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On Far and Near Ends of Closed and Convex Sets

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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We first introduce for a closed and convex set two classes of subsets: the near and far ends relative to a point, and give some full characterizations for these end sets by virtue of the face theory of closed and convex sets. We also provide some connections between closedness of the far (near) end and the relative continuity of the gauge (cogauge) for closed and convex sets. Moreover, motivated by these connections, we clarify Conjecture 6.1 of K.-W. Meng, V. Roshchina and X.-Q. Yang [*On local coincidence of a convex set and its tangent cone*, J. Optim. Theory Appl. 164 (2015) 123–137] in the sense that the gauge of a closed and convex set containing 0 is continuous relative to its domain if it is relatively continuous at 0, holds always in the two-dimensional case, but is not true in general by constructing a three-dimensional counterexample.

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1. Introduction

In this paper we introduce two subsets for a closed and convex set C: the near end and the far end. The near end of C relative to a given point x_0, denoted by $\mathrm{near}(C, x_0)$, consists of all the points x in C such that, among the points in the intersection of C with the closed half-line $\{x_0 + t(x - x_0) \mid t \geq 0\}$ emanating from x_0, the nearest one to x_0 is x, while in the same manner the far end of C relative to x_0, denoted by $\mathrm{far}(C, x_0)$, consists of the farthest ones. Roughly speaking, the subset $\mathrm{near}(C, x_0)$ of C is nothing else but its visible part seen from x_0, while the

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subset $\mathrm{far}(C, x_0)$ of C is nothing else but its farthest part seen from x_0. Note that the far end relative to the origin was referred to as the end set in [6, 7, 10], and that the notation and terminology of near and far ends relative to a point not necessarily being the origin appear here for the first time.

The reasons for a systematic investigation of the near and far ends of closed and convex sets are twofold. The first reason is that many important properties appeared in variational analysis and optimization have been expressed by the near and far ends of closed and convex sets. For instance, it was shown in [18, Proposition 2.5] that for an inequality system defined by finitely many differentiable convex functions in Banach space, the basic constraint qualification (BCQ) and the strong BCQ are the same, of which the proof relies on a property that the distance of 0 from the far end of the subdifferential set relative to 0 is always greater than 0. The latter property was then utilized to characterize the general difference between BCQ and strong BCQ for an inequality system defined by a proper convex function, and to investigate local and global error bounds for a convex inequality defined by a proper convex function in a Banach space in [6] and [7] respectively.

This property was also shown to be equivalent to a number of interesting properties such as the exact tangent approximation property for a closed and convex set, the relative continuity of some gauge function, and the global error bound property for some support function in [12]. Recently, it has been shown in [10] that the local error bound modulus of a locally Lipschitz and regular function is upper bounded by the distance of 0 from the far end of the subdifferential set relative to 0. See [9, 5] for results on some lower estimates of the local error bound moduli in terms of the outer limiting subdifferential set, which have some connections with the far end of the subdifferential set relative to 0.

The second one is that, despite the usefulness of the near and far ends in variational analysis and optimization just mentioned, the investigation about their structures in connection with the facial structure of closed and convex sets is limited. In the case of C being a convex polytope it is clear to see from [2, 3, 10] that, $\mathrm{far}(C, 0)$ equals to the subset consisting of all the faces F of C such that F can be exposed by some vector w at which the support function σ_C of C takes a positive value, while in the case of C being a general compact convex set, the subset constructed in the same manner just mentioned is merely a dense subset of $\mathrm{far}(C, 0)$ as can be seen from [10, Theorem 2.1 and Example 2.1]. So it is desirable to use general faces instead of exposed faces only to fully characterize the near and far ends of a general closed and convex set, with or without a polyhedral structure or the boundedness assumption.

For the reasons listed above, we conduct in this paper a systematic investigation of the near and far ends of closed and convex sets. In Section 2, by virtue of the face theory [14, Section 18], we give some full characterizations for the near and far ends of a closed and convex set. In Section 3, we provide some connections between closedness of the far (near) end and the relative continuity of the gauge (cogauge) for closed and convex sets. We end the paper by pointing out that the conjecture [12, Conjecture 6.1] that the gauge of a closed and convex set containing 0 is continuous relative to its domain if it is relatively continuous at 0, though holds always in the two-dimensional case, but is not true in general by constructing a three-dimensional counterexample.

Throughout the paper we use the standard notation in variational analysis and optimization; see the seminal book [15] by Rockafellar and Wets. The *Euclidean norm* of a vector x is denoted by $||x||$, the *transpose* of a vector x is denoted by x^T, and the *inner product* of vectors x and y is denoted by $\langle x, y\rangle$. The portion of the line corresponding to $\lambda x_0 + (1-\lambda)x_1$ with $\lambda \in [0,1]$ is called the *closed line segment* joining x_0 and x_1, denoted by $[x_0, x_1]$. This terminology is also used if x_0 and x_1 coincide, although the line segment reduces then to a single point.

We think of the *direction* of x, denoted by $\operatorname{dir} x$, as an attribute associated with x under the rule that $\operatorname{dir} x = \operatorname{dir} x'$ means $x' = \lambda x \neq 0$ for some $\lambda > 0$. The zero vector is regarded as having no direction; $\operatorname{dir} 0$ is undefined. As half-lines can be viewed as infinite line segments joining ordinary points with direction points, we use the notation $[x, \operatorname{dir} y) := \{x + ty \mid t \geq 0\}$ to denote the closed half-line emanating from x and going along the direction of y. For the sake of simplicity, we use the following notation for a point x and a set Y throughout the paper:

$$[x,\, Y] := \bigcup_{y\in Y} [x,\, y], \qquad [x,\, \operatorname{dir} Y) := \bigcup_{y\in Y} [x,\, \operatorname{dir} y).$$

Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R}\cup\{\pm\infty\}$ be an extended-real-valued function and let $A \subset \mathbb{R}^n$. The *domain* of f is denoted by $\operatorname{dom} f := \{x \mid f(x) < \infty\}$. We say that f is *continuous* at some $\bar{x} \in A$ relative to A if, for each $x_k \to \bar{x}$ with $x_k \in A$ for all k, it holds that $f(x_k) \to f(\bar{x})$. If f is continuous at every $x \in A$ relative to A, we say that f is continuous relative to A. We denote the *interior*, the *relative interior*, the *closure*, the *boundary*, the *convex hull* and the *positive hull* of A respectively by $\operatorname{int} A$, $\operatorname{ri} A$, $\operatorname{cl} A$, $\operatorname{bd} A$, $\operatorname{co} A$ and $\operatorname{pos} A$. The *horizon cone* of A is defined by

$$A^\infty := \{x \mid \exists t_n \to 0+, x_n \in A, \text{ with } t_n x_n \to x\}$$

if $A \neq \emptyset$, and $A^\infty := \{0\}$ if $A = \emptyset$. The *support function* $\sigma_A\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ of A is defined by

$$\sigma_A(w) := \sup_{x\in A} \langle x, w\rangle.$$

The *distance* from x to A is defined by

$$d(x, A) := \inf_{y\in A} \|y - x\|.$$

Let $C \subset \mathbb{R}^n$ be a nonempty and convex set. A *face* of C is a convex subset C' of C such that every closed line segment in C with a relative interior point in C' has both endpoints in C'. An *exposed face* of C is the intersection of C and a non-trivial supporting hyperplane to C. See [14, Section 18] for more results on faces of convex sets.

Definition 1.1. For a closed and convex set $C \subset \mathbb{R}^n$ and some $x_0 \notin C$, the *shadow* of C cast by a light source at x_0 is defined by

$$\operatorname{shad}(C, x_0) := \bigcup_{x \in C} \{x_0 + t(x - x_0) \mid t \geq 1\}.$$

The *gauge* of a closed and convex set C with $0 \in C$ is the function $\gamma_C \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\gamma_C(x) := \inf\{\lambda \geq 0 | x \in \lambda C\}, \tag{1}$$

where the convention $\inf \emptyset = +\infty$ is used. This function is nonnegative, lower semicontinuous and sublinear with $\operatorname{dom} \gamma_C = \operatorname{pos} C$. See [15, Example 3.50] for more properties of the gauge function γ_C. For a nonempty closed and convex set $C \subset \mathbb{R}^n$ such that $0 \notin C$ and $\lambda C \subset C$ for all $\lambda \geq 1$, the cogauge of C is the function $\nu_C \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\nu_C(x) := \begin{cases} \sup\{t \geq 0 \mid x \in tC\}, & \text{if } x \in \operatorname{cl}(\operatorname{pos} C), \\ -\infty, & \text{otherwise}, \end{cases} \tag{2}$$

where the convention $\sup \emptyset = 0$ is used. This function is upper semicontinuous and suplinear with $\operatorname{dom}(-\nu_C) = \operatorname{cl}(\operatorname{pos} C)$. See [16, 13, 4, 1, 17].

2. Characterizations of far and near ends of closed and convex sets

We begin with the formal definition of near and far ends of a closed and convex set.

Definition 2.1. Consider a closed and convex set $C \subset \mathbb{R}^n$ and a point $x_0 \in \mathbb{R}^n$. The *far end* of C relative to x_0 is defined by

$$\operatorname{far}(C, x_0) := \{x \in C \mid x_0 + t(x - x_0) \notin C \, \forall t > 1\},$$

while the *near end* of C relative to x_0 is defined by

$$\operatorname{near}(C, x_0) := \begin{cases} \{x \in C \mid x_0 + t(x - x_0) \notin C \, \forall t < 1\} & \text{if } x_0 \notin C, \\ \{x_0\} & \text{if } x_0 \in C. \end{cases}$$

Remark 2.2. It is clear to see that $\operatorname{near}(C, x_0)$ consists of all the points $x \in C$ such that among the points in the intersection of C with the closed half-line $\{x_0 + t(x - x_0) \mid t \geq 0\}$ emanating from x_0, the nearest one to x_0 is x, while in the same manner $\operatorname{far}(C, x_0)$ consists of the farthest ones. With the help of $\operatorname{near}(C, x_0)$ and $\operatorname{far}(C, x_0)$, we can decompose C into distinct (infinite) line segments (having a common endpoint x_0 if $x_0 \in C$). That is, in the case of $x_0 \in C$, C consists of line segments $[x_0, y]$ with $y \in \operatorname{far}(C, x_0)$, and closed half-lines $[x_0, \operatorname{dir} y)$ with $y \in C^\infty$, while in the case of $x_0 \notin C$, C consists of line segments $[x, y]$ with $x \in \operatorname{near}(C, x_0)$, $y \in \operatorname{far}(C, x_0)$ and $\operatorname{dir}(x - x_0) = \operatorname{dir}(y - x_0)$, and closed half-lines $[x, \operatorname{dir} y)$ with $x \in \operatorname{near}(C, x_0)$, $y \in \operatorname{pos}(C - x_0) \cap C^\infty$ and $\operatorname{dir}(x - x_0) = \operatorname{dir} y$.

Basic properties of near and far ends are summarized in the following lemmas.

Lemma 2.3. *For a nonempty closed and convex set $C \subset \mathbb{R}^n$, the following hold:*

(a) $\operatorname{far}(C, x_0) = \operatorname{near}(C, x_0) = C$ *for all* $x_0 \notin \operatorname{aff} C$.

(b) $\operatorname{far}(C, x_0) \cap \operatorname{ri} C = \emptyset$ *for all* $x_0 \in C$.

(c) $\operatorname{far}(C, x_0) \cap \operatorname{ri} C = \operatorname{near}(C, x_0) \cap \operatorname{ri} C = \emptyset$ *for all* $x_0 \in \operatorname{aff} C \backslash C$.

(d) $\operatorname{near}(C, x_0) \neq \emptyset$ *for all* $x_0 \in \mathbb{R}^n$.

(e) *For any* $x_0 \in \mathbb{R}^n$, $\operatorname{far}(C, x_0) \neq \emptyset$ *if and only if* $\operatorname{pos}(C - x_0) \backslash C^\infty \neq \emptyset$.

Lemma 2.4. *Consider a nonempty closed and convex set $C \subset \mathbb{R}^n$ and a subset $E \subset C$. If $x_0 \in C$, then* $C = [x_0,\ E] \cup [x_0, \operatorname{dir} C^\infty)$ *if and only if* $\operatorname{far}(C, x_0) \subset E$. *If $x_0 \notin C$, then* $\operatorname{shad}(C, x_0) = \operatorname{shad}(E, x_0)$ *if and only if* $\operatorname{near}(C, x_0) \subset E$.

Lemma 2.5. *For a closed and convex set $C \subset \mathbb{R}^n$ and a point $x_0 \notin C$, the following hold:*

(a) $\operatorname{far}(C, x_0) = \operatorname{far}\left(\operatorname{cl}(\operatorname{co}(C \cup \{x_0\})), x_0\right)$.

(b) $\operatorname{near}(C, x_0) = \operatorname{near}\left(\operatorname{shad}(C, x_0), x_0\right)$.

In the following theorems, we will give full characterizations for the far and near ends of closed and convex sets by virtue of their (exposed) faces.

Theorem 2.6. *Consider a nonempty closed and convex set $C \subset \mathbb{R}^n$ and a point $x_0 \in \mathbb{R}^n$. If $x_0 \in C$, the following hold:*

(a) $\operatorname{far}(C, x_0)$ *consists of all the faces F of C such that $x_0 \notin F$.*

(b) *The set consisting of all the exposed faces F of C such that $x_0 \notin F$, is a dense subset of* $\operatorname{far}(C, x_0)$, *or in other words,*

$$\bigcup_{\sigma_C(w) > \langle x_0, w\rangle} \partial \sigma_C(w) \subset \operatorname{far}(C, x_0) \subset \operatorname{cl}\left(\bigcup_{\sigma_C(w) > \langle x_0, w\rangle} \partial \sigma_C(w)\right). \tag{3}$$

If $x_0 \notin C$, the following hold in terms of $C' := \operatorname{cl}(\operatorname{co}(C \cup \{x_0\}))$:

(a) $\operatorname{far}(C, x_0)$ *consists of all the common faces of C and C'.*

(b) *The set consisting of all the common exposed faces of C and C', is a dense subset of* $\operatorname{far}(C, x_0)$, *or in other words, the inclusions in* (3) *hold.*

Proof. (a) Let $\mathcal{F}$ be the collection of all the faces of C, $\mathcal{F}^+(x_0)$ be the collection of all the faces F of C such that $x_0 \notin F$, and $\mathcal{F}^0(x_0)$ be the collection of all the faces F of C such that $x_0 \in F$. Clearly, we have $\mathcal{F} = \mathcal{F}^+(x_0) \cup \mathcal{F}^0(x_0)$ and $\mathcal{F}^+(x_0) \cap \mathcal{F}^0(x_0) = \emptyset$.

Let $F \in \mathcal{F}^+(x_0)$. Suppose by contradiction that the inclusion $F \subset \operatorname{far}(C, x_0)$ is invalid, i.e., there is some $x \in F$ such that $x \notin \operatorname{far}(C, x_0)$. Since $x_0 \notin F$, we have $x \neq x_0$. There exists some $t_0 > 1$ such that $x_0 + t_0(x - x_0) \in C$ for otherwise by definition $x \in \operatorname{far}(C, x_0)$. This entails that $x \in \operatorname{ri}[x_0, x_0 + t_0(x - x_0)] \cap F$. By [14, Theorem 18.1], we have $[x_0, x_0 + t_0(x - x_0)] \subset F$, contradicting to the fact $x_0 \notin F$.

This implies that

$$\bigcup_{F\in\mathcal{F}^+(x_0)} F \subset \operatorname{far}(C,x_0). \tag{4}$$

Let $F \in \mathcal{F}^0(x_0)$. By Lemma 2.3, we have $\operatorname{ri} F \cap \operatorname{far}(F,x_0) = \emptyset$. Since it follows from the definition that $\operatorname{far}(C,x_0)\cap F \subset \operatorname{far}(F,x_0)$, we thus have $\operatorname{ri} F \cap \operatorname{far}(C,x_0)\cap F = \emptyset$. As it always holds that $\operatorname{ri} F \subset F$, we have $\operatorname{far}(C,x_0)\cap \operatorname{ri} F = \emptyset$. This implies that

$$\operatorname{far}(C,x_0) \cap \left(\cup_{F\in\mathcal{F}^0(x_0)} \operatorname{ri} F\right) = \emptyset. \tag{5}$$

By [14, Theorem 18.2], we have $C = \bigcup_{F\in\mathcal{F}^+(x_0)\cup\mathcal{F}^0(x_0)} \operatorname{ri} F$. In view of (5) and the fact that $\operatorname{far}(C,x_0) \subset C$, we have

$$\operatorname{far}(C,x_0) \subset \bigcup_{F\in\mathcal{F}^+(x_0)} \operatorname{ri} F,$$

which together with (4) implies (a).

(b) Let $S^+ := \bigcup_{F\in\mathcal{S}^+(x_0)} F$ and $S^0 := \bigcup_{F\in\mathcal{S}^0(x_0)} F$, where $\mathcal{S}^+(x_0)$ is the collection of all the exposed faces F of C such that $x_0 \notin F$, and $\mathcal{S}^0(x_0)$ is the collection of all the exposed faces F of C such that $x_0 \in F$ and $F \neq C$. Clearly, we have $S^+ \cup S^0 \subset C$. Since $\mathcal{S}^+(x_0) \subset \mathcal{F}^+(x_0)$, it follows from (**a**) that $S^+ \subset \operatorname{far}(C,x_0)$.

Now let $x \in \operatorname{far}(C,x_0)$ be given arbitrarily. By Lemma 2.3, we have $x \notin \operatorname{ri} C$. It then follows from [14, Theorem 11.6] that there exists a non-trivial supporting hyperplane H to C at x. This entails that $F := C \cap H$ is a non-trivial exposed face of C containing x. Then we have either $F \in \mathcal{S}^+(x_0)$ or $F \in \mathcal{S}^0(x_0)$. So we have $x \in S^+ \cup S^0$ and hence the inclusion $\operatorname{far}(C,x_0) \subset S^+ \cup S^0$. Then by Lemma 2.4, we have

$$C = [x_0,\ S^+ \cup S^0] \cup [x_0, \operatorname{dir} C^\infty) = [x_0,\ S^+] \cup [x_0,\ S^0] \cup [x_0, \operatorname{dir} C^\infty). \tag{6}$$

Since each $F \in \mathcal{S}^0(x_0)$ is a non-trivial exposed face of C containing x_0, we have $[x_0, F] = F$ and $F \cap \operatorname{ri} C = \emptyset$. So we have $[x_0,\ S^0] = S^0$ and $S^0 \cap \operatorname{ri} C = \emptyset$. In view of (6), we have

$$\operatorname{ri} C \subset [x_0,\ S^+] \cup [x_0, \operatorname{dir} C^\infty).$$

Due to $C = \operatorname{cl}(\operatorname{ri} C)$ and $[x_0, \operatorname{dir} C^\infty)$ being closed, we have

$$C \subset \operatorname{cl}[x_0,\ S^+] \cup [x_0, \operatorname{dir} C^\infty). \tag{7}$$

Let $y \in C\backslash[x_0, \operatorname{dir} C^\infty)$ be given arbitrarily. In view of (7), we have $y \in \operatorname{cl}[x_0,\ S^+]$. Then there exist some $y_k \to y$ and $\{\lambda_k\} \subset [1, +\infty)$ such that $x_0+\lambda_k(y_k-x_0) \in S^+$ for all k. Let $x_k := x_0 + \lambda_k(y_k - x_0)$ for all k. Clearly, we have $x_k \in S^+ \subset C$ for all k. If $\{\lambda_k\}$ is unbounded, then $\frac{1}{\lambda_k}x_k = y_k - x_0 + \frac{1}{\lambda_k}x_0 \to y - x_0$, implying by definition that $y - x_0 \in C^\infty$ and hence that $y \in [x_0, \operatorname{dir} C^\infty)$, a contradiction to our assumption that $y \in C\backslash[x_0, \operatorname{dir} C^\infty)$. So we may assume that $\lambda_k \to \lambda^* \geq 1$ by taking a subsequence if necessary. Then, we have $x_k = x_0 + \lambda_k(y_k - x_0) \to x_0 + \lambda^*(y - x_0) \in \operatorname{cl} S^+$ and hence

$$y \in [x_0, x_0 + \lambda^*(y - x_0)] \subset [x_0, \operatorname{cl} S^+].$$

As $y \in C \backslash [x_0, \operatorname{dir} C^\infty)$ is given arbitrarily, we have $C \backslash [x_0, \operatorname{dir} C^\infty) \subset [x_0, \operatorname{cl} S^+]$, and hence

$$C \subset [x_0,\ \operatorname{cl} S^+] \cup [x_0, \operatorname{dir} C^\infty).$$

Since $S^+ \subset C$ and C is closed, we have $\operatorname{cl} S^+ \subset C$. Then we actually have

$$C = [x_0,\ \operatorname{cl} S^+] \cup [x_0, \operatorname{dir} C^\infty).$$

By Lemma 2.4 again, we have $\operatorname{far}(C, x_0) \subset \operatorname{cl} S^+$. In view of the inclusion $S^+ \subset \operatorname{far}(C, x_0)$ we obtained above, we get the desired result that S^+ is a dense subset of $\operatorname{far}(C, x_0)$. It remains to show that S^+ can be rewritten as the union in (3). To see why F is an exposed face of C such that $x_0 \notin F$ if and only if $F = \partial \sigma_C(w)$ for some $w \in \mathbb{R}^n$ with $\sigma_C(w) > \langle x_0, w \rangle$, it suffices to observe that $\partial \sigma_C(w) = \arg\max_{x \in C} \langle x, w \rangle$ for all $w \in \mathbb{R}^n$ with $\sigma_C(w) := \sup_{x \in C} \langle x, w \rangle < +\infty$, and that $x_0 \notin \arg\max_{x \in C} \langle x, w \rangle$ if and only if $\sigma_C(w) > \langle x_0, w \rangle$.

(c) By Lemma 2.5, we have $\operatorname{far}(C, x_0) = \operatorname{far}(C', x_0)$. In view of (a) and the fact that C' is a closed and convex set containing x_0, we deduce that $\operatorname{far}(C, x_0)$ consists of all the faces F of C' such that $x_0 \notin F$. It remains to show that F is a face of C' such that $x_0 \notin F$ if and only if F is a common face of C' and C. The 'if' part follows easily from a simple fact that $x_0 \notin C$, implying that all the faces of C do not contain x_0. To see the 'only if' part, we observe that if F is a face of C' such that $x_0 \notin F$, then $F \subset \operatorname{far}(C, x_0) \subset C \subset C'$ and F is a fortiori a face of C.

(d) By the same argument used in the proof of (**c**), the set consisting of all the exposed faces F of C' such that $x_0 \notin F$, is a dense subset of $\operatorname{far}(C, x_0) = \operatorname{far}(C', x_0)$, and moreover, it can be written as the union of all $\partial \sigma_{C'}(w)$ such that $\sigma_{C'}(w) > \langle x_0, w \rangle$. Since it follows from the definition that $\sigma_{C'}(w) = \max\{\sigma_C(w), \langle x_0, w \rangle\}$ for all $w \in \mathbb{R}^n$, we have $\partial \sigma_{C'}(w) = \partial \sigma_C(w)$ whenever $\sigma_{C'}(w) > \langle x_0, w \rangle$ or equivalently $\sigma_C(w) > \langle x_0, w \rangle$. This justifies the validity of the inclusions in (3) in the circumstance that $x_0 \notin C$.

It remains to show that F is an exposed face of C' such that $x_0 \notin F$ if and only if F is a common exposed face of C and C'. The 'if' part follows easily from a simple fact that $x_0 \notin C$, implying that all the exposed faces of C do not contain x_0. To see the 'only if' part, we observe that if F is an exposed face of C' such that $x_0 \notin F$, then $F \subset \operatorname{far}(C, x_0) \subset C \subset C'$ and F is a fortiori an exposed face of C. The proof is completed. □

Theorem 2.7. *For a nonempty closed and convex set $C \subset \mathbb{R}^n$ and a point $x_0 \notin C$, the following hold in terms of $C' := \operatorname{shad}(C, x_0)$:*

(a) $\operatorname{near}(C, x_0)$ *consists of all common faces F of C and C' such that $x_0 \notin \operatorname{aff} F$.*

(b) *The set consisting of all common exposed faces F of C and C' such that $x_0 \notin \operatorname{aff} F$, is a dense subset of* $\operatorname{near}(C, x_0)$*, or in other words,*

$$\bigcup_{\sigma_C(w) < \langle x_0, w \rangle} \partial \sigma_C(w) \subset \operatorname{near}(C, x_0) \subset \operatorname{cl} \left(\bigcup_{\sigma_C(w) < \langle x_0, w \rangle} \partial \sigma_C(w) \right). \tag{8}$$

Proof. In view of Lemma 2.5, we have $\mathrm{near}(C, x_0) = \mathrm{near}(C', x_0)$. Our argument is first applied to C' and then to C. By the same techniques used in the proof of Theorem 2.6, we can get all the results readily. So we omit the proof. $\square$

3. On closedness of near and far ends

Many properties of a closed and convex set can be expressed in terms of its far (near) end, see [6, 7, 12, 10] for more details. In this section, we will study the continuity properties of the gauge and the cogauge of closed and convex sets in connection with closedness of their far and near ends.

From Definition 2.1 and the definition of the gauge γ_C (cf. (1)) for a closed and convex set C with $0 \in C$, it follows that

$$\mathrm{far}(C, 0) = \{x \in \mathbb{R}^n \mid \gamma_C(x) = 1\};$$

and from Definition 2.1 and the definition of the cogauge ν_C (cf. (2)) for a closed and convex set C with $0 \notin C$ and $\lambda C \subset C$ for all $\lambda \geq 1$, it follows that

$$\mathrm{near}(C, 0) = \{x \in \mathbb{R}^n \mid \nu_C(x) = 1\}.$$

These equalities shed light on the relations between closedness of the far (near) end and the continuity property of the gauge (cogauge) function as can be seen from the following theorems.

Theorem 3.1. *For a nonempty closed and convex set $C \subset \mathbb{R}^n$ such that $0 \notin C$ and $\lambda C \subset C$ for all $\lambda \geq 1$, we have*

$$\mathrm{dom}(-\nu_C) = C^\infty = (\mathrm{near}(C, 0))^\infty \cup (\mathrm{pos}\, C), \tag{9}$$

and the following hold:

(a) ν_C *is continuous at every* $x \in \{0\} \cup (\mathrm{ri}\, C^\infty) \cup (C^\infty \setminus \mathrm{pos}\, C)$ *relative to* C^∞.

(b) ν_C *is continuous relative to* C^∞ *if and only if* $\mathrm{near}(C, 0)$ *is closed and*

$$(\mathrm{near}(C, 0))^\infty \cap \mathrm{pos}\, C = \{0\}. \tag{10}$$

(c) *If* C^∞ *is a polyhedral cone, as is true in particular when* $\dim C \leq 2$, *then* ν_C *is continuous relative to* C^∞.

Proof. By the definition of the cogauge function ν_C, we have $\mathrm{dom}(-\nu_C) = \mathrm{cl}(\mathrm{pos}\, C)$. In view of [15, Exercise 3.48], we have $\mathrm{cl}(\mathrm{pos}\, C) = (\mathrm{pos}\, C) \cup C^\infty$. Since $\lambda C \subset C$ for all $\lambda \geq 1$, we have $x + tx \in C$ for all $t \geq 0$ and $x \in C$. It then follows from [15, Theorem 3.6] that $x \in C^\infty$ and hence that $C \subset C^\infty$. Thus we have $(\mathrm{pos}\, C) \subset C^\infty$ and hence $\mathrm{cl}(\mathrm{pos}\, C) = C^\infty$. This entails that $\mathrm{dom}(-\nu_C) = C^\infty$.

Since $\mathrm{near}(C, 0) \subset C$, we have $(\mathrm{near}(C, 0))^\infty \subset C^\infty$ and hence

$$(\mathrm{near}(C, 0))^\infty \cup (\mathrm{pos}\, C) \subset C^\infty.$$

Let $x \in C^\infty \setminus (\mathrm{pos}\, C) = \mathrm{cl}(\mathrm{pos}\, C) \setminus (\mathrm{pos}\, C)$. Clearly, we have $x \neq 0$ and $\nu_C(x) = 0$. Then there exists some $x_n \to x$ with $x_n \in (\mathrm{pos}\, C) \setminus \{0\}$ for all n. We claim that $\nu_C(x_n) \to 0+$, for otherwise we deduce from the upper semi-continuity of ν_C that $\nu_C(x) \geq \limsup_{n \to +\infty} \nu_C(x_n) > 0$, a contradiction.

Observing that $\nu_C(x_n)\frac{x_n}{\nu_C(x_n)} \to x$ and that $\frac{x_n}{\nu_C(x_n)} \in \mathrm{near}(C,0)$ for all n, we have $x \in (\mathrm{near}(C,0))^\infty$ by the definition of horizon cones. This entails the equality $(\mathrm{near}(C,0))^\infty \cup (\mathrm{pos}\, C) = C^\infty$ as expected.

(a) By the continuity properties of convex functions [15, Theorem 2.35], the lsc sublinear function $-\nu_C$ is continuous relative to $\mathrm{ri}(\mathrm{dom}(-\nu_C)) = \mathrm{ri}\, C^\infty$, and so is ν_C. Now let $x \in C^\infty \backslash (\mathrm{pos}\, C)$ or $x = 0$, and $x_n \to x$ with $x_n \in C^\infty$ for all n. In view of the facts that $\nu_C(x) = 0$ and $\nu_C(x_n) \geq 0$ for all n, and that ν_C is upper semi-continuous, we have $0 \leq \liminf_{n\to+\infty} \nu_C(x_n) \leq \limsup_{n\to+\infty} \nu_C(x_n) \leq \nu_C(x) = 0$. That is, $\lim_{n\to+\infty} \nu_C(x_n) = \nu_C(x)$. This entails that ν_C is continuous at every $x \in \{0\} \cup (\mathrm{ri}\, C^\infty) \cup (C^\infty \backslash \mathrm{pos}\, C)$ relative to C^∞.

(b) Assume that ν_C is continuous relative to C^∞. Suppose by contradiction that (10) does not hold, i.e., there exists some $x \in (\mathrm{near}(C,0))^\infty \cap \mathrm{pos}\, C$ with $x \neq 0$. By the definition of horizon cones, there exist some $t_n \to 0+$ and $x_n \in \mathrm{near}(C,0)$ such that $t_n x_n \to x$. Clearly, we have $t_n x_n \in \mathrm{pos}\, C \subset C^\infty$ for all n. Since ν_C is by assumption continuous relative to C^∞, we have $\nu_C(t_n x_n) \to \nu_C(x)$. In view of $\nu_C(t_n x_n) = t_n \nu_C(x_n) = t_n$ and $t_n \to 0+$, we have $\nu_C(x) = 0$, contradicting to the fact that $x \in (\mathrm{pos}\, C)\backslash\{0\}$. This contradiction indicates that (10) holds. To show the closedness of $\mathrm{near}(C,0)$, let $x_n \to x^*$ with $x_n \in \mathrm{near}(C,0)$ for all n. Clearly, we have $\nu_C(x_n) = 1$ and $x_n \in C \subset \mathrm{pos}\, C$ for all n, implying that $x^* \in C \subset \mathrm{pos}\, C$. By the continuity assumption we have $\nu_C(x_n) \to \nu_C(x^*)$. This entails that $\nu_C(x^*) = 1$ or equivalently $x^* \in \mathrm{near}(C,0)$. That is, $\mathrm{near}(C,0)$ is closed.

Conversely, we assume that $\mathrm{near}(C,0)$ is closed and that equation (10) holds. Let $x^* \in (\mathrm{pos}\, C)\backslash (\{0\} \cup \mathrm{ri}(\mathrm{pos}\, C))$ be given arbitrarily. In view of (a), we need only prove that ν_C is continuous at x^* relative to C^∞. Let $x_n \to x^*$ with $x_n \in C^\infty \backslash \{0\}$ for all n. Without loss of generality, we may assume that $x_n \in \mathrm{pos}\, C \backslash \{0\}$ for all n, for otherwise by taking a subsequence if necessary we have $x_n \in \mathrm{near}(C,0)^\infty$ for all n (due to (9)), implying that $x^* \in \mathrm{near}(C,0)^\infty$ (due to $\mathrm{near}(C,0)^\infty$ being closed), and hence that $x^* \notin \mathrm{pos}\, C$ (due to (10)), a contradiction to the setting for x^*.

We claim that the sequence $\{\nu_C(x_n)\} \subset (0,+\infty)$ is bounded, for otherwise by taking a subsequence if necessary we have $\nu_C(x_n) \to +\infty$, implying that $\frac{x_n}{\nu_C(x_n)} \to 0$ with $\frac{x_n}{\nu_C(x_n)} \in \mathrm{near}(C,0) \subset C$ for all n and hence that $0 \in C$, a contradiction to the setting for C. Let β be any clustering point of the sequence $\{\nu_C(x_n)\}$. Clearly, we have $0 \leq \beta < +\infty$.

We claim that $\beta = 0$ is impossible, for otherwise by taking a subsequence if necessary we have $\nu_C(x_n) \to 0+$ and $\frac{x_n}{\nu_C(x_n)} \in \mathrm{near}(C,0)$ for all n, implying that $x^* \in \mathrm{near}(C,0)^\infty$ and hence that $x^* \notin \mathrm{pos}\, C$, which is a contradiction to the setting for x^* again. So we have $0 < \beta < +\infty$. As $\frac{x_n}{\nu_C(x_n)} \in \mathrm{near}(C,0)$ for all n and $\mathrm{near}(C,0)$ is closed by assumption, by taking a subsequence if necessary we have $\frac{x_n}{\nu_C(x_n)} \to \frac{x^*}{\beta} \in \mathrm{near}(C,0)$, and hence $\beta = \nu_C(x^*)$. This entails that $\nu_C(x_n) \to \nu_C(x^*)$, i.e., ν_C is continuous at x^* relative to C^∞.

(c) In the case of C^∞ being a polyhedral cone, the relative continuity of ν_C relative to its domain C^∞ is a simple consequence of polyhedral convexity and local simpliciality, see [14]. From the fact that every closed cone in two-dimensional space is polyhedral, it follows that C^∞ is always a polyhedral cone whenever $\dim C \leq 2$. This completes the proof. □

Theorem 3.2. *For a closed, convex set $C \subset \mathbb{R}^n$ with $0 \in C$, the following hold:*

(a) *γ_C is continuous at 0 relative to $\operatorname{pos} C$ if and only if $d(0, \operatorname{far}(C, 0)) > 0$.*

(b) *γ_C is continuous relative to its domain $\operatorname{pos} C$ if and only if $\operatorname{far}(C, 0)$ is closed.*

(c) *If $\operatorname{pos} C$ is a polyhedral cone, as is true when $\dim C \leq 2$ and $d(0, \operatorname{far}(C, 0)) > 0$, then $\operatorname{far}(C, 0)$ is closed.*

Proof. Property (a) can be found in [12, Theorem 4.1]. Property (c) can be proved in a similar way as Theorem 3.1 (c). Property (b) can be found in [10, Lemma 2.4] except for the 'if' part in the circumstance that C may be unbounded, whose proof not relying the boundedness assumption is now given below.

Assume that $\operatorname{far}(C, 0)$ is closed. Observing that $0 \notin \operatorname{far}(C, 0)$ by definition, we have $d(0, \operatorname{far}(C, 0)) > 0$. By property (a), γ_C is continuous at 0 relative to $\operatorname{pos} C$. Let $x \in \operatorname{pos} C \backslash \{0\}$ and $x_k \to x$ with $x_k \in \operatorname{pos} C \backslash \{0\}$ for all k. We claim that the sequence $\{\gamma_C(x_k)\}$ is bounded, for otherwise by taking a subsequence if necessary we have $\gamma_C(x_k) \to +\infty$, implying that $\frac{x_k}{\gamma_C(x_k)} \to 0$ with $\frac{x_k}{\gamma_C(x_k)} \in \operatorname{far}(C, 0)$ for all k and hence that $d(0, \operatorname{far}(C, 0)) = 0$, a contradiction. Let β be any clustering point of the sequence $\{\gamma_C(x_k)\}$. Clearly, $0 \leq \beta < +\infty$.

We first consider the case that $\gamma_C(x) > 0$. As γ_C is lower semi-continuous, we have $\beta \geq \gamma_C(x) > 0$ and $\gamma_C(x_k) > 0$ for all k large enough. By taking a subsequence if necessary we have $\frac{x_k}{\gamma_C(x_k)} \to x/\beta$ with $\frac{x_k}{\gamma_C(x_k)} \in \operatorname{far}(C, 0)$ for all k (due to $\gamma_C(\frac{x_k}{\gamma_C(x_k)}) = 1$). As $\operatorname{far}(C, 0)$ is assumed to be closed, we have $x/\beta \in \operatorname{far}(C, 0)$. Thus, we have $\gamma_C(x/\beta) = 1$ and hence $\beta = \gamma_C(x)$. This entails that $\gamma_C(x_k) \to \gamma_C(x)$ whenever $\gamma_C(x) > 0$.

Next we consider the case that $\gamma_C(x) = 0$. In this case, β cannot be a positive number, for otherwise by the same argument used earlier we deduce that $\beta = \gamma_C(x) > 0$, a contradiction. This entails that $\gamma_C(x_k) \to \gamma_C(x)$ whenever $\gamma_C(x) = 0$. To sum up, γ_C is continuous at every $x \in \operatorname{pos} C$ relative to $\operatorname{pos} C$ provided that $\operatorname{far}(C, 0)$ is closed. This completes the proof. □

Recall that the gauge γ_C for a closed and convex set C containing 0 is an lsc and sublinear function with $\operatorname{pos} C$ as its domain, which is not necessarily the whole space. From [11, Proposition 2.1.6] it is well known that a sublinear function having the whole space as its domain, is continuous everywhere if it is continuous at 0. This motivates the proposal for a conjecture [12, Conjecture 6.1] on the relative continuity property of the gauge function γ_C, which claims that γ_C is continuous relative to the domain if it is continuous at 0 relative to the domain. This conjecture, according to Theorem 3.2, can be restated as saying that $d(0, \operatorname{far}(C, 0)) > 0$ if and only if $\operatorname{far}(C, 0)$ is closed.

This conjecture turns out to be true in the two-dimensional case as can been seen from Theorem 3.2 (b) and (c). However, in Example 3.3 below, we are able to construct a three-dimensional compact and convex set to demonstrate that [12, Conjecture 6.1] is in general not true by showing that the closedness of $\mathrm{far}(C,0)$ cannot be always guaranteed in the circumstance that $d(0,\mathrm{far}(C,0))>0$. Moreover by using the same example, we demonstrate that the conditions given in Theorem 3.1(b) to describe the relative continuity of the cogauge function ν_C, may not hold simultaneously, and that the characterizations presented in Section 2 are helpful for calculating the near and far ends.

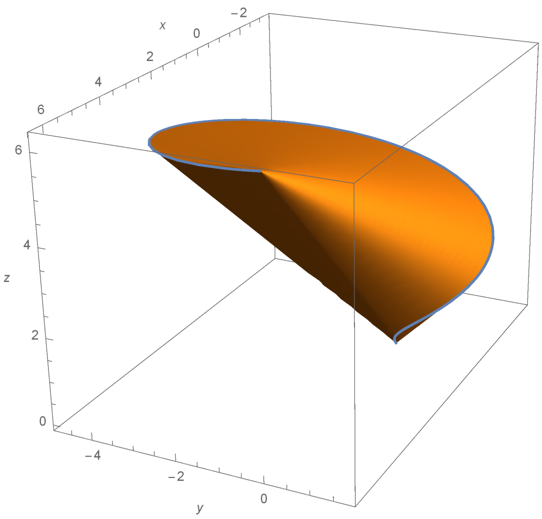

Figure 3.1: The compact convex set $C=\mathrm{co}(\{\gamma(t)\mid t\in[0,2\pi]\})$ is shown with the boundary curve being the one defined by $\gamma(t)=(t\cos t,t\sin t,t)^T$.

Example 3.3. Let $C:=\mathrm{co}(\{\gamma(t)\mid t\in[0,2\pi]\})$, $Q:=\{x\in C\mid x_3\geq\pi\}$, and $x_0=\omega\gamma(2\pi)$ for some $\omega>1$, where $\gamma\colon[0,2\pi]\to\mathbb{R}^3$ be a three-dimensional curve defined by $\gamma(t):=(t\cos t,t\sin t,t)^T$. The following hold:

(a) $\mathrm{far}(C,0)=\cup_{0<t<2\pi}[\gamma(t),\gamma(2\pi)]$ and $\mathrm{cl}(\mathrm{far}(C,0))=\cup_{0\leq t\leq 2\pi}[\gamma(t),\gamma(2\pi)]$, implying that $\mathrm{far}(C,0)$ is not closed, and that $d(0,\mathrm{far}(C,0))=0$.

(b) $\mathrm{far}(Q,0)=\{x\in\mathrm{far}(C,0)|x_3\geq\pi\}$ and $\mathrm{cl}(\mathrm{far}(Q,0))=\{x\in\mathrm{cl}(\mathrm{far}(C,0))|x_3\geq\pi\}$, implying that $d(0,\mathrm{far}(Q,0))>0$ and that $\mathrm{far}(Q,0)$ is not closed.

(c) $\mathrm{near}(C,x_0)=\mathrm{far}(C,0)$ and $\mathrm{cl}(\mathrm{near}(C,x_0))=\mathrm{cl}(\mathrm{far}(C,0))$, implying that $\mathrm{near}(C,x_0)$ is not closed, though the following holds:

$$(\mathrm{near}(C,x_0))^\infty\cap\mathrm{pos}(C-x_0)=\{0\}. \tag{11}$$

Proof. The set C and the curve γ are shown in Figure 3.1, while the convex hull $\mathrm{co}(Q\cup\{0\})$ and the curve γ are shown in Figure 3.2. Let $A:=\bigcup_{0<t<2\pi}[\gamma(t),\gamma(2\pi)]$ and $B:=\bigcup_{0\leq t\leq 2\pi}[\gamma(t),\gamma(2\pi)]$. It is straightforward to verify that $\mathrm{cl}\,A=B$, and that $(B\backslash A)\cap\mathrm{far}(C,0)=[\gamma(0),\gamma(2\pi))\cap\mathrm{far}(C,0)=\emptyset$.

(a) Let $w\in\mathbb{R}^n$ be such that $\sigma_C(w)>0$. In view of the facts that $\partial\sigma_C(w)=\arg\max_{v'\in C}\langle v',w\rangle$ and that C is the convex hull of $\{\gamma(t)\mid t\in[0,2\pi]\}$, we have $\partial\sigma_C(w)=\mathrm{co}\{\gamma(t)\mid t\in T\}$, where $T=\arg\max_{t\in[0,2\pi]}f(t)$ with $f(t):=\langle\gamma(t),w\rangle$.

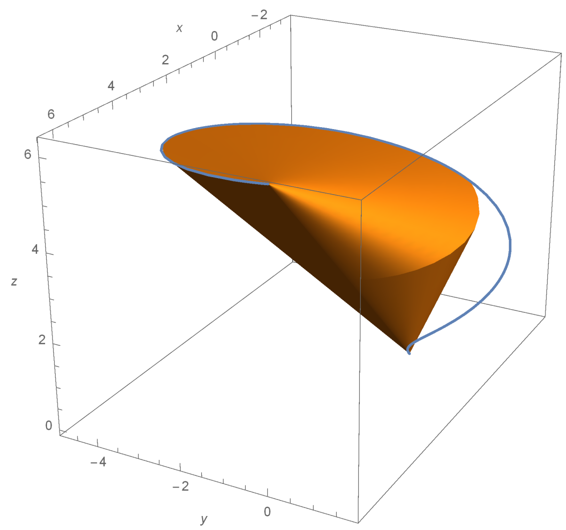

Figure 3.2: The compact convex set $\mathrm{co}(Q \cup \{0\})$ is shown with $Q = \{x \in C \mid x_3 \geq \pi\}$ and $C = \mathrm{co}(\{\gamma(t) \mid t \in [0, 2\pi]\})$, while the curve shown in the figure is the one defined by $\gamma(t) = (t \cos t, t \sin t, t)^T$.

Clearly, $0 \notin T$, for otherwise we have $\sigma_C(w) = \max_{v' \in C} \langle v', w \rangle = \langle \gamma(0), w \rangle = 0$, contradicting to the fact that $\sigma_C(w) > 0$. Let $t_1, t_2 \in (0, 2\pi)$ with $t_1 \neq t_2$. We claim that it is impossible for $\{t_1, t_2\}$ being a subset of T, for otherwise we have

$$f'(t_1) = 0, \quad f'(t_2) = 0, \quad f(t_1) = f(t_2),$$

or explicitly, $Mw = 0$, where

$$M = \begin{pmatrix} \cos t_1 - t_1 \sin t_1 & \sin t_1 + t_1 \cos t_1 & 1 \\ \cos t_2 - t_2 \sin t_2 & \sin t_2 + t_2 \cos t_2 & 1 \\ t_2 \cos t_2 - t_1 \cos t_1 & t_2 \sin t_2 - t_1 \sin t_1 & t_2 - t_1 \end{pmatrix}.$$

But the determinant of M equals $2t_1t_2[1 - \cos(t_2 - t_1)][\frac{t_2 - t_1}{2} \mathrm{ctg} \frac{t_2 - t_1}{2} - \frac{1}{2}(\frac{t_1}{t_2} + \frac{t_2}{t1})]$, which is a negative number. This implies that $w = 0$, contradicting again to the fact that $\sigma_C(w) > 0$. Therefore, we have either $T = \{t\}$ with $t \in (0, 2\pi]$ or $T = \{t, 2\pi\}$ with $t \in (0, 2\pi)$. In either case, we have $\partial\sigma_C(w) = \mathrm{co}\{\gamma(t) \mid t \in T\} \subset A$, and hence $\cup_{\sigma_C(w) > 0} \partial\sigma_C(w) \subset A$. In view of the fact that C is a compact and convex set with $0 \in C$, we deduce from Theorem 2.6 that $\mathrm{far}(C, 0) \subset \mathrm{cl}(\bigcup_{\sigma_C(w) > 0} \partial\sigma_C(w))$, and hence that

$$\mathrm{far}(C, 0) \subset \mathrm{cl}\, A = B = A \cup (B \backslash A).$$

Since $(B \backslash A) \cap \mathrm{far}(C, 0) = \emptyset$, we have $\mathrm{far}(C, 0) \subset A$.

To show the reverse inclusion, let $v \in A$ be given arbitrarily. Then by the definition of A there exist some $t \in (0, 2\pi)$ and $\lambda \in [0, 1]$ such that $v = \lambda\gamma(2\pi) + (1 - \lambda)\gamma(t)$. Suppose by contradiction that $v \notin \mathrm{far}(C, 0)$, i.e., there exists some $\tau > 1$ such that $\tau v \in \mathrm{far}(C, 0) \subset A$. Then by the definition of A, there exist some $\kappa \in [0, 1]$ and $t' \in (0, 2\pi)$ such that $\tau v = \kappa\gamma(2\pi) + (1 - \kappa)\gamma(t')$. Thus, we have

$$(\tau\lambda - \kappa)\gamma(2\pi) + \tau(1 - \lambda)\gamma(t) = (1 - \kappa)\gamma(t'). \tag{12}$$

Clearly, we have $t' \neq t$, for otherwise we have

$$(\tau\lambda - \kappa)\gamma(2\pi) + (\tau - 1 - \tau\lambda + \kappa)\gamma(t) = 0,$$

which is possible only if $\tau\lambda - \kappa = \tau - 1 - \tau\lambda + \kappa = 0$ (due to $\gamma(2\pi)$ and $\gamma(t)$ being linearly independent), implying that $\tau = 1$, a contradiction. So the vectors $\gamma(t), \gamma(t'), \gamma(2\pi)$ are three distinct extreme directions of

$$K := \{(x, y, z) \mid z \geq \sqrt{x^2 + y^2}\},$$

the Lorentz cone (also known as the ice-cream cone). This together with (12) implies that $\tau\lambda - \kappa = \tau(1 - \lambda) = (1 - \kappa) = 0$ and hence that $\tau = \lambda = \kappa = 1$, a contradiction. This contradiction indicates that $v \in \mathrm{far}(C, 0)$ and hence $A \subset \mathrm{far}(C, 0)$. Thus, we get the equality $\mathrm{far}(C, 0) = A$ and all other assertions in (a) then follow readily.

(b) Let $D := \{x \in \mathbb{R}^3 \mid x_3 \geq \pi\}$. Then by definition we have

$$\mathrm{far}(Q, 0) = \mathrm{far}(C \cap D, 0) = \mathrm{far}(C, 0) \cap D,$$

implying that $d(0, \mathrm{far}(Q, 0)) \geq d(0, D) = \pi > 0$. Let $v \in (\mathrm{cl}(\mathrm{far}(C, 0)) \setminus \mathrm{far}(C, 0)) \cap D$. Equivalently, we have $v = \lambda\gamma(2\pi))$ with $\lambda \in [\frac{1}{2}, 1)$. Clearly, we have $v \notin \mathrm{far}(Q, 0)$. Let $t_k \to 0+$ and $\lambda_k \to \lambda$ with $\lambda_k \in [\frac{\pi - t_k}{2\pi - t_k}, 1]$ for all k. Then, we have

$$v_k := \lambda_k\gamma(2\pi) + (1 - \lambda_k)\gamma(t_k) \in \mathrm{far}(C, 0) \cap D$$

and $v_k \to v$. This implies that $v \in \mathrm{cl}(\mathrm{far}(C, 0) \cap D) = \mathrm{cl}(\mathrm{far}(Q, 0))$. That is, $\mathrm{far}(Q, 0)$ is not closed. Moreover, we have

$$(\mathrm{cl}(\mathrm{far}(C, 0)) \setminus \mathrm{far}(C, 0)) \cap D \subset \mathrm{cl}(\mathrm{far}(C, 0) \cap D)$$

and hence $\mathrm{cl}(\mathrm{far}(Q, 0)) = \mathrm{cl}(\mathrm{far}(C, 0) \cap D) = \mathrm{cl}(\mathrm{far}(C, 0)) \cap D$.

(c) Since C is compact and $\mathrm{near}(C, x_0) \subset C$, we have $(\mathrm{near}(C, x_0))^\infty = \{0\}$ and hence (11) follows. From the proof of (a), it is clear that C has no two-dimensional exposed faces, and that $[\gamma(2\pi), \gamma(t)]$ with $t \in (0, 2\pi)$ is an one-dimensional exposed face of C. To identify such a face by a supporting hyperplane to C, we need to find a vector $w \in \mathbb{R}^3$ such that $\arg\max_{t \in [0, 2\pi]} f(t) = \{t, 2\pi\}$, where $f(t) := \langle \gamma(t), w \rangle$. According to the necessary optimality condition, we have $f'(t) = 0$ and also $f(t) = f(2\pi) > f(0) = 0$, which gives

$$w_t = \beta\Big((-\frac{t^2}{2\pi} + t)\cos(t) + \sin(t),\\ (-\frac{t^2}{2\pi} + t)\sin(t) - \cos(t) + 1,\ \frac{t^2}{2\pi} - \sin(t) - t\cos(t)\Big)^T,$$

where $\beta > 0$. For such vectors w_t, we have $0 < \sigma_C(w_t) = \langle \gamma(2\pi), w_t \rangle < \langle x_0, w_t \rangle$, and $\partial\sigma_C(w_t) = [\gamma(t), \gamma(2\pi)]$. Moreover, each line segment $[\gamma(0), \gamma(t)]$ with $t \in (0, 2\pi]$ is also an one-dimensional exposed face of C, which can be exposed by the vector $u_t := \beta(\cos t, \sin t, -1)^T$ with $\beta > 0$. That is, we have

$$0 = \sigma_C(u_t) = \langle u_t, \gamma(t) \rangle \geq \langle u_t, x_0 \rangle = \langle u_t, \omega\gamma(2\pi) \rangle = 2\pi\omega\beta(\cos t - 1),$$

and $\partial\sigma_C(u_t) = [\gamma(0), \gamma(t)]$.

Then, we have

$$\bigcup_{\sigma_C(w)<\langle x_0,w\rangle} \partial\sigma_C(w) = \bigcup_{0<t<2\pi} [\gamma(t),\gamma(2\pi)].$$

In view of Theorem 2.7, we have

$$\bigcup_{0<t<2\pi} [\gamma(t),\gamma(2\pi)] \subset \mathrm{near}(C,x_0) \subset \mathrm{cl}(\bigcup_{0<t<2\pi} [\gamma(t),\gamma(2\pi)]).$$

By the same argument used in (a), we can easily obtain the remaining results in (c). This completes the proof. □

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