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On Formulae for the Ioffe Geometric Subdifferential of a Supremum Function*

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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We study the first order variational behavior of the supremum of an arbitrary family of lower semicontinuous functions over a weakly compactly generated β-smooth Banach space. In this context, we present new upper-estimations for the viscosity subdifferential and the Ioffe geometric subdifferential.

Keywords: Supremum function, Ioffe geometric subdifferential, β-smooth property, first order analysis, fuzzy calculus.

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1. Introduction

Let X be a Banach space and $\{f_t \colon X \to \overline{\mathbb{R}} \;:\; t \in T\}$ be a family of functions indexed by a nonempty set T, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. We define the *supremum function* $f \colon X \to \overline{\mathbb{R}}$ as

$$f(x) := \sup_{t \in T} f_t(x).$$

The supremum functions arise naturally in different topics of optimization and nonsmooth analysis, and therefore several authors have contributed to the subject since the sixties. We refer the reader to [7, 21, 22, 33, 42, 46, 50, 51, 54, 57].

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A key point in these studies is the understanding of the first variational behavior of the function f, for which a preliminary idea could be the study of the gradient of f. Nevertheless, the nonsmoothness of the supremum function forces the study of generalized notions of differentiability instead of simple derivatives.

From its beginning, subdifferential calculus has been one of the fundamental topics in nonsmooth analysis and optimization. The most well-known object in this topic is the subdifferential of convex functions, also known as the Moreau-Rockafellar subdifferential. Nowadays, it can be found in many monographs of optimization, and it is without any doubt an essential topic in any course of convex analysis (see, e.g., [4, 49, 53, 58]). However, several models in science impose the study of non-convex problems, and so many generalizations of the convex subdifferential have been introduced. Moreover, this necessity of nonconvex and nonsmooth analysis has driven some authors to establish an axiomatic theory for subdifferentiability; we refer to [29, 34, 37] for more details.

Most of the works related to the study of the subdifferential of the supremum function are given in the convex case, that is, when the data functions f_t are convex proper and lower-semicontinuous. In these cases, the authors have computed the subdifferential at the point of interest using the subdifferential of the nominal data. Among the references, we highlight [22], where the authors showed that the supremum functional plays a fundamental role in the whole theory of convex subdifferential calculus. In [7, 33, 50] the reader can found extensions of the work mentioned above.

In the nonconvex case, many of the results in the literature consider that the nominal data are uniformly locally Lipschitz at the point of interest $\bar{x}$, which means that there exist a neighborhood U of $\bar{x}$ and a real constant $L > 0$ such that for all $t \in T$ and all $y, z \in U$

$$|f_t(y) - f_t(z)| \leq L\|y - z\|.$$

We refer to [5, 46, 47] and the references therein for more details. In these works, diverse upper-estimation for the Mordukhovich subdifferential and Clarke subdifferential have been obtained, using the subdifferential of the data at points near the point of interest.

Recently, in [51] the author studies the Fréchet and Mordukhovich subdifferentials of the supremum function in Asplund spaces with arbitrary lower-semicontinuous data functions f_t. However, typical applications of the supremum function emerge in *semi-infinite programming*, and in this setting, we naturally encounter spaces as $\ell^\infty(T)$ and $C(T)$ (the space of bounded sequences and continuous functions over T, respectively), which are not Asplund and consequently the theory of the Fréchet and Mordukhovich subdifferentials is not adequate in such a framework (see, e.g., [2, 4, 25, 26, 27, 49]).

This reason motivated us to extend the results of [51] to a broader class of Banach spaces using Viscosity subdifferentials and the *Ioffe geometric subdifferential* [27].

The contributions of Alexander Ioffe in subdifferential calculus have been a cornerstone in the development of the field. However, his wide production is not limited to this: he has contributed to the variational analysis and optimization in several subjects, like metric regularity [29, 35, 36, 39], semialgebraic and tame optimization [13, 31], separable reduction [17, 18, 19, 32], integral functionals [23, 24, 30, 38], second-order conditions [1, 40, 41] and optimal control [28, 31], among others. Our contribution in the present work is to provide to this vast spectrum a compliment concerning suprema of nonsmooth functions.

The rest of the paper is organized as follows: Section 2 is concerned with some preliminaries and notation needed for the sequel. In Section 3, we present some basic properties of the Viscosity subdifferential, and we give some fuzzy calculus rules for it. Section 4 contains our main results, which are upper estimates for the Ioffe geometric subdifferential and approximate subdifferential. We finish the article in Section 5 with some final comments and perspectives.

2. Preliminaries

As general notation we write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, and for a set A, we denote by $\operatorname{Card}(A)$ its *cardinality*.

In what follows, X denotes a Banach space with norm $\|\cdot\|$, dual space X^*, and duality product $\langle \cdot, \cdot \rangle$. We also denote by $\|\cdot\|$ the dual norm, and the difference between it and the primal one is given by the context. We also write $w(X, X^*)$ (or simply w) and $w^*(X^*, X)$ (or simply w^*) to denote the *weak topology* on X given by X^* and the *weak* topology* on X^* given by X, respectively.

As usual, $\mathbb{B}_X$ and $\mathbb{S}_X$ stand for the *unit ball* and the *unit sphere* of X, respectively. Similarly, for a point $x \in X$ and $r > 0$ we write $B_X(x, r)$ and $B_X[x, r]$ to denote the open and closed *balls* centered in x with radius r, respectively. If there is no ambiguity, we may omit the space subindex in the latter notation. For a point $x \in X$ and a topology τ over X, we write $\mathcal{N}_X(x, \tau)$ the set of all τ-neighborhoods of x in X.

Let $f\colon X \to \overline{\mathbb{R}}$ be a function over X, $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, and x be a point in X. For a topology τ on X, the notation $x_n \xrightarrow{\tau} x$ means that (x_n) τ-converges to x. Also, the notation $x_n \xrightarrow{\tau,f} x$ means that (x_n) τ-converges to x and $(f(x_n))$ converges to $f(x)$ (under the usual topology on $\overline{\mathbb{R}}$). If there is no ambiguity, we will omit the topology τ in the above notation. Finally, if $f(x) \in \mathbb{R}$, we define for $r > 0$ the sets

$$B_X(x, f, r) := \{y \in X \ : \ \|x - y\| < r \text{ and } |f(x) - f(y)| < r\},$$
$$B_X[x, f, r] := \{y \in X \ : \ \|x - y\| \le r \text{ and } |f(x) - f(y)| \le r\}.$$

Again, we will omit the subindex X, if there is no confusion.

A function $\phi\colon X \to \mathbb{R}$ is said to be a *bump function* if

$$\operatorname{supp} \phi := \{x \in X : \phi(x) \neq 0\}$$

is nonempty and bounded. A *bornology* β on X is a collection of closed, symmetric and bounded subsets of X which covers X and satisfies the following properties:

- For any two elements $A, B \in \beta$ there exists $C \in \beta$ such that $A \cup B \subseteq C$; and
- $aU \in \beta$ whenever $U \in \beta$ and $a > 0$.

For a bornology β, the *β^*-topology* on X^* is the topology generated by the basis of neighborhoods of zero $\{U^\circ : U \in \beta\}$, where U° stands for the (negative) polar set of U, that is, $U^\circ := \{x^* \in X^* : \langle x^*, x\rangle \leq 1\}$. Two of the most useful bornologies considered in the literature are the *Fréchet bornology* F consisting of all closed symmetric bounded sets, and the *Hadamard bornology* H consisting of all symmetric norm-compact sets.

For a bornology β on X, a function $\phi\colon U \to \mathbb{R}$ (where U is an open set) is said to be *β-differentiable* at $x \in U$ if there exists $x^* \in X^*$ such that for every $B \in \beta$

$$\lim_{s\to 0^+} \sup_{h\in B} \left| \frac{\phi(x+sh)-\phi(x)}{s} - \langle x^*, h\rangle \right| = 0. \tag{1}$$

In such a case, x^* is unique and consequently we denote $\nabla_\beta \phi(x) = x^*$. Besides, ϕ is said to be *β-smooth* if it is β-differentiable at every $x \in U$ and the derivative $\nabla_\beta \phi$ is a continuous function from $(U, \|\cdot\|)$ to (X^*, β^*).

A Banach space X is said to be β-smooth if there is a Lipschitz continuous, bump β-smooth function from X to $\mathbb{R}$. Each separable Banach space, or more general every *weakly compactly generated* (WCG) Banach space, admits an H-smooth renorming. Moreover, if the dual is also separable (that is, if the separable space is Asplund), then the space admits an F-smooth renorming (see, e.g., [11, 16, 52]). Particularly, if T is a metric space, $\ell^\infty(T)$ is H-smooth when T is separable, and $C(T)$ is H-smooth when T is compact.

Let β be a bornology on X and $f\colon X \to \overline{\mathbb{R}}$ be a lower semicontinuous (lsc) function, finite at $x \in X$. For $k > 0$ we set

$$\partial_{\beta,k} f(x) := \left\{ x^* \in X^* \colon \begin{array}{c} \text{there is a } \beta\text{-smooth function } \phi, \text{ locally} \\ k\text{-Lipschitz continuous near } x, \text{ such that} \\ \nabla_\beta \phi(x) = x^* \text{ and } f - \phi \text{ attains a local minimum at } x \end{array} \right\}.$$

By locally k-Lipschitz continuous near x, we mean that there exists a neighborhood U of x such that $\phi\big|_U\colon U \to \mathbb{R}$ is Lipschitz continuous with constant k. The *(viscosity) β-subdifferential* of f at x is defined by

$$\partial_\beta f(x) := \bigcup_{k>0} \partial_{\beta,k} f(x)$$

(see, e.g., [2] and references therein). For any set S, the *(viscosity) β-normal cone* of S at x is denoted and defined as $N_\beta(S, x) := \partial_\beta \delta_S(x)$, where δ_S is the indicator function of S with $\delta_S(x) = 0$ if $x \in S$ and $\delta_S(x) = +\infty$ if $x \in X \setminus S$.

Consider $S \subseteq X$ and $x \in S$. A functional $x^* \in X^*$ is said to be *G-normal* to S at x if there exists $k > 0$ such that for any $\varepsilon > 0$ and any finite-dimensional subspace $F \subseteq X$, there are $y \in B_X(x, \varepsilon)$ and $y^* \in X^*$ such that

$$\langle y^* - x^*, h\rangle \leq \varepsilon \|h\| \text{ and } \langle y^*, h\rangle \leq k d_S^-(x, h), \ \forall h \in F,$$

where $d_S(y) := \inf_{s\in S} \|s - y\|$ and $d_S^-(x, h) := \liminf_{r\to 0^+} r^{-1}(d_S(x + rh) - d_S(x))$.

The set of all G-normals is denoted by $N_G(S,x)$ (see [27]) and it is called the Ioffe G-normal cone (also known as the geometric normal cone). The *Clarke normal cone* to S at x can be defined as $N_C(S,x) := \overline{\mathrm{co}}^{w^*} N_G(S,x)$ (see [27, Proposition 3.4]). Hence, for a function f and $x \in \operatorname{dom} f$ the *G-subdifferential*, the *singular G-subdifferential* and the *Clarke subdifferential* can be defined as

$$\begin{aligned}
\partial_G f(x) :=&\{x^* \in X^* : (x^*, -1) \in N_G(\operatorname{epi} f, (x, f(x)))\},\\
\partial_G^\infty f(x) :=&\{x^* \in X^* : (x^*, 0) \in N_G(\operatorname{epi} f, (x, f(x)))\},\\
\partial_C f(x) :=&\{x^* \in X^* : (x^*, -1) \in N_C(\operatorname{epi} f, (x, f(x)))\},
\end{aligned}$$

respectively. Observe that, as for the β-subdifferential, one also has that for any subset S of X, $N_G(S,x) = \partial_G \delta_S(x)$ and $N_C(S,x) = \partial_C \delta_S(x)$.

If $|f(x)| = +\infty$, any of the previous subdifferentials is set, by convention, as empty. It is important to recall that whenever f is convex proper and lower semicontinuous, all of these subdifferentials coincide with the classical subdifferential of convex analysis, that is,

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y - x\rangle \le f(y) - f(x), \forall y \in X\}.$$

The construction of the Clarke and the G-subdifferential is implicit and therefore it is useful to have sequential representation formulae for these mathematical objects. The following result was proved in [43, Theorem 1.6 and Theorem 1.8] for β-smooth Banach spaces. We refer to [43] and the references therein for previous results in the same line.

Proposition 2.1. *Assume that the space X is β-smooth. Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $x \in \operatorname{dom} f$. Then*

$$\begin{aligned}
\partial_G f(x) =& \bigcup_{k>0} \mathrm{cl}^{w^*} \left\{ w^*\text{-}\lim x_n^* : x_n^* \in \partial_{\beta,k} f(x_n), \text{ and } x_n \xrightarrow{f} x \right\},\\
\partial_G^\infty f(x) =& \bigcup_{k>0} \mathrm{cl}^{w^*} \left\{ w^*\text{-}\lim \nu_n \cdot x_n^* : x_n^* \in \partial_{\beta,\nu_n^{-1}k} f(x_n), \text{ and } x_n \xrightarrow{f} x,\ \nu_n \to 0^+ \right\},\\
\partial_C f(x) =& \overline{\mathrm{co}}^{w^*} \left\{ \tilde{\partial}_G f(x) + \tilde{\partial}_G^\infty f(x) \right\},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\partial}_G f(x) :=& \bigcup_{k>0} \left\{ w^*\text{-}\lim x_n^* : x_n^* \in \partial_{\beta,k} f(x_n), \text{ and } x_n \xrightarrow{f} x \right\},\\
\tilde{\partial}_G^\infty f(x) :=& \bigcup_{k>0} \left\{ w^*\text{-}\lim \nu_n \cdot x_n^* : x_n^* \in \partial_{\beta,\nu_n^{-1}k} f(x_n), \text{ and } x_n \xrightarrow{f} x,\ \nu_n \to 0^+ \right\}.
\end{aligned}$$

are called the core of the G-subdifferential and the core of the singular G-subdifferential, respectively.

The following result is a well-known approximation property for horizontal subgradients of the β-normal cone of the epigraph of a function. We refer to [43, Proposition 1.4] for the proof.

Proposition 2.2. *Let X be a β-smooth Banach space. Let $f\colon X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function and $(x^*, 0) \in N_\beta(\operatorname{epi} f, (x, f(x)))$. Then for any $\varepsilon > 0$ and β^*-neighborhood U^* of zero in X^* there exist $y \in X$ and $(y^*, \lambda) \in N_\beta(\operatorname{epi} f, (y, f(y)))$ such that $\lambda \in (-\varepsilon, 0)$, $\|y - x\| \leq \varepsilon$, $|f(y) - f(x)| < \varepsilon$ and $y^* \in x^* + U^*$.*

Let T be a nonempty index set, and let $\{f_t\}_{t\in T}$ be a family of functions from X to $\overline{\mathbb{R}}$. We will consider the pointwise supremum function of the family as

$$f\colon X \to \overline{\mathbb{R}}, \quad x \mapsto f(x) := \sup_{t\in T} f_t(x). \tag{2}$$

The symbol $\mathcal{P}_{\mathtt{f}}(T)$ denotes the set of all $F \subseteq T$ such that F is finite.
For $F \in \mathcal{P}_{\mathtt{f}}(T)$ we define $f_F(x) := \max_{s\in F} f_s(x)$.

Following the notation of [46], $\mathbb{R}^T$ is defined as the space of all multipliers $\lambda = (\lambda_t)$ and $\mathbb{R}^{(T)}$ denotes the set of all $\lambda \in \mathbb{R}^T$ such that $\lambda_t \neq 0$ for finitely many $t \in T$; for $\lambda \in \mathbb{R}^T$, we denote the support of λ by $\operatorname{supp}\lambda := \{t \in T \ : \ \lambda_t \neq 0\}$.

We define the *generalized simplex on T* as

$$\Delta(T) := \{\lambda \in \mathbb{R}^{(T)} : \ \lambda \geq 0 \text{ and } \textstyle\sum_{t\in T} \lambda_t = 1\}.$$

If $\operatorname{Card}(T) = n$, we simply identify $\Delta(T)$ with the usual simplex set in $\mathbb{R}^n$, denoted by Δ_n. For a point $\bar{x} \in X$ and $\varepsilon \geq 0$, we define

- the set of *ε-active indexes at $\bar{x}$* by
$$T_\varepsilon\big(\{f_t\}_{t\in T}, \bar{x}\big) := \{t \in T : f(\bar{x}) \leq f_t(\bar{x}) + \varepsilon\};$$
- the set of *ε-active sets at $\bar{x}$* by
$$\mathcal{T}_\varepsilon\big(\{f_t\}_{t\in T}, \bar{x}\big) := \{F \in \mathcal{P}_{\mathtt{f}}(T) : f(\bar{x}) \leq f_F(\bar{x}) + \varepsilon\};$$
- The set of *ε-multipliers at $\bar{x}$* by
$$\Delta(T, \{f_t\}_{t\in T}, \bar{x}, \varepsilon) := \left\{ (\lambda_t) \in \mathbb{R}^{(T)} : \begin{array}{c} (\lambda_t) \geq 0 \text{ for all } t \in T, \\ \lambda_t \leq \varepsilon, \ \forall t \in T \backslash T_\varepsilon\big(\{f_t\}_{t\in T}, \bar{x}\big) \\ \text{and } |\sum_{t\in T} \lambda_t - 1| \leq \varepsilon \end{array} \right\}. \tag{3}$$

If there is no ambiguity, we may omit the family $\{f_t\}_{t\in T}$ in the above definitions, simply writing $T_\varepsilon(\bar{x})$, $\mathcal{T}_\varepsilon(\bar{x})$ and $\Delta(T, \bar{x}, \varepsilon)$ instead. Also, if $\operatorname{Card}(T) = n \in \mathbb{N}$, we write $\Delta_n(\bar{x}, \varepsilon)$ instead of $\Delta(T, \bar{x}, \varepsilon)$.

3. Fuzzy calculus rules for the viscosity subdifferential of the supremum function

In this section, we present the results on the viscosity subdifferential we will need to deduce our main results. This preliminary work is needed since the aforementioned subdifferential is the core element to construct the Ioffe geometric subdifferential, as the reader can appreciate from the definitions given in the preceding section. First, we develop some basic properties and calculus estimations, and secondly, we give some fuzzy calculus rules. Finally, the main results of this article, which are Theorems 4.4 and 4.5, will be presented in the next section. In what follows, we always assume that X is a β-smooth Banach space, for some bornology β.

The next proposition contains the main properties we will use in our results. Most of them are known, and consequently, the corresponding references are given in the proof.

Proposition 3.1. *The viscosity β-subdifferential ∂_β satisfies the following properties:*

(i) *(Finite-dimensional representation) Consider an lsc function $f\colon X \to \overline{\mathbb{R}}$ and $x^* \in \partial_\beta f(x)$. Then, for every $V \in \mathcal{N}_{X^*}(0, w^*)$ and every finite-dimensional subspace $L \subseteq X$ there exist $\varepsilon > 0$ and a continuous function $\phi\colon X \to \mathbb{R}$ such that $\partial_\beta \phi(B[x, \varepsilon]) \subseteq x^* + V^*$ and $f - \phi + \delta_{L \cap B[x,\varepsilon]}$ attains its minimum at x.*

(ii) *(Calculus estimation) For every $\varepsilon > 0$, any point $x \in X$ and every finite-dimensional subspace L of X, we have*

$$\partial_\beta \delta_{B[x,\varepsilon] \cap L}(x') \subseteq L^\perp, \ \forall x' \in B(x, \varepsilon).$$

(iii) *(Enhanced Fuzzy Sum Rule) Consider an lsc function f, a convex Lipschitz function g and a point $x \in X$. If x is a local minimum of $f + g$ with $f(x) \in \mathbb{R}$, then there are sequences $(x_n, x_n^*)_{n \in \mathbb{N}}$ such that $x_n^* \in \partial_\beta f(x_n)$, $x_n \xrightarrow{f} x_0$, $x_n^* \xrightarrow{\|\cdot\|} x_0^*$ with $-x_0^* \in \partial_\beta g(x)$.*

(iv) *(Fuzzy Sum Rule) Consider a finite family of lsc functions $f_j\colon X \to \overline{\mathbb{R}}$ (with $j \in J$) and $x^* \in \partial_\beta (\sum_{j \in J} f_j)(x)$. Then, there are nets $(x_{\alpha,j}, x^*_{\alpha,j})_{\alpha \in \mathbb{D}}$ such that $x^*_{\alpha,j} \in \partial_\beta f_j(x_{\alpha,j})$, $x_{\alpha,j} \xrightarrow{f} x$ and $\sum_{j \in J} x^*_{\alpha,j} \xrightarrow{w^*} x^*$.*

(v) *For every finite family of lsc functions $f_j\colon X \to \overline{\mathbb{R}}$ (with $j \in J$) and every $x \in X$, one has*

$$\partial_\beta f_J(x) \subseteq \bigcap_{\varepsilon > 0} \mathrm{cl}^{w^*} \left\{ \sum \lambda_t \partial_\beta f_j(x_j) : x_j \in B[x, f_j, \varepsilon], \ \lambda \in \Delta(J, x, \varepsilon) \right\}. \quad (4)$$

Moreover, all properties are also valid for the convex subdifferential in arbitrary Banach spaces considering convex proper lsc functions.

Proof. The properties (i) and (ii) follow from the definition of the β-subdifferential. Property (iii) is the well-known *Enhanced Fuzzy Sum Rule* (see, e.g., [55, 56, 2, 43, 45, 6]) and property (iv) is an equivalence of property (iii) (see [44]). Finally, we must prove property (v); To do so, it is enough to prove the result for the pointwise supremum of two functions, that is, with $J = \{1, 2\}$.

Let us denote $g := \max\{f_1, f_2\}$. Let $x^* \in \partial_\beta g(x)$, $\varepsilon \in (0, 1)$, $V^* \in \mathcal{N}_{X^*}(0, w^*)$ and let L be a finite-dimensional subspace of X satisfying that $L^\perp \subset V^*$. Applying property (i), there exist $\gamma \in (0, \varepsilon)$ and a function $\phi\colon X \to \mathbb{R}$ such that $\partial_\beta \phi(B[x, \gamma]) \subset x^* + V^*$ and $g - \phi + \delta_{L \cap B[x,\gamma]}$ attains its minimum at x.

By lower semicontinuity, and shrinking γ if necessary, we may assume that

$$f_i(u) > f_i(x) - \varepsilon, \text{ for all } u \in B[x,\gamma], i = 1,2. \tag{5}$$

Now, consider the function $h\colon X \times \mathbb{R}^2 \to \overline{\mathbb{R}}$ given by

$$h(w,\alpha_1,\alpha_2) := m(\alpha_1,\alpha_2) + \delta_{\operatorname{epi} f_1}(w,\alpha_1) + \delta_{\operatorname{epi} f_2}(w,\alpha_2) - \phi(w) + \delta_{L\cap B[x,\varepsilon]}(w),$$

where $m(\alpha_1,\alpha_2) := \max\{\alpha_1,\alpha_2\}$. Clearly, h has a local minimum at the point $(x, f_1(x), f_2(x))$, and so, by property (iv) we can choose $(\alpha_1,\alpha_2), (q_1,q_2) \in \mathbb{R}^2$, $(w_i,s_i) \in X \times \mathbb{R}$ $(i = 1,2)$ and $(w_i^*,\lambda_i) \in X^* \times \mathbb{R}$ $(i = 1,2)$ satisfying for $i = 1,2$ that

- $|f_i(x) - \alpha_i| \leq \gamma/2$ and
  $(q_1,q_2) \in \partial m(\alpha_1,\alpha_2) = \{(p_1,p_2) \in \Delta_2 \,:\, p_i = 0 \text{ if } \alpha_i < m(\alpha_1,\alpha_2)\}$;
- $(w_i,s_i) \in B[x,f_i,\gamma/2]$;
- $(w_i^*,\lambda_i) \in \partial_\beta \delta_{\operatorname{epi} f_i}(w_i,s_i)$;
- $w_1^* + w_2^* \in x^* + V^* + V^*$ and $|q_i + \lambda_i| < \gamma/2$.

By (5) and property (iv), we deduce that $(w_i, f(w_i)) \in B[x,f_i,\varepsilon]$, for each $i = 1,2$, and following classic arguments (see, e.g., [2, 6, 43, 45]) we further deduce that $(w_i^*,\lambda_i) \in \partial_\beta \delta_{\operatorname{epi} f_i}(w_i, f(w_i))$ and $\lambda_i \leq 0$.

Observe that $|\lambda_1+\lambda_2+1| = |\lambda_1+\lambda_2+(q_1+q_2)| \leq \varepsilon$. Moreover, if $f_i(x) < g(x)$ (for $i = 1,2$), then (by shrinking γ if necessary) we may assume that $\alpha_i < m(\alpha_1,\alpha_2)$ and so in such a case $q_i = 0$ and consequently $|\lambda_i| \leq \varepsilon$. These two points yield that $(-\lambda_1,-\lambda_2) \in \Delta_2(x,\varepsilon)$, according to the definition (3).

Without loss of generality, we may assume that $\lambda_i \neq 0$ for $i = 1,2$. Otherwise, we may apply Proposition 2.2 to replace (w_i,s_i) and (w_i^*,λ_i) by arbitrarily close approximations (w_i',s_i') and $(w_i'^*,\lambda_i')$, satisfying all the properties we have observed so far and where the multiplier $\lambda_i' \neq 0$. Thus, for $i = 1,2$, we can define $x_i = w_i$ and $x_i^* := \lambda_i^{-1} w_i^* \in \partial_\beta f_i(x_i)$.

With this construction, it is not hard to see that $\lambda_1 x_1^* + \lambda_2 x_2^* \in x^* + V^* + V^*$ and since V^* is an arbitrary w^*-neighborhood of zero, we deduce that

$$\partial_\beta g(x) \subseteq \bigcap_{\varepsilon>0} \operatorname{cl}^{w^*} \left\{ \lambda_1 \partial_\beta f_1(x_1) + \lambda_2 \partial_\beta f_2(x_2) : \begin{array}{c} x_1 \in B[x,f_1,\varepsilon],\, x_2 \in B[x,f_2,\varepsilon], \\ \text{and } \lambda \in \Delta(J,x,\varepsilon) \end{array} \right\}.$$

The proof is complete. $\square$

In what follows, T will be an arbitrary index set and $f_t\colon X \to \overline{\mathbb{R}}$ will be a family of lsc functions. We recall that f is defined as the supremum function of the family, that is, as in (2). We say that T is a directed set ordered by $\preceq$ if $(T,\preceq)$ is an ordered set and for every $t_1,t_2 \in T$ there exists $t_3 \in T$ such that $t_1 \preceq t_3$ and $t_2 \preceq t_3$. Moreover, we say that the family $\{f_t\}_{t\in T}$ is increasing provided that for all $t_1,t_2 \in T$

$$t_1 \preceq t_2 \implies f_{t_1}(x) \leq f_{t_2}(x),\ \forall x \in X.$$

The next definition is an adaptation of the notion of *robust infimum* or *decoupled infimum* used in subdifferential theory to get *fuzzy calculus rules* (see, e.g., [4, 34, 45, 49]).

Definition 3.2. (Robust infimum) For a subset B of X, we will say that the family $\{f_t\}_{t\in T}$ has a *robust infimum* on $B \subseteq X$ provided that

$$\inf_{x\in B} f(x) = \sup_{t\in T} \inf_{x\in B} f_t(x). \tag{6}$$

In addition, if there exists some $\bar{x} \in B$ such that

$$\sup_{t\in T} \inf_{x\in B} f_t(x) = f(\bar{x}),$$

then we will say that the family $\{f_t\}_{t\in T}$ has a *robust minimum* on $B \subseteq X$. Finally, if B is some neighborhood of $\bar{x}$, then we say that $\{f_t\}_{t\in T}$ has a robust local minimum of at $\bar{x}$. □

The following lemma gives a sufficient condition for robust local minimum. Let us recall that for $\varepsilon > 0$ and a set C we define

$$\varepsilon\text{-}\underset{C}{\operatorname{argmin}}(f) := \begin{cases} \{x \in C : f(x) \le \inf_C f + \varepsilon\}, & \text{if } \inf_C f > -\infty, \\ \left\{x \in C : f(x) \le -\frac{1}{\varepsilon}\right\}, & \text{if } \inf_C f = -\infty. \end{cases}$$

Lemma 3.3. *Let X be a Banach space and $B \subseteq X$. Suppose that $(T, \preceq)$ is directed, $\{f_t\}_{t\in T}$ is an increasing family of τ-lower semicontinuous functions, B is τ-closed and there exists some t_0 such that f_{t_0} is τ-infcompact on B, with τ some topology coarser (weaker or smaller) than the norm topology. Then the family $\{f_t\}_{t\in T}$ has a robust minimum on B.*

Proof. It is clear that $\eta := \sup_{t\in T} \inf_{x\in B} f_t(x) \le \inf_{x\in B} \sup_{t\in T} f_t(x).$

So we focus only on the opposite inequality. For this purpose, we consider $\varepsilon_t \in (0,1)$ such that $\varepsilon_t \to 0$ and points $x_t \in B$ with $x_t \in \varepsilon_t\text{-}\operatorname{argmin}_B\{f_t\}$. According to [8], Lemma 4, it is enough to prove that the set $\{x_t : t \succeq t_0\}$ is τ-relatively compact. Indeed, since the family $\{f_t\}_{t\in T}$ is increasing, we have that for all $t \succeq t_0$,

$$\inf_{x\in B} f_t(x) \ge \inf_{x\in B} f_{t_0}(x) > -\infty.$$

Moreover, since $f_{t_0}(x_t) \le f_t(x_t) \le \eta + \varepsilon_t$, we get that $x_t \in \{x : f_{t_0}(x_t) \le \eta + 1\}$, which yields that the set $\{x_t : t \succeq t_0\}$ is τ-relatively compact. □

Remark 3.4. The hypothesis of inf-compactness in the above lemma is required, even if the supremum function f is inf-compact. Indeed, let us consider the functions

$$f_n(x) = n^2x^2 - x^4 \quad x \in \mathbb{R}.$$

Then, it is clear that the family $\{f_n : n \in \mathbb{N}\}$ is increasing and $f = \delta_{\{0\}}$. However, $\inf_{\mathbb{R}} f_n = -\infty$ and $\inf_{\mathbb{R}} f = 0$.

The following result is a necessary condition for the existence of robust local minimum of the supremum function in terms of the data f_t's.

Proposition 3.5. *Let $(T, \preceq)$ be a directed set and $\{f_t\}_{t\in T}$ be an increasing family of lower semicontinuous functions. If $\{f_t : t \in T\}$ has a robust local minimum at $\bar{x}$, then*

$$0 \in \bigcap_{\varepsilon>0} \mathrm{cl}^{\|\cdot\|} \left\{ \bigcup \{\partial_\beta f_t(x) : x \in B[\bar{x}, f_t, \varepsilon],\ t \in T_\varepsilon(\bar{x})\} \right\}. \tag{7}$$

Proof. Assume that $\{f_t : t \in T\}$ has a robust minimum at $\bar{x}$ on $B := B(\bar{x}, \eta)$, and pick $\varepsilon \in (0, 1)$ and $\gamma \in (0, \min\{\eta/2, \varepsilon/2\})$. Since $\bar{x}$ is a robust minimum, there exists $t \in T$ such that $\inf_B f_t \geq f(\bar{x}) - \gamma^2 \geq f_t(\bar{x}) - \gamma^2$. Therefore, $|f_t(\bar{x}) - f(\bar{x})| \leq \gamma^2$ and $\bar{x}$ is a γ^2-minimum of $f_t + \delta_B$. Hence, by *Ekeland's variational principle* (see, e.g., [4]) there exists $x_\gamma \in B(\bar{x}, \gamma)$ such that $|f_t(x_\gamma) - f_t(\bar{x})| \leq \gamma^2$ and x_γ is a minimum of the function $f(\cdot) + \delta_B(\cdot) + \gamma\| \cdot - x_\gamma\|$, which implies that $f(\cdot) + \gamma\| \cdot - x_\gamma\|$ attains a local minimum at x_γ. By Proposition 3.1.(iii), there exist sequences $(x_n, x_n^*) \in X \times X^*$ such that $x_n^* \in \partial_\beta f_t(x_n)$, $x_n \xrightarrow{f_t} x_\gamma$, $x_n^* \xrightarrow{\|\cdot\|} \bar{x}^*$ with $\bar{x}^* \in \gamma\mathbb{B}_{X^*}$. Then, take $n \in \mathbb{N}$ such that $|f_t(x_n) - f_t(x_\gamma)| \leq \gamma$, $\|x_n - x_\gamma\| \leq \gamma$ and $0 \in \partial_\beta f_t(x_n) + 2\gamma\mathbb{B}_{X^*}$. Therefore, $x_n \in B[\bar{x}, f_t, \varepsilon]$, $|f_t(\bar{x}) - f_t(x_n)| \leq \varepsilon$, $|f(\bar{x}) - f_t(x_n)| \leq \varepsilon$ and $0 \in \partial_\beta f_t(x_n) + \varepsilon\mathbb{B}_{X^*}$, which implies

$$0 \in \bigcup \{\partial_\beta f_t(x) : x \in B[\bar{x}, f_t, \varepsilon],\ t \in T_\varepsilon(\bar{x})\} + \varepsilon\mathbb{B}_{X^*}.$$

The conclusion follows. □

Now, we present the following upper-estimation for the viscosity subdifferential of the supremum function of an increasing family.

Proposition 3.6. *Let $(T, \preceq)$ be directed and $\{f_t\}_{t\in T}$ be an increasing family of lower semicontinuous functions. Then, for all $\bar{x} \in X$*

$$\partial_\beta f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*} \bigcup \left\{ \partial_\beta f_t(x) : x \in B[\bar{x}, f_t, \varepsilon],\ t \in T_\varepsilon(\bar{x}) \right\}. \tag{8}$$

Proof. Fix $x^* \in \partial_\beta f(\bar{x})$, $V \in \mathcal{N}_{X^*}(0, w^*)$ convex and symmetric, $\varepsilon > 0$ and L a finite-dimensional subspace of X such that $L^\perp \subseteq V$, so by Proposition 3.1.(i) there exist a ball $B := B[\bar{x}, \eta]$ and a continuous function $\phi\colon X \to \mathbb{R}$ such that $\partial_\beta\phi(B) \subseteq x^* + V$ and $\tilde{f} := f - \phi + \delta_{L\cap B}$ attains a local minimum at $\bar{x}$. Hence, consider the family of functions $\tilde{f}_t := f_t - \phi + \delta_{L\cap B}$. It is easy to see that the family is increasing, that $\tilde{f} = \sup_T \tilde{f}_t$ and that there exists some $t \in T$ such that $\tilde{f}_t$ is inf-compact. Whence, Lemma 3.3 yields that the family $\{\tilde{f}_t : t \in T\}$ has a robust local minimum at $\bar{x}$, and so Proposition 3.5 implies

$$0 \in \bigcap_{\gamma>0} \mathrm{cl}^{w^*} \left\{ \bigcup \{\partial_\beta \tilde{f}_t(x) : x \in B[\bar{x}, \tilde{f}_t, \gamma],\ t \in T_\gamma\big(\{\tilde{f}_t\}_{t\in T}, \bar{x}\big)\} \right\}. \tag{9}$$

Now take $\nu \in (0, \min\{\varepsilon/3, \eta/3\})$ small enough such that $|\phi(w) - \phi(\bar{x})| \leq \varepsilon/3$ for all $w \in B[\bar{x}, \nu]$, so by (9) there exist $t \in T_\nu(\{\tilde{f}_t\}_{t\in T}, \bar{x})$, $x \in B[\bar{x}, \tilde{f}_t, \nu]$ and $w^* \in \partial_\beta \tilde{f}_t(x) = \partial_\beta(f - \phi + \delta_{L\cap B})(x)$ such that $w^* \in x^* + V$. It implies that $x \in B[\bar{x}, f_t, \nu + \varepsilon/3]$ and $t \in T_{\nu+\varepsilon/3}(\{f_t\}_{t\in T}, \bar{x})$.

Now applying Propositions 3.1.(ii) and 3.1.(iv) to the sum of functions $f-\phi+\delta_{L\cap B}$ we get the existence of points $u \in X$ and $u^* \in X^*$ such that $u^* \in \partial_\beta f_t(u)$, $u \in B[x, f_t, \nu]$ and $u^* \in w^* + L^\perp + V = w^* + V + V$. Therefore $t \in T_\varepsilon(\{f_t\}_{t\in T}, \bar{x})$, $u \in B[\bar{x}, f_t, \varepsilon]$ and $x^* \in u^* + V + V + V$. The proof is now complete. □

Using the family of finite subsets of T and the supremum function, we have the following upper-estimation of the subdifferential of the supremum function of an arbitrary family of lsc functions.

Theorem 3.7. *Let $\{f_t : t \in T\}$ be an arbitrary family of lsc functions. Then for every $\bar{x} \in X$*

$$\partial_\beta f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*} \left\{ \bigcup_{\substack{F\in\mathcal{T}_\varepsilon(\bar{x})\\ x'\in B[\bar{x}, f_F, \varepsilon]}} \bigcap_{\gamma>0} \mathrm{cl}^{w^*} \left\{ \sum_{t\in F} \lambda_t \partial_\beta f_t(x_t) : \begin{array}{c} x_t \in B[x', f_t, \gamma], \\ \lambda \in \Delta(F, x', \gamma). \end{array} \right\} \right\} \quad (10)$$

Proof. Consider the set $\tilde{T} := \mathcal{P}_{\mathfrak{f}}(T)$, ordered by the direct inclusion $\subseteq$, and the family of functions $\{f_F : F \in \tilde{T}\}$ (recall that $f_F = \max_{s\in F} f_s$). Then it is easy to see that the family $\{f_F : F \in \tilde{T}\}$ is an increasing family of functions and $\sup_{F\in\tilde{T}} f_F = f$. Let $x^* \in \partial_\beta f(\bar{x})$. Then, by Proposition 3.6

$$x^* \in \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*} \left\{ \bigcup \left\{ \partial_\beta f_F(x') : x' \in B[\bar{x}, f_F, \varepsilon],\ F \in \tilde{T}_\varepsilon(\bar{x}) \right\} \right\}.$$

Now, if $w^* \in \partial_\beta f_F(x')$ for some $x' \in B[\bar{x}, f_F, \varepsilon]$ and $F \in \tilde{T}_\varepsilon(\bar{x})$, we get $x' \in B(\bar{x}, \varepsilon)$ and $F \in \mathcal{T}_\varepsilon(\bar{x})$, so using Proposition 3.1 (v) we get

$$w^* \in \bigcap_{\gamma>0} \mathrm{cl}^{w^*} \left\{ \sum \lambda_t \partial_\beta f_t(x_t) :\ x_t \in B[x', f_t, \gamma], \lambda \in \Delta(F, x', \gamma). \right\},$$

and so, inclusion (10) holds. □

It is worth mentioning that all the results enunciated in this section are also valid for any *abstract* subdifferential operator on a Banach space X, satisfying axiomatic properties such as the ones described in Proposition 3.1.(i)-(v).

4. Ioffe geometric subdifferential of the pointwise supremum function

The next result extends the *fuzzy intersection rule for Fréchet normals to intersections of cones*, see [48, Theorem 5.2] and [51, Theorem 4.5], for the β-subdifferential.

Theorem 4.1. *Let $\{\Lambda_t\}_{t\in T}$ be an arbitrary family of closed subsets of X and $\Lambda := \bigcap_{t\in T}\Lambda_t$. Then given, $\bar{x}\in X$, $x^*\in N_\beta(\Lambda,\bar{x})$, $\varepsilon>0$ and $V^*\in\mathcal{N}_{X^*}(0,w^*)$, there are $F\in\mathcal{P}_{\mathtt{f}}(T)$, $w_t\in B[\bar{x},\varepsilon]$ and $w_t^*\in N_\beta(\Lambda_t,w_t)$ such that*

$$x^*\in\sum_{t\in F}w_t^*+V^*. \tag{11}$$

Consequently, if $\{\Lambda_t\}_{t\in T}$ is a family of closed cones we have $N_\beta(\Lambda_t,w_t)\subseteq N_G(\Lambda_t,0)$ for all $t\in T$ and thus

$$N_\beta(\Lambda,\bar{x})\subseteq \mathrm{cl}^{w^*}\left\{\sum_{t\in F}w_t^*\;\middle|\; w_t^*\in N_G(\Lambda_t,0) \text{ and } t\in F\in\mathcal{P}_{\mathtt{f}}(T)\right\}. \tag{12}$$

Proof. The first part corresponds to a straightforward application of Theorem 3.7. Now if one considers a closed cone $K\subseteq X$ and $u\in K$ one has that

$$N_\beta(K,u)\subseteq N_\beta(K,n^{-1}u),\ \forall n\in\mathbb{N}.$$

Therefore, by virtue of Proposition 2.1, $N_\beta(\Lambda_t,u)\subseteq N_G(\Lambda_t,0)$ for every $t\in T$ and $u\in\Lambda_t$. Consequently, (11) implies (12). □

Before presenting our main result, we need to adapt the notion of *sequential normal epi-compactness* (SNEC) of functions defined for the limiting/Mordukhovich subdifferential (see, e.g., [45, Definition 1.116 and Corollary 2.39]). We use the same definition in order to extend similar results for the Ioffe geometric subdifferential.

Definition 4.2. An extended real valued function f finite at x is said to be SNEC at x with respect to ∂_β if for any sequences $(\lambda_k,x_k,x_k^*)\in[0,+\infty)\times X\times X^*$ satisfying $\lambda_k\to 0$, $x_k\xrightarrow{f}x$, $x_k^*\in\partial_\beta f(x_k)$ and $\lambda_k x_k^*\xrightarrow{*}0$, one has $\|\lambda_k x_k^*\|\to 0$. Moreover, a family of functions $\{f_t\}_{t\in T}$ is said to be SNEC in a neighborhood of a point $\bar{x}$ if there exists a neighborhood U of $\bar{x}$ such that for all $x\in U$, all but at most one of these functions are SNEC at x. □

We also say that the family of functions $\{f_t\}_{t\in T}$ satisfied the *limiting condition* with respect to ∂_G^∞ on a neighborhood of a point $\bar{x}$ if there exists a neighborhood U of $\bar{x}$ such that for all $x\in U$ and all $F\in\mathcal{P}_{\mathtt{f}}(T)$

$$w_t^*\in\partial_G^\infty f_t(x),\ t\in F \text{ and } \sum_{t\in F}w_t^*=0 \text{ implies } w_t^*=0,\ \text{for all } t\in F. \tag{13}$$

The next theorem is the main result of the paper and provides a fuzzy calculus rule for the Ioffe geometric subdifferential of the pointwise supremum function in the context of β-smooth weakly compactly generated (WCG) Banach spaces. To prove it, we use a technique called *separable reduction*, which has been employed by several authors in the recent years. Particularly, Ioffe and Fabian [18, 17] used it to obtain calculus rules for the Fréchet subdifferential.

To use this technique, let us introduce the following notation: by the symbol $\mathcal{S}(X \times X^*)$ we understand the family of set $U \times Y$ where U and Y are (norm-) separable closed linear subspaces of X and X^*. A set $\mathcal{A} \subseteq \mathcal{S}(X \times X^*)$ is called a *rich family* (see, e.g., [9] and the references therein) if

(i) for every $U \times Y \in \mathcal{S}(X \times X^*)$, there exists $V \times Z \in \mathcal{A}$ such that $U \subseteq V$ and $Y \subseteq Z$;

(ii) $\overline{\bigcup_{n\in\mathbb{N}} U_n} \times \overline{\bigcup_{n\in\mathbb{N}} Y_n} \in \mathcal{A}$, whenever the sequence $(U_n \times Y_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ satisfies $U_n \subseteq U_{n+1}$ and $Y_n \subseteq Y_{n+1}$.

We will also need the next lemma, whose proof follows directly from [9, Proposition 10] taking into account that *every WCG Banach space admits a projectional generator with domain X^** (see [15, pages 125,153]).

Lemma 4.3. *Let X be a WCG Banach space. Then, there exists a rich family $\mathcal{A} \subset \mathcal{S}(X \times X^*)$ such that for every $Z \times Y \in \mathcal{A}$ the following condition holds:*

$$Z^{\perp} \cap \mathrm{cl}_{w^*}^{\mathrm{seq}}(Y) = \{0\}, \tag{14}$$

where $\mathrm{cl}_{w^*}^{\mathrm{seq}}(Y)$ stands for the sequential w^*-closure of Y.

Now, we present the main result of the article.

Theorem 4.4. *Consider a family of lower semicontinuous functions $\{f_t\}_{t\in T}$ over a β-smooth WCG Banach space X. If the family $\{f_t\}_{t\in T}$ is SNEC with respect to ∂_β and satisfies the limiting condition* (13) *with respect to $\tilde{\partial}_G^\infty$ on a neighborhood of $\bar{x}$. Then*

$$\tilde{\partial}_G f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*}\left(\mathcal{S}(\bar{x},\varepsilon)\right), \quad \textit{and} \quad \tilde{\partial}_G^\infty f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*}\left([0,\varepsilon]\cdot\mathcal{S}(\bar{x},\varepsilon)\right), \quad \textit{where}$$

$$\mathcal{S}(\bar{x},\varepsilon) := \left\{ \sum_{t\in F} \lambda_t \circ \tilde{\partial}_G f_t(x') : \begin{array}{c} F \in \mathcal{P}_{\mathtt{f}}(T), x' \in B[\bar{x},\varepsilon], \\ |f_F(x') - f(\bar{x})| \le \varepsilon,\ \lambda \in \Delta(F) \textit{ and} \\ f_t(x') = f_F(x') \textit{ for all } t' \in \operatorname{supp}\lambda \end{array} \right\}, \tag{15}$$

and
$$\lambda \circ \tilde{\partial}_G f_t(x) := \begin{cases} \lambda \tilde{\partial}_G f_t(x) & \textit{if } \lambda > 0, \\ \tilde{\partial}_G^\infty f_t(x) & \textit{if } \lambda = 0, \end{cases}$$

Proof. We will only prove the inclusion for $\tilde{\partial}_G f$, since the one for $\tilde{\partial}_G^\infty f$ is essentially the same. Consider $\varepsilon > 0$ and $V \in \mathcal{N}_{X^*}(0, w^*)$. Pick $x^* \in \tilde{\partial}_G f(\bar{x})$. Hence, by definition of $\tilde{\partial}_G$, there exist a sequence $x_n \xrightarrow{f} \bar{x}$ and a selection $x_n^* \in \partial_\beta f(x_n)$ such that $x_n^* \xrightarrow{w^*} x^*$. Now, let us consider $j_0 \in \mathbb{N}$ large enough such that $x^* \in x_{j_0}^* + V$ and $x_{j_0} \in B[\bar{x}, f, \varepsilon]$. By Theorem 3.7, there exists an ε-active set $F \in \mathcal{T}_\varepsilon(x_{j_0})$ and a point $x' \in B[x_{j_0}, f_F, \varepsilon]$ such that $x_{j_0}^* = u^* + v^*$ with $v^* \in V$ and

$$u^* \in \bigcap_{\gamma>0} \mathrm{cl}^{w^*} \left\{ \sum_{t\in F} \lambda_t \partial_\beta f_t(x_t) : x_t \in B[x', f_t, \gamma],\ (\lambda_t) \in \Delta(F, x', \gamma) \right\}. \tag{16}$$

By construction, it is clear that $x' \in B(\bar{x}, 2\varepsilon)$ and $|f_F(x') - f(\bar{x})| \leq 3\varepsilon$. Now, we only need to show that

$$u^* \in \mathcal{S}(\bar{x}, 3\varepsilon) \tag{17}$$

Indeed, if this is the case, we will get that $x^* \in \mathcal{S}(\bar{x}, 3\varepsilon) + 2V$ and so, the desired inclusion will hold due to the arbitrariness of ε and V.

Let $(\gamma_n)_{n\in\mathbb{N}}$ be a decreasing sequence converging to 0 and let $\mathcal{A}$ be the rich family given by Lemma 4.3. The aim is to build a sequence $(Z_n \times Y_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{A}$, a countable set $\{e(k,i)\}_{k,i\in\mathbb{N}}$, a family of tuples $(x_t(k,p), x_t^*(k,p), \lambda_t(k,p)) \in X \times X^* \times \mathbb{R}_+$ indexed by $t \in F$, $k \in \mathbb{N}$ and $p \in \mathbb{N}$, and neighborhoods $W(k,p) \in \mathcal{N}_{X^*}(0, w^*)$ with elements $v^*(k,p) \in W(k,p)$ indexed by $k \in \mathbb{N}$ and $p \in \mathbb{N}$, satisfying, for each $n \in \mathbb{N}$, the following properties:

(i) $Z_1 \times Y_1 \subset Z_2 \times Y_2 \subset \cdots \subset Z_n \times Y_n$;

(ii) $\{e(n,i)\}_{i\in\mathbb{N}}$ is a dense subset of $\mathbb{B} \cap Z_n$;

(iii) $x_t(n,p) \in B[x', f_t, \gamma_p]$, $x_t^*(n,p) \in \partial_\beta f_t(x(n,p))$ and $\lambda_t(n,p) \in \Delta(F, x', \gamma_p)$;

(iv) $W(n,p) = \{y^* \in X^* \ : \ |\langle y^*, e(k,i)\rangle| \leq \gamma_p, \forall k = 1, \ldots, n, \forall i = 1, \ldots, p\}$;

(v) For every $p \in \mathbb{N}$, $x_t(n,p) \in Z_{n+1}$ and $x_t^*(n,p) \in Y_{n+1}$; and

(vi) $u^* = \left(\sum_{t\in F} \lambda_t x_t^*(n,p)\right) + v^*(n,p)$, for every $n \in \mathbb{N}$ and every $p \in \mathbb{N}$.

For $n = 1$, choose $Z_1 \times Y_1 \in \mathcal{A}$ such that $(x', u^*) \in Z_1 \times Y_1$. Since Z_1 is separable, there exists a countable set $\{e(1,i)\}_{i\in\mathbb{N}} \subset \mathbb{B} \cap Z_1$ dense in $\mathbb{B} \cap Z_1$. For every $p \in \mathbb{N}$, we consider the neighborhood $W(1,p) \in \mathcal{N}_{X^*}(0, w^*)$ given by

$$W(1,p) := \{y^* \in X^* \ : \ |\langle y^*, e(1,i)\rangle| \leq \gamma_p, \forall i = 1, \ldots, p\}.$$

By inclusion (16), we can find couples $(x_t(1,p), x_t^*(1,p))_{t\in F}$ for every $p \in \mathbb{N}$ with $x_t(1,p) \in B[x', f_t, \gamma_p]$ and $x_t^*(1,p) \in \partial_\beta f_t(x_t(1,p))$, ε-multipliers $(\lambda_t)_{t\in F} \in \Delta(F, x', \gamma_p)$ and an element $v^*(1,p) \in W(1,p)$ such that

$$u^* = \left(\sum_{t\in F} \lambda_t x_t^*(1,p)\right) + v^*(1,p).$$

Suppose now that we have constructed a sequence $Z_1 \times Y_1, \ldots, Z_n \times Y_n$ (with $n \geq 1$) of elements in $\mathcal{A}$, a countable set $\{e(k,i)\}_{k\leq n, i\in\mathbb{N}}$, a family of tuples $(x_t(k,p), x_t^*(k,p), \lambda_t(k,p)) \in X \times X^* \times \mathbb{R}_+$ indexed by $t \in F$, $k = 1, \ldots, n$ and $p \in \mathbb{N}$, and neighborhoods $W(k,p) \in \mathcal{N}_{X^*}(0, w^*)$ with elements $v^*(k,p) \in W(k,p)$ indexed by $k = 1, \ldots, n$ and $p \in \mathbb{N}$ satisfying properties $(i) - (vi)$, with the observation that property (v) only holds until $n-1$.

Now, since $\mathcal{A}$ is a rich family, we can find $Z_{n+1} \times Y_{n+1} \in \mathcal{A}$ containing $Z_n \times Y_n$ satisfying that for each $p \in \mathbb{N}$ and each $t \in F$,

$$x_t(n,p) \in Z_{n+1} \quad \text{and} \quad x_t^*(n,p) \in Y_{n+1}.$$

Then, we replicate the construction we have done for $n = 1$ to find the elements $\{e(n+1,i)\}_{i\in\mathbb{N}}$, the tuples $(x_t(n+1,p), x_t^*(n+1,p), \lambda_t(n+1,p)) \in X \times X^* \times \mathbb{R}_+$

indexed by $t \in F$, and $p \in \mathbb{N}$, and the neighborhoods $W(n+1,p) \in \mathcal{N}_{X^*}(0,w^*)$ with elements $v^*(n+1,p) \in W(n+1,p)$ indexed by $p \in \mathbb{N}$. By induction, this finishes the construction.

Now, let us define $Z \times Y = \overline{\bigcup_{n\in\mathbb{N}} Z_n} \times \overline{\bigcup_{n\in\mathbb{N}} Y_n}$ which belongs to $\mathcal{A}$, since the family is rich. Also, for each $n \in \mathbb{N}$ and each $t \in F$, let us define $x_t(n) := x_t(n,n)$, $x_t^*(n) := x_t^*(n,n)$ and $\lambda_t(n) := \lambda_t(n,n)$.

Since F is finite and $(\lambda_t(n))_{n\in\mathbb{N}}$ is bounded for each $t \in F$, we may assume without losing any generality, that $(\lambda_t(n))_{t\in F} \to (\lambda_t)_{t\in F}$. Furthermore, since $(\lambda_t(n))_{t\in F} \in \Delta(F, x', \gamma_n)$, we get that

$$\sum_{t\in F} \lambda_t = \lim_n \sum_{t\in F} \lambda_t(n) = 1,$$

and moreover, $f_s(x') = f_F(x')$ for each $s \in \operatorname{supp}(\lambda_t)_{t\in F}$.

Similarly, since $x_t(n) \in B[x', f_t, \gamma_n]$, we get that $x_t(n) \xrightarrow{f_t} x'$.

Now, by property (vi), we know that for each $n \in \mathbb{N}$,

$$u^* = \sum_{t\in F} \lambda_t(n) x_t^*(n) + v^*(n).$$

So, to conclude, we need to study the convergence of $(x_t^*(n))_n$ and $(v^*(n))_n$. To do so, we claim that there exists $M > 0$ such that for each $n \in \mathbb{N}$ and each $t \in F$,

$$\|\lambda_t(n) x_t^*(n)\| \leq M.$$

If not, we can define $\eta_n := \max_{t\in F} \|\lambda_t(n)x_t^*(n)\| \to +\infty$. Then, we would get that

$$\eta_n^{-1} u^* = \sum_{t\in F} \eta_n^{-1} \lambda_t(n) x_t^*(n) + \eta_n^{-1} v^*(n) \to 0.$$

Since for each $t \in F$ the sequence $(\eta_n^{-1}\lambda_t(n)x_t^*(n))_{n\in\mathbb{N}}$ is bounded, it w^*-converges, up to a subsequence (because X is WCG), to a point $y_t^* \in \tilde{\partial}_G^\infty f_t(x')$. Furthermore, this yields that $(\eta_n^{-1} v^*(n))_{n\in\mathbb{N}}$ is also bounded and so w^*-convergent up to a subsequence. Finally, noting that

$$\langle \eta_n^{-1} v^*(n), e(k,i)\rangle \to 0, \quad \forall k, i \in \mathbb{N},$$

and applying the orthogonality condition (14), we conclude that any w^*-convergent subsequence of $(\eta_n^{-1} v^*(n))$ must converge to 0. Then,

$$0 = \sum_{t\in F} y_t^*,$$

which yields by the limiting condition (13) that each $y_t^* = 0$. However, since F is finite, it is not hard to see that there exists $t \in F$ and a subsequence $(\eta_{n_k}^{-1}\lambda_t(n_k)x_t^*(n_k))_k$ such that

$$\|\eta_{n_k}^{-1}\lambda_t(n_k)x_t^*(n_k)\| = 1, \qquad \forall k \in \mathbb{N}.$$

Furthermore, since $\eta_n^{-1}u^* \to 0$, the above condition ensures the existence of a second $s \in F \setminus \{t\}$ for which

$$\liminf \|\eta_{n_k}^{-1}\lambda_s(n_k)x_s^*(n_k)\| > 0.$$

This is a contradiction with the SNEC property given in Definition 4.2. The claim is then proved.

Now, since X is WCG, for each $t \in F$ the (bounded) sequence $(\lambda_t(n)x_t^*(n))_{n\in\mathbb{N}}$ admits a w^*-convergent subsequence. Without losing any generality, we may and do assume that for each $t \in F$, $(\lambda_t(n)x_t^*(n))$ w^*-converges to some point u_t^*.

Moreover, we again conclude that $v^*(n)$ also w^*-converges to 0. Indeed, by construction, we know that

$$\langle v^*(n), e(k,i)\rangle \to 0, \quad \forall k, i \in \mathbb{N},$$

and then, since $(v^*(n))$ is bounded, $\{e(k,i)\}_{k\in\mathbb{N}, i\in\mathbb{N}}$ is dense in $\mathbb{B} \cap Z$ and $\mathcal{A}$ is the rich family given by Lemma 4.3, we deduce applying the orthogonality condition (14), that $v^*(n) \xrightarrow{w^*} 0$. Then,

$$u_t^* = \begin{cases} \lambda_t x_t^* \in \lambda_t \tilde{\partial}_G f_t(x') & \text{if } t \in \operatorname{supp}(\lambda_s)_{s\in F} \\ u_t^* \in \partial_G^\infty f_t(x') & \text{otherwise.} \end{cases}$$

Finally, applying again property (v), we conclude that

$$u^* = \sum_{t\in F} u_t^* \in \mathcal{S}(\bar{x}, 3\varepsilon),$$

and so, the proof is complete. □

The next theorem is due to Mordukhovich-Nghia when X is an Asplund space [46]. We make an extension of this result using the Ioffe geometric subdifferential in an arbitrary Banach space and also for locally convex spaces using the *approximate subdifferential* and the *singular approximate subdifferential*, which are defined for a point $x \in \operatorname{dom} f$ as

$$\partial_A f(x) := \bigcap_{L\in\mathcal{F}} \limsup_{u \xrightarrow{f} x} \partial^- f_{u+L}(u), \quad \partial_A^\infty f(x) = \bigcap_{L\in\mathcal{F}} \limsup_{\substack{u \xrightarrow{f} x \\ \nu \to 0}} \nu\partial^- f_{u+L}(u),$$

respectively, where $f_{u+L} := f + \delta_{u+L}$, ∂^- is the *Haddamard subdifferential*, given by

$$\partial^- g(x) = \left\{ x^* \in X^* : \langle x^*, h\rangle \le \liminf_{s\to 0^+,\, w\to s} \frac{g(x+sw) - g(w)}{s},\ \forall h \in X \right\},$$

and $\mathcal{F}$ denotes the collection of all finite-dimensional subspaces of X. Here the $\limsup$ is considered with respect to the w^*-topology in X^*. It has been proved in [26] that the family $\mathcal{F}$ can be changed for the family subspaces with the *weak trustworthiness property* using an enlargement of ∂^- (see, e.g., [26, Theorem 5.1]).

It is worth mentioning that the Ioffe geometric subdifferential and the approximate subdifferential coincides in the finite-dimensional setting (see Proposition 2.1), and moreover both coincide in arbitrary Banach spaces for Lipschitz functions (see, e.g., [27, Proposition 3.3], or [37, Chapter 4]). We refer to [25, 26, 27], where the theory of the approximate subdifferential was developed in finite dimensions, locally convex spaces and Banach spaces, respectively.

Theorem 4.5. *Let $\{f_t\}_{t\in T}$ be a family of locally Lipschitz functions on a neighborhood of a point $\bar{x} \in \operatorname{dom} f$. Assume that one of the following conditions holds*

(a) *X is a Banach space and ∂ is the G-subdifferential.*

(b) *X is a locally convex space and ∂ is the approximate subdifferential.*

Then

$$\partial f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \operatorname{cl}^{w^*}\left(\mathcal{S}(\bar{x},\varepsilon)\right), \quad \textit{and} \quad \partial^\infty f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \operatorname{cl}^{w^*}\left([0,\varepsilon]\cdot\mathcal{S}(\bar{x},\varepsilon)\right), \tag{18}$$

where $\mathcal{S}(\bar{x},\varepsilon)$ was defined in (15)*. In addition, if the family $\{f_t\}_{t\in T}$ is uniformly locally Lipschitz at $\bar{x}$, then*

$$\partial f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \operatorname{cl}^{w^*}\left\{\sum_{t\in F}\lambda_t\partial f_t(x') : \begin{array}{c} F\in\mathcal{P}_{\mathtt{f}}(T_\varepsilon(\bar{x})), x'\in B[\bar{x},\varepsilon], \\ \lambda\in\Delta(F) \textit{ and} \\ f_t(x') = f_F(x') \textit{ for all } t\in F \end{array}\right\}. \tag{19}$$

Proof. Consider $V \in \mathcal{N}_{X^*}(0,w^*)$, $\varepsilon>0$, a finite-dimensional subspace $L \ni \bar{x}$ such that $L^\perp \subseteq V$ and $x^* \in \partial f(x)$ (respectively, $y^* \in \partial^\infty f(\bar{x})$). Let $P\colon X \to L$ be a continuous linear projection and define $W = (P^*)^{-1}(V)$. Hence, $x^*_{|_L} \in \partial f_{|_L}(x)$ (respectively, $y^*_{|_L} \in \partial^\infty f_{|_L}(x)$) and, since L is finite-dimensional, the subdifferentials described in (a) and (b) are equal for Lipschitz functions (see comments before the statement of Theorem 4.5) to the Ioffe geometric subdifferential. Hence, we apply Theorem 4.4 and we conclude the existence of some $F \in \mathcal{T}_\varepsilon(\bar{x})$, $x' \in B[\bar{x},\varepsilon]$, $\lambda\in\Delta(F)$ such that $x^*_{|_L} \in \sum_{t\in F}\lambda_t\partial(f_{|_L})_t(x') + W$ and $(f_{|_L})_{t'}(x') = (f_{|_L})_{t''}(x')$ for all $t', t'' \in \operatorname{supp}\lambda$, then

$$P^*(x^*_{|_L}) \in \sum_{t\in F}\lambda_t\partial(f_t+\delta_L)(x') + V = \sum_{t\in F}\lambda_t\partial f_t(x') + L^\perp + V,$$

where the last equality follows from the sum rule for Lipschitz functions (see [26, 27, 45]). Therefore $x^* = P(x^*_L) + x^* - P(x^*_L) \in \sum_{t\in F}\lambda_t\partial f_t(x') + V$, which implies $x^* \in \mathcal{S}(\bar{x},\varepsilon) + V$. Similarly one concludes that $y^* \in [0,\varepsilon]\cdot\mathcal{S}(\bar{x},\varepsilon) + V$ for $y^*_{|_L} \in \partial^\infty f_{|_L}(x)$. Since $\varepsilon > 0$ and $V \in \mathcal{N}_{X^*}(0,w^*)$ are arbitrary, we conclude the proof of (18).

Finally to prove (19) we notice that if the functions are uniformly locally Lipschitz at $\bar{x}$ with constant K, then, assuming that $\varepsilon > 0$ is small enough, we have that for any $t \in T$, $x \in B[\bar{x}, \varepsilon]$ and $|f_t(x) - f(\bar{x})| \leq \varepsilon$ we also have $f_t(\bar{x}) \geq f(\bar{x}) - (K+1)\varepsilon$, which means $t \in T_{(K+1)\varepsilon}(\bar{x})$. $\square$

5. Final comments

In this work we extended the results of [51] to WCG Banach spaces using the Ioffe geometric subdifferential. In particular, Theorem 4.4 allows us to provide formulae for the supremum function under lower-semicontinuous data satisfying the SNEC and the limiting condition (13). Furthermore, in Theorem 4.5 we showed that the results are also valid in arbitrary Banach spaces, and in arbitrary locally convex spaces using the geometric subdifferential and the Aproximate subdifferential, respectively.

Here, it worth mentioning that the main motivation to extend the aforementioned results come from the fact that applications in semi-infnite programming, typically uses spaces $\ell^\infty(T)$, and $C(T)$, which are not necessarily Asplund spaces. We hope that this class of formulae may open a gate to extend results in semi-infinite programming.

The key point in the proof of our main result (Theorem 4.4) is Lemma 4.3, which shows the existence of a rich family in WCG Banach spaces providing the orthogonality condition (14). However, as we already mention, this result only needs the existence of a *projectional generator with domain* X^*. Nevertheless, it is an open question to know which Banach spaces admit such a rich family. It is easy to see that a Banach space satisfying the existence of such rich family is necessarily a *Banach space with weak***-sequentially compact dual balls* (see, e.g., [12, Chapter XIII] for more details about the property of weak*-sequentially compact dual balls).

If one aims to extend the results of this article for general β-smooth Banach spaces, a delicate work would be needed to replace subsequences by subnets, which may present further difficulties. The main obstruction to do so is to find an alternative result to conclude that the sequence of perturbations $(v^*(n))$ in the proof of Theorem 4.4 would converge to 0. Such a result would be of high interest enlarging the applicability of the Ioffe geometric subdifferential.

In general, the Ioffe geometric subdifferential is the well-adapted first order tool when we work outside the Asplund setting. It is well-known that in such Banach spaces, the Fréchet (and limiting) subdifferential is no longer trustworthy [20, 14]. If one wants to work with the Clarke subdifferential instead, another difficulty may arise: It is possible to construct Lipschitz continuous functions where the Clarke subdifferential is everywhere saturated [3, 10].

However, we don't know whether both problems may arise at the same time, that is, if there exists a continuous function such that its Fréchet subdifferential is everywhere empty and its Clarke subdifferential is everywhere saturated. If such an example could be built in a β-smooth space, it would be very interesting to study the behavior of its Ioffe geometric subdifferential, which may further illustrate both, the reach and the limitations of this object.

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