivalued map with complete graph. Let $U \subset X$ and $V \subset Y$. Then*

$$\operatorname{sur}_m F(U|V) = \sup \left\{ r > 0 : \begin{array}{c} \exists\, \xi > 0 \text{ such that } \forall (x, v) \in \operatorname{Gr} F,\ x \in U,\ y \in V \\ \text{satisfying } 0 < d(y, v) < r m_U(x),\ \exists (u, w) \in \operatorname{Gr} F \\ \text{such that } d(y, w) < d(y, v) - r d_\xi((x, v), (u, w)) \end{array} \right\}. \tag{3}$$

Proof. We denote by s_1 the left hand side of equation (3), i.e. $s_1 := \operatorname{sur}_m F(U|V)$. In other words,

$$s_1 = \sup\{r \ge 0 : B(v, rt) \cap V \subset F(B(x, t)),\ \forall (x, v) \in \operatorname{Gr} F,\ x \in U, t < m_U(x)\}.$$

Denote by s_2 the right hand side of equation (3). We need to show that $s_1 = s_2$.

First, we will show that $s_1 \le s_2$.

If $s_1 = 0$ we have nothing to prove.

Let $s_1 > 0$. Take $0 < r < r' < s_1$. Let $x \in U$, $v \in F(x)$ be fixed. Let $y \in V$ be such that $0 < d(y, v) < rm_U(x)$. In particular $0 < d(y, v) < r'm_U(x)$. Set $t := d(y, v)/r'$. Then $t < m_U(x)$. By the inequality $r' < s_1 = \mathrm{sur}_m F(U|V)$ and by the definition of $\mathrm{sur}_m F(U|V)$ it holds that $y \in B(v, r't) \cap V \subset F(B(x, t))$, i.e. $y \in F(B(x, t))$. So, there exists $u \in B(x, t)$ such that $y \in F(u)$.

Fix ξ such that $0 < \xi r' < 1$. Then

$$d_\xi((x, v), (u, y)) = \max\{d(x, u), \xi d(v, y)\} \le \max\{t, \xi r't\} = t \max\{1, \xi r'\} = t,$$

so
$$r'd_\xi((x, v), (u, y)) \le r't = d(y, v).$$

Observe that $d_\xi((x, v), (u, y)) > 0$ since $d(v, y) > 0$. The latter and the inequality $r' > r$ yield

$$rd_\xi((x, v), (u, y)) < r't = d(y, v),$$

hence
$$0 < d(y, v) - rd_\xi((x, v), (u, y)).$$

Since $0 = d(y, y)$ we get that

$$d(y, y) < d(y, v) - rd_\xi((x, v), (u, y))$$

and (3) holds with $w = y$ as $(u, y) \in \mathrm{Gr}\, F$. This means that $r \le s_2$. Finally, $s_1 \le s_2$.

Second, we will prove that $s_2 \le s_1$.

If $s_2 = 0$ we have nothing to prove.

Let now $s_2 > 0$. Let $0 < r < s_2$. Let us fix $x_0 \in U$, $v_0 \in F(x_0)$ and $0 < t < m_U(x_0)$.

Fix $y \in V$ such that $d(y, v_0) \le rt$, i.e. $y \in B(v_0, rt) \cap V$. Let $M := \mathrm{Gr}\, F$, and let $\xi > 0$ correspond to r in the definition of s_2. It is clear that (M, d_ξ) is a complete metric space.

Consider the function $f\colon M \to \mathbb{R}$ defined as $f(u, w) := d(w, y)$.

Then $f \ge 0$ and it is continuous on M. Since $f(x_0, v_0) = d(v_0, y) \le rt$, by Ekeland Variational Principle there exists $(x_1, v_1) \in M$ such that

(i) $f(x_1, v_1) \le f(x_0, v_0) - rd_\xi((x_1, v_1), (x_0, v_0))$;

(ii) $d_\xi((x_1, v_1), (x_0, v_0)) \le t$;

(iii) $f(u, w) + rd_\xi((u, w), (x_1, v_1)) \ge f(x_1, v_1)$, for all $(u, w) \in M$.

Or, equivalently

(i) $d(v_1, y) \le d(v_0, y) - rd_\xi((x_1, v_1), (x_0, v_0)) \le rt - rd_\xi((x_1, v_1), (x_0, v_0))$;

(ii) $d(x_1, x_0) \le t, \quad \xi d(v_1, v_0) \le t$;

(iii) $d(w, y) + rd_\xi((u, w), (x_1, v_1)) \ge d(v_1, y)$, for all $(u, w) \in M$.

Set $p := d(v_1, y)$. Assume that $p > 0$. Take t' such that $t < t' < m_U(x_0)$.

For $x \in B\left(x_1, \frac{p}{r} + t' - t\right)$ we have that

$$\begin{array}{rcl} d(x, x_0) & \leq & d(x, x_1) + d(x_1, x_0) \\ & \leq & \frac{p}{r} + t' - t + d(x_1, x_0) \\ \text{(using (i))} & \leq & \frac{rt - rd(x_1, x_0)}{r} + t' - t + d(x_1, x_0) \\ & = & t - d(x_1, x_0) + t' - t + d(x_1, x_0) \\ & = & t'. \end{array}$$

Hence $B\left(x_1, \frac{p}{r} + t' - t\right) \subset B(x_0, t') \subset U$. Then $\frac{p}{r} + t' - t \leq m_U(x_1)$, and $\frac{p}{r} < m_U(x_1)$ because $t' - t > 0$. Hence, $0 < d(v_1, y) < rm_U(x_1)$. But now (3) contradicts (iii).

Therefore, $p = 0$ and then $y = v_1 \in F(x_1)$. Since by (ii) $x_1 \in B(x_0, t)$, we have $y \in F(B(x_0, t)) \cap V$.

Since $x_0 \in U$, $v_0 \in F(x_0)$, $y \in B(v_0, rt) \cap V$ and $0 < t < m_U(x_0)$ were arbitrary, this means that $r \leq s_1$. Since $0 < r < s_2$ was arbitrary, $s_2 \leq s_1$, and the proof is completed. □

In the definitions of regularity properties it is not required that $F(x) \subset V$. Such requirements can be included in the definitions as follows.

Definition 2.7. (Restricted regularity, [6, Definition 2.35])
Set $F^V(x) := F(x) \cap V$. We define *restricted γ-openness at linear rate* and *restricted γ-metric regularity* on (U, V) by replacing F by F^V.

The equivalence Theorem 2.3 also holds for the restricted versions of the properties. The case is the same with Theorem 2.6, where the proof needs only small adjustments when working with F^V instead of F.

3. Proof of the main result

The proof of our main result relies on the following Lemma.

Lemma 3.1. *Let (X, d) be a metric space and $(Y, \|\cdot\|)$ be a Banach space, let $U \subset X$ and $V \subset Y$ be non-empty sets and let $F\colon X \rightrightarrows Y$ be a multi-valued map. If for some $r > 0$ it holds that*

$$F^{(1)}(x, v) \supset rB_Y \quad \textit{for all } (x, v) \in \operatorname{Gr} F \cap (U \times V),$$

then for any $0 < r' < r$ and any $\xi \in (r^{-1}, (r')^{-1})$ it holds that for any $x \in U$ and any $v \in F^V(x)$ and $y \in V \setminus \{v\}$ there is $(u, w) \in \operatorname{Gr} F$ such that

$$\|y - w\| < \|y - v\| - r' d_\xi((x, v), (u, w)).$$

Proof. Let $r' \in (0, r)$ be fixed. Fix $\xi > 0$ such that $(r')^{-1} > \xi > r^{-1}$. Take (x, v) such that $(x, v) \in \operatorname{Gr} F \cap (U \times V)$. Fix $y \in V$ such that $0 < \|y - v\|$.

Set $\bar{v} := r\dfrac{y - v}{\|y - v\|}$. Obviously $\|\bar{v}\| = r$.

By assumption, $F^{(1)}(x, v) \ni \bar{v}$. By definition of the contingent variation there exist $t_n \downarrow 0$, $u_n \in X$ as well as $w_n \in Y$ and $z_n \in Y$ such that $w_n \in F(u_n)$, $d(x, u_n) \le t_n$, $\|z_n\| \to 0$ and

$$v + t_n \bar{v} = w_n + t_n z_n. \tag{4}$$

Note first that for n large enough

$$\xi\|w_n - v\| > t_n \ge d(x, u_n) \Rightarrow d_\xi((x, v), (u_n, w_n)) = \xi\|w_n - v\|. \tag{5}$$

Indeed, $\|w_n - v\| = t_n\|\bar{v} - z_n\| \ge t_n(r - \|z_n\|)$ and, since $\xi(r - \|z_n\|) \to \xi r > 1$ as $n \to \infty$, we have $\xi\|w_n - v\| > t_n$ for n large enough.

From (4) we have

$$y - w_n = y - v - t_n\bar{v} + t_n z_n. \tag{6}$$

Since

$$y - v - t_n\bar{v} = \left(1 - t_n r\|y - v\|^{-1}\right)(y - v),$$

and since $1 - t_n r\|y - v\|^{-1} > 0$ for n large enough, we have for such n that

$$\|y - v - t_n\bar{v}\| = \left(1 - t_n r\|y - v\|^{-1}\right)\|y - v\| = \|y - v\| - t_n r.$$

Combining the latter with (6) we get for n large enough

$$\begin{aligned} \|y - w_n\| &= \|y - v - t_n\bar{v} + t_n z_n\| \\ &\le \|y - v - t_n\bar{v}\| + t_n\|z_n\| \\ &= \|y - v\| - t_n(r - \|z_n\|). \end{aligned} \tag{7}$$

On the other hand, (4) can be rewritten as $w_n - v = t_n\bar{v} - t_n z_n$, hence

$$\|w_n - v\| = t_n\|\bar{v} - z_n\| \le t_n(r + \|z_n\|),$$

and using this estimate we obtain that

$$\liminf_{n\to\infty} \frac{t_n(r - \|z_n\|)}{r'\xi\|v - w_n\|} \ge \liminf_{n\to\infty} \frac{t_n(r - \|z_n\|)}{r'\xi t_n(r + \|z_n\|)} = \frac{1}{r'\xi} > 1.$$

From this and (7) we have that for large n

$$\|y - w_n\| < \|y - v\| - r'\xi\|v - w_n\|.$$

Using (5) we finally obtain that for all n large enough

$$\|y - w_n\| < \|y - v\| - r' d_\xi((x, v), (u_n, w_n))$$

and the claim follows. $\square$

Proving our main result is now straightforward.

Proof of Theorem 1.1. Let $F^{(1)}(x,v) \supset rB_Y$ for all $(x,v) \in \operatorname{Gr} F \cap (U \times V)$. From Lemma 3.1 and Theorem 2.6 it follows that $\operatorname{sur}_m F^V(U|V) \geq r$.

Conversely, let $\operatorname{sur}_m F^V(U|V) \geq r > 0$. This means that

$$B(v,rt) \cap V \subset F(B(x,t))$$

whenever $(x,v) \in \operatorname{Gr} F$, $x \in U$, $v \in V$ and $t < m_U(x)$.

Take arbitrary $(x,v) \in \operatorname{Gr} F^V$, $x \in U$ and note that $m_U(x) > 0$ because U is open. Take positive t such that $t < m_U(x)$.

For any $y \in rB_Y$ it holds that $v + ty \in B(v,rt)$. Moreover, $v + ty \in V$ will be true for small t because V is open. Then, by assumption, $v + ty \in F(B(x,t))$, so

$$y \in \frac{F(B(x,t)) - v}{t}$$

which means that $y \in F^{(1)}(x,v)$. Hence, $F^{(1)}(x,v) \supset rB_Y$. □

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