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Stability Analysis of Partial Differential Set-Valued Variational Inequalities on Banach Spaces

Wei Li
Geomathematics Key Laboratory of Sichuan Province,
Chengdu University of Technology, Chengdu, 610059, P.R. China
lovelylw@126.com

Xianfu Wang
Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada
shawn.wang@ubc.ca

Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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We consider the partial differential set-valued variational inequality (PDSVI) on real Banach spaces. We give conditions for the existence and uniqueness of mild solution of the PDSVI, and establish closedness and continuity of the mild solution set mapping of the PDSVI. The proofs of the main results are based on the theory of semigroups, set-valued mappings, and variational inequality.

Keywords: Partial differential set-valued variational inequality, stability, mild solution, continuity.

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1. Introduction

Variational inequality (VI) modeling, analysis and computations are important for many applications, see [1, 5, 7, 8, 9, 12, 27, 34, 35]. Recently, researchers have paid great attention to differential variational inequalities (DVIs), which is a fruitful modeling paradigm for many problems arising in economics, operations research and mathematical programming, see [3, 4, 7, 28, 29, 31]. Two main research areas are: existence of solutions and stability of solutions.

Let us first review some results on existence of solutions about DVIs. Pang and Stewart [28] systematically discussed DVIs on Euclidean spaces. Li and Huang [18] studied the existence of Carathéodory weak solutions of differential mixed variational inequalities (DMVIs) on finite dimensional spaces, which generalized the corresponding results in [28]. When the ordinary differential equation is replaced by a partial differential equation, such problems are called as partial differential variational inequalities (PDVIs) in Liu et al. [23]. Liu et al. studied the existence of mild solution of partial differential mixed variational inequality on real

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Banach spaces in [22], and the existence of mild solutions of partial differential variational inequalities (PDVIs) involving nonlocal boundary conditions on real Banach spaces in [21]. For more results about the existence of solutions of DVIs or PDVIs, we refer the reader to [16, 17, 24, 25, 32].

Next, we review the solution stability results of DVIs and PDVIs. Pang and Stewart [29] studied the continuous dependence of solutions of a DVI with respect to its initial conditions on finite dimensional spaces. Wang et al. [33] studied the upper semicontinuity and continuity of the Carathéodory weak solution set mapping of a DMVI when both the mappings involved in the mixed VI and the constraint set are perturbed on finite dimensional spaces. On Hilbert spaces, Gwinner [7] obtained the relationship between a DVI and a projected dynamic system, studied the stability of the slow solution set of a DVI, and gave a novel upper set convergence result with respect to perturbations in the data. Li and Liu et al. [19] studied the sensitivity analysis of optimal control problems for a class of systems governed by a differential inclusion on a separable Hilbert space and a hemivariational inequality on a reflexive real Banach space. Li et al. [20] studied the exponential stability of differential hemivariational inequalities on real Banach spaces. Solution stability is also important in optimization; see [10, 11, 15] and references therein.

However, up to now, there are few research results on the existence of solutions of the PDVIs on infinite dimensional spaces with the mapping in variational inequality being set-valued and on the stability of the DVIs or PDVIs on finite dimensional or infinite dimensional spaces. Nevertheless, many problems arising in engineering, operational research and economical dynamics are described by PDVIs on infinite dimensional spaces.

The goal of this paper is to study the existence of the mild solution and the stability of PDSVI(1) on real Banach spaces with the constraint set K and the set-valued mapping F being perturbed by different parameters. Our results about existence of mild solution of PDSVI(1) generalize Liu, Zeng and Motreanu's some results in [23] for the case that the mixed part in VI in [23] is an indicator function. While the mapping in VI in [23] is single-valued, the mapping in VI in our PDSVI(1) is set-valued. Our stability results about PPDSVI(2) generalize Wang, Li and Huang's some results in [33] from finite dimensional spaces to infinite dimensional spaces, from ordinary differential variational inequality to partial differential variational inequality, for the case where the mixed part in VI in [33] is an indicator function. To the best of our knowledge, the results are new. Our technique relies on set-valued analysis, VI, and semigroup, which has been studied in [22, 23].

The paper is organized as follows. Section 2 contains some useful definitions and lemmas. In Section 3, the existence and uniqueness results of mild solution of PDSVI(1) are discussed. Furthermore, the continuity of mild solution set mapping is obtained when the constraint set K is perturbed by p in a metric space Z_1 and the set-valued mapping F is perturbed by another different parameter v in the metric space Z_2. An application on the Nash equilibrium is given.

2. Preliminaries

In this section, we give the precise formulation of PDSVI, and collect some facts on set-valued mappings, variational inequalities and semigroups.

Throughout the paper, we assume that E and E_1 are two real Banach spaces with E_1^* being the dual space of E_1, $\langle\cdot,\cdot\rangle$ being the dual pair between E_1^* and E_1. For a nonempty convex set $K \subset E_1$, its *barrier cone* is defined by

$$\mathrm{barr}(K) = \{x^* \in X^* : \sup_{x\in K}\langle x^*, x\rangle < \infty\}.$$

2.1. PDSVI and PPDSVI formulations

Let $A\colon D(A) \subset E \to E$ be the infinitesimal generator of a C_0 semigroup $\tilde{T}(t)$ on E, $f\colon [0,T]\times E\times E_1 \to E$ be a single valued mapping, $F\colon [0,T]\times E\times K \rightrightarrows E_1^*$ be a set-valued mapping, and $\bar{x} \in D(A)$. We consider the following partial differential set-valued variational inequality (PDSVI) on real Banach spaces: find functions $x\colon [0,T] \to E$ and $u\colon [0,T] \to E_1$ such that

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)) & \text{for a.e. } t \in [0,T], \\ u(t) \in SOL(K, F(t, x(t), \cdot)) & \text{for a.e. } t \in [0,T], \\ x(0) = \bar{x}, \end{cases} \tag{1}$$

where $\dot{x}(t)$ stands for the derivative of function x with respect to time variable t and $SOL(K, F(t, x(t), \cdot))$ denotes the solution set of the following time-dependent set-valued variational inequality problem: find $u\colon [0,T] \to K$ such that for any $t \in [0,T]$, there exists $u^*(t) \in F(t, x(t), u(t))$ and

$$\langle u^*(t), y - u(t)\rangle \geq 0, \text{ for all } y \in K.$$

To describe the perturbed problem, let (Z_1, d_1) and (Z_2, d_2) be two metric spaces. The perturbed mapping $F\colon [0,T]\times E\times K \rightrightarrows E_1^*$ and the perturbed set $K \subset E_1$ in PDSVI(1) will be denoted by $F\colon [0,T]\times E\times K\times Z_2 \rightrightarrows E_1^*$ and $L\colon Z_1 \rightrightarrows E_1$ respectively. The perturbed problem for the PDSVI(1), which is referred as the perturbed partial differential set-valued variational inequality (PPDSVI), is as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)) & \text{a.e. } t \in [0,T], \\ u(t) \in SOL(L(p), F(t, x(t), \cdot, v)) & \text{a.e. } t \in [0,T], \\ x(0) = \bar{x}, \end{cases} \tag{2}$$

where the perturbation parameter $(p, v) \in Z_1 \times Z_2$. Let $L^2((0,T); E_1)$ consist of all Lebesgue measurable functions $w\colon (0,T) \to E_1$ satisfying

$$\|w\|_{L^2((0,T);E_1)} = \sqrt{\int_0^T \|w(t)\|_{E_1}^2 dt} < +\infty,$$

$C([0,T]; E) = \{x : x \text{ is continuous on [0,T]}\}$, and $\|x\|_{C([0,T];E)} = \sup_{t\in[0,T]} \|x(t)\|_E$.

Definition 2.1. A pair of functions (x, u) is called a *mild solution* of PDSVI(1), if $x \in C([0,T]; E)$, $u\colon [0,T] \to E_1$ is measurable such that

$$x(t) = \tilde{T}(t)\bar{x} + \int_0^t \tilde{T}(t-s) f(s, x(s), u(s)) ds \quad \text{for all } t \in [0,T]$$

and $$u(t) \in SOL(L(p), F(t, x(t), \cdot, v)) \text{ for a.e. } t \in [0,T].$$

2.2. Set-valued analysis

For any nonempty subset X of a topological space, we denote by $P(X)$ the collection of its nonempty subsets, $P_K(X) = \{D \in P(X) : D \text{ is nonempty and compact}\}$, and $P_C(X) = \{D \subset X : D \text{ is nonempty and closed}\}$.

Definition 2.2. Let Y, Z be topological spaces and $H\colon Y \rightrightarrows Z$ be a set-valued mapping with nonempty values. H is said to be

(i) *upper semicontinuous* at $x_0 \in Y$ if, for any neighborhood $N(H(x_0))$ of $H(x_0)$, there exists a neighborhood $N(x_0)$ of x_0 such that $H(x) \subset N(H(x_0))$ for all $x \in N(x_0)$;

(ii) *lower semicontinuous* at $x_0 \in Y$ if, for any $y_0 \in H(x_0)$ and any neighborhood $N(y_0)$ of y_0, there exists a neighborhood $N(x_0)$ of x_0 such that $H(x) \bigcap N(y_0) \neq \emptyset$ for all $x \in N(x_0)$. It is evident that H is lower semicontinuous at $x_0 \in Y$ if and only if, for any sequence $x_n \to x_0$ and $y_0 \in H(x_0)$, there exists a sequence $\{y_n\} \subset Z$ with $y_n \in H(x_n)$ such that $y_n \to y_0$.

Definition 2.3. Let X, Y be metric spaces, $H\colon X \rightrightarrows Y$ be a set-valued mapping. We say that H is *closed* at $p \in X$, if $X \ni p_n \to p$ and $H(p_n) \ni x_n \to x$, then $x \in H(p)$.

Lemma 2.4. ([27, page 53]) *Let X, Y be metric spaces and $H\colon X \rightrightarrows Y$ be a set-valued mapping. Then the following statements are equivalent:*

(i) *H is upper semicontinuous.*

(ii) *For every $C \subset Y$ closed, $H^-(C) = \{x \in X : H(x) \cap C \neq \emptyset\}$ is closed in X.*

(iii) *If $x \in X$, $\{x_n\} \subset X$ with $x_n \to x$ and $V \subset Y$ is an open set such that $H(x) \subset V$, then we can have $n_0 \in N$, depending on V, such that $H(x_n) \subset V$ for all $n \geq n_0$.*

Lemma 2.5. ([27, page 54]) *Assume that X and Y are metric spaces. Then the following statements hold:*

(i) *If a set-valued mapping $H\colon X \rightrightarrows P_C(Y)$ is upper semicontinuous, then H is closed.*

(ii) *If a set-valued mapping $H\colon X \rightrightarrows P_C(Y)$ is closed and locally compact, then H is upper semicontinuous.*

(iii) *The set-valued mapping $H\colon X \rightrightarrows P_K(Y)$ is upper semicontinuous if and only if for every $x \in X$ and every sequence $(x_n, y_n) \in graph(H)$ with $x_n \to x$ in X, there exists a converging subsequence of $\{y_n\}$ whose limit belongs to $H(x)$.*

Lemma 2.6. (Arzelà-Ascoli) ([14, page 57]) *Let N be a compact subset of a metric space, and let E be a Banach space. Let ϕ be a subset of the space of continuous maps $C(N,E)$ with sup norm. Then ϕ is relatively compact in $C(N,E)$ if and only if the following two conditions are satisfied:*

(i) *ϕ is equicontinuous.*

(ii) *For each $x \in N$, the set $\phi(x)$ consisting of all values $f(x)$ for $f \in \phi$ is relatively compact.*

2.3. Well-known results on VI and semigroups

Definition 2.7. Let E and E_1 be Banach spaces and let an interval $I \subseteq R$. We say that $F\colon I \times E \to P(E_1)$ is *superpositionally measurable*, if for any measurable set-valued mapping $Q\colon I \to P_K(E)$, the composition $\Phi\colon I \to P(E_1)$ given by $\Phi(t) = F(t, Q(t))$ is measurable.

Lemma 2.8. ([13, page 28]) *If $U\colon [0,T] \times E \to P_K(E_1)$ satisfies the Carathéodory condition or U is upper or lower semicontinuous, then U is superpositionally measurable.*

Definition 2.9. Let E_1 be a Banach space with topological dual E_1^* and let $H\colon K \subset E_1 \rightrightarrows E_1^*$ be a set-valued mapping with nonempty values. H is said to be

(i) *monotone* on K if, for all $(x,x^*) \in graph(H), (y,y^*) \in graph(H)$,

$$\langle y^* - x^*, y - x\rangle \geq 0;$$

(ii) *pseudomonotone* on K if, for all $(x,x^*) \in graph(H)$, $(y,y^*) \in graph(H)$

$$\langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0;$$

(iii) *upper hemicontinuous* on K if the restriction of H to every line segment of K is upper semicontinuous with respect to the weak topology in E_1^*.

Lemma 2.10. ([6, Theorem 3.3]) *Let (Z_1,d_1) and (Z_2,d_2) be two metric spaces, $u_0 \in Z_1$ and $v_0 \in Z_2$ be given points. Let X be a reflexive Banach space, $L\colon Z_1 \rightrightarrows X$ be a continuous set-valued mapping with nonempty closed convex values and* $\mathrm{int}(\mathrm{barr}(L(u_0))) \neq \emptyset$. *Suppose that there exists a neighborhood $U \times V$ of (u_0,v_0) such that $F\colon \tilde{M} \times V \rightrightarrows X^*$ is a lower semicontinuous set-valued mapping with nonempty weakly compact convex values for $\tilde{M} = \bigcup_{u\in U} L(u)$. Consider the generalized variational inequality problem GVIP$(L(u), F(\cdot,v))$ for $(u,v) \in Z_1 \times Z_2$: find $x \in L(u)$ and $x^* \in F(x,v)$ such that*

$$\langle x^*, y - x\rangle \geq 0, \ \textit{for all } y \in K.$$

Assume that

(i) For each $v \in V$, the mapping $x \to F(x,v)$ is upper hemicontinuous and pseudomonotone on $\tilde{M}$.

(ii) The solution of the generalized variational inequality problem GVIP$(L(u_0)$, $F(\cdot,v_0))$ is nonempty and bounded.

Then there exists a neighborhood $U' \times V'$ of (u_0, v_0) with $U' \times V' \subset U \times V$ such that, for every $(u, v) \in U' \times V'$, the solution set of $GVIP(L(u), F(\cdot, v))$ is nonempty and bounded.

Lemma 2.11. ([30, page 4]) *Let E be a Banach space and $\tilde{T}(t)$ be a C_0 semigroup on E. Then there exist constants $w > 0$ and $M \geq 1$ such that*

$$\|\tilde{T}(t)\| \leq Me^{wt} \quad \textit{for any } 0 \leq t < \infty.$$

Lemma 2.12. ([30, page 4]) *Let E be a Banach space and $\tilde{T}(t)$ be a C_0 semigroup on E. Define*

$$Ax = \lim_{t \downarrow 0} \frac{\tilde{T}(t)x - x}{t} \quad \textit{and} \quad D(A) = \{x \in E : \lim_{t \downarrow 0} \frac{\tilde{T}(t)x - x}{t} \ \textit{exists}\}.$$

Then for any $x \in D(A)$, we have

$$\tilde{T}(t)x - \tilde{T}(s)x = \int_s^t \tilde{T}(\tau)Axd\tau.$$

3. Main results

In this section, we prove the existence of mild solution for PDSVI(1) and the stability of the mild solution of PPDSVI(2). An application on the Nash equilibrium illustrates our results.

3.1. Existence of mild solution for unperturbed PDSVI(1)

Define $U\colon [0, T] \times E \to P(K)$ by

$$U(t, x) = \{u \in K : \exists u^* \in F(t, x, u) \text{ such that } \langle u^*, y - u\rangle \geq 0, \forall y \in K\}.$$

In order to prove the existence of the mild solution of PDSVI(1), we need to show that $U(t, x)$ is nonempty closed and convex for any $(t, x) \in [0, T] \times E$.

Lemma 3.1. *Let K be a nonempty compact convex set on a real Banach space E_1. Suppose that $U(t, x)$ is nonempty, $F(t, x, \cdot)$ is upper semicontinuous with nonempty compact values and pseudomonotone on K for all $(t, x) \in [0, T] \times E$. Then $U(t, x)$ is nonempty closed and convex for all $(t, x) \in [0, T] \times E$.*

Proof. First, we show that $U(t, x)$ is convex. Let u_1, $u_2 \in U(t, x)$. That means $u_1, u_2 \in K$ and there exist $u_1^* \in F(t, x, u_1), u_2^* \in F(t, x, u_2)$ such that

$$\langle u_1^*, y - u_1\rangle \geq 0, \text{ for all } y \in K \tag{3}$$

and
$$\langle u_2^*, y - u_2\rangle \geq 0, \text{ for all } y \in K. \tag{4}$$

Since for any $(t, x) \in [0, T] \times E$, $F(t, x, \cdot)$ is pseudomonotone on K, it follows from (3) and (4) that for any $y \in K$ and $y^* \in F(t, x, y)$,

$$\langle y^*, y - u_1\rangle \geq 0, \quad \text{and} \quad \langle y^*, y - u_2\rangle \geq 0.$$

Furthermore, from the above two inequalities, we get for any $\lambda \in (0,1)$,

$$\langle y^*, \lambda(y-u_1)+(1-\lambda)(y-u_2)\rangle \geq 0, \text{ for all } y \in K,$$

which immediately implies that

$$\langle y^*, y-(\lambda u_1+(1-\lambda)u_2)\rangle \geq 0, \text{ for all } y \in K. \tag{5}$$

Now we let $\tilde{u} = \lambda u_1 + (1-\lambda)u_2$. Applying the convexity of K, it is clear that $\tilde{u} \in K$. Let $v \in K$ and $\beta \in (0,1]$.Taking $y = \tilde{u} + \beta(v-\tilde{u}) = \beta v + (1-\beta)\tilde{u} \in K$ in (5), we obtain that for any $y^*_\beta \in F(t,x,\tilde{u}+\beta(v-\tilde{u}))$, it is easy to see that

$$\langle y^*_\beta, \tilde{u}+\beta(v-\tilde{u})-\tilde{u}\rangle \geq 0, \quad \text{which means} \quad \langle y^*_\beta, \beta(v-\tilde{u})\rangle \geq 0,$$

$$\text{and so} \qquad \langle y^*_\beta, v-\tilde{u}\rangle \geq 0. \tag{6}$$

Letting $\beta \to 0^+$ and applying Lemma 2.5(iii), we can get there exists a subsequence of $\{y^*_\beta\}$, denoted again by $\{y^*_\beta\}$, such that $y^*_\beta \to \tilde{y}^*$ with $\tilde{y}^* \in F(t,x,\tilde{u})$. Thus taking the limit in (6) means that $\langle \tilde{y}^*, v-\tilde{u}\rangle \geq 0$ for all $v \in K$.

So $\tilde{u} \in U(t,x)$ and $U(t,x)$ is convex.

Next, we show that $U(t,x)$ is closed. Let $\{u_n\} \subset U(t,x)$ with $u_n \to u$. From the definition of $U(t,x)$, we get $u_n \in K$ and there exists $u^*_n \in F(t,x,u_n)$ such that

$$\langle u^*_n, y-u_n\rangle \geq 0, \text{ for all } y \in K. \tag{7}$$

Since K is a compact set, it is easy to see that $u \in K$. Recall that F is upper semicontinuous with nonempty compact value. So there exists a subsequence of $\{u^*_n\}$, denoted again by $\{u^*_n\}$, with $u^*_n \to u^* \in F(t,x,u)$. Letting $n \to \infty$ and it follows from (7) that there exists $u^* \in F(t,x,u)$ such that

$$\langle u^*, y-u\rangle \geq 0, \text{ for all } y \in K.$$

That means that $u \in U(t,x)$ and so $U(t,x)$ is closed. □

Remark 3.2. If we have $E = R^m$, $E_1 = R^n$, $F(t,x,u) = c(t,x) + H(u)$, where $c\colon [0,T] \times R^m \to R^n$ is a mapping and $H\colon R^n \rightrightarrows R^n$ is a set-valued mapping, then Lemma 3.1 reduces to Lemma 3.1 in [33] for the case that φ is an indicator function on K. □

Now we establish some properties of $U(t,x)$, which will be used to prove the existence of the mild solution for PDSVI(1) and PPDSVI(2). Let us point out that the proof of Lemma 3.3 below is similar to the method of Theorem 3.4 in [23], but the set $U(t,x)$ is different.

Lemma 3.3. *Assume that E_1 is a Banach space, $K \subset E_1$ a compact set, and $F\colon [0,T] \times E \times E_1 \rightrightarrows E^*$ upper semicontinuous with nonempty compact value. Assume that $U(t,x)$ is nonempty. Then*

(i) *U is upper semicontinuous.*

(ii) *U is superpositionally measurable.*

Proof. (i) According to Lemma 2.4, in order to prove U is upper semicontinuous we need to show that for every closed set $C \subset K$, the set

$$U^{-}(C) = \{(t, x) \in [0, T] \times E : U(t, x) \cap C \neq \emptyset\}$$

is closed in $[0, T] \times E$.

Let $(t_n, x_n) \in U^{-}(C)$ with $(t_n, x_n) \to (t, x)$ in $[0, T] \times E$. So we can choose $u_n \in U(t_n, x_n) \cap C$. This means $u_n \in C \subset K$ and $u_n \in U(t_n, x_n)$ means there exists $u_n^* \in F(t_n, x_n, u_n)$ such that

$$\langle u_n^*, y - u_n \rangle \geq 0, \text{ for all } y \in K. \tag{8}$$

Applying the compactness of K, we know there exists a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that $u_n \to u$ for some $u \in C$. Since F is upper semicontinuous with nonempty compact values, by Lemma 2.5 (iii), we know there exists a convergent subsequence of $\{u_n^*\}$, whose limit u^* belongs to $F(t, x, u)$. Then letting $n \to \infty$ in (8), we get

$$\langle u^*, y - u \rangle \geq 0, \text{ for all } y \in K.$$

That means $u \in U(t, x)$ and so $(t, x) \in U^{-}(C)$.

(ii) Since U is upper semicontinuous and K is compact, by Lemma 2.8, it is clear that U is superpositionally measurable. □

In the rest of this paper, we impose the following assumptions on the Banach space E and the mapping $f\colon [0, T] \times E \times E_1 \to E$ in PDSVI(1) and PPDSVI(2):

Assumption 3.4. We assume that E is a real separable Banach space and f satisfies the following assumptions:

(f_1) For every $(t, x) \in [0, T] \times E$ and every convex set $D \subset K$, the set $f(t, x, D)$ is convex in E;

(f_2) there exists $\alpha > 0$ such that

$$\|f(t, x, u)\|_E \leq \alpha(1 + \|x\|_E), \forall (t, x, u) \in [0, T] \times E \times K;$$

(f_3) $f(\cdot, x, u)\colon [0, T] \to E$ is measurable for every $(x, u) \in E \times E_1$;

(f_4) $f(t, \cdot, \cdot)\colon E \times E_1 \to E$ is continuous for a.e. $t \in [0, T]$;

(f_5) there exists $k \in L^1([0, T]; R)$ such that

$$\|f(t, x_0, u) - f(t, x_1, u)\|_E \leq k(t)\|x_0 - x_1\|_E,$$

for a.e. $t \in [0, T]$, for all $x_0, x_1 \in E$, and for all $u \in K$.

Remark 3.5. While the assumptions $(f_1), (f_3) - (f_5)$ are the same as the assumptions in [23], assumption (f_2) is a special case of the corresponding assumption in [23]. □

Now we give new conditions under which PDSVI(1) has a mild solution.

Theorem 3.6. *Let $K \subset E_1$ be a nonempty compact convex set and F be upper semicontinuous with nonempty compact value. Assume that $U(t, x)$ is nonempty and $(f_1) - (f_5)$ hold. Then* PDSVI(1) *has a mild solution.*

Proof. By Lemma 3.1, for any $(t, x) \in [0, T] \times E$, $U(t, x)$ is nonempty compact convex. Lemma 3.3 shows that U is upper semicontinuous and superpositionally measurable. It follows from Theorem 4.4 in [23] that PDSVI(1) has a mild solution. □

Remark 3.7. If (x, u) is a pair of mild solution of PDSVI(1), the mild variational trajectory u is not only measurable, but also belongs to $L^2((0, T); E_1)$, due to the compactness of K. □

Our next result gives conditions under which the mild solution set is a singleton.

Theorem 3.8. *Suppose the conditions of Theorem* 3.6 *hold. In addition, for any* $t \in [0, T], x_1, x_2 \in E$, $u_1, u_2 \in E_1$, *we assume that*

(i) *there exists* $\alpha_F > 0$ *such that for any* $u_1^* \in F(t, x_1, u_1), u_2^* \in F(t, x_1, u_2)$,

$$\langle u_1^* - u_2^*, u_1 - u_2 \rangle \geq \alpha_F \|u_1 - u_2\|_{E_1}^2;$$

(ii) *there exists* $\beta_F > 0$ *such that for any* $v_1^* \in F(t, x_1, u_1), v_2^* \in F(t, x_2, u_1)$,

$$\|v_1^* - v_2^*\|_{E_1^*} \leq \beta_F \|x_1 - x_2\|_E;$$

(iii) $f(t, \cdot, \cdot)$ *is Lipschitz continuous with Lipschitz constant* $L_f > 0$, *i.e.*

$$\|f(t, x_1, u_1) - f(t, x_2, u_2)\|_E \leq L_f(\|x_1 - x_2\|_E + \|u_1 - u_2\|_{E_1}).$$

Then the mild solution set of PDSVI(1) is a singleton.

Proof. By Theorem 3.6, the mild solution set of PDSVI(1) is nonempty. We only need to show the uniqueness. Let (x_1, u_1) and (x_2, u_2) be two mild solutions of PDSVI(1). Then we have $x_i \in C([0, T]; E)$, $u_i \in L^2((0, T); E_1)$ and $\exists u_i^*(t) \in F(t, x_i(t), u_i(t))$ such that

$$\langle u_i^*(t), y - u_i(t) \rangle \geq 0, \text{ for all } y \in K, \text{ a.e. } t \in [0, T], \tag{9}$$

$$x_i(t) = \tilde{T}(t)\bar{x} + \int_0^t \tilde{T}(t - s) f(s, x_i(s), u_i(s)) ds, \quad \text{for all } t \in [0, T], \tag{10}$$

for $i = 1, 2$. Setting $y = u_2(t)$ in (9) for $i = 1$ we get

$$\langle u_1^*(t), u_2(t) - u_1(t) \rangle \geq 0, \text{ for a.e. } t \in [0, T].$$

Setting $y = u_1(t)$ in (9) for $i = 2$, we obtain that

$$\langle u_2^*(t), u_1(t) - u_2(t) \rangle \geq 0, \text{ for a.e. } t \in [0, T].$$

Furthermore, it follows from the above two inequalities that

$$\langle u_1^*(t), u_1(t) - u_2(t) \rangle \leq \langle u_2^*(t), u_1(t) - u_2(t) \rangle, \text{ for a.e. } t \in [0, T]. \tag{11}$$

From (11), by the assumptions (i) and (ii), we get for any $v_{12}^*(t) \in F(t, x_1(t), u_2(t))$,

$$\begin{aligned} \alpha_F \|u_1(t) - u_2(t)\|_{E_1}^2 &\leq \langle u_1^*(t) - v_{12}^*(t), u_1(t) - u_2(t) \rangle \\ &\leq \langle u_2^*(t) - v_{12}^*(t), u_1(t) - u_2(t) \rangle \\ &\leq \beta_F \|x_1(t) - x_2(t)\|_E \|u_1(t) - u_2(t)\|_{E_1}, \end{aligned} \tag{12}$$

for a.e. $t \in [0,T]$. This inequality leads us to

$$\|u_1(t) - u_2(t)\|_{E_1} \leq \frac{\beta_F}{\alpha_F}\|x_1(t) - x_2(t)\|_E, \quad \text{for a.e.} \quad t \in [0,T]. \tag{13}$$

So $$\int_0^t \|u_1(s) - u_2(s)\|_{E_1} ds \leq \int_0^t \frac{\beta_F}{\alpha_F}\|x_1(s) - x_2(s)\|_E ds, \text{ for all } t \in [0,T]. \tag{14}$$

On the other hand, by Lemma 2.11, there exist $M > 0$ and $w > 0$ such that $\|\tilde{T}(t)\| \leq Me^{wt}$ for all $t \in [0,T]$. So by assumption (iii), (10) implies that for all $t \in [0,T]$,

$$\begin{aligned}
\|x_1(t) - x_2(t)\|_E &= \int_0^t \|\tilde{T}(t-s)(f(s, x_1(s), u_1(s)) - f(s, x_2(s), u_2(s)))\|_E \, ds \\
&\leq L_f \int_0^t \|\tilde{T}(t-s)\|(\|x_1(s) - x_2(s)\|_E + \|u_1(s) - u_2(s)\|_{E_1}) ds \\
&\leq Me^{wT} L_f \int_0^t (\|x_1(s) - x_2(s)\|_E + \|u_1(s) - u_2(s)\|_{E_1}) ds. \qquad (15)
\end{aligned}$$

Combining (14) with (15) we have that there exists a constant $C > 0$ such that

$$\|x_1(t) - x_2(t)\|_E \leq C \int_0^t \|x_1(s) - x_2(s)\|_E ds.$$

Applying the Gronwall inequality [26, page 45], we can get $x_1(t) = x_2(t)$ for all $t \in [0,T]$, So $x_1 = x_2$ in $C([0,T]; E)$. Then it follows from (13) that $u_1(t) = u_2(t)$ in E_1 for *a.e.* $t \in [0,T]$ and so $u_1 = u_2$ in $L^2((0,T); E_1)$. □

3.2. Stability of the mild solution set mapping

In this section, we consider the continuity of the mild solution set mapping of PPDSVI(2). In the rest of this paper, in addition to the Assumption 3.4, we also assume E_1 is a reflexive Banach space and $L\colon Z_1 \rightrightarrows E_1$ is a continuous set-valued mapping with nonempty compact convex values, $U(x_0)$, $U(u_0)$, $U(p_0)$, $U(v_0)$ are neighborhoods of $x_0 \in C([0,T]; E)$, $u_0 \in L^2((0,T); E_1)$ and $p_0 \in Z_1$, $v_0 \in Z_2$, respectively.

Let $\mathcal{S}(p,v)$ denote the mild solution set of PPDSVI(2). We first show that the mild solution set mapping $(p,v) \to \mathcal{S}(p,v)$ is closed.

Theorem 3.9. *Let* $\operatorname{int}(\operatorname{barr}(L(p_0))) \neq \emptyset$, $(f_1) - (f_5)$ *hold for any* $p \in U(p_0)$ *and* $K = L(p)$. *Let* $SOL(L(p_0), F(t,x,u,v_0)) \neq \emptyset$ *for any* $(t,x,u) \in [0,T] \times E \times E_1$. *Suppose that*

(i) *F is upper semicontinuous set-valued mapping with nonempty compact value on $[0,T] \times E \times E_1 \times Z_2$, and there exists $u_n^* \in F(Id, x_n, u_n, v_n)$ such that $\{u_n^*\}$ is convergent in $L^2((0,T); E_1^*)$ for any*

$$(x_n, u_n, v_n) \in C([0,T]; E) \times L^2((0,T); E_1) \times U(v_0),$$

where Id is the identity mapping on $[0,T]$;

(ii) *there exists a neighborhood* $P \times V$ *of* (p_0, v_0) *and* $M = \bigcup_{p \in P} L(p)$ *such that* $F(t, x, \cdot, \cdot) \colon M \times V \rightrightarrows E_1^*$ *is lower semicontinuous;*

(iii) *for each* $p \in Z_1$ *and* $v \in Z_2$*, the mapping* $u \to F(t, x, u, v)$ *is pseudomonotone on* M.

Then $\mathcal{S}$ *is closed at* (p_0, v_0).

Proof. We prove the result in two steps.

Step 1. There exists a neighborhood $P' \times V'$ of (p_0, v_0) such that $\mathcal{S}(p, v) \neq \emptyset$ for $(p, v) \in P' \times V'$.

By Lemma 2.10, there exists a neighborhood $P' \times V'$ of (p_0, v_0) with $P' \times V' \subset U(p_0) \times U(v_0)$, such that for every $(p, v) \in P' \times V'$ and $(t, x) \in [0, T] \times E$, the solution set $SOL(L(p), F(t, x, \cdot, v))$ is nonempty and bounded. Furthermore, from Lemma 3.1 the solution set $SOL(L(p), F(t, x, \cdot, v))$ is also convex. Then by Theorem 3.6 we know $\mathcal{S}(p_n, v_n)$ is nonempty for any $(p_n, v_n) \in P' \times V'$. Assume $(x_n, u_n) \in \mathcal{S}(p_n, v_n)$. Then there exist $x_n \in C([0, T]; E)$, $u_n \in L^2((0, T); E_1)$ and $u_n^*(t) \in F(t, x_n(t), u_n(t), p_n)$ such that

$$x_n(t) = \tilde{T}(t)\bar{x} + \int_0^t \tilde{T}(t-s) f(s, x_n(s), u_n(s)) ds, \text{ for all } t \in [0, T], \tag{16}$$

$$\langle u_n^*(t), y - u_n(t) \rangle \geq 0 \quad \forall y \in L(p_n), \text{ for a.e. } t \in [0, T]. \tag{17}$$

Step 2. If $P' \times V' \ni (p_n, v_n) \to (p_0, v_0)$ and $\mathcal{S}(p_n, v_n) \ni (x_n, u_n) \to (x_0, u_0)$, then $(x_0, u_0) \in \mathcal{S}(p_0, v_0)$.

Let $(p_n, v_n) \to (p_0, v_0)$ in $Z_1 \times Z_2$ and $(x_n, u_n) \to (x_0, u_0)$ in $C([0, T]; E) \times L^2((0, T); E_1)$ with $v_n \in U(v_0), x_n \in U(x_0), u_n \in U(u_0)$. It is clear that $x_0 \in C([0, T]; E)$ by the compleness of $C([0, T]; E)$ and there exists a convergent subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that $u_n(t) \to u_0(t)$ for a.e. $t \in [0, T]$. According to condition (f_4), it is true that $f(t, x_n(t), u_n(t)) \to f(t, x_0(t), u_0(t))$ for a.e. $t \in [0, T]$. Furthermore, applying (f_2) we know that there exists $C > 0$ with

$$(\forall n) \ \|f(t, x_n(t), u_n(t))\|_E \leq \alpha(1 + \|x_n\|_{C([0,T];E)}) \leq C$$

for a.e. $t \in [0, T]$. Thus, by the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} x_n(t) &= \tilde{T}(t)\bar{x} + \int_0^t \tilde{T}(t-s) f(s, x_n(s), u_n(s)) ds \\ &\longrightarrow \ \tilde{T}(t)\bar{x} + \int_0^t \tilde{T}(t-s) f(s, x_0(s), u_0(s)) ds \end{aligned}$$

for all $t \in [0, T]$. By applying $x_n \to x_0$ in $C([0, T]; E)$ we get for all $t \in [0, T]$

$$x_0(t) = \tilde{T}(t)\bar{x} + \int_0^t \tilde{T}(t-s) f(s, x(s), u_0(s)) ds. \tag{18}$$

On the other hand, from $(x_n, u_n) \in \mathcal{S}(p_n, v_n)$ we know that $u_n(t) \in L(p_n)$. Since $p_n \to p_0$ and L is a upper semicontinuous set-valued mapping with nonempty compact values, then by Lemma 2.5(iii) we know $u(t) \in L(p_0)$ for a.e. $t \in [0, T]$.

From assumption (i), there exists a subsequence of $u_n^* \in F(Id, x_n, u_n, v_n)$, denoted again by u_n^*, such that $u_n^* \to u_0^*$ in $L^2((0,T); E_1^*)$. By Lemma 2.5(iii) again, we obtain $u_0^*(t) \in F(t, x_0(t), u_0(t), v)$. for a.e. $t \in [0,T]$. Next we prove that for any $y \in L(p)$,

$$\langle u_0^*(t), y - u_0(t)\rangle \geq 0, \text{ for a.e. } t \in [0,T]. \tag{19}$$

Applying the lower semicontinuity of L at p_0, we get for any sequence $p_n \to p_0$ and $y \in L(p_0)$, there exists a sequence $y_n \in L(p_n)$ such that $y_n \to y$. Setting $y = y_n$ in (17) we get for any $y_n \in L(p_n)$,

$$\langle u_n^*(t), y_n - u_n(t)\rangle \geq 0, \text{ for a.e. } t \in [0,T].$$

Taking limit in the above inequality, we obtain (19). It is clear that (18) and (19) together mean that $(x_0, u_0) \in \mathcal{S}(p_0, v_0)$. So $\mathcal{S}$ is closed at (p_0, v_0). □

Now we assume that $\mathcal{S}_x(p,v)$ and $\mathcal{S}_u(p,v)$ are all collections of x and u in the mild solution $(x,u) \in \mathcal{S}(p,v)$, respectively. Let

$$L(U_{p_0}) = \bigcup_{p \in U(p_0)} L(p) = \{u \in L(p) \subseteq E_1 : p \in U(p_0)\},$$

$$S_1 = \{w \in \mathcal{S}_u(p,v) \subseteq L^2((0,T); E_1) : (p,v) \in U(p_0) \times U(v_0)\}, \tag{20}$$

and $$S_2 = \{y(t) \in E : y \in \mathcal{S}_x(p,v) \text{ and } (p,v) \in U(p_0) \times U(v_0)\}. \tag{21}$$

The following Theorem 3.10 is the main result of the stability of PPDSVI (2) with the constraint set and the mapping perturbed by different parameters.

Theorem 3.10. *Let the conditions in Theorem* 3.9 *hold with* $\alpha T < \frac{1}{Me^{\omega T}}$, *where* M, ω *are the constants in Lemma* 2.11. *Assume that*

(i) *There exists* $\hat{\alpha}_F > 0$ *such that* $\langle u_1^* - u_2^*, u_1 - u_2\rangle \geq \hat{\alpha}_F \|u_1 - u_2\|_{E_1}^2$, *for all* $(t, x, u_i, v) \in [0,T] \times E \times L(U_{p_0}) \times U(v_0)$ *and* $u_1^* \in F(t, x, u_1, v)$, $u_2^* \in F(t, x, u_2, v)$, *where* $i = 1, 2$.

(ii) *There exists* $\rho_F > 0$ *such that* $\|v_1^* - v_2^*\|_{E_1^*} \leq \rho_F(|t_1 - t_2| + \|x_1 - x_2\|_E)$, *for all* $(t_i, x_i, u, v) \in [0,T] \times E \times L(U_{p_0}) \times U(v_0)$ *and* $v_1^* \in F(t_1, x_1, u, v)$, $v_2^* \in F(t_2, x_2, u, v)$, *where* $i = 1, 2$.

(iii) S_1 *is compact on* $L^2((0,T); E_1)$ *and for any* $t \in [0,T]$, S_2 *is relatively compact on* E.

(iv) $\|f(t_1, x(t_1), u(t_1)) - f(t_2, x(t_1), u(t_1))\|_E \leq \gamma_f(|t_1 - t_2| + \|x(t_1) - x(t_2)\|_E + \|u(t_1) - u(t_2)\|_{E_1})$ *for any* $t_1, t_2 \in [0,T]$ *and* $(x,u) \in \bigcup_{(p,v) \in U(p_0) \times U(v_0)} \mathcal{S}(p,v)$.

Then $(p,v) \to \mathcal{S}(p,v)$ *is continuous at* $(p_0, v_0) \in Z_1 \times Z_2$.

Proof. From Step 1 in Theorem 3.9 we know there exists a neighborhood $P' \times V'$ of (p_0, v_0) such that $\mathcal{S}(p,v) \neq \emptyset$ for any $(p,v) \in P' \times V'$. Take $(p_n, v_n) \in P' \times V'$, with $(p_n, v_n) \to (p_0, v_0)$. So there exists the mild solution $(x_n, u_n) \in \mathcal{S}(p_n, v_n)$ with $x_n \in C([0,T]; E)$ and $u_n \in L^2((0,T); E_1)$. As in Theorem 3.8, we can obtain

$\mathcal{S}(p,v)$ is a singleton for any $(p,v) \in P' \times V'$. From the definition of the mild solution, we know

$$\begin{cases} u_n(t) \in L(p_n), \quad \exists\ u_n^*(t) \in F(t, x_n(t), u_n(t), v_n) \text{ subject to} \\ \langle u_n^*(t), y - u_n(t)\rangle \geq 0, \text{ a.e. } t \in [0,T], \forall y \in L(p_n), \\ x_n(t) = \tilde{T}(t)\bar{x} + \displaystyle\int_0^t \tilde{T}(t-s) f(s, x_n(s), u_n(s))ds, \text{ for all } t \in [0,T]. \end{cases} \tag{22}$$

We prove the result in three steps.

Step 1. $\{x_n\}$ is uniformly bounded. Using (22) and (f_2), we deduce that

$$\begin{aligned} \|x_n\|_{C([0,T];E)} &= \sup_{t\in[0,T]} \|x_n(t)\|_E \\ &= \sup_{t\in[0,T]} \|\tilde{T}(t)\bar{x} + \int_0^t \tilde{T}(t-s) f(s, x_n(s), u_n(s))ds\|_E \\ &\leq Me^{wT}\|\bar{x}\|_E + Me^{wT} \sup_{t\in[0,T]} \int_0^t \|f(s, x_n(s), u_n(s))\|_E ds \\ &\leq Me^{wT}\|\bar{x}\|_E + Me^{wT} \int_0^T \alpha(1 + \|x_n\|_{C([0,T];E)})ds \\ &= Me^{wT}\|\bar{x}\|_E + Me^{wT}(\alpha T + \alpha T\|x_n\|_{C([0,T];E)}). \end{aligned} \tag{23}$$

It is clear that $$\|x_n\|_{C([0,T];R^m)} \leq \frac{Me^{wT}(\|\bar{x}\|_E + \alpha T)}{1 - Me^{wT}\alpha T},$$

which means $\{x_n\}$ is uniformly bounded.

Step 2. $\{x_n\}$ is an equicontinuous family of functions. For any $0 \leq t \leq t+\delta \leq T$, we have

$$\begin{aligned} &x_n(t+\delta) - x_n(t) \\ &= \tilde{T}(t+\delta)\bar{x} - \tilde{T}(t)\bar{x} + \int_0^{t+\delta} \tilde{T}(t+\delta-s) f(s, x_n(s), u_n(s))ds - \int_0^t \tilde{T}(t-s) f(s, x_n(s), u_n(s))ds \\ &= \tilde{T}(t+\delta)\bar{x} - \tilde{T}(t)\bar{x} - \int_0^{\delta} \tilde{T}(t+\delta-s) f(s, x_n(s), u_n(s))ds - \int_0^t \tilde{T}(t-s) f(s, x_n(s), u_n(s))ds \\ &+ \int_0^{\delta} \tilde{T}(t+\delta-s) f(s, x_n(s), u_n(s))ds + \int_0^{t+\delta} \tilde{T}(t+\delta-s) f(s, x_n(s), u_n(s))ds \\ &= \tilde{T}(t+\delta)\bar{x} - \tilde{T}(t)\bar{x} + \int_{\delta}^{t+\delta} \tilde{T}(t+\delta-s) f(s, x_n(s), u_n(s))ds \\ &+ \int_0^{\delta} \tilde{T}(t+\delta-s) f(s, x_n(s), u_n(s))ds - \int_0^t \tilde{T}(t-s) f(s, x_n(s), u_n(s))ds. \end{aligned} \tag{24}$$

Letting $\tau = s - \delta$, we note that

$$\int_{\delta}^{\delta+t} \tilde{T}(t+\delta-s) f(s, x_n(s), u_n(s))ds = \int_0^t \tilde{T}(t-\tau) f(\tau+\delta, x_n(\tau+\delta), u_n(\tau+\delta))d\tau,$$

and so $\displaystyle\int_{\delta}^{\delta+t}\tilde{T}(t+\delta-s)f(s,x_n(s),u_n(s))ds-\int_0^t\tilde{T}(t-s)f(s,x_n(s),u_n(s))ds$

$$=\int_0^t\tilde{T}(t-s)[f(s+\delta,x_n(s+\delta),u_n(s+\delta))-f((s,x_n(s),u_n(s))]ds. \tag{25}$$

Combing (25) with (24), we see that

$$\begin{aligned}&\|x_n(t+\delta)-x_n(t)\|_E\\&\le\|\tilde{T}(t+\delta)\bar{x}-\tilde{T}(t)\bar{x}\|_E+\int_0^{\delta}\|\tilde{T}(t+\delta-s)f(s,x_n(s),u_n(s))\|_E ds\\&+\int_0^t\|\tilde{T}(t-\tau)\|_E\|f(s+\delta,x_n(s+\delta),u_n(s+\delta))-f(s,x_n(s),u_n(s))\|_E ds.\end{aligned} \tag{26}$$

By virtue of (iv),

$$\begin{aligned}&\|f(s+\delta,x_n(s+\delta),u_n(s+\delta))-f(s,x_n(s),u_n(s))\|_E\\&\le\gamma_f(\delta+\|x_n(s+\delta)-x_n(s)\|_E+\|u_n(s+\delta)-u_n(s)\|_{E_1}).\end{aligned} \tag{27}$$

Since $\{x_n\}$ is uniformly bounded, by Lemma 2.12, from (26) and (3.2) we get there exist $C_1,C_2,C_3>0$ such that

$$\begin{aligned}&\|x_n(t+\delta)-x_n(t)\|_E\\&\le C_1\delta+C_2\delta+Me^{wT}\int_0^t\gamma_f(\delta+\|x_n(s+\delta)-x_n(s)\|_E+\|u_n(s+\delta)-u_n(s)\|_{E_1})ds\\&\le C_1\delta+C_2\delta+C_3\delta+Me^{wT}\gamma_f\int_0^t\|x_n(s+\delta)-x_n(s)\|_E ds\\&\quad+Me^{wT}\gamma_f\int_0^t\|u_n(s+\delta)-u_n(s)\|_{E_1}ds.\end{aligned} \tag{28}$$

Next we show that there exists a constant $\tilde{C}>0$ such that

$$\int_0^t\|u_n(s+\delta)-u_n(s)\|_{E_1}ds\le\tilde{C}\int_0^t\|x_n(s+\delta)-x_n(s)\|_E ds.$$

(22) implies that there exists a subset $I\subset[0,T]$ with $m(I)=0$ (where $m(I)$ denotes the Lebesgue measure of I) such that for any $t\in[0,T]\setminus I$, there exists $u_n^*(t)\in F(t,x_n(t),u_n(t),v_n)$ subject to

$$\langle u_n^*(t),y-u_n(t)\rangle\ge0,\ \forall y\in L(p_n).$$

So for $s,s+\delta\in[0,T]\setminus I$, there exist $u_n^*(s)\in F(s,x_n(s),u_n(s),v_n)$ and $u_n^*(s+\delta)\in F(s+\delta,x_n(s+\delta),u_n(s+\delta),v_n)$ with $u_n(s),u_n(s+\delta)\in L(p_n)$ such that for any $y\in L(p_n)$,

$$\langle u_n^*(s),y-u_n(s)\rangle\ge0,\quad\text{and}\quad\langle u_n^*(s+\delta),y-u_n(s+\delta)\rangle\ge0.$$

Taking $y = u_n(s+\delta)$ and $y = u_n(s)$ in above inequality respectively, we have

$$\langle u_n^*(s), u_n(s+\delta) - u_n(s)\rangle \geq 0, \quad \text{and} \quad \langle u_n^*(s+\delta), u_n(s) - u_n(s+\delta)\rangle \geq 0.$$

Upon adding these two inequalities, we obtain

$$\langle u_n^*(s+\delta), u_n(s+\delta) - u_n(s)\rangle \leq \langle u_n^*(s), u_n(s+\delta) - u_n(s)\rangle. \tag{29}$$

By conditions (i) and (ii), for $s, s+\delta \in [0,T] \setminus I$, and $w^*_{n(s,\delta)} \in F(s+\delta, x_n(s+\delta), u_n(s), v_n)$, we have

$$\begin{aligned}\hat{\alpha}_F \|u_n(s+\delta) - u_n(s)\|^2_{E_1} &\leq \langle u_n^*(s+\delta) - w^*_{n(s,\delta)}, u_n(s+\delta) - u_n(s)\rangle \\ &\leq \langle u_n^*(s) - w^*_{n(s,\delta)}, u_n(s+\delta) - u_n(s)\rangle \\ &\leq \rho_F(\delta + \|x_n(s+\delta) - x_n(s)\|_E)\|u_n(s+\delta) - u_n(s)\|_{E_1}.\end{aligned} \tag{30}$$

It is easy to see that for $s, s+\delta \in [0,T] \setminus I$,

$$\|u_n(s+\delta) - u_n(s)\|_{E_1} \leq \frac{\rho_F}{\hat{\alpha}_F}(\delta + \|x_n(s+\delta) - x_n(s)\|_E). \tag{31}$$

Furthermore, for any $0 \leq t \leq t+\delta \leq T$, we let $I_1 = \{s \in [0,t] : s, s+\delta \in [0,T]\setminus I\}$. It is can be proved that $m([0,t]\setminus I_1) = 0$. So it follows from (31) that

$$\begin{aligned}&\int_0^t \|u_n(s+\delta) - u_n(s)\|_{E_1} ds \\ = &\int_{I_1} \|u_n(s+\delta) - u_n(s)\|_{E_1} ds + \int_{[0,t]\setminus I_1} \|u_n(s+\delta) - u_n(s)\|_{E_1} ds \\ \leq &\int_0^t \frac{\rho_F}{\hat{\alpha}_F}(\delta + \|x_n(s+\delta) - x_n(s)\|_E) ds.\end{aligned} \tag{32}$$

We therefore deduce from (28) and (31) that there exist $C_4, C_5 > 0$ such that for any $0 \leq t \leq t+\delta \leq T$,

$$\|x_n(t+\delta) - x_n(t)\|_E \leq C_4\delta + C_5 \int_0^t \|x_n(s+\delta) - x_n(s)\|_E ds, \ \text{for all } t \in [0,T]. \tag{33}$$

Applying the Gronwall inequality [26, page 45] we know for any $0 \leq t \leq t+\delta \leq T$, there exists $C_6 > 0$ such that

$$\|x_n(t+\delta) - x_n(t)\|_E \leq C_6\delta, \tag{34}$$

which means that $\{x_n\}$ is an equicontinuous family of functions.

Step 3. $\mathcal{S}(p_n, v_n) \to \mathcal{S}(p_0, v_0)$ in $C([0,T]; E) \times L^2((0,T); E_1)$.

By Lemma 2.6, using the assumption (iii) we know that $\{x_n\}$ has a convergent subsequence, denoted again by $\{x_n\}$, with $x_n \to x_0 \in C([0,T]; E)$. Since $(x_n, u_n) = \mathcal{S}(p_n, v_n)$, using assumption (iii) we get that $\{u_n\}$ has a convergent subsequence, denoted again by $\{u_n\}$, such that $u_n \to u_0$ in $L^2((0,T); E_1)$. By Theorem 3.9, $(x_0, u_0) \in \mathcal{S}(p_0, v_0)$. So $(x_n, u_n) = \mathcal{S}(p_n, v_n) \to (x_0, u_0) = \mathcal{S}(p_0, v_0)$ in $C([0,T]; E) \times L^2((0,T); E_1)$. $\square$

Remark 3.11. When $E = R^m, E_1 = R^n$, Theorem 3.10 still holds without assumption (iii). □

Proof. Recalling the proof of Theorem 3.10, we need to show that $\{x_n\}$ has a convergent subsequence in $C([0,T];R^m)$ and $\{u_n\}$ has a convergent subsequence in $L^2((0,T);R^n)$. First, we show that $\{x_n\}$ is uniformly bounded. By Lemma 2.11, using (22), we deduce that

$$\begin{aligned}
\|x_n\|_{C([0,T];R^m)} &= \sup_{t\in[0,T]} \|x_n(t)\|_{R^m} \\
&= \sup_{t\in[0,T]} \left\|\tilde{T}(t)\bar{x} + \int_0^t \tilde{T}(t-s)f(s,x_n(s),u_n(s))ds\right\|_{R^m} \\
&\leq Me^{wT}\|\bar{x}\|_{R^m} + Me^{wT}\sup_{t\in[0,T]}\int_0^t \|f(s,x_n(s),u_n(s))\|_{R^m}ds \\
&\leq Me^{wT}\|\bar{x}\|_{R^m} + Me^{wT}\int_0^T \alpha(1+\|x_n\|_{C([0,T];R^m)})ds \\
&= Me^{wT}\|\bar{x}\|_{R^m} + Me^{wT}(\alpha T + \alpha T\|x_n\|_{C([0,T];R^m)}). \qquad (35)
\end{aligned}$$

It is clear that $$\|x_n\|_{C([0,T];R^m)} \leq \frac{Me^{wT}(\|\bar{x}\|_E + \alpha T)}{1 - Me^{wT}\alpha T},$$

which means $\{x_n\}$ is uniformly bounded. Using the same proof as in Theorem 3.10, we can obtain $\{x_n\}$ is equicontinuous. So by Lemma 2.6, $\{x_n\}$ has a convergent subsequence. Next, we show that $\{u_n\}$ has a convergent subsequence. For any $\varepsilon > 0$, taking $\delta = (\sqrt{\varepsilon}\hat{\alpha}_F)(\rho_F\sqrt{T + 2C_6T + T(C_6)^2})^{-1}$ in (31), by (34) we have for any $0 \leq t \leq T$,

$$\begin{aligned}
&\int_0^t \|u_n(s+\delta) - u_n(s)\|_{R^n}^2 ds \leq \int_0^t (\frac{\rho_F}{\hat{\alpha}_F})^2(\delta + \|x_n(s+\delta) - x_n(s)\|_{R^m})^2 ds \\
&\leq \int_0^t (\frac{\rho_F}{\hat{\alpha}_F})^2(\delta^2 + 2C_6\delta^2 + C_6^2\delta^2)ds \leq T(\frac{\rho_F}{\hat{\alpha}_F})^2(1 + 2C_6 + {C_6}^2)\delta^2 = \varepsilon. \qquad (36)
\end{aligned}$$

The Kolmogorov-Riesz-Fréchet Theorem [2, page 111] shows that $\{u_n\}$ has a convergent subsequence. □

3.3. An example on the Nash equilibrium

In this subsection, we will give an example of PPDVI about the Nash equilibrium problem. Pang and Stewart [28] studied differential Nash games with DVIs. They consider a noncooperative Nash equilibrium problem with N players where each player solves an optimal control problem. We restate it here with a concise version and refer the readers to the reference [28] for more details. Let $(y^v, u^v) \in R^{n_v+m_v}$, denote player $v's$ pair of state and control variables, respectively. Let $K^v \subseteq R^{m_v}$ be the closed convex set of player $v's$ admissible controls, let $y = (y^v)_{v=1}^N \in R^n$, and $u = (u^v)_{v=1}^N \in R^m$, where $n = \sum_{n=1}^N n_v$ and $m = \sum_{v=1}^N m_v$. Parameterized by rival player's strategies, player $v's$ optimization problem is to compute an optimal trajectory (y^v, u^v). Under appropriate assumptions, finding a linear quadratic differential Nash equilibrium solution (y^v, u^v) is equivalent to find the solution of the following problem:

$$\begin{cases} \begin{pmatrix} \dot{\lambda}(t) \\ \dot{y}(t) \end{pmatrix} = d(t) + A(t) \begin{pmatrix} \dot{\lambda}(t) \\ \dot{y}(t) \end{pmatrix} + B(t)u(t), \\ u(t) \in SOL(K, F(t, y(t), \lambda(t), \cdot)), \\ y(0) = y_0, \lambda(T) = \tilde{C} + Wy(T), \end{cases} \tag{37}$$

where $d(t) \in R^{2n}$, $A(t) \in R^{2n\times 2n}$, $B(t) \in R^{2n\times m}$,

$$F(t, y, \lambda, u) = \nabla_{u^v} H_v(t, y, u, \lambda^v)_{v=1}^N = q(t) + C(t) \begin{pmatrix} \lambda(t) \\ y(t) \end{pmatrix} + S(t)(\cdot)$$

and $H_v(t, y, u, \lambda^v)$ is player $v's$ Hamiltonian function, where λ^v is the adjoint variable of the ODE constraint in player $v's$ control problem, $q(t) \in R^m$, $C(t) \in R^{m\times 2n}$, $S(t) \in R^{m\times m}$, $\tilde{C} \in R^n$, $W \in R^{n\times n,}$ $K = \prod_{v=1}^N K^v \subseteq R^m, y_0 \in R^n$. For the details about this problem we refer the reader to [28].

Pang and Stewart [28] obtained an equivalent relation between a linear quadratic differential Nash equilibrium problem and a DVI. But they did not consider the stability of the problem, we continue to consider the problem here. In generalized differential Nash games, each player's constraints can depend on his rival's changes. As an illustration, let us consider the case where this occurs with the set of controls, thus the constraint set K is replaced by the set $L(p)$, where p is a parameter and L is a set-valued mapping. In real life, the strategy can be influenced by various factors existed in some specific area and any minute change in the proportion of each strategy seen will lead to a change in strategy and the breakdown of the equilibrium. Based on this fact, we assume that the integral function, a constituent part of the cost function, is perturbed by a parameter v, which will lead to the perturbation of F. Thus, under this assumption that $\lambda(0) = \lambda_0 \in R^n$, the differential Nash equilibrium problem (37) can be transformed to the following model:

$$\begin{cases} \begin{pmatrix} \dot{\lambda}(t) \\ \dot{y}(t) \end{pmatrix} = d(t) + A(t) \begin{pmatrix} \lambda(t) \\ y(t) \end{pmatrix} + B(t)u(t), \\ u(t) \in SOL(L(p), F(t, y(t), \lambda(t), \cdot, v)), \\ \begin{pmatrix} \lambda(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ y_0 \end{pmatrix}, \lambda(T) = \tilde{C} + Wy(T). \end{cases} \tag{38}$$

This differential Nash equilibrium problem (38) is a special form of PPDVI(2) with $A = 0$, $x(t) = \begin{pmatrix} \lambda(t) \\ y(t) \end{pmatrix}$, $f(t, x(t), u(t)) = d(t) + A(t)x(t) + B(t)u(t)$. Applying Theorem 3.10 we obtain:

Corollary 3.12. *Let $A(t) \in L((0,T); R^+)$, $q\colon [0,T] \to R^m$, $C\colon [0,T] \to R^{m\times 2n}$, $S\colon [0,T] \to R^{m\times m}$ be continuous and F satisfy the conditions in Theorem* 3.10*. Let $(p_0, v_0) \in Z_1 \times Z_2$, $L\colon Z_1 \rightrightarrows R^m$ be a continuous set-valued mapping with nonempty compact convex values and $int(barr L(p_0)) \neq \emptyset$, where Z_1, Z_2 are metric spaces. Suppose that*

(i) *There exists $C_p > 0$ such that $\|d(t)\|_{R^{2n}} + \|k\|_{R^m} + \|A(t)\|_{R^{2n\times 2n}} \leq C_p$ for any $t \in [0,T], k \in L(p)$;*

(ii) *$g(\cdot) = d(\cdot) + A(\cdot)x + B(\cdot)u$ is measurable for any $(x,u) \in R^{2n} \times R^m$ and $h(x,u) = d(t) + A(t)x + B(t)u$ is continuous on $R^{2n} \times R^m$ for any $t \in [0,T]$;*

(iii) *there exists $\gamma > 0$ such that*

$$\begin{aligned}&\|d(t_1) + A(t_1)x(t_1) + B(t_1)u(t_1) - d(t_2) + A(t_2)x(t_2) + B(t_2)u(t_2)\|_{R^{2n}}\\ &\leq \gamma\left(|t_1 - t_2| + \|x(t_1) - x(t_2)\|_{R^{2n}} + \|u(t_1) - u(t_2)\|_{R^m}\right)\end{aligned} \tag{39}$$

for any $t_1, t_2 \in [0,T]$ and $(x,u) \in \bigcup_{(p,v)\in U(p_0)\times U(v_0)} \mathcal{S}(p,v)$, where $\mathcal{S}(p,v)$ is the mild solution set of (38)*.*

Then $(p,v) \to \mathcal{S}(p,v)$ is continuous at (p_0,v_0).

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