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Co-Radiant Set-Valued Mappings

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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We introduce and study a notion of co-radiantness for set-valued mappings between nonnegative orthants of Euclidean spaces. We analyze them from an abstract convexity perspective. Our main results consist in representations, in terms of intersections of graphs, of the increasing co-radiant set-valued mappings that take closed normal values, by means of elementary members in the class of such set-valued mappings.

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1. Introduction

In this work we introduce and study a notion of co-radiantness for set-valued mappings from $\mathbb{R}^n_+$ into $\mathbb{R}^m_+$. The main aim is to extend the results of [6] on single-valued increasing co-radiant functions, especially their abstract convexity representations by elementary functions. Such functions arise in economic theory, where they are used to model single output production with decreasing returns to scale [5]. Similarly, our co-radiant set-valued vector mappings have obvious potential applications to model economies of scale in the multiple output case, which have been considered in [8]. Having such an application in mind, we will restrict our attention to set-valued mappings taking only normal values, a quite natural property when dealing with production technologies. Moving from the single-valued case to a set-valued vector setting provides much more flexibility to model production technologies. Our approach departs from the classical one in production theory, which is mostly based on ordinary convexity, in that our main tools belong to abstract convexity theory. We will assume the values of our set-valued mappings to be normal, but not necessarily convex, thus making our approach radically different from the one based on convex processes [7, 9].

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The results we present in this paper constitute nontrivial extensions of the existing ones on single-valued functions. They belong to an abstract convexity framework; our main contribution consists in identifying suitable elementary set-valued mappings within the class of increasing co-radiant set-valued mappings which generate the whole class when taking intersections of their graphs. For simplicity, our developments are presented in a finite dimensional setting, but extensions to real topological vector spaces, as those in [3] for the single-valued case, are certainly possible.

The structure of the paper is as follows. In Section 2 we present the fundamental definitions and results our developments will be based on. We recall there the main notions of abstract convexity and set-valued mappings we need. In Section 3 we introduce and study co-radiant set-valued mappings. Section 4 contains our main results on the representation of increasing co-radiant set-valued mappings by means of suitably defined elementary members in the class of such set-valued mappings.

Notations

We mostly follow the standard notation in set-valued analysis [2] and abstract convexity theory [11]. We denote by $\mathbb{R}^n_+$ the nonnegative orthant in $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n_+$, by $x \leq y$ we mean that $y - x \in \mathbb{R}^n_+$. The closure and the interior of a set C are $\overline{C}$ and $int(C)$, respectively. We set $\mathbb{R}^n_{++} := int\left(\mathbb{R}^n_+\right)$. The domain and the graph of a set-valued mapping $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ are $dom(F) := \left\{x \in \mathbb{R}^n_+ : F(x) \neq \emptyset\right\}$ and $gr(F) := \left\{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ : y \in F(x)\right\}$, respectively. By $\|\cdot\|_\infty$ we denote the maximum norm, defined on $\mathbb{R}^n$ by $\|x\|_\infty := \max_{i=1,\ldots,n} |x_i|$ for $x := (x_1, ..., x_n)$. Its associated unit ball is $\mathbf{B}_\infty$.

2. Preliminaries

Our notion of co-radiant set-valued function is a generalization of the corresponding one for single-valued functions, which is in turn based on the fundamental concepts of radiant and co-radiant sets.

Definition 2.1. [12] A nonempty set $C \subset \mathbb{R}^n_+$ is called

(i) *radiant*, if $x \in C, t \in (0, 1] \Longrightarrow tx \in C$.

(ii) *co-radiant*, if $x \in C, \lambda \geq 1 \Longrightarrow \lambda x \in C$.

Definition 2.2. [11, 13] A function $f \colon \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is called *co-radiant*, if

$$f(tx) \geq tf(x),\ \forall x \in \mathbb{R}^n_+,\ t \in (0, 1].$$

It is easy to see that f is co-radiant if, and only if,

$$f(\lambda x) \leq \lambda f(x),\ \forall x \in \mathbb{R}^n_+,\ \lambda \geq 1.$$

It is also a simple exercise to prove that a function $f \not\equiv +\infty$ is co-radiant if, and only if, the set $hypo(f) := \{(x, \alpha) \in \mathbb{R}^n_+ \times \mathbb{R}_+ : f(x) \geq \alpha\}$ is radiant.

Co-radiant functions with the additional property of being increasing have been intensively studied in [13], where applications to a class of optimization problems are discussed.

Definition 2.3. (see [14, p. 106]). A set $C \subset \mathbb{R}^n_+$ is called *normal* if it satisfies the following property:

$$x \in C,\, 0 \leq y \leq x \Rightarrow y \in C.$$

We next recall the basic properties of normal sets (see [17], [11] and [15]):

Proposition 2.4.

(i) Both $\mathbb{R}^n_+$ and $\emptyset$ are normal sets.

(ii) For any $y \in \mathbb{R}^n_+$, the set $[0, y] := \{x \in \mathbb{R}^n_+ : x \leq y\}$ is normal.

(iii) If $\{C_i\}_{i \in I}$ is an arbitrary collection of normal sets in $\mathbb{R}^n_+$, then $\bigcup_{i \in I} C_i$ and $\bigcap_{i \in I} C_i$ are normal sets, too.

(iv) If C is a normal set in $\mathbb{R}^n_+$, then $\overline{C}$ is normal, too.

(v) If C is a normal set in $\mathbb{R}^n_+$, then the following equivalence holds true:

$$C \cap \mathbb{R}^n_{++} \neq \emptyset \Leftrightarrow int(C) \neq \emptyset.$$

As in [15], we consider the coupling function $\gamma : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ defined by

$$\gamma(\ell, x) := \begin{cases} \min\limits_{i \in I_+(\ell)} \ell_i x_i, & \text{if } \ell \neq 0 \\ 0, & \text{if } \ell = 0, \end{cases}$$

with $\ell = (\ell_1, \cdots \ell_n)$ and $I_+(\ell) = \{i \in \{1, \cdots, n\} : \ell_i > 0\}$. Functions of the type $\gamma(\ell, \cdot)$ are called $min - type$ functions. Their level sets $\{x \in \mathbb{R}^n_+ : \gamma(\ell, x) \leq \alpha\}$ $(\alpha \geq 0)$ are closed normal sets. From the following proposition it immediately follows that every closed normal set is the intersection of a collection of such level sets.

Proposition 2.5. [15, Proposition 2.3] *For a subset C of $\mathbb{R}^n_+$, the following conditions are equivalent:*

(i) *C is normal and closed.*

(ii) *For each $x \in \mathbb{R}^n_+ \setminus C$ there exists $\ell \in \mathbb{R}^n_+$ such that*

$$\gamma(\ell, y) \leq 1 < \gamma(\ell, x), \quad \forall y \in C.$$

The following notion of support function, which is useful for dealing with closed normal sets, was essentially introduced in [14, p. 108].

Definition 2.6. The *support function* of $C \subset \mathbb{R}^n_+$ is $\sigma_C : \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$, defined by

$$\sigma_C(\ell) := \sup\{\gamma(\ell, x) \ : \ x \in C\},$$

with the convention $\sup \emptyset = 0$.

We next present some basic properties of σ_C.

Proposition 2.7. *For every $C \subset \mathbb{R}^n_+$, one has:*

(i) $\sigma_C(0) = 0$.

(ii) σ_C *is positively homogeneous, that is, for every* $\alpha > 0$ *and* $\ell \in \mathbb{R}^n_+$ *one has* $\sigma_C(\alpha\ell) = \alpha\sigma_C(\ell)$.

(iii) σ_C *is increasing.*

(iv) σ_C *is lower semicontinuous.*

(v) $\sigma_C = \sigma_{\overline{C}}$.

(vi) *If* $\lambda > 0$ *is positive, then* $\sigma_{\lambda C} = \lambda\sigma_C$.

(vii) *If A and B are nonempty subsets of* $\mathbb{R}^n_+$, *then*

$$\sigma_{A+B} \geq \sigma_A + \sigma_B. \tag{1}$$

(viii) *If A and B are subsets of $\mathbb{R}^n_+$ and B is normal and closed, the following equivalence holds true:*

$$A \subset B \Leftrightarrow \sigma_A \leq \sigma_B.$$

Proof. Statements (i) (ii), (iii) and (vi) follow immediately from the definition of σ_C. Statements (iv) and (v) are easy consequences of the continuity of the min-type functions $\gamma(\ell, \cdot)$. Statement (vii) follows from the superadditivity of the min-type functions $\gamma(\ell, x)$. The implication $\Longrightarrow$ in (viii) is obvious. Conversely, assume that $\sigma_A \leq \sigma_B$. Then, if $x \in \mathbb{R}^n_+ \setminus B$, by Proposition 2.5 there exists $\ell \in \mathbb{R}^n_+$ such that $\gamma(\ell, y) \leq 1 < \gamma(\ell, x)$ for every $y \in B$; hence $\sigma_A(\ell) \leq \sigma_B(\ell) \leq 1 < \gamma(\ell, x)$, which shows that $x \notin A$. This proves the inclusion $A \subset B$. □

We observe that equality does not necessarily hold in (1), even if A and B are normal. Consider, for instance, in $\mathbb{R}^2_+$ the segments A and B joining the origin with the points $(1, 0)$ and $(0, 1)$, respectively; for $\ell := (1, 1)$ one has $\sigma_A(\ell) = 0$, $\sigma_B(\ell) = 0$ and $\sigma_{A+B}(\ell) = 1$.

Corollary 2.8. *If A and B are normal and closed subsets of $\mathbb{R}^n_+$, the following equivalence holds true:*

$$A = B \Leftrightarrow \sigma_A = \sigma_B.$$

Proof. It follows from Proposition 2.7(viii). □

Corollary 2.9. *A set $C \subset \mathbb{R}^n_+$ is normal and closed if, and only if,*

$$C = \{\, x \in \mathbb{R}^n_+ : \gamma(\ell, x) \leq \sigma_C(\ell), \quad \forall \ell \in \mathbb{R}^n_+ \,\}. \tag{2}$$

Proof. The "if" statement is an immediate consequence of Proposition 2.7(iv) (closedness of C) together with the fact that the left hand sides of the inequalities in (2) are increasing (normality of C). To prove the converse, let D be the right hand side of (2). It is immediate that $C \subset D$ and $\sigma_D \leq \sigma_C$; from this inequality and Proposition 2.7(viii), one gets the opposite inclusion $D \subset C$. □

3. Increasing co-radiant (ICR) set-valued mappings

The notion of increasingness we will use for set-valued mappings is provided by the following definition.

Definition 3.1. A set-valued mapping $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ is called *increasing* if

$$x \leq y \Rightarrow F(x) \subset F(y).$$

We next extend the notion of co-radiant function [11] to set-valued mappings.

Definition 3.2. A set-valued mapping $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ with nonempty graph will be called *co-radiant*, if

$$F(tx) \supset tF(x), \quad \forall x \in \mathbb{R}^n_+, t \in (0,1].$$

Let us observe that the latter condition can be equivalently written as

$$F(\lambda x) \subset \lambda F(x) \qquad \forall x \in \mathbb{R}^n_+, \lambda \geq 1.$$

The preceding definitions are generalizations of the corresponding ones for real-valued functions. Indeed, one can associate to $f : \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$ the set-valued mapping $[f] : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}_+$ defined by

$$[f](x) := [0, f(x)],$$

with the convention $[0,+\infty] := \mathbb{R}_+$. Then, it is easy to see that f is increasing if, and only if, $[f]$ is increasing, and f is co-radiant if, and only if, $[f]$ is co-radiant. Notice that $gr([f]) = hypo(f)$.

For a function $f : \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$, one can easily check that the function $\widehat{f} : \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$ defined by $\widehat{f}(x) := \sup_{\lambda \geq 1} \frac{f(\lambda x)}{\lambda}$ is the smallest co-radiant majorant of f. Similarly, for set-valued mappings we have the following proposition.

Proposition 3.3. *For a set-valued mapping $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ with nonempty graph, the set-valued mapping $\widehat{F} : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ defined by*

$$\widehat{F}(x) := \bigcup_{\lambda \geq 1} \frac{F(\lambda x)}{\lambda} \tag{3}$$

is the pointwise smallest (in the sense of inclusion) co-radiant majorant of F. If F takes only normal values, then $\widehat{F}$ takes only normal values, too.

Proof. Let $x \in \mathbb{R}^n_+$. Taking $\lambda := 1$ in (3), we see that $F(x) \subset \widehat{F}(x)$; thus $\widehat{F}$ is a majorant of F. The set-valued mapping $\widehat{F}$ is also co-radiant, since for $\beta \geq 1$ we have

$$\widehat{F}(\beta x) = \bigcup_{\lambda \geq 1} \frac{F(\lambda \beta x)}{\lambda} = \beta \bigcup_{\lambda \geq 1} \frac{F(\lambda \beta x)}{\lambda \beta} = \beta \bigcup_{\mu \geq \beta} \frac{F(\mu x)}{\mu} \subset \beta \bigcup_{\mu \geq 1} \frac{F(\mu x)}{\mu} = \beta \widehat{F}(x).$$

If G is a co-radiant majorant of F, then for every $\lambda \geq 1$ we have

$$\widehat{F}(x) = \bigcup_{\lambda \geq 1} \frac{F(\lambda x)}{\lambda} \subset \bigcup_{\lambda \geq 1} \frac{G(\lambda x)}{\lambda} \subset \bigcup_{\lambda \geq 1} \frac{\lambda G(x)}{\lambda} = G(x);$$

this proves that $\widehat{F}$ is the smallest co-radiant majorant of F. Finally, if F takes only normal values then, since each set $F(\lambda x)/\lambda$ $(\lambda \geq 1)$ is normal, their union $\widehat{F}(x)$ is normal, too; so $\widehat{F}$ takes only normal values. $\square$

For $f\colon \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$, a straightforward computation shows that, for every $x \in \mathbb{R}^n_+$, one has

$$\widehat{[f]}(x) = \begin{cases} \left[\widehat{f}\right](x) & \text{if either } \sup_{\lambda\geq 1} \frac{f(\lambda x)}{\lambda} \text{ is attained or is } +\infty, \\ \left[0, \widehat{f}(x)\right) & \text{otherwise.} \end{cases}$$

Hence
$$\left[\widehat{f}\right](x) = cl\widehat{[f]}(x).$$

The following proposition generalizes the above mentioned equivalence of co-radiantness of a function $f : \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$ and radiantness of its hypograph.

Proposition 3.4. *Let $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$. Then F is co-radiant if, and only if, the set $gr(F)$ is radiant.*

Proof. Assume first that F is co-radiant. Let $(x, y) \in gr(F)$ and $t \in (0, 1]$, then $ty \in tF(x) \subset F(tx)$, therefore $t(x, y) = (tx, ty) \in gr(F)$. Conversely, assume that $gr(F)$ is radiant and let $y \in F(x)$; then $(x, y) \in gr(F)$ and, for $t \in (0, 1]$, we have $(tx, ty) = t(x, y) \in gr(F)$ and thus $ty \in F(tx)$. This proves that $tF(x) \subset F(tx)$. $\square$

We are going to consider increasing co-radiant (briefly, ICR) set-valued mappings $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$. We first present a simple result on the domains of such functions.

Proposition 3.5. *Let $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ be ICR. If $\mathbb{R}^n_{++} \cap dom(F) \neq \emptyset$, then $\mathbb{R}^n_{++} \subset dom(F)$.*

Proof. For $x, y \in \mathbb{R}^n_{++}$, setting $t := \max\left\{\max_i \frac{x_i}{y_i}, 1\right\}$, we have $x \leq ty$; hence $F(x) \subset F(ty) \subset tF(y)$. Therefore, if $x \in dom(F)$, then $y \in dom(F)$, too. $\square$

We are going to use the following notion of Lipschitz set-valued mapping.

Definition 3.6. (see [2]). Let $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ and $K \subset dom(F)$. One says that F is Lipschitz on K if there exists $M > 0$ such that

$$F(x) \subset F(y) + M\|x - y\|_\infty \mathbf{B}_\infty \qquad \forall x, y \in K. \tag{4}$$

Notice that, in the preceding definition, one can replace $\|\cdot\|_\infty$ and $\mathbf{B}_\infty$ with any other norm and its corresponding unit ball, respectively, since all norms in $\mathbb{R}^n$ are equivalent.

It is easy to check that a function $f : \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is Lipschitz on a set $K \subset \mathbb{R}^n_+$ if and only if its set-valued version $[f]$ is Lipschitz on K.

The following proposition generalizes [16, Remark 4.4].

Proposition 3.7. *Let $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ be an ICR set-valued mapping which takes only normal values. If $F(\overline{x})$ is bounded for some $\overline{x} := (\overline{x}_1, \cdots, \overline{x}_n) \in \mathbb{R}^n_{++}$, then F is Lipschitz on every compact set $K \subset \mathbb{R}^n_{++}$.*

Proof. We will first consider the case when F is positively homogeneous on $\mathbb{R}^n_{++}$. Let $x \in \mathbb{R}^n_{++}$ and $y = (y_1, \cdots, y_n) \in K$. By the boundedness of $F(\overline{x})$, there exists $L > 0$ such that $F(\overline{x}) \subset L\mathbf{B}_\infty$. Since $x \leq \left(\max_i \frac{x_i}{y_i}\right) y$, $y \leq \left(\max_i \frac{y_i}{\overline{x}_i}\right) \overline{x}$ and, for every $i = 1, ..., n$,

$$\frac{x_i}{y_i} \leq 1 + \frac{\max_i |x_i - y_i|}{y_i} \leq 1 + \frac{\|x-y\|_\infty}{\min_j y_j},$$

we have

$$\begin{aligned} F(x) &\subset \left(\max_i \frac{x_i}{y_i}\right) F(y) \subset \left(1 + \frac{\|x-y\|_\infty}{\min_j y_j}\right) F(y) \subset F(y) + \frac{\|x-y\|_\infty}{\min_j y_j} F(y) \\ &\subset F(y) + \frac{\|x-y\|_\infty}{\min_j y_j} (\max_j \frac{y_j}{\overline{x}_j}) F(\overline{x}) \subset F(y) + \frac{\max_j \frac{y_j}{\overline{x}_j}}{\min_j y_j} \|x-y\|_\infty L\mathbf{B}_\infty; \end{aligned}$$

thus (4) holds with $M := L \max_{y \in K} \dfrac{\max_j \frac{y_j}{\overline{x}_j}}{\min_j y_j}$.

In the general case, we extend F to a set-valued mapping $\widetilde{F} : \mathbb{R}^{n+1}_+ \rightrightarrows \mathbb{R}^m_+$ defined by

$$\widetilde{F}(x, \lambda) := \begin{cases} \lambda F(\frac{x}{\lambda}), & \text{if } \lambda > 0, \\ \emptyset, & \text{if } \lambda = 0. \end{cases}$$

The set-valued mapping $\widetilde{F}$ is positively homogeneous on $\mathbb{R}^n_{++}$, increasing, and takes only normal values. Hence, as $\widetilde{F}(\overline{x}, 1) = F(\overline{x})$ is nonempty and bounded, the result follows by applying the first part of the proof to $\widetilde{F}$ and the compact set $K \times \{1\}$. $\square$

The folllowing (semi)continuity notions are standard in set-valued analysis.

Definition 3.8. (see [2]). (i) One says that $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ is upper semicontinuous (u.s.c.) at $x \in dom(F)$ if for every open set V in $\mathbb{R}^m_+$ such that $F(x) \subset V$, there exists an open neighborhood U of x such that $F(U) \subset V$.

(ii) F is lower semicontinuous (l.s.c.) at $x \in dom(F)$ if for every open set V in $\mathbb{R}^m_+$ such that $V \cap F(x) \neq \emptyset$, there exists an open neighborhood U of x such that $V \cap F(x') \neq \emptyset$ for every $x' \in U$.

(iii) F is continuous at $x \in dom(F)$ if it is both u.s.c. and l.s.c. at x.

It is worth observing that a function $f : \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is upper (lower) semicontinuous at a point $x \in \mathbb{R}^n_+$ if, and only if, its set-valued counterpart $[f]$ is u.s.c. (resp., l.s.c.) at this point.

A useful sequential characterization of upper semicontinuity is stated in the next proposition.

Proposition 3.9. [4, Theorem 1] *Let $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ take only compact values. Then it is u.s.c. at $x \in dom(F)$ if, and only if, for every sequence $\{x^k\}$ in $\mathbb{R}^n_+$ converging to x and every sequence $\{y^k\}$ in $\mathbb{R}^m_+$ such that $y^k \in F(x^k)$ for every k there exists a subsequence of $\{y^k\}$ which converges to a point in $F(x)$.*

For $x \in X$ and $A \subset X$, we define $d_\infty(x, A) := \inf_{a\in A} \|x - a\|_\infty$. The following corollary is immediate.

Corollary 3.10. *Let $F\colon \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ take only compact values. Then it is u.s.c. at $x \in dom(F)$ if, and only if, for every sequence $\{x^k\}$ in $\mathbb{R}^n_+$ converging to x and every sequence $\{y^k\}$ in $\mathbb{R}^m_+$ such that $y^k \in F(x^k)$ for every k, one has $\lim\limits_{k\to+\infty} d_\infty(y^k, F(x)) = 0$.*

We next present generalizations of some results obtained in [6].

Proposition 3.11. *If $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ is an ICR set-valued mapping and takes only normal values and there exists $\overline{x} \in \mathbb{R}^n_{++}$ such that $F(\overline{x})$ is bounded, then $F(x)$ is bounded for every $x \in \mathbb{R}^n_+$. Furthermore, if $F(\overline{x}) = \{0\}$ ($F(\overline{x}) = \emptyset$) then $F(x) = \{0\}$ ($F(x) = \emptyset$, respectively) for every $x \in \mathbb{R}^n_+$.*

Proof. For every $x \in \mathbb{R}^n_+$ there exists $\lambda \geq 1$ such that $x \leq \lambda\overline{x}$, hence $F(x) \subset F(\lambda\overline{x}) \subset \lambda F(\overline{x})$. □

Proposition 3.12. *If $F\colon \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ is an ICR set-valued mapping and takes only normal values, then*

(i) *F is l.s.c. at every $x \in dom(F) \cap \mathbb{R}^n_{++}$.*

(ii) *If F takes only compact values, then it is continuous at every point $x \in dom(F) \cap \mathbb{R}^n_{++}$.*

Proof. (i) Let V be an open set in $\mathbb{R}^m_+$ such that $V \cap F(x) \neq \emptyset$. Take $y \in V \cap F(x)$. For small enough $\delta > 0$ and $U := \{ty : t \in [1-\delta, 1]\}$, we have $U \subset V$. Set $V := (1-\delta)\,x + \mathbb{R}^n_{++}$, then V is open and $x = (1-\delta)\,x + \delta x \in (1-\delta)\,x + \mathbb{R}^n_{++} = V$; moreover, for every $x' \in V$, we have

$$(1-\delta)\,y \in U \cap (1-\delta)\,F(x) \subset V \cap F((1-\delta)x) \subset V \cap F(x'),$$

which shows that $V \cap F(x') \neq \emptyset$.

(ii) Let $\{x^k\}$ be a sequence in $\mathbb{R}^n_+$ converging to x and $\{y^k\}$ be a sequence in $\mathbb{R}^m_+$ such that $y^k \in F(x^k)$ for every k. Fix $\epsilon > 0$, and let $M > \epsilon$ be the radius of an open ball centered at the origin which contains $F(x)$. Since $\{x^k\}$ converges to x, for sufficiently large k we have $x^k \leq \frac{M}{M-\epsilon}x$; hence, as F is increasing and co-radiant, we deduce that $F(x^k) \subset \frac{M}{M-\epsilon}F(x)$, which implies that $y^k \in \frac{M}{M-\epsilon}F(x)$. Therefore

$$d_\infty(y^k, F(x)) \leq d(y^k, \frac{M-\epsilon}{M}y^k) = \frac{\epsilon}{M}\|y^k\| < \epsilon,$$

which proves that $\lim_{k\to+\infty} d_\infty(y^k, F(x)) = 0$. Hence, by Corollary 3.10, the set-valued mapping F is u.s.c. at x. Continuity follows from statement (i). $\square$

The proofs of the following propositions are immediate.

Proposition 3.13. *If* $F\colon \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ *is co-radiant, then the set-valued mappings* $\overline{F}$ *and* $\mathbf{B}_\epsilon(F) : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ *defined by*

$$\overline{F}(x) := \overline{F(x)} \quad \textit{and} \quad \mathbf{B}_\epsilon(F)(x) := \{y \in \mathbb{R}^m \,:\, d_\infty(y, F(x)) \le \epsilon\},$$

respectively, are co-radiant.

As is well known, the class of co-radiant functions is closed both under pointwise infimum and pointwise supremum. The following proposition provides set-valued generalizations of these facts.

Proposition 3.14. *If* $\{F^i\}_{i\in\mathcal{I}}$ *is a family of co-radiant set-valued mappings, then the set-valued mappings* $\bigcap_{i\in\mathcal{I}} F^i$ *and* $\bigcup_{i\in\mathcal{I}} F^i$ *defined by*

$$\left(\bigcap_{i\in\mathcal{I}} F^i\right)(x) := \bigcap_{i\in\mathcal{I}} F^i(x) \quad \textit{and} \quad \left(\bigcup_{i\in\mathcal{I}} F^i\right)(x) := \bigcup_{i\in\mathcal{I}} F^i(x)$$

are co-radiant.

We are going to deal with pointwise Painlevé-Kuratowski limits of sequences of set-valued mappings. We first recall the convergence notions in the Painlevé-Kuratowski sense for sequences of subsets.

Definition 3.15. Let $\{C_k\}$ be a sequence of subsets of $\mathbb{R}^n$. The sets

$$\mathrm{Limsup}_k C_k := \left\{x \in \mathbb{R}^n \;: \lim\inf_{k\to+\infty} d_\infty(x, C_k) = 0\right\}$$

and

$$\mathrm{Liminf}_k C_k := \left\{x \in \mathbb{R}^n \;:\; \lim_{k\to+\infty} d_\infty(x, C_k) = 0\right\}$$

are called the *upper limit* and the *lower limit* of $\{C_k\}$, respectively.

It is easy to see that these sets do not change if one replaces the sets C_k with their closures and that $\mathrm{Liminf}_k C_k \subset \mathrm{Limsup}_k C_k$.

The following representations of Liminf and Limsup are very useful.

Proposition 3.16. [2, p. 21][10, Exercise 4.2(b)] *For a sequence* $\{C_k\}$ *of subsets of* $\mathbb{R}^n$*, one has:*

(i) $\mathrm{Limsup}_k C_k = \bigcap_{l\in\mathbb{N}} \overline{\bigcup_{k\ge l} C_k}$.

(ii) $\mathrm{Liminf}_k C_k = \bigcap_{I\in\aleph} \overline{\bigcup_{i\in I} C_i}$.

Here $\aleph$ denotes the set of infinite subsets of $\mathbb{N}$.

From Proposition 3.16, it directly follows that the sets $\mathrm{Liminf}_k C_k$ and $\mathrm{Limsup}_k C_k$ are closed.

Definition 3.17. [10, Chapter 5] Let $\{F^k\}$ be a sequence of set-valued mappings from $\mathbb{R}^n_+$ into $\mathbb{R}^m_+$.

(i) The *pointwise lower limit* of $\{F^k\}$ is the set-valued mapping $\mathrm{Liminf}_k F^k \colon \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ defined by $(\mathrm{Liminf}_k F^k)(x) := \mathrm{Liminf}_k F^k(x)$.

(ii) The *pointwise upper limit* of $\{F^k\}$ is the set-valued mapping $\mathrm{Limsup}_k F^k \colon \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ defined by $(\mathrm{Limsup}_k F^k)(x) := \mathrm{Limsup}_k F^k(x)$.

The following proposition collects some basic properties of pointwise lower and upper limits of set-valued mappings.

Proposition 3.18. *For a sequence $\{F^k\}$ of set-valued mappings from $\mathbb{R}^n_+$ into $\mathbb{R}^m_+$, one has:*

(i) $\mathrm{Liminf}_k F^k$ *and* $\mathrm{Limsup}_k F^k$ *only take closed values.*

(ii) *If each F^k only takes normal values, then* $\mathrm{Liminf}_k F^k$ *and* $\mathrm{Limsup}_k F^k$ *only take normal values, too.*

(iii) *If each F^k is increasing, then* $\mathrm{Liminf}_k F^k$ *and* $\mathrm{Limsup}_k F^k$ *are increasing, too.*

(iv) *If each F^k is co-radiant, then* $\mathrm{Liminf}_k F^k$ *and* $\mathrm{Limsup}_k F^k$ *are co-radiant, too.*

Proof. Statement (i) directly follows from the definitions of Liminf and Limsup. Statements (ii) and (iii) follow from Proposition 3.16. Finally, statement (iv) follows from Propositions 3.13, 3.14 and 3.16. $\square$

We next give several propositions providing characterizations of some properties of set-valued mappings in terms of their graphs.

Proposition 3.19. *If $F \colon \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ takes only normal values, then*

$$F \text{ is increasing} \Leftrightarrow gr(F) + \mathbb{R}^n_+ \times \{0\} = gr(F)$$

Proof. $\Rightarrow$: Obviously, $gr(F) \subset gr(F) + \mathbb{R}^n_+ \times \{0\}$. To prove the opposite inclusion, let $(x, y) \in gr(F)$ and $p \in \mathbb{R}^n_+$; then $y \in F(x) \subset F(x+p)$, so we conclude that $(x, y) + (p, 0) \in gr(F)$.

$\Leftarrow$: Let $x' \geq x \in \mathbb{R}^n_+$ and $y \in F(x)$, then

$$(x', y) = (x, y) + (x' - x, 0) \in gr(F) + \mathbb{R}^n_+ \times \{0\} = gr(F),$$

therefore $y \in F(x')$. $\square$

Proposition 3.20. *Let $F \colon \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$. Then*

$$F \text{ takes only normal values} \Leftrightarrow \big(gr(F) + \{0\} \times (-\mathbb{R}^m_+)\big) \cap \mathbb{R}^n_+ \times \mathbb{R}^m_+ = gr(F).$$

Proof. $\Rightarrow$: Obviously, $gr(F) \subset \big(gr(F) + \{0\} \times (-\mathbb{R}^m_+)\big) \cap \mathbb{R}^n_+ \times \mathbb{R}^m_+$. To prove the opposite inclusion, let $(x, y) \in gr(F)$ and $p \in \mathbb{R}^m_+$ be such that $y - p \in \mathbb{R}^m_+$, then, by the normality of $F(x)$, one has $(x, y - p) \in gr(F)$.

$\Leftarrow$: If $x \in \mathbb{R}^n_+$ and $0 \leq y' \leq y \in F(x)$, we have $(x, y') = (x, y) + (0, y' - y) \in \left(gr(F) + \{0\} \times (-\mathbb{R}^m_+)\right) \cap \mathbb{R}^n_+ \times \mathbb{R}^m_+ = gr(F)$. Hence $y' \in F(x)$, which proves that $F(x)$ is normal. $\square$

Proposition 3.21. *Let $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$. Then F is increasing and takes only normal values if, and only if, for every $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \setminus gr(F)$ one has*

$$gr(F) \cap ((x, y) + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+) = \emptyset. \tag{5}$$

Proof. If F is increasing and takes only normal values, the existence of $(p_1, p_2) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$ satisfying $(x, y) + (-p_1, p_2) \in gr(F)$, that is, $y + p_2 \in F(x - p_1)$, by the increasingness of F would imply $y + p_2 \in F(x)$; hence, by the normality of $F(x)$, we would have $y \in F(x)$, that is, $(x, y) \in gr(F)$, a contradiction. To prove the converse implication, let $x, x' \in \mathbb{R}^n_+$ be such that $x \leq x'$, and let $y \in \mathbb{R}^m_+ \setminus F(x')$. Since $(x, y) = (x', y) + (x - x', 0) \in (x, y) + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+$, by (5) we have $(x, y) \notin gr(F)$, that is, $y \notin F(x)$, which shows that F is increasing. To prove that F takes only normal values, let $x \in \mathbb{R}^n_+$ and $y, y' \in \mathbb{R}^m_+$ be such that $y \leq y'$ and $y \in \mathbb{R}^m_+ \setminus F(x)$. Since $(x, y') = (x, y) + (0, y' - y) \in (x, y) + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+)$, by (5) we have $(x, y') \notin gr(F)$, that is, $y' \notin F(x)$, which shows that $F(x)$ is normal. $\square$

4. Representations of ICR set-valued mappings as intersections of elementary set-valued mappings

In this section we will introduce two notions of elementary ICR set-valued mappings, and we will show that they generate all the ICR set-valued mappings that satisfy some suitable additional properties.

The following separation result of radiant sets by convex cones will be used to separate graphs of co-radiant set-valued mappings.

Lemma 4.1. (see [18, Proposition 3.4]) *A nonempty closed set $A \subset \mathbb{R}^p_+$ is radiant if, and only if, for every $x \in \mathbb{R}^p_+ \setminus A$ there exists a cone $K_{A,x} \subset \mathbb{R}^p_+$ such that $x \in K_{A,x}$ and $A \cap (x + K_{A,x}) = \emptyset$. One can take $K_{A,x}$ convex and such that $K_{A,x} \setminus \{0\}$ is open in $\mathbb{R}^p_+$, namely*

$$K_{A,x} := \left\{ \sum_{i=1}^{p} \lambda_i \left(x + r_{A,x} e_i\right) : \begin{array}{l} \lambda_i > 0 \text{ for } i \in I_+(x), \\ \lambda_i \geq 0 \text{ for } i \in I \setminus I_+(x) \end{array} \right\} \cup \{0\}, \tag{6}$$

with the $e_i^{'}s$ denoting the unit vectors and $r_{A,x} := d_\infty(x, A)$.

Proof. Assume first that A is radiant. For $x \in \mathbb{R}^p_+ \setminus A$, the cone $K_{A,x}$ is convex, $K_{A,x} \setminus \{0\}$ is open in $\mathbb{R}^p_+$, and $x \in K_{A,x}$; indeed, take $\lambda_i := x_i / \left(\sum_{j=1}^p x_j + r_{A,x}\right)$. Assume that $x + \sum_{i=1}^p \lambda_i (x + r_{A,x} e_i) \in A$ for some $\lambda_i > 0$ ($i \in I_+(x)$) and $\lambda_i \geq 0$ ($i \in I \setminus I_+(x)$). Then, as A is radiant, we have

$$x + \frac{r_{A,x}}{1 + \sum\limits_{i=1}^{p} \lambda_i} \sum_{i=1}^{p} \lambda_i e_i = \frac{1}{1 + \sum\limits_{i=1}^{p} \lambda_i} \left(x + \sum_{i=1}^{p} \lambda_i \left(x + r_{A,x} e_i\right) \right) \in A,$$

which contradicts the definition of $r_{A,x}$. This proves that $A \cap (x + K_{A,z}) = \emptyset$.

Conversely, let $a \in A \setminus \{0\}$ and $t \in (0,1]$. If $ta \notin A$, by assumption there exists a cone $K_{A,ta} \subset \mathbb{R}^p_+$ such that $ta \in K_{A,ta}$ and $A \cap (ta + K_{A,ta}) = \emptyset$. However this is impossible, since $a = ta + \frac{1-t}{t} ta \in ta + K_{A,ta}$. Hence $ta \in A$, which proves that A is radiant. □

Proposition 4.2. *If $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ is co-radiant and $gr(F)$ is closed, then*

(a) *For every $(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \setminus gr(F)$ there exists a convex cone K in $\mathbb{R}^n_+ \times \mathbb{R}^m_+$ containing (x,y) such that $K \setminus \{(0,0)\}$ is open in $\mathbb{R}^n_+ \times \mathbb{R}^m_+$ and*

$$gr(F) \cap ((x,y) + K \setminus \{(0,0)\}) = \emptyset, \tag{7}$$

namely one can take $K := K_{gr(F),(x,y)}$, the cone defined in (6), with $p := n+m$, $z := (x,y)$, $A := gr(F)$ and $r_{gr(F),(x,y)} := d_\infty((x,y), gr(F))$.

(b) *If F takes only normal values, then for every $(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \setminus gr(F)$ there exists a convex cone K in $\mathbb{R}^n_+ \times \mathbb{R}^m_+$ containing (x,y) such that $K \setminus \{(0,0)\}$ is open in $\mathbb{R}^n_+ \times \mathbb{R}^m_+$ and*

$$gr(F) \cap ((x,y) + K + \{0\} \times \mathbb{R}^m_+) = \emptyset, \tag{8}$$

namely one can take $K := K_{gr(F),(x,y)}$, the cone considered in (a).

(c) *If F is increasing and takes only normal values, then for every $(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \setminus gr(F)$ there exists a convex cone K in $\mathbb{R}^n_+ \times \mathbb{R}^m_+$ containing (x,y) such that $K \setminus \{(0,0)\}$ is open in $\mathbb{R}^n_+ \times \mathbb{R}^m_+$ and*

$$gr(F) \cap \left((x,y) + K + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+\right) = \emptyset, \tag{9}$$

namely one can take $K := K_{gr(F),(x,y)}$, the cone considered in (a).

Proof. (a) It is an immediate consequence of Lemma 4.1.

(b) For $K := K_{gr(F),(x,y)}$, equality (8) holds, since the existence of $(k_1,k_2) \in K_{gr(F),(x,y)}$ and $p \in \mathbb{R}^m_+$ satisfying $(x,y) + (k_1,k_2) + (0,p) \in gr(F)$, that is, $y + k_2 + p \in F(x+k_1)$, by the normality of $F(x+k_1)$ would imply $y + k_2 \in F(x+k_1)$, that is, $(x,y) + (k_1,k_2) \in gr(F)$, a contradiction with (7).

(c) In view of Proposition 3.21, the convex cone $K := K_{gr(F),(x,y)}$ satisfies (9). □

Motivated by Proposition 4.2(c), given a set $A \subset \mathbb{R}^n_+ \times \mathbb{R}^m_+$ and a point $(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \setminus A$ we introduce the set-valued mapping $E_{A,(x,y)} : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ defined by

$$gr(E_{A,(x,y)}) := \mathbb{R}^n_+ \times \mathbb{R}^m_+ \setminus \left((x,y) + K_{A,(x,y)} \setminus \{(0,0)\} + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+\right),$$

with $K_{A,(x,y)}$ being the cone defined in Proposition 4.2(a).

Proposition 4.3. *For $A \subset \mathbb{R}^n_+ \times \mathbb{R}^m_+$ and $(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \setminus A$, one has:*

(i) *$gr(E_{A,(x,y)})$ is closed.*

(ii) *$dom(E_{A,(x,y)}) = \mathbb{R}^n_+$ if, and only if, $y \neq 0$.*

(iii) *$E_{A,(x,y)}$ is ICR and takes only normal values.*

Proof. (i) It is a consequence of the fact that $K_{A,(x,y)} \setminus \{(0,0)\}$ is open in $\mathbb{R}^n_+ \times \mathbb{R}^m_+$.

(ii) If $y \neq 0$ and $u \in \mathbb{R}^n_+$, then $\left(u, \frac{1}{2}y\right) \notin (x,y) + K_{A,(x,y)} \setminus \{(0,0)\} + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+$, since $K_{A,(x,y)} \subset \mathbb{R}^n_+ \times \mathbb{R}^m_+$. Therefore $\frac{1}{2}y \in E_{A,(x,y)}(u)$, and thus $u \in dom(E_{A,(x,y)})$. This proves that $dom(E_{A,(x,y)}) = \mathbb{R}^n_+$. Conversely, assume that $y = 0$, and let $v \in \mathbb{R}^m_+$. Since

$$(x,v) = (x,0) + (x,0) + (-x,v) \in (x,0) + K_{A,(x,y)} \setminus \{(0,0)\} + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+,$$

it follows that $(x,v) \notin gr(E_{A,(x,y)})$, which shows that $x \notin dom(E_{A,(x,y)})$. Thus $dom(E_{A,(x,y)}) \neq \mathbb{R}^n_+$. This proves the "only if" statement.

(iii) Let $(u,v) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \setminus gr(E_{A,(x,y)})$ and $(u',v') \in (u,v) + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+$. Since

$$\begin{aligned}(u',v') &= (u,v) + (u'-u, v'-v) \\ &\in (x,y) + K_{A,(x,y)} \setminus \{(0,0)\} + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+ + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+ \\ &= (x,y) + K_{A,(x,y)} \setminus \{(0,0)\} + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+,\end{aligned}$$

we have $(u',v') \notin gr(E_{A,(x,y)})$, which proves that

$$gr(E_{A,(x,y)}) \cap \left((u,v) + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+\right) = \emptyset.$$

Hence, by Proposition 3.21, the set-valued mapping $E_{A,(x,y)}$ is increasing and takes only normal values.

Let $(u,v) \in (x,y) + K_{A,(x,y)} \setminus \{(0,0)\} + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+$ and $t \geq 1$. Since $K_{A,(x,y)}$ is a convex cone and contains the point (x,y), we have

$$\begin{aligned}t\,(u,v) &\in t(x,y) + K_{A,(x,y)} \setminus \{(0,0)\} + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+ \\ &= (x,y) + (t-1)(x,y) + K_{A,(x,y)} \setminus \{(0,0)\} + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+ \\ &\subset (x,y) + K_{A,(x,y)} \setminus \{(0,0)\} + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+,\end{aligned}$$

which proves that $(x,y) + K_{A,(x,y)} \setminus \{(0,0)\} + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+$ is co-radiant. Hence $gr(E_{A,(x,y)})$ is radiant and therefore, by Proposition 3.4, the set-valued mapping $E_{A,(x,y)}$ is co-radiant. □

We will denote by $\mathcal{A}$ the class of nonempty closed radiant sets $A \subset \mathbb{R}^n_+ \times \mathbb{R}^m_+$ such that $A + \mathbb{R}^n_+ \times \{0\} = A$ and $\left(A + \{0\} \times (-\mathbb{R}^m_+)\right) \cap \mathbb{R}^n_+ \times \mathbb{R}^m_+ = A$, and we set $\mathcal{E} := \left\{E_{A,(x,y)} : A \in \mathcal{A},\ (x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \setminus A\right\}$.

The following theorem is one of the main results of this paper.

Theorem 4.4. *A set-valued mapping $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ is ICR, takes only normal values and has a closed graph if, and only if, there exists a set $A \in \mathcal{A}$ such that*

$$F = \bigcap_{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \setminus A} E_{A,(x,y)},$$

namely one can take

$$A := gr(F). \tag{10}$$

Proof. The "if" statement is an immediate consequence of Proposition 4.3 and the fact that the properties of being increasing, co-radiant, taking only normal values and having a closed graph are preserved by intersections. To prove the converse, we will actually prove the equivalent equality

$$gr(F) = \bigcap_{(x,y)\in\mathbb{R}^n_+\times\mathbb{R}^m_+\setminus A} gr\left(E_{A,(x,y)}\right). \tag{11}$$

Assume that F is ICR, takes only normal values and has a closed graph, and define A by (10). By Propositions 3.19, 3.4 and 3.20, we have $A \in \mathcal{A}$. One clearly has the inclusion $\subset$ in (11). To prove the opposite inclusion, let $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \setminus gr(F)$, and take $\alpha \in (0, 1)$ such that $\alpha(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \setminus A$. Then, as $(x, y) = \alpha(x, y) + \frac{1-\alpha}{\alpha}\alpha(x, y) + (0, 0) \in \alpha(x, y) + K_{A,\alpha(x,y)} \setminus \{(0, 0)\} + (-\mathbb{R}^n_+) \times \mathbb{R}^m_+$, we have $(x, y) \notin gr(E_{A,\alpha(x,y)})$. This proves the inclusion $\supset$ in (11) and hence the equality. □

Given $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$, for each $\ell \in \mathbb{R}^m_+ \setminus \{0\}$ we define $\psi_\ell : \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$\psi_\ell(x) := \sigma_{F(x)}(\ell).$$

We will need the following version of the Maximum Theorem.

Lemma 4.5. [2, Theorem 1.4.16] *Let $C\colon \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ be u.s.c. and such that it takes only nonempty compact values, and $f\colon gr(C) \to \mathbb{R}$ be upper semicontinuous. Then the function $f^{\max}\colon \mathbb{R}^n_+ \to \mathbb{R}$ defined by $f^{\max}(x) := \max_{y\in C(x)} f(x, y)$ is upper semicontinuous.*

Proposition 4.6. *Let $F\colon \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ and $\ell \in \mathbb{R}^m_+ \setminus \{0\}$.*

(i) *If F is co-radiant, then ψ_ℓ is co-radiant. The converse holds true if F has a nonempty graph and takes only closed normal values.*

(ii) *If F is increasing, then ψ_ℓ is increasing. The converse holds true if F takes only closed normal values.*

(iii) *If F is u.s.c. and takes only nonempty compact values, then ψ_ℓ is finite-valued and upper semicontinuous.*

Proof. (i) For $x \in \mathbb{R}^n_+$ and $t \in (0, 1]$, one has $F(tx) \supset tF(x)$ and therefore

$$\psi_\ell(tx) = \sigma_{F(tx)}(\ell) \geq \sigma_{tF(x)}(\ell) = t\sigma_{F(x)}(\ell) = t\psi_\ell(x).$$

Conversely, assume that F has a nonempty graph and takes only closed normal values and ψ_ℓ is co-radiant, and let $x \in \mathbb{R}^n_+$ and $\lambda \geq 1$. Using Proposition 2.7(vi), we obtain

$$\sigma_{F(\lambda x)}(l) = \psi_\ell(\lambda x) \leq \lambda\psi_\ell(x) = \lambda\sigma_{F(x)}(l) = \sigma_{\lambda F(x)}(l),$$

that is, $\sigma_{F(\lambda x)} \leq \sigma_{\lambda F(\lambda x)}$; hence, since $\ell \in \mathbb{R}^m_+ \setminus \{0\}$ is arbitrary, by Proposition 2.7(viii) we have $F(\lambda x) \subset \lambda F(x)$, which proves that F is co-radiant.

(ii) For $x, x' \in \mathbb{R}^n_+$ such that $x \leq x'$, one has $F(x) \subset F(x')$ and therefore $\psi_\ell(x) = \sigma_{F(x)}(\ell) \leq \sigma_{F(x')}(\ell) = \psi_\ell(x')$. Conversely, assume that F takes only closed normal values and ψ_ℓ is increasing, and let $x, x' \in \mathbb{R}^n_+$ be such that $x \leq x'$. For every $l \in \mathbb{R}^n_+ \setminus \{0\}$, we have

$$\sigma_{F(x)}(l) = \psi_\ell(x) \leq \psi_\ell(x') = \sigma_{F(x')}(l),$$

that is, $\sigma_{F(x)} \leq \sigma_{F(x')}$; hence, by Proposition 2.7(viii), we have $F(x) \subset F(x')$, which proves that F is increasing.

(iii) Apply Lemma 4.5 with $C := F$ and $f := \gamma$. □

Following [11], for $k \in \mathbb{R}^n_+$ and $c \in \mathbb{R}_+$, we set

$$h^+_{k,c}(x) := \max\{\max_{i \in I} k_i x_i,\, c\}.$$

For $l \in \mathbb{R}^m_+ \setminus \{0\}$, $k \in \mathbb{R}^n_+$ and $c \in \mathbb{R}_+$, we introduce the set-valued mapping $\Delta_{l,k,c} : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ by

$$\Delta_{l,k,c}(x) := \{y \in \mathbb{R}^m_+ \,:\, \max\left\{\max_{i \in I} y_i, \gamma(\ell, y)\right\} \leq h^+_{k,c}(x)\}.$$

In the case when $m = 1$, it can immediately be seen that $\Delta_{l,k,c} = [h^+_{\alpha(k,\ell),\alpha(c,\ell)}]$ with $\alpha(t, \ell) := t/\max\{\ell, 1\}$.

Proposition 4.7. *For every $l \in \mathbb{R}^m_+ \setminus \{0\}$, $k \in \mathbb{R}^n_+$ and $c \in \mathbb{R}_+$, the set-valued mapping $\Delta_{l,k,c}$ is ICR, u.s.c. and takes only nonempty compact normal values.*

Proof. Since $h^+_{k,c}$ is increasing and co-radiant, $\Delta_{l,k,c}$ is ICR. Since the set-valued mapping $y \mapsto \max\{\max_{i \in I} y_i, \gamma(\ell, y)\}$ is increasing and continuous, the sets $\Delta_{l,k,c}(x)$ are normal and closed. Since they are contained in the cube $\left[0, h^+_{k,c}(x)\right]^m$, they are also bounded, hence compact. It is clear that they contain the origin, so they are nonempty. To see that $\Delta_{l,k,c}$ is u.s.c., notice that it is the pointwise intersection of the set-valued mappings $\Delta^1_{l,k,c}$ and $\Delta^2_{k,c}$ defined by

$$\Delta^1_{l,k,c}(x) := \{y \in \mathbb{R}^m_+ \,: \gamma(\ell, y) \leq h^+_{k,c}(x)\}$$

and

$$\Delta^2_{k,c}(x) := \left[0, h^+_{k,c}(x)\right]^m,$$

respectively. Since the functions $\gamma(\ell, \cdot)$ and $h^+_{k,c}$ are continuous, the set-valued mapping $\Delta^1_{l,k,c}$ has a closed graph. The continuity of $h^+_{k,c}$ also implies the upper semicontinuity of $\Delta^2_{k,c}$. Hence, since $\Delta^2_{k,c}$ is compact-valued, by [1, Theorem 17.25.2] the set-valued mapping $\Delta_{l,k,c}$ is u.s.c.. □

The second main result of this paper is the following generalization of [6, Theorem 5.2].

Theorem 4.8. *A set-valued mapping $F : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ is ICR, u.s.c. and takes only nonempty compact normal values if, and only if, there exists a nonempty set $T \subset \left(\mathbb{R}^m_+ \setminus \{0\}\right) \times \mathbb{R}^n_+ \times \mathbb{R}_+$ such that*

$$F = \bigcap_{(l,k,c)\in T} \Delta_{l,k,c}. \tag{12}$$

Proof. Assume that F is ICR, u.s.c. and takes only nonempty compact normal values, and let $x \in \mathbb{R}^n_+$. By Corollary 2.9, we have

$$\begin{aligned} F(x) &= \{y \in \mathbb{R}^m_+ \,:\, \gamma(\ell, y) \le \sigma_{F(x)}(\ell),\ \forall \ell \in \mathbb{R}^m_+ \setminus \{0\}\} \\ &= \{y \in \mathbb{R}^m_+ \,:\, \gamma(\ell, y) \le \psi_\ell(x), \forall \ell \in \mathbb{R}^m_+ \setminus \{0\}\} \\ &= \bigcap_{\ell\in\mathbb{R}^m_+\setminus\{0\}} \{y \in \mathbb{R}^m_+ \,: \gamma(\ell, y) \le \psi_\ell(x)\}. \end{aligned} \tag{13}$$

On the other hand, by Proposition 4.6 and [6, Theorem 5.2], for every $\ell \in \mathbb{R}^m_+\setminus\{0\}$ there exists a nonempty set $T_\ell \subset \mathbb{R}^n_+ \times \mathbb{R}_+$ such that

$$\psi_\ell(x) := \inf_{(k,c)\in T_\ell} h^+_{k,c}(x).$$

From this equality and (13), we obtain that $F(x) = \bigcap_{\ell\in\mathbb{R}^m_+\setminus\{0\}} \bigcap_{(k,c)\in T_\ell} \Delta^1_{l,k,c}(x)$. Hence $F(x) \subset \bigcap_{i=1}^m \bigcap_{(k,c)\in T_{e^i}} \Delta^1_{e^i,k,c}(x) = \Delta^2_{k,c}(x)$; here we denote by e^i the i-th unit vector.We therefore have

$$\begin{aligned} F(x) &= F(x) \cap \Delta^2_{k,c}(x) = \bigcap_{\ell\in\mathbb{R}^m_+\setminus\{0\}} \bigcap_{(k,c)\in T_\ell} \left(\Delta^1_{l,k,c}(x) \cap \Delta^2_{k,c}(x)\right) \\ &= \bigcap_{\ell\in\mathbb{R}^m_+\setminus\{0\}} \bigcap_{(k,c)\in T_\ell} \Delta_{l,k,c}(x). \end{aligned}$$

From this equality we immediately obtain (12), with

$$T := \left\{(l,k,c) \in \left(\mathbb{R}^m_+ \setminus \{0\}\right) \times \mathbb{R}^n_+ \times \mathbb{R}_+ : (k,l) \in T_l\right\}.$$

To prove the converse, assume that (12) holds for some nonempty set $T \subset \left(\mathbb{R}^m_+ \setminus \{0\}\right) \times \mathbb{R}^n_+ \times \mathbb{R}_+$. Then, using Proposition 4.7, it is easy to prove that the set-valued mapping F is ICR and takes only nonempty compact normal values. Its upper semicontinuity follows from Proposition 4.7 and [1, Theorem 17.25.3]. □

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