Journal of Convex Analysis
Volume 27 (2020), No. 2, 583–622

Ellipsoidal Cones

Alberto Seeger
Dép. de Mathématiques, Université d'Avignon, 33 rue Louis Pasteur,
84000 Avignon, France
alberto.seeger@univ-avignon.fr

Mounir Torki
CERI, Université d'Avignon, 339 chemin de Meinajaries,
84911 Avignon, France
mounir.torki@univ-avignon.fr

Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

Received: April 18, 2018
Revised manuscript received: January 28, 2019
Accepted: February 01, 2019

Ellipsoidal cones are more general than revolution cones but they still have a simple structure. Such a compromise between simplicity and sufficient degree of generality make them very useful in practice. Ellipsoidal cones have applications in optimization, statistics, control of linear dynamical systems, and in many other fields. The purpose of this survey paper is to gather in a single place a rich variety of results on ellipsoidal cones disseminated in the literature. A few selected examples of applications are provided to show the importance of this particular class of convex cones.

Keywords: Ellipsoidal cone, Lorentz cone, volume of a convex cone, cross section, axial symmetry, semiaxes lengths of an ellipsoidal cone, critical angles, cone-invariance.

2010 Mathematics Subject Classification: 15A18, 15A63, 52A20, 52A38, 52A40.

1. Introduction

Ellipsoidality is a mathematical concept with a long history and with applications in almost all branches of physical science. A set with ellipsoidal shape has a number of favorable geometric and analytic properties: symmetry, strict convexity, smoothness, and so on. In modern terms, an ellipsoid in the Euclidean space $\mathbb{R}^n$ is a set of the form

$$f(\mathbb{B}_n) := \{f(x) : x \in \mathbb{B}_n\},$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is an invertible affine transformation and $\mathbb{B}_n$ is the closed unit ball of $\mathbb{R}^n$. An ellipsoid in a hyperplane H of $\mathbb{R}^n$ is defined in a similar way: it suffices to consider instead an invertible affine transformation from $\mathbb{R}^{n-1}$ to H. Theoretical and practical reasons lead us to inquire into the meaning of ellipsoidality when such a concept refers to a closed convex cone in $\mathbb{R}^n$. Throughout this work, the dimension n is assumed to be greater than or equal to three. There is no universal agreement on the definition of an ellipsoidal cone, so we review different

ISSN 0944-6532 / $ 2.50

approaches proposed in the literature. We also comment on the geometric and analytic consequences of ellipsoidality in a conic setting. The overall exposition is restricted to proper cones in finite dimensions. Recall that a closed convex cone in $\mathbb{R}^n$ is proper if it has nonempty interior and contains no line. The first condition in the definition of properness is called solidity and the second one is called pointedness. The set of proper cones in $\mathbb{R}^n$ is denoted by Π_n.

A natural way of recognizing whether a proper cone is ellipsoidal or not is to proceed *à la manière* of the Ancient Greeks, i.e., to cut the cone with a hyperplane and observe the shape of the cross section. Formally speaking, a cross section of a proper cone K in $\mathbb{R}^n$ is an $(n-1)$-dimensional compact convex set of the form

$$S(K,y) := \{x \in K : \langle y, x\rangle = 1\},$$

where $y \in \mathbb{R}^n$ is a nonzero vector and $\langle \cdot, \cdot \rangle$ is the usual inner product of $\mathbb{R}^n$. The dimension of a closed convex set is understood as the dimension of its affine hull. The closed convex set $S(K,y)$ is required to be bounded and of dimension $n-1$ in order to be called a cross section of K. Hence, a cross section $S(K,y)$ is a convex body in the hyperplane

$$y^\diamond := \{x \in \mathbb{R}^n : \langle y, x\rangle = 1\}.$$

The definition of ellipsoidal cone that we use is the same as the one considered by Jerónimo-Castro and Mcallister [38].

Definition 1.1. $K \in \Pi_n$ is an ellipsoidal cone if there exists a nonzero vector $y_0 \in \mathbb{R}^n$ such that $S(K,y_0)$ is an ellipsoid in the hyperplane $y_0^\diamond$.

In plain English, an ellipsoidal cone is a proper cone that has at least one ellipsoidal cross section. Definition 1.1 has the merit of being geometrically intuitive and easily understandable. A preliminary question that needs clarification is how to select a nonzero vector y_0 for testing whether $S(K,y_0)$ is an ellipsoid or not. A blind choice of y_0 may lead to a set $S(K,y_0)$ that is empty or unbounded. The next result suggests to focus on the interior of the dual cone

$$K^* := \{y \in \mathbb{R}^n : \langle y, x\rangle \geq 0 \text{ for all } x \in K\}.$$

Proposition 1.2 is part of the folklore of the theory of convex cones and can be found for instance in Gigena [18, p. 191]. The symbol $\mathrm{int}(\Omega)$ stands for the interior of a set Ω.

Proposition 1.2. *Let $K \in \Pi_n$ and $y \in \mathbb{R}^n$ be a nonzero vector. Then $S(K,y)$ is a cross section of K if and only if $y \in \mathrm{int}(K^*)$.*

Everything is now ready to start our survey work on ellipsoidal cones. The organization of the paper is as follows:

Section 2 presents a good dozen of geometric characterizations of ellipsoidality in proper cones. Most of these characterizations are consistent with geometric intuition, but a few of them are rather abstract and not so easy to grasp.

Section 3 presents several alternative parametric representations of an ellipsoidal cone. The "parameter" that represents an ellipsoidal cone is usually a matrix (as for instance in the Lorentzian representation) or a matrix coupled with a vector (as for instance in the semiaxial representation). Depending on the context, each choice of representation has its own advantages and disadvantages. We explain how to pass from one representation to another.

Section 4 is about semiaxes lengths in ellipsoidal cones. We also discuss about the related concept of critical angle. Semiaxes lenghts and critical angles can be defined for arbitrary proper cones, but they are of special relevance for ellipsoidal cones.

Section 5 explains how to define and evaluate the volume of an ellipsoidal cone. There are essentially two choices for consideration: the ball-truncated volume and the canonical volume. The latter choice has numerous advantages over the former. Section 5 explains also how to measure other geometric properties like degree of pointedness and degree of solidity.

Section 6 presents miscellaneous results involving ellipsoidal cones. By way of example, we consider the notions of copositivity and exponential nonnegativity on ellipsoidal cones. We present also the conic version of the celebrated Loewner-John theorem. In particular, we say some words on inscribed and circumscribed ellipsoidal cones.

Proofs are given only if they help to introduce an interesting idea or if they facilitate the flow of the exposition. Appropriate historical and bibliographical comments are given whenever possible. Finally, some words on general notation. The unit sphere of $\mathbb{R}^n$ is denoted by $\mathbb{S}_n$. The symbol $\mathbb{M}_n$ stands for the linear space of (square) matrices of order n. Each element of $\mathbb{M}_n$ can be seen as a linear transformation from $\mathbb{R}^n$ into itself. We write $\mathbb{GL}(n)$ and $\mathbb{O}(n)$ to indicate respectively the set of invertible and orthogonal matrices of order n. The identity matrix of order n is denoted by I_n.

2. Geometric characterizations of an ellipsoidal cone

Any revolution cone is an ellipsoidal cone. Conversely, any ellipsoidal cone is a revolution cone up to isomorphism. These and related statements are the object of the next theorem. Some comments on terminology are in order. Isomorphism between subsets of a common linear space is not always understood in the same way in the literature. The definition that we adopt is as follows: two sets in $\mathbb{R}^n$ are isomorphic if one of them is the image of the other under an invertible linear transformation. In particular, $K_0, K_1 \in \Pi_n$ are isomorphic if $K_1 = A(K_0)$ for some $A \in \mathbb{GL}(n)$. An isomorphism between sets is of course an equivalence relation. A revolution cone in $\mathbb{R}^n$ is a proper cone of the form

$$\mathrm{rev}(s, b) := \{x \in \mathbb{R}^n : s\|x\| \le \langle b, x\rangle\}$$

with $0 < s < 1$ and $b \in \mathbb{S}_n$. The dual of a revolution cone is a revolution cone.

Up to orthogonal transformation, the Lorentz cone

$$\mathbb{L}_n := \left\{x \in \mathbb{R}^n : [x_1^2 + \ldots + x_{n-1}^2]^{1/2} \leq x_n\right\}$$

is the only self-dual revolution cone. The Lorentz cone is also known as ice cream cone or second-order cone.

Theorem 2.1. *Let $K \in \Pi_n$. Then the following conditions are equivalent:*

(a) *K is an ellipsoidal cone.*
(b) *K is isomorphic to $\mathbb{L}_n$.*
(c) *K is isomorphic to a revolution cone.*
(d) *K^* is an ellipsoidal cone.*

Proof. (a) $\Leftrightarrow$ (b). Let y_0 be as in Def. 1.1. Then $S(K, y_0) = \{Mz + c : z \in \mathbb{B}_{n-1}\}$ for some $c \in \mathbb{R}^n$ in the hyperplane $y_0^\diamond$ and some matrix M of size $n \times (n-1)$ whose columns form a basis of the linear subspace

$$y_0^\perp := \{x \in \mathbb{R}^n : \langle y_0, x\rangle = 0\}.$$

Since the cross section $S(K, y_0)$ is a base of K, we have

$$K = \{t\,(Mz + c) : t \geq 0,\ z \in \mathbb{B}_{n-1}\}\,.$$

This can be rewritten in the form

$$K \;=\; \left\{[M, c]\begin{bmatrix} \xi \\ t \end{bmatrix} : \|\xi\| \leq t\right\} \;=\; \{Ax : x \in \mathbb{L}_n\}\,,$$

where $A := [M, c]$ is invertible because c is not orthogonal to y_0. This shows that K is isomorphic to $\mathbb{L}_n$. Conversely, let $K = A(\mathbb{L}_n)$ for some $A \in \mathbb{GL}(n)$. Consider the vector $y_0 := A^{-\top} e_n$, where $A^{-\top}$ is the transpose of the inverse of A and e_n is the last column of the identity matrix I_n. A direct computation shows that $y_0 \in \mathrm{int}(K^*)$ and that

$$S(K, y_0) \;=\; A\,(\mathbb{L}_n \cap e_n^\diamond) \;=\; \left\{A\begin{bmatrix} z \\ 1 \end{bmatrix} : z \in \mathbb{B}_{n-1}\right\}$$

is an ellipsoid in $y_0^\diamond$.

(b) $\Leftrightarrow$ (c). Isomorphism between proper cones is a transitive relation and any revolution cone is isomorphic to $\mathbb{L}_n$. Indeed,

$$\mathrm{rev}(s, b) = \left\{U\begin{bmatrix} \frac{\sqrt{1-s^2}}{s} I_{n-1} & 0 \\ 0 & 1 \end{bmatrix} x : x \in \mathbb{L}_n\right\}$$

for any $U \in \mathbb{O}(n)$ such that $Ue_n = b$.

(a) $\Leftrightarrow$ (d). The Lorentz cone being self-dual, we readily get

$$[A(\mathbb{L}_n)]^* = A^{-\top}(\mathbb{L}_n) \tag{1}$$

for all $A \in \mathbb{GL}(n)$. This formula proves that K is isomorphic to $\mathbb{L}_n$ if and only if K^* is isomorphic to $\mathbb{L}_n$. But we know already that (a) and (b) are equivalent. Hence, K is an ellipsoidal cone if and only if K^* is an ellipsoidal cone. $\square$

It is not easy to attribute authorship to each part of Theorem 2.1. That a revolution cone is isomorphic to the Lorentz cone is a known fact, cf. Eichfelder and Povh [12, Lemma 4.4]. The next result corresponds to Corollary 2.1 in Stern and Wolkowicz [58]. It says that the set of ellipsoidal cones is stable under invertible linear transformations.

Corollary 2.2. *Let $K \in \Pi_n$ be an ellipsoidal cone. Then any proper cone isomorphic to K is an ellipsoidal cone.*

We say that $K \in \Pi_n$ is sectionally uniform if the cross sections of K are pairwisely isomorphic, i.e., for any pair S_0, S_1 of cross sections of K, there exists $A \in \mathbb{GL}(n)$ such that $S_1 = A(S_0)$. It turns out that sectional uniformity is the same property as homogeneity. Recall that $K \in \Pi_n$ is an homogeneous cone if for all $x, y \in \operatorname{int}(K)$, there exists $A \in \mathbb{GL}(n)$ such that $A(K) = K$ and $Ax = y$. General material on homogeneous cones can be found in the book of Faraut and Korányi [14].

Proposition 2.3. *Let $K \in \Pi_n$. Then the following conditions are equivalent:*

(a) *K is sectionally uniform.*

(b) *K is homogeneous.*

Furthermore, these conditions are necessary for K to be an ellipsoidal cone.

Proof. Let K be homogeneous. Let $S_0 := S(K, y_0)$ and $S_1 := S(K, y_1)$ with $y_0, y_1 \in \operatorname{int}(K^*)$. Since K is homogeneous, also K^* is homogeneous. Hence, there exists $B \in \mathbb{GL}(n)$ such that $B(K^*) = K^*$ and $By_0 = y_1$. It follows that $S_1 = B^{-\top}(S_0)$. This proves that the cross sections of an homogeneous proper cone are pairwisely isomorphic. Conversely, let K be sectionally uniform. Pick $y_0, y_1 \in \operatorname{int}(K^*)$ and define S_0, S_1 as before. Hence, there exists $A \in \mathbb{GL}(n)$ such that $S_1 = A(S_0)$. In such a case,

$$K = \{tu : t \geq 0, u \in S_1\} = A(\{tv : t \geq 0, v \in S_0\}) = A(K)$$

and, a posteriori, $A^{-\top}(K^*) = K^*$. Moreover,

$$y_1^\diamond = \operatorname{aff}(S_1) = A(\operatorname{aff}(S_0)) = A(y_0^\diamond),$$

where $\operatorname{aff}(S)$ is the affine hull of a set S. This implies that $y_1 = A^{-\top} y_0$ and proves that K^* is homogeneous. Hence, K is homogeneous. Finally, we mention that the Lorentz cone is known to be homogenous. But the set of homogeneous proper cones is stable under invertible linear transformations. Hence, ellipsoidality implies homogeneity. □

Ellipsoidality implies homogeneity but not conversely. For instance, a simplicial cone is homogeneous but not ellipsoidal. Similarly, a Cartesian product of ellipsoidal cones is homogeneous but not ellipsoidal. The next theorem shows that ellipsoidality of one particular cross section of K implies ellipsoidality of every cross section. In other words, the choice of y_0 in Definition 1.1 is irrelevant as long as this vector belongs to $\operatorname{int}(K^*)$. Theorem 2.4 shows also that every ellipsoidal cone is axially symmetric. Axial symmetry means stability with respect to reflections onto a line. More precisely, $K \in \Pi_n$ is axially symmetric if it contains a unit vector σ such that $R_\sigma(K) \subseteq K$, where R_σ denotes the reflection matrix onto the line generated by σ. In case of existence, such a vector σ is unique, it is denoted by σ_K and called the axial symmetry center of K. The line generated by σ_K is said to be the symmetry axis of K and

$$S(K, \sigma_K) = \{x \in K : \langle \sigma_K, x\rangle = 1\}$$

is called the perpendicular cross section of K (at unit distance from the origin). The set of axially symmetric proper cones is stable under passage to dual cones and under orthogonal transformations, but not under general invertible linear transformations.

Theorem 2.4. *Let $K \in \Pi_n$. Then the following conditions are equivalent:*
(a) *K is an ellipsoidal cone.*
(b) *For each $y \in \mathrm{int}(K^*)$, the cross section $S(K, y)$ is an ellipsoid in $y^\diamond$.*
(c) *K is axially symmetric and $S(K, \sigma_K)$ is an ellipsoid in $\sigma_K^\diamond$.*

The implication (a) $\Rightarrow$ (b) in Theorem 2.4 is a consequence of the fact that any ellipsoidal cone is sectionally uniform. That an ellipsoidal cone is axially symmetric is consistent with geometric intuition, but proving axial symmetry by relying only on Definition 1.1 or Theorem 2.1 is not as straightforward as one may think at first sight. The proof of axial symmetry is however trivial if the ellipsoidal cone under consideration is given by its semiaxial representation, cf. Section 3.4.

Central symmetry enters now into action. A bounded set Ω in $\mathbb{R}^n$ is centrally symmetric if there exists a point $c \in \mathbb{R}^n$ such that $\Omega - c = c - \Omega$. In case of existence, such a point c is unique and called the center of symmetry of Ω. For instance, ellipsoids in $\mathbb{R}^n$ (or in some hyperplane of $\mathbb{R}^n$) are centrally symmetric bounded sets. From Theorem 2.4 we know that every cross section of an ellipsoidal cone is centrally symmetric. This property characterizes in fact the class of ellipsoidal cones.

Theorem 2.5. (Jerónimo-Castro and Mcallister [38]) *Let $K \in \Pi_n$. Then the following conditions are equivalent:*
(a) *K is an ellipsoidal cone.*
(b) *Every cross section of K is centrally symmetric.*

The three dimensional version of this nice result was established several decades ago in Olovjanischnikoff [41]. The proof of Theorem 2.5 relies on deep results of the theory of convex bodies. The next theorem states two by-products of Theorem 2.5. To the best of our knowledge, these by-products have not been mentioned before in the literature.

Theorem 2.6. *Let $K \in \Pi_n$. Then the following conditions are equivalent:*
(a) *K is an ellipsoidal cone.*
(b) *Any proper cone isomorphic to K is axially symmetric.*
(c) *K is axially symmetric and homogeneous.*

Proof. We prove first that (c) implies (a). Let K be axially symmetric. Then its perpendicular cross section is centrally symmetric, cf. Seeger and Torki [54, Lemma 3.7]. Assume, in addition, that K is homogenous. Then every cross section of K is centrally symmetric because it is isomorphic to the perpendicular cross section of K. It follows from Theorem 2.5 that K is an ellipsoidal cone. The proof of the implication (b) ⇒ (a) runs as follows. Suppose that $A(K)$ is axially symmetric for all $A \in \mathbb{GL}(n)$. Pick any $y_0 \in \mathrm{int}(K^*)$. We claim that $S(K, y_0)$ is centrally symmetric. As a particular instance of a general formula of Seeger and Torki [53, Lemma 3.3] for the interior of a proper cone, we have

$$\mathrm{int}(K^*) = \{A^\top \sigma_{A(K)} : A \in \mathbb{GL}(n)\}.$$

Hence, there exists $A_0 \in \mathbb{GL}(n)$ such that $y_0 = A_0^\top \sigma_{A_0(K)}$. Let $P := A_0(K)$. In such a case,

$$\begin{aligned} S(K, y_0) &= K \cap \left(A_0^\top \sigma_P\right)^\diamond = K \cap A_0^{-1}(\sigma_P^\diamond) \\ &= A_0^{-1}\left(P \cap \sigma_P^\diamond\right) = A_0^{-1}\left[S(P, \sigma_P)\right]. \end{aligned}$$

Since P is axially symmetric, its perpendicular cross section $S(P, \sigma_P)$ is centrally symmetric. Hence, the isomorphic set $S(K, y_0)$ is centrally symmetric as well. Since y_0 was an arbitrary vector taken from $\mathrm{int}(K^*)$, we deduce from Theorem 2.5 that K is an ellipsoidal cone. □

As mentioned before, an homogeneous proper cone is not necessarily an ellipsoidal cone. Condition (c) of Theorem 2.6 is a particular instance of a statement of the abstract type

$$\text{ellipsoidality} = \text{homogeneity} + \text{something else}.$$

Two other results in the same vein are stated in the next theorem. Recall that $K \in \Pi_n$ is said to be strictly convex if every face of K, other than K itself and $\{0\}$, is a half-line. Smoothness is defined as the dual concept of strict convexity: if a proper cone is strictly convex then its dual cone is smooth, and viceversa. Ellipsoidal cones are both strictly convex and smooth, cf. Stern and Wolkowicz [58, p. 82] or Seeger [48, Section 3]. Ito and Lourenço [31] have shown recently that homogeneity plus strict convexity implies ellipsoidality.

Theorem 2.7. *Let $K \in \Pi_n$. Then the following conditions are equivalent:*

(a) *K is an ellipsoidal cone.*

(b) *K is homogeneous and strictly convex.*

(c) *K is homogeneous and smooth.*

The proof of (b) ⇒ (a) suggested in Ito and Lourenço [31] relies on the T-algebra machinery. We wonder whether it is possible to write a simpler proof based uniquely on geometric and standard linear algebra arguments. The implication (c) ⇒ (a) can be proved afterwards by using duality arguments. It suffices to observe that if K is homogeneous and smooth, then K^* is homogeneous and strictly convex.

The exposition goes on with the ellipso-generational characterization of ellipsoidal cones. For economy of language, a set that does not contain the origin is called *origin-free*. The *conic hull* of an origin-free convex set Ω is the convex cone

$$\text{cone}\,(\Omega) := \{tu : t \geq 0, u \in \Omega\}. \tag{2}$$

The set (2) is known also as the convex cone generated by Ω. If Ω is an origin-free convex body in $\mathbb{R}^n$, then (2) is a proper cone in $\mathbb{R}^n$. Any revolution cone can be represented as a ball-generated cone. Indeed, the revolution cone $\text{rev}(s, b)$ is generated by a ball of center $b \in \mathbb{S}_n$ and radius $[1 - s^2]^{1/2}$, cf. Henrion and Seeger [30, Lemma 2.1]. In particular,

$$\mathbb{L}_n \;=\; \text{cone}\left(e_n + \frac{1}{\sqrt{2}}\,\mathbb{B}_n\right) \;=\; \text{cone}\left(\sqrt{2}\,e_n + \mathbb{B}_n\right).$$

In the same vein, any ellipsoidal cone can be represented as an ellipsoid-generated cone.

Theorem 2.8. *Let $K \in \Pi_n$. Then the following conditions are equivalent:*

(a) *K is an ellipsoidal cone.*

(b) *K is the conic hull of some origin-free ellipsoid in $\mathbb{R}^n$.*

If K is an ellipsoidal cone, then the origin-free ellipsoid Ω_0 that serves to generate K is not unique and, contrarily to the cross section $S(K, y_0)$ mentioned in Definition 1.1, it is full dimensional in the underlying space $\mathbb{R}^n$. In order to prove the implication (a) $\Rightarrow$ (b) in Theorem 2.8 we can take for instance

$$\Omega_0 \;:=\; A\left(\sqrt{2}\,e_n + \mathbb{B}_n\right) \;=\; \sqrt{2}\,Ae_n + A(\mathbb{B}_n), \tag{3}$$

where A is any invertible matrix such that $K = A(\mathbb{L}_n)$. Note that (3) is an origin-free ellipsoid in $\mathbb{R}^n$ and

$$K \;=\; A\left[\text{cone}\left(\sqrt{2}e_n + \mathbb{B}_n\right)\right] \;=\; \text{cone}\,(\Omega_0)\,.$$

In connection with Theorem 2.8, we mention below an interesting characterization of ellipsoids. If Ω is a convex body in $\mathbb{R}^n$ and x is a point not in Ω, then

$$\Upsilon_\Omega(x) := \text{cone}\,(\Omega - x)$$

is a proper cone called the supporting cone to Ω at x. The definition of $\Upsilon_\Omega(x)$ is identical to that of the tangent cone to Ω at x, except that x is now in the exterior of Ω and not on the boundary.

Proposition 2.9. (Gruber and Ódor [24]) *Let Ω be convex body in $\mathbb{R}^n$. Then the following conditions are equivalent:*

(a) *Ω is an ellipsoid.*

(b) *For all $x \notin \Omega$, $\Upsilon_\Omega(x)$ is an axially symmetric proper cone.*

(c) *For all $x \notin \Omega$, $\Upsilon_\Omega(x)$ is an ellipsoidal cone.*

In fact, only the equivalence (a) $\Leftrightarrow$ (b) is established in [24]. That (a) implies (c) is implicit in Theorem 2.8. Note that Proposition 2.9 characterizes ellipsoidality in convex bodies in terms of ellipsoidality in proper cones. Usually, it is the other way around. We continue the exposition with a second characterization of ellipsoidality in proper cones proposed by Jerónimo-Castro and Mcallister [38, Theorem 1.4]. The second proposal of these authors involves the so-called *Flat Boundary Intersection (FBI) property.* We say that $K \in \Pi_n$ satisfies the FBI property if, for all $w \in \mathrm{int}(K)$, the set

$$\tau(K, w) := \mathrm{bd}(K) \cap \mathrm{bd}(2w - K)$$

is flat, i.e., contained in a hyperplane. Here, $\mathrm{bd}(\Omega)$ stands for the boundary of a set Ω. We mention in passing that $\tau(K, w)$ is compact and admits w as center of symmetry. The set of proper cones satisfying the FBI property is stable under invertible linear transformations and under passage to dual cones. We give below an easy example of proper cone satisfying the FBI property.

Example 2.10. Consider the Lorentz cone $\mathbb{L}_n$. Pick $w \in \mathrm{int}(\mathbb{L}_n)$. Then x belongs to $\tau(\mathbb{L}_n, w)$ if and only if $x_n \in [0, 2w_n]$ and

$$\begin{aligned} x_1^2 + \ldots + x_{n-1}^2 &= x_n^2\,, \\ (2w_1 - x_1)^2 + \ldots + (2w_{n-1} - x_{n-1})^2 &= (2w_n - x_n)^2. \end{aligned}$$

In particular, $\tau(\mathbb{L}_n, w)$ lies on the hyperplane

$$w_1 x_1 + \ldots + w_{n-1} x_{n-1} - w_n x_n = (w_1^2 + \ldots + w_{n-1}^2) - w_n^2\,.$$

Hence, $\mathbb{L}_n$ satisfies the FBI property. A posteriori, any ellipsoidal cone satisfies the FBI property.

Theorem 2.11. (Jerónimo-Castro and Mcallister [38]) *Let $K \in \Pi_n$. Then the following conditions are equivalent:*
(a) *K is an ellipsoidal cone.*
(b) *K satisfies the FBI property.*

The geometric characterizations of ellipsoidality in proper cones that we are aware of have been collected in Theorems 2.1 to 2.11. We stated 13 geometric characterizations in all, but some of them can be easily derived from the others. We now shift the attention to matrix algebraic characterizations.

3. Matrix representation of an ellipsoidal cone

The definition of a mathematical object is often times motivated by a specific geometric property, but the definition itself is not readily applicable when it comes to carry out numerical computations. This is precisely what happens with Definition 1.1. If we know that $K \in \Pi_n$ admits $S(K, y_0)$ as ellipsoidal cross section, how can we compute its axial symmetry center, its volume, its maximal angle, and so on? In order to take advantage of numerical linear algebra techniques, the ellipsoidal cone under analysis should be represented in terms of finitely many real parameters, to be stacked in a matrix of appropriate size or in a finite collection of vectors. A parametric representation of that sort is what we call a "parametrization" of the ellipsoidal cone.

3.1. Lorentzian and pre-Lorentzian representations

We saw in Theorem 2.1 that a proper cone in $\mathbb{R}^n$ is an ellipsoidal cone if and only if it can be expressed in the form

$$A(\mathbb{L}_n) := \{Ax : x \in \mathbb{L}_n\} \tag{4}$$

for some $A \in \mathbb{GL}(n)$. We say that (4) is a Lorentzian representation of the ellipsoidal cone under consideration. The expression (4) is considered in numerous works because, among other reasons, it intervenes in the definition of the automorphism group of the Lorentz cone:

$$\mathrm{Aut}(\mathbb{L}_n) := \{A \in \mathbb{GL}(n) : A(\mathbb{L}_n) = \mathbb{L}_n\}.$$

The emphasis in the definition of $\mathrm{Aut}(\mathbb{L}_n)$ is not however on ellipsoidality but on group theoretic properties associated to the Lorentz cone. The set (4) is a cone and therefore it remains unaffected under positive scalar multiplication. Note that two non-collinear matrices $A_0, A_1 \in \mathbb{GL}(n)$ may perfectly well produce the same ellipsoidal cone.

Proposition 3.1. *Let $A_0, A_1 \in \mathbb{GL}(n)$ and consider the block decomposition*

$$A_1^{-1}A_0 = \begin{bmatrix} N & z \\ q^\top & t \end{bmatrix},$$

where the block N is a matrix of order $n-1$. Then the following conditions are equivalent:

(a) $A_0(\mathbb{L}_n) = A_1(\mathbb{L}_n)$.

(b) $A_1^{-1}A_0 \in \mathrm{Aut}(\mathbb{L}_n)$.

(c) $\|z\| < t$, $N^\top z = tq$, *and* $N^\top N - qq^\top = (t^2 - \|z\|^2)I_{n-1}$.

The equivalence (a) $\Leftrightarrow$ (b) in Proposition 3.1 is obvious. That (b) is equivalent to (c) is a consequence of Theorem 2.4 in Loewy and Schneider [40]. Note that (c) serves to characterize $\mathrm{Aut}(\mathbb{L}_n)$. Any ellipsoidal cone in $\mathbb{R}^n$ can be represented in the Lorentzian format (4) with $A \in \mathbb{M}_n$ satisfying the "normalization" condition $\det A = 1$. Hence, without loss of generality, the matrix A representing the ellipsoidal cone can be sought on a closed subset of $\mathbb{GL}(n)$. The change of variables $B = A^{-1}$ in (4) leads to a pre-Lorentzian representation

$$\Phi(B) := \{x \in \mathbb{R}^n : Bx \in \mathbb{L}_n\}. \tag{5}$$

In other words, an ellipsoidal cone can be expressed as inverse image of $\mathbb{L}_n$ under an invertible linear transformation. The ellipsoidal cone $\Phi(B)$ can be seen also as the image of $\mathbb{L}_n$ under B^{-1}, but this is not what we want to emphasize when we write the right-hand side of (5). On some occasions the pre-Lorentzian representation has some numerical advantages over the Lorentzian one. Of course, the formulas $\Phi(B) = B^{-1}(\mathbb{L}_n)$ and $A(\mathbb{L}_n) = \Phi(A^{-1})$ allow to pass from a Lorentzian to a pre-Lorentzian representation, and viceversa.

The next proposition explains how to compute the axial symmetry center of $\Phi(B)$. We use the notation

$$J_n := I_n - 2e_n e_n^\top \quad \text{and} \quad B^\sharp := B^\top J_n B\,.$$

Recall that e_n is the last column of the identity matrix I_n. The inertia of a symmetric matrix is a triplet indicating respectively the number of positive eigenvalues, the number of eigenvalues equal to zero, and the number of negative eigenvalues. For instance, J_n has inertia $(n-1, 0, 1)$. If $B \in \mathbb{GL}(n)$, then Silvester's inertia theorem ensures that the symmetric matrix $B^\sharp$ has the same inertia as J_n. In particular, the smallest eigenvalue of $B^\sharp$ is simple, i.e., the corresponding eigenspace

$$E_{\min}(B^\sharp) := \left\{x \in \mathbb{R}^n : B^\sharp x = \lambda_{\min}(B^\sharp)\, x\right\}$$

has dimension one.

Proposition 3.2. (Seeger and Torki [52]) *Let $B \in \mathbb{GL}(n)$. Then $\Phi(B)$ has the line $E_{\min}(B^\sharp)$ as symmetry axis. In particular, $\sigma_{\Phi(B)}$ is the unique solution $x \in \mathbb{R}^n$ to the system*

$$B^\sharp x = \lambda_{\min}(B^\sharp)\, x, \quad \|x\| = 1, \quad \langle B^\top e_n, x\rangle \geq 0. \tag{6}$$

The axial symmetry center of $\Phi(B)$ is then a special eigenvector of $B^\sharp$. To compute the dual cone of $\Phi(B)$ is essentially a matter of carrying out a matrix inversion followed by a transposition:

$$[\Phi(B)]^* = \Phi(B^{-\top}). \tag{7}$$

Formula (7) is the pre-Lorentzian counterpart of the Lorentzian analog (1).

3.2. Stern-Wolkowicz representation

The notation $\mathrm{Sym}(n)$ indicates the linear space of symmetric matrices of order n. In Stern and Wolkowicz [58, p. 82], ellipsoidal cones are defined as proper cones whose cross sections are ellipsoids. We know already that this is equivalent to Definition 1.1. In page 83 of [58] we find the following characterization of ellipsoidality in proper cones.

Theorem 3.3. (Stern and Wolkowicz [58]) *$K \in \Pi_n$ is an ellipsoidal cone if and only if K admits the representation*

$$\mathrm{sw}(C, b) := \{x \in \mathbb{R}^n : \langle x, Cx\rangle \leq 0,\ \langle b, x\rangle \geq 0\}\,, \tag{8}$$

where $C \in \mathrm{Sym}(n)$ has inertia $(n-1, 0, 1)$ and $b \in \mathbb{S}_n$ is an eigenvector of C associated to the unique negative eigenvalue.

The meaning of the acronym "sw" is obvious. The ellipsoidal cone $\mathrm{sw}(C,b)$ remains unchanged if C is multiplied by a positive scalar. Hence, the matrix C can be rescaled so that $\lambda_{\min}(C) = -1$. If this extra normalization assumption is in force, then we say that (8) is a normalized Stern-Wolkowicz representation. For instance,

$$\mathbb{L}_n = \mathrm{sw}(J_n, e_n)$$

is a normalized Stern-Wolkowicz representation of the Lorentz cone. The model (8) is quite popular in the literature and it is frequently used as definition of ellipsoidal cone. Such a model has been adopted for instance in the control theory community, cf. Bhattacharya et al. [5] and Grussler and Rantzer [25]. Except for minor cosmetic changes, the parametrization of an ellipsoidal cone suggested in Aliluiko and Mazko [1] is the same as in Theorem 3.3. If we drop the linear inequality constraint in (8), then we get the set

$$\mathrm{e}(C) := \{x \in \mathbb{R}^n : \langle x, Cx\rangle \leq 0\} \tag{9}$$

which is no longer convex. In view of the formula

$$\mathrm{e}(C) = \mathrm{sw}(C,b) \cup -\,\mathrm{sw}(C,b),$$

the set (9) is rightly called an *ellipsoidal double-cone*. The boundary of $\mathrm{e}(C)$ is given by the equation $\langle x, Cx\rangle = 0$ and, therefore, it is a particular instance of a quadric surface. The literature dealing with ellipsoidal double-cones is too vast to be quoted here. The hyperplane $b^\perp$ cuts $\mathrm{e}(C)$ into two opposite ellipsoidal cones and, of course, we keep only one of them. As pointed out by Stern and Wolkowicz [58, Proposition 2.1], the ellipsoidal cone (8) remains unchanged if b is replaced by any vector $b_0 \in \mathbb{S}_n$ such that

$$b_0^\perp \cap \mathrm{e}(C) = \{0\}, \quad \langle b, b_0\rangle \geq 0.$$

The fact of choosing b as a special eigenvector of C is not fortuitous. The next proposition reveals the *raison d'être* of this particular choice.

Proposition 3.4. *Let* (C,b) *be as in Theorem* 3.3. *Then:*

(a) $\mathrm{e}(C)$ *is symmetric with respect to the line* $\mathbb{R}b$ *and* $\sigma_{\mathrm{sw}(C,b)} = b$.

(b) *The pair* (C^{-1}, b) *is also as in Theorem* 3.3 *and* $[\mathrm{sw}(C,b)]^* = \mathrm{sw}(C^{-1}, b)$.

Computing the dual cone of (8) is then a matter of inverting the matrix C. The duality formula mentioned in (b) is proven for instance in Grussler and Rantzer [25, Lemma 1]. According to Aliluiko and Mazko [1, p. 1638] we can also write $[\mathrm{sw}(C,b)]^* = -\,C\,[\mathrm{sw}(C,b)]$, but the set on the right-hand side of this equality is not in the Stern-Wolkowicz format.

3.3. Bishop-Phelps and Seeger-Torki representations

In what follows the notation $R \succeq 0$ indicates that R is a positive semidefinite symmetric matrix, the order being understood from the context. Similarly, $R \succ 0$ indicates that R is a positive definite symmetric matrix. We introduce the notation

$$\mathbb{P}(n) := \{R \in \mathrm{Sym}(n) : R \succ 0\}.$$

The Bishop-Phelps and Seeger-Torki representations of an ellipsoidal cone are both of the form

$$\Gamma(R, q) := \left\{x \in \mathbb{R}^n : \langle x, Rx\rangle^{1/2} \leq \langle q, x\rangle\right\}, \tag{10}$$

but the assumptions on the matrix $R \in \mathrm{Sym}(n)$ and the vector $q \in \mathbb{R}^n$ are different in each case.

Theorem 3.5. *Let $K \in \Pi_n$. Then the following statements are equivalent:*

(a) *K is an ellipsoidal cone.*

(b) *K is expressible as in* (10) *for some $(R, q) \in \mathrm{Sym}(n) \times \mathbb{R}^n$ satisfying the Bishop-Phelps assumption*

$$R \succ 0, \quad \langle q, R^{-1}q\rangle > 1. \tag{11}$$

(c) *K is expressible as in* (10) *for some $(R, q) \in \mathrm{Sym}(n) \times \mathbb{R}^n$ satisfying the Seeger-Torki assumption*

$$R \succeq 0, \quad \mathrm{rank}(R) = n - 1, \quad Rq = 0, \|q\| = 1. \tag{12}$$

The differences between (11) and (12) are strong enough to justify a separate treatment for each representation. We start with the Bishop-Phelps representation which comes first historically and in terms of popularity. The Bishop-Phelps representation is considered in the optimization literature for modeling constraints on decision variables, cf. Belloni and Freund [3], Tian et al. [61], and Guo et al. [27]. If $R \succ 0$, then $\|x\|_R := \langle x, Rx\rangle^{1/2}$ is a norm and $\Gamma(R, q)$ becomes a particular instance of a general Bishop-Phelps cone, cf. Jahn [37, Definition 2.1]. Clearly, $\Gamma(R, q)$ contains no line passing through the origin. The second condition in (11) is exactly what is needed to ensure that $\Gamma(R, q)$ has nonempty interior. Cutting the ellipsoidal cone $\Gamma(R, q)$ with the hyperplane $q^\diamond$ does not yield necessarily the perpendicular cross section of that cone. In other words, the symmetry axis of $\Gamma(R, q)$ may not be the line $\mathbb{R}q$. This is perhaps the main inconvenience of the Bishop-Phelps representation.

Proposition 3.6. *Let (R, q) be as in* (11)*. Then:*

(a) *$R^q := R - qq^\top$ has inertia $(n - 1, 0, 1)$ and the axial symmetry center of $\Gamma(R, q)$ is the unique solution $x \in \mathbb{R}^n$ to the system*

$$R^q x = \lambda_{\min}(R^q)\, x, \quad \|x\| = 1, \quad \langle q, x\rangle \geq 0. \tag{13}$$

(b) *In particular, $\Gamma(R, q)$ is symmetric with respect to the line $\mathbb{R}q$ if and only if*

$$Rq = \left[\lambda_{\min}(R - qq^\top) + \|q\|^2\right] q.$$

Proposition 3.6 (a) is obtained by combining Proposition 3.4 (a) and Belloni and Freund [3, Theorem 1(i)]. As we can see from (13), the axial symmetry center of $\Gamma(R, q)$ is a special eigenvector of the symmetric matrix R^q. Concerning the computation of the dual cone of $\Gamma(R, q)$, we rely on the next result proven first in Belloni and Freund [3, Corollary 1] and then, a few years later, in Guo et al. [27, Lemma 2.2]. The duality formula of Proposition 3.7 can also be obtained as a particular instance of Corollary 2 in Chua [9].

Proposition 3.7. *Let* (R, q) *be as in* (11). *Then* $[\Gamma(R, q)]^* = \Gamma(R_*, q_*)$*, where the pair*

$$R_* := R^{-1}, \quad q_* := \left[\langle q, R^{-1}q\rangle - 1\right]^{-1/2} R^{-1}q$$

is also as in (11).

From Proposition 3.7 we see that q belongs to the interior of $[\Gamma(R, q)]^*$. Hence, the set

$$\Gamma(R, q) \cap q^\diamond = \{x \in \mathbb{R}^n : \langle x, Rx\rangle \leq 1, \ \langle q, x\rangle = 1\} \tag{14}$$

is a cross section of $\Gamma(R, q)$. Such a cross section is possibly oblique, i.e., non-perpendicular to the symmetry axis of $\Gamma(R, q)$. On the other hand, the vector q_* is in general different from q. Thus, the duality formula of Proposition 3.7 is not entirely similar to the Stern-Wolkowicz counterpart stated in Proposition 3.4 (b). Note that $q_* = q$ if and only if q satisfies the eigenvector condition

$$Rq = \left[\langle q, R^{-1}q\rangle - 1\right]^{-1/2} q.$$

We now shift the attention to the Seeger-Torki representation. An important advantage of assumption (12) is that the axial symmetry center of $\Gamma(R, q)$ becomes readily available. Furthermore, for computing the dual cone of $\Gamma(R, q)$ we keep the parameter q unchanged and proceed to a Moore-Penrose inversion of the matrix R. Under assumption (12) the Moore-Penrose inverse of R admits the characterization

$$R^\dagger = (R - qq^\top)^{-1} + qq^\top.$$

The next result is borrowed from Seeger and Torki [54, Section 5.1].

Proposition 3.8. *Suppose that* (R, q) *satisfies assumption* (12)*. Then:*

(a) $\sigma_{\Gamma(R,q)} = q$*. In particular,* (14) *is the perpendicular cross section of* $\Gamma(R, q)$*.*

(b) $(R^\dagger, q)$ *satisfies assumption* (12) *and* $[\Gamma(R, q)]^* = \Gamma(R^\dagger, q)$*.*

If we compare Proposition 3.8 and Proposition 3.4, we see that the Seeger-Torki representation share many things in common with the Stern-Wolkowicz representation. An application of the Seeger-Torki representation is given in Section 6.4.1.

3.4. Semiaxial representation

The semiaxes lengths of an ellipsoid determine the volume, the diameter, the condition number, and many other geometric coefficients of the ellipsoid. It is then natural to introduce the concept of semiaxis length also for ellipsoidal cones.

Definition 3.9. Let $K \in \Pi_n$ be an ellipsoidal cone. The *semiaxes lengths* of K, which for convenience we arrange in nondecreasing order

$$\gamma_1(K) \leq \ldots \leq \gamma_{n-1}(K),$$

are defined as the semiaxes lengths of the perpendicular cross section $S(K, \sigma_K)$.

Since $S(K,\sigma_K)$ is an ellipsoid in the hyperplane $\sigma_K^\diamond$, it has $n-1$ semiaxes lengths in all, counting possible repetitions. By way of example, consider the particular ellipsoidal cone

$$\mathcal{E}_a := \left\{x \in \mathbb{R}^n : \left[(x_1/a_1)^2 + \ldots + (x_{n-1}/a_{n-1})^2\right]^{1/2} \leq x_n\right\}, \tag{15}$$

where a is an $(n-1)$-dimensional vector with positive components. The axial symmetry center of $\mathcal{E}_a$ is the vector e_n. Hence, the perpendicular cross section is obtained by setting $x_n = 1$ and the semiaxes lengths of $\mathcal{E}_a$ are simply the parameters $a_1, \ldots, a_{n-1}$. The ellipsoidal cone (15) is not as particular as one may think at first sight. In fact, every ellipsoidal cone can be brought to this form by using a suitable orthogonal transformation.

Theorem 3.10. *$K \in \Pi_n$ is an ellipsoidal cone if and only if K admits the representation*

$$U(\mathcal{E}_a) := \left\{Ux : \left[(x_1/a_1)^2 + \ldots + (x_{n-1}/a_{n-1})^2\right]^{1/2} \leq x_n\right\} \tag{16}$$

with $U \in \mathbb{O}(n)$ and $a \in \operatorname{int}(\mathbb{R}^{n-1}_+)$.

Without loss of generality we may arrange the components of a in nondecreasing order, but in practice we seldom need to bother with such a reordering operation. Seeger and Vidal-Nuñez [56] refer to (16) as a semiaxial representation because the semiaxes lengths play a prominent role. Note that $U(\mathcal{E}_a)$ has the same semiaxes lengths as $\mathcal{E}_a$. Another advantage of the representation (16) is that the axial symmetry center and the dual cone are both readily available.

Proposition 3.11. *Let (U,a) be as in Theorem 3.10. Then:*

(a) *The axial symmetry center of $U(\mathcal{E}_a)$ is equal to the last column of U.*

(b) $[U(\mathcal{E}_a)]^* = U(\mathcal{E}_{a^{-1}})$ *with* $a^{-1} := (1/a_1, \ldots, 1/a_{n-1})^\top$.

In view of all the advantages mentioned above, the semiaxial representation (16) is one of our favorite ways of parametrizing an ellipsoidal cone. A model of ellipsoidal cone slightly more general than (15) is

$$\mathcal{E}(W) := \left\{(z,t) \in \mathbb{R}^n : \langle z, Wz\rangle^{1/2} \leq t\right\}, \tag{17}$$

where $W \succ 0$ is not necessarily diagonal. The set (17) is the epigraph of the norm $\|\cdot\|_W$ and, therefore, it is a particular instance of a topheavy cone in the sense of Fiedler and Haynsworth [15]. Since the dual of a topheavy cone is equal to the topheavy cone associated to the dual norm, we get

$$[\mathcal{E}(W)]^* = \mathcal{E}(W^{-1}). \tag{18}$$

Formula (18) is proposed as exercise in the book of Facchinei and Pang [13]. The model (17) has been used in the literature for a variety of purposes, cf. Henrion and Seeger [29, 30], Iusem and Seeger [34, 35, 36], Seeger [47, 48], Seeger and Sossa [51]. The following corollary is an easy consequence of Theorem 3.10. Recall that a norm is said to be Hilbertian if it derives from an inner product or, what is equivalent, if it satisfies the parallelogram law.

Corollary 3.12. *Let $K \in \Pi_n$. Then the following conditions are equivalent:*

(a) *K is an ellipsoidal cone.*

(b) *Up to orthogonal transformation, K is the epigraph of some Hilbertian norm.*

Proof. The implication (a) $\Rightarrow$ (b) is contained in Theorem 3.10. For proving the reverse implication we note that any Hilbertian norm on $\mathbb{R}^{n-1}$ is of the form $\|\cdot\|_W$ with $W \in \mathbb{P}(n-1)$. Furthermore, for all $U \in \mathbb{O}(n)$, we have

$$U(\mathcal{E}(W)) \;=\; U \begin{bmatrix} V & 0 \\ 0 & 1 \end{bmatrix} (\,\mathcal{E}_a)\,, \tag{19}$$

where $a \in \mathbb{R}^{n-1}$ is formed with the eigenvalues of W and $V \in \mathbb{O}(n-1)$ is an orthogonal matrix whose columns are corresponding eigenvectors. With the help of the diagonalization formula (19) we are back at the semiaxial representation (16). $\square$

3.5. Ellipso - generational representation

Theorem 2.8 asserts that $K \in \Pi_n$ is an ellipsoidal cone if and only if K admits the representation

$$\Theta(G,c) := \operatorname{cone}(c + G(\mathbb{B}_n))\,, \tag{20}$$

where $G \in \mathbb{GL}(n)$ and $c \in \mathbb{R}^n$ are such that

$$\|G^{-1}c\| > 1. \tag{21}$$

By an obvious reason, we refer to (20) as an ellipso-generational representation. Condition (21) is exactly what is needed to ensure that the ellipsoid $c + G(\mathbb{B}_n)$ is origin-free. Ellipsoidal cones represented in the format (20) have been considered for instance in Seeger [49]. The reader should be aware that the line generated by c is not necessarily the symmetry axis of $\Theta(G,c)$. This is a clear inconvenience of the ellipso-generational parametrization. In order to compute the axial symmetry center of $\Theta(G,c)$ we resort to the following calculus rule.

Proposition 3.13. *Let (G,c) be as in (21). Then $G^c := GG^\top - cc^\top$ has inertia $(n-1,0,1)$ and the axial symmetry center of $\Theta(G,c)$ is the unique solution $x \in \mathbb{R}^n$ to the system*

$$G^c x = \lambda_{\min}(G^c)\, x, \quad \|x\| = 1, \quad \langle c, x\rangle \geq 0. \tag{22}$$

In particular, $\Theta(G,c)$ is symmetric with respect to the line $\mathbb{R}c$ if and only if

$$GG^\top c = \left[\lambda_{\min}(GG^\top - cc^\top) + \|c\|^2\right] c.$$

Proof. If $K \in \Pi_n$ is axially symmetric, then K^* is axially symmetric and has the same axial symmetry center as K. We compute then the axially symmetry center of $[\Theta(G,c)]^*$. We have

$$\begin{aligned} y \in [\Theta(G,c)]^* &\Leftrightarrow\; \langle y, t(c+Gu)\rangle \geq 0 \;\text{ for all }\; t \geq 0,\; u \in \mathbb{B}_n \\ &\Leftrightarrow\; \langle y, c\rangle \geq \langle -G^\top y, u\rangle \;\text{ for all }\; u \in \mathbb{B}_n \;\Leftrightarrow\; \langle y, c\rangle \geq \|G^\top y\|. \end{aligned}$$

Hence, $(GG^\top, c)$ satisfies the Bishop-Phelps assumption (11) and

$$[\Theta(G,c)]^* = \Gamma(GG^\top, c). \tag{23}$$

It suffices now to apply Proposition 3.6 to the right-hand side of (23). □

The above proof shows that ellipso-generational and Bishop-Phelps representations are related through duality. The next result is obtained by combining (23) and Proposition 3.7.

Proposition 3.14. *Let* (G, c) *be as in* (21). *Then* $[\Theta(G,c)]^* = \Theta(\widetilde{G}, \tilde{c})$, *where the pair*

$$\widetilde{G} := G^{-\top}, \quad \tilde{c} := \left[\langle c, (GG^\top)^{-1}c\rangle - 1\right]^{-1/2} (GG^\top)^{-1}c$$

is also as in (21).

3.6. Ellipso-lifting representation

A natural way of constructing a proper cone in $\mathbb{R}^n$ is by starting with a convex body E in a space of one dimension less, say $\mathbb{R}^{n-1}$. If we "lift" the set $E \times \{0\}$ to the hyperplane $e_n^\diamond := \{x \in \mathbb{R}^n : x_n = 1\}$ and form the conic hull, then we get the proper cone

$$\mathrm{lift}(E) := \mathrm{cone}\,(E \times \{1\})\,. \tag{24}$$

We say that (24) is obtained by lifting the convex body E. For instance, the Lorentz cone $\mathbb{L}_n$ is obtained by lifting the unit ball $\mathbb{B}_{n-1}$. By construction, $\mathrm{lift}(E)$ is upwardly oriented in the sense that each nonzero vector of $\mathrm{lift}(E)$ belongs to the upper open halfspace $\{x \in \mathbb{R}^n : x_n > 0\}$. This choice of orientation is dictated only for notational convenience.

Theorem 3.15. *A set in* $\mathbb{R}^n$ *is an upwardly oriented ellipsoidal cone if and only it it admits the representation*

$$\Lambda(N, \varrho) := \mathrm{lift}(\varrho + N(\mathbb{B}_{n-1})), \tag{25}$$

where $N \in \mathbb{GL}(n-1)$ *and* $\varrho \in \mathbb{R}^{n-1}$.

Despite appearances, the definition of $\Lambda(N, \varrho)$ is quite different from that of $\Theta(G, c)$. We underline that $\Lambda(N, \varrho)$ is generated by an ellipsoid in the hyperplane $e_n^\diamond$ and not by a full dimensional ellipsoid in $\mathbb{R}^n$. The set (25) can be rewritten in the more algebraic format

$$\Lambda(N, \varrho) = \{(z,t) \in \mathbb{R}^n : \|N^{-1}(z - t\varrho)\| \le t\},$$

from where it is easier to compute its axial symmetry center.

Proposition 3.16. (Seeger and Vidal-Nuñez [56]) *Let* $N \in \mathbb{GL}(n-1)$ *and* $\varrho \in \mathbb{R}^{n-1}$. *Then the axial symmetry center of* $\Lambda(N, \varrho)$ *is the unique solution* $x \in \mathbb{R}^n$ *to the system*

$$N_\varrho x = \lambda_{\max}(N_\varrho)\, x, \;\; \|x\| = 1, \;\; \langle e_n, x\rangle \ge 0, \tag{26}$$

where $N_\varrho \in \mathrm{Sym}(n)$ is a matrix with inertia $(1, 0, n-1)$ given by

$$N_\varrho := \left[\begin{array}{cc} \varrho\varrho^\top - NN^\top & \varrho \\ \varrho^\top & 1 \end{array} \right].$$

In particular, the line generated by $(\varrho, 1)$ is the symmetry axis of $\Lambda(N, \varrho)$ if and only if $\varrho = 0$.

Ellipso-lifting representations are in general badly suited to deal with duality issues. A direct computation shows that the dual cone of (25) is given by

$$[\Lambda(N, \varrho)]^* = \{(z, t) \in \mathbb{R}^n : \|N^\top z\| - \langle \varrho, z\rangle \le t\}. \tag{27}$$

This set is of course an ellipsoidal cone, but not necessarily upwardly oriented. An additional hypothesis is required to ensure the upward orientation of (27).

Proposition 3.17. *Let $N \in \mathbb{GL}(n-1)$ and $\varrho \in \mathbb{R}^{n-1}$ be such that $\|N^{-1}\varrho\| < 1$. Then the symmetric matrix $S := NN^\top - \varrho\varrho^\top$ is positive definite and*

$$[\Lambda(N, \varrho)]^* = \Lambda(N_0, \varrho_0) \tag{28}$$

with $N_0 N_0^\top := (\langle \varrho, S^{-1}\varrho\rangle + 1)S^{-1}$ *and* $\varrho_0 := S^{-1}\varrho$.

Proof. Besides implying that $S \succ 0$, the hypothesis $\|N^{-1}\varrho\| < 1$ ensures that e_n belongs to the interior of $K := \Lambda(N, \varrho)$. Hence,

$$S(K^*, e_n) = \{z \in \mathbb{R}^{n-1} : \|N^\top z\| - \langle \varrho, z\rangle \le 1\} \times \{1\}$$

is a cross section of K^*. Since

$$E := \{z \in \mathbb{R}^{n-1} : \|N^\top z\| - \langle \varrho, z\rangle \le 1\}$$

is an ellipsoid in $\mathbb{R}^{n-1}$, it is possible to find $N_0 \in \mathbb{GL}(n-1)$ and $\varrho_0 \in \mathbb{R}^{n-1}$ such that $E = \varrho_0 + N_0(\mathbb{B}_{n-1})$ or, equivalently, such that

$$\|N^\top z\| \le \langle \varrho, z\rangle + 1 \quad \Leftrightarrow \quad \|N_0^{-1}(z - \varrho_0)\| \le 1 \tag{29}$$

for all $z \in \mathbb{R}^{n-1}$. Squaring on each inequality of (29) and identifying terms we get a nonlinear system

$$\left\{ \begin{array}{l} (N_0 N_0^\top)^{-1} = \mu_0\, S\,, \\ (N_0 N_0^\top)^{-1}\varrho_0 = \mu_0\, \varrho\,, \\ \mu_0 = 1 - \langle \varrho_0, (N_0 N_0^\top)^{-1}\varrho_0\rangle\,, \end{array} \right.$$

from where we determine the unknowns N_0 and ϱ_0. □

Note that $\|N_0^{-1}\varrho_0\| < 1$, i.e., the new pair (N_0, ϱ_0) satisfies the same hypothesis as (N, ϱ).

3.7. Rules for passing from one parametrization to another

Identifying the axial symmetry center of an ellipsoidal cone may be easy or not depending on the way the cone is parametrized. The same remark applies to the computation of the dual cone. Table 3.1 summarizes the situation for the 8 parametrization methods presented from Section 3.1 to Section 3.6. The last column of Table 3.1 concerns the particular case of the Lorentz cone. The notation $\mathbf{1}_n$ refers to the n-dimensional vector of ones. As we can see, the same ellipsoidal cone can be parametrized in manifold ways.

Parametrization	K	σ_K	K^*	Case of $\mathbb{L}_n$
Pre-Lorentzian	$\Phi(B)$	sol. to (6)	$\Phi(B^{-\top})$	$\Phi(I_n)$
Lorentzian	$A(\mathbb{L}_n)$	sol. to (6) with $B = A^{-1}$	$A^{-\top}(\mathbb{L}_n)$	$I_n(\mathbb{L}_n)$
Stern-Wolkowicz (SW)	$\mathrm{sw}(C, b)$	b	$\mathrm{sw}(C^{-1}, b)$	$\mathrm{sw}(J_n, e_n)$
Bishop-Phelps (BP)	$\Gamma(R, q)$	sol. to (13)	$\Gamma(R_*, q_*)$, Prop. 3.7	$\Gamma(I_n, \sqrt{2}e_n)$
Seeger-Torki (ST)	$\Gamma(R, q)$	q	$\Gamma(R^\dagger, q)$	$\Gamma(I_n - e_n e_n^\top, e_n)$
Semiaxial	$U(\mathcal{E}_a)$	Ue_n	$U(\mathcal{E}_{a^{-1}})$, Prop. 3.11	$I_n(\mathcal{E}_{\mathbf{1}_{n-1}})$
Ellipso-generat. (EG)	$\Theta(G, c)$	sol. to (22)	$\Theta(\widetilde{G}, \tilde{c})$, Prop. 3.14	$\Theta(I_n, \sqrt{2}e_n)$
Ellipso-lifting (EL)	$\Lambda(N, \varrho)$	sol. to (26)	see (27), Prop. 3.17	$\Lambda(I_{n-1}, 0)$

Table 3.1: Axial symmetry center and dual of an ellipsoidal cone.

The choice of parametrization method used in practice depends on the context (control of linear systems, modeling of constrains in optimization theory, etc), but also on what we want to do with the ellipsoidal cone under consideration (to compute its collection of critical angles, to approximate a polyhedral cone, etc). In any case, it is helpful to know how to pass quickly from one parametrization to another. Listing the transformation rules for the $56 = 8 \times 7$ possible cases would be too space consuming, so we select just a few examples. Besides the obvious transformation rules bewteen the Lorentzian and pre-Lorentzian representations, other easy cases are

$$\mathrm{SW} \rightleftarrows \mathrm{ST} \quad \begin{cases} \mathrm{sw}(C, b) = \Gamma\left(bb^\top - (\langle b, Cb\rangle)^{-1}C, b\right), \\ \Gamma(R, q) = \mathrm{sw}(R - qq^\top, q), \\ (C, b) \text{ as in Thm. 3.3, } (R, q) \text{ as in (12)} \end{cases}$$

and

$$\mathrm{EG} \rightleftarrows \mathrm{BP} \quad \begin{cases} \Theta(G, c) = \Gamma\left(\widetilde{G}(\widetilde{G})^\top, \tilde{c}\right), \\ \Gamma(R, q) = \Theta\left(R^{-1/2}, q_*\right), \\ (G, c) \text{ as in (21), } (R, q) \text{ as in (11).} \end{cases} \tag{30}$$

The formulas in (30) are obtained by combining (23), Proposition 3.7, and Proposition 3.14. As we saw in (3), the relation

$$A(\mathbb{L}_n) = \Theta\left(A, \sqrt{2}Ae_n\right)$$

allows us to pass from a Lorentzian to an ellipso-generational representation. Further examples of transformation rules are given in the next four propositions.

Proposition 3.18. *Stern-Wolkowicz and pre-Lorentzian representations are linked as follows:*

(a) *Let $B \in \mathbb{GL}(n)$. Then $\Phi(B) = \mathrm{sw}\left[B^\sharp, \sigma_{\Phi(B)}\right]$.*

(b) *Let (C, b) be as in Theorem 3.3. The spectral decomposition $C = UDU^\top$ is considered, where the diagonal matrix D contains the eigenvalues of C arranged in nonincreasing order and the orthogonal matrix $U = [u_1, \ldots, u_{n-1}, b]$ is formed with corresponding eigenvectors. Then*

$$\mathrm{sw}(C, b) = \Phi\left((J_n D)^{1/2} U^\top\right).$$

Part (b) of Proposition 3.18 is due to Stern and Wolkowicz [58, Proposition 2.3], except that we are bringing $\mathrm{sw}(C, b)$ to a pre-Lorentzian format and not to a Lorentzian format. The diagonal matrix $J_n D$ is positive definite and therefore its square root is well defined.

Proposition 3.19. *Semiaxial and pre-Lorentzian representations are linked as follows:*

(a) *Let $B \in \mathbb{GL}(n)$. Consider the spectral decomposition $B^\sharp = UDU^\top$, where the diagonal matrix D contains the eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ of $B^\sharp$ arranged in nonincreasing order and the columns of $U \in \mathbb{O}(n)$ are corresponding eigenvectors. As last column of U we take $\sigma_{\Phi(B)}$. Then $\Phi(B) = U(\mathcal{E}_a)$ with*

$$a := \left(\sqrt{-\lambda_n/\lambda_1}, \ldots, \sqrt{-\lambda_n/\lambda_{n-1}}\right)^\top.$$

(b) *Let (U, a) be as in Theorem 3.10. Then*

$$U(\mathcal{E}_a) = \Phi\left(\mathrm{Diag}\,(1/a_1, \ldots, 1/a_{n-1}, 1)\, U^\top\right).$$

Proposition 3.20. *Bishop-Phelps and pre-Lorentzian representations are linked as follows:*

(a) *Let $B \in \mathbb{GL}(n)$. Then $\Phi(B) = \Gamma\left(B^\top B, \sqrt{2} B^\top e_n\right)$.*

(b) *Let (R, q) be as in (11). Then*

$$\Gamma(R, q) = \Phi\left(\begin{bmatrix} \kappa I_{n-1} & 0 \\ 0 & 1 \end{bmatrix} U^\top R^{1/2}\right),$$

where $\kappa := \left[\langle q, R^{-1} q\rangle - 1\right]^{-1/2}$ and U is any orthogonal matrix with last column equal to $\|R^{-1/2} q\|^{-1} R^{-1/2} q$.

Proposition 3.21. *Let $N \in \mathbb{GL}(n-1)$ and $\varrho \in \mathbb{R}^{n-1}$. Then*

$$\Lambda(N, \varrho) = \mathrm{sw}[-N_\varrho^{-1}, \sigma_{\Lambda(N,\varrho)}]$$

with N_ϱ as in Proposition 3.16.

The transformation rules stated above can be combined to produce new transformation rules by transitivity. We have covered in this way all the possible combinations, except for the passage toward the ellipso-lifting representation. The latter passage is possible only if the ellipsoidal cone is upwardly oriented, a situation that requires extra assumptions on the starting parametrization.

4. Semiaxes lengths and critical angles

The collection of semiaxes lengths is a very valuable information on a given ellipsoidal cone. This statement is supported by Theorem 4.2. We need first to recall the concept of orthogonal invariance of a function on Π_n.

Definition 4.1. $\mathfrak{S}\colon \Pi_n \to \mathbb{R}$ is *orthogonally invariant* if $\mathfrak{S}(U(K)) = \mathfrak{S}(K)$ for all $K \in \Pi_n$ and $U \in \mathbb{O}(n)$.

For the sake of simplicity we assume that $\mathfrak{S}$ is real-valued, but we could equally well consider extended-real-valued functions. All interesting functions on Π_n turn out to be orthogonally invariant: any reasonable measure of volume, all pointedness and solidity indices in the axiomatic sense of Iusem and Seeger [32], the statistical dimension as defined in Amelunxen et al. [2], the maximal angle function, the solid angle, etc. An orthogonal invariant function can be extended beyond Π_n or, on the contrary, restricted to a subset of Π_n. Of special importance to us is of course the subset

$$\Xi_n := \{K \in \Pi_n : K \text{ is an ellipsoidal cone}\}.$$

Now we associate to each $K \in \Xi_n$ the vector whose components are the semiaxes lengths of K arranged in nondecreasing order and counting repetitions, i.e.,

$$\gamma(K) := (\gamma_1(K), \ldots, \gamma_{n-1}(K))^\top . \tag{31}$$

The next theorem asserts that any orthogonal invariant function on Ξ_n has the composite form $\mathfrak{S} = g \circ \gamma$ for a suitable function g. Such a result can be proven by following the same steps as in Seeger [50, Lemma 3.1].

Theorem 4.2. $\mathfrak{S}\colon \Xi_n \to \mathbb{R}$ *is orthogonally invariant if and only if there exists a permutation invariant function* $g\colon \mathrm{int}(\mathbb{R}^{n-1}_+) \to \mathbb{R}$ *such that*

$$\mathfrak{S}(K) = g(\gamma(K)) \quad \textit{for all } K \in \Xi_n. \tag{32}$$

In case of existence, such a function g is unique and given by $g(a) = \mathfrak{S}(\mathcal{E}_a)$.

It is convenient for practical purposes to have efficient calculus rules for computing the semiaxes lengths of an ellipsoidal cone. If the ellipsoidal cone K under consideration is given from the outset in the semiaxial format (16), then we are done because $\gamma(K)$ is formed with the parameters a_i's arranged in nondecreasing order. Otherwise, we have to convert the original parametrization of K into a semiaxial one by using appropriate transformation rules. By way of example, the next proposition identifies the semiaxes lenghts of an ellipsoidal cone given in ellipso-generational format.

Proposition 4.3. *Let* (G, c) *be as in* (21). *Then*

$$\gamma(\Theta(G,c)) = \left(\sqrt{-\lambda_{n-1}/\lambda_n}\,, \ldots, \sqrt{-\lambda_1/\lambda_n}\right)^\top,$$

where $\lambda_1 \geq \ldots \geq \lambda_n$ *are the eigenvalues of* $GG^\top - cc^\top$ *arranged in nonincreasing order.*

Intuitively speaking, if the semiaxes lengths of an ellipsoidal cone are small, then the semiaxes lengths of the dual ellipsoidal cone are large, and viceversa. The next duality result, which is a direct consequence of Proposition 3.11 (b), expresses this idea in a more precise manner.

Proposition 4.4. *Let* K *be an ellipsoidal cone in* $\mathbb{R}^n$. *Then*

$$\gamma(K^*) = \left(\frac{1}{\gamma_{n-1}(K)}, \ldots, \frac{1}{\gamma_1(K)}\right)^\top.$$

Let us come back to Theorem 4.2. It is clear that the composition formula (32) serves as starting point to build orthogonally invariant functions on Ξ_n.

Example 4.5. There is no consensus in the literature on what is to be understood as the size of an ellipsoid. For instance, Fisher [16, Section 3] and Calafiore [7, Section V.A] suggest to use the sum of squared semiaxes lengths as size criterion. This choice leads to consider

$$\mathfrak{S}_2(K) := \sum_{i=1}^{n-1} [\gamma_i(K)]^2$$

as measure of size of an ellipsoidal cone K. Other measures of size in the same spirit are the arithmetic mean and the geometric mean of the semiaxes lengths:

$$\mathfrak{S}_{\mathrm{am}}(K) := \frac{1}{n-1}\sum_{i=1}^{n-1} \gamma_i(K)\,, \quad \text{and} \quad \mathfrak{S}_{\mathrm{gm}}(K) := \left[\prod_{i=1}^{n-1} \gamma_i(K)\right]^{\frac{1}{n-1}}.$$

The functions $\mathfrak{S}_{\mathrm{gm}}$, $\mathfrak{S}_{\mathrm{am}}$, and $\mathfrak{S}_2$, are orthogonally invariant because $\Pi_{i=1}^{n-1} a_i$, $\Sigma_{i=1}^{n-1} a_i$, and $\Sigma_{i=1}^{n-1} a_i^2$, are permutation invariant.

The composition formula (32) has yet many other applications. The following proposition is a conic formulation of a result by Carlson [8] on the electrostatic capacity of an ellipsoid; see Tee [60] for a gentle introduction to this topic. The electrostatic capacity of an ellipsoidal cone K in $\mathbb{R}^n$, which we denote by $\mathrm{cap}(K)$, is defined as the electrostatic capacity of its perpendicular cross section $S(K, \sigma_K)$. Hence, for n larger than three we have

$$\mathrm{cap}(K) = \left[\frac{1}{2}\int_0^\infty \frac{1}{\left[(a_1^2 + t)\ldots(a_{n-1}^2 + t)\right]^{1/2}}\,dt\right]^{-1}, \tag{33}$$

where the a_i's are the semiaxes lengths of K. The function cap: $\Xi_n \to \mathbb{R}$ is orthogonally invariant because the integral in (33) remains unchanged if we permute the order of the a_i's.

Proposition 4.6. *Let K be an ellipsoidal cone in $\mathbb{R}^n$, with $n \geq 4$. Then*

$$\mathfrak{S}_{\rm gm}(K) \leq \left[\frac{\operatorname{cap}(K)}{n-3}\right]^{\frac{1}{n-3}} \leq \mathfrak{S}_{\rm am}(K). \tag{34}$$

The sandwich (34) *becomes a chain of equalities if and only if K is a revolution cone.*

Proposition 4.6 is consistent with the geometric-arithmetic inequality, according to which the geometric mean of positive numbers is always less than or equal to the arithmetic mean.

4.1. Connection with the theory of critical angles

The need of evaluating the maximal angle of a proper cone arises in various fields of applied mathematics, see for instance Peña and Renegar [42], Iusem and Seeger [33, 35], and Clarke et al. [10, 11]. By definition, the maximal angle of $K \in \Pi_n$ is the number

$$\theta_{\max}(K) := \max_{u,v\,\in\, K\cap\mathbb{S}_n} \arccos\langle u, v\rangle. \tag{35}$$

The geometric interpretation of (35) is clear and does not need further explanations. By contrast, defining the minimal angle of a proper cone in a meaningful way is less trivial.

Definition 4.7. Let $K \in \Pi_n$. We say that (u, v) is a *critical pair* of K if u and v are distinct unit vectors in K satisfying

$$v - \langle u, v\rangle u \in K^*\,, \quad u - \langle u, v\rangle v \in K^*. \tag{36}$$

The angle formed by a critical pair is called a critical angle of K. The minimal angle of K, denoted by $\theta_{\min}(K)$, is defined as the smallest critical angle of K.

The criticality conditions mentioned in (36) are necessary optimality conditions for the nonconvex optimization problem (35). Every polyhedral proper cone has finitely many critical angles, but a non-polyhedral proper cone could have infinitely many of them. The proposition stated below is a result of Iusem and Seeger [36, Theorem 5] on critical angles of ellipsoidal cones written in the format (17). As shown in such proposition, computing the critical angles of $\mathcal{E}(W)$ is essentially a matter of computing the eigenvalues of W. The critical pairs of $\mathcal{E}(W)$ are obtained by using the eigenvectors of W.

Proposition 4.8. *Let $W \in \mathbb{P}(n-1)$. The vectors $(z, t) \in \mathbb{R}^n$ and $(w, s) \in \mathbb{R}^n$ form a critical pair of $\mathcal{E}(W)$ if and only if the following conditions hold:*

$w = -z, \quad s = t = \sqrt{1 - \|z\|^2} = \sqrt{\langle z, Wz\rangle}, \quad$ *z is an eigenvector of W.*

In that case, the corresponding critical angle is given by $\theta = 2\arctan\left(1/\sqrt{\mu}\,\right)$, where μ is the eigenvalue of W associated to z.

The critical angles of a proper cone remain unchanged if the cone undergoes an orthogonal transformation. Hence, without loss of generality, we may concentrate on the diagonal case

$$W = \mathrm{Diag}(\mu_1, \ldots, \mu_{n-1}) \tag{37}$$

with diagonal entries arranged in nonincreasing order. In such a case, the semiaxes lengths and critical angles arranged in nondecreasing order are given by

$$a_1 = \frac{1}{\sqrt{\mu_{n-1}}}, \quad \ldots \quad , a_{n-1} = \frac{1}{\sqrt{\mu_1}},$$

$$\theta_1 = 2\arctan\left(\frac{1}{\sqrt{\mu_{n-1}}}\right), \quad \ldots \quad , \theta_{n-1} = 2\arctan\left(\frac{1}{\sqrt{\mu_1}}\right).$$

In both cases we count repetitions. For each $i \in \{1, \ldots, n-1\}$, we say that a_i has multiplicity p if the term μ_i appears exactly p times on the diagonal matrix (37). In the same way we define the multiplicity of θ_i. By analogy with (31) we introduce the vector

$$\theta(K) := (\theta_1(K), \ldots, \theta_{n-1}(K))^\top$$

whose components are the critical angles of K arranged in nondecreasing order. The next proposition compares semiaxes lengths and critical angles in an arbitrary ellipsoidal cone, not necessarily given in the format (17).

Proposition 4.9. *Let K be an ellipsoidal cone in $\mathbb{R}^n$ with semiaxes lengths $a_1 \leq \ldots \leq a_{n-1}$ and critical angles $\theta_1 \leq \ldots \leq \theta_{n-1}$. Then $\theta_i = 2\arctan a_i$ for all $i \in \{1, \ldots, n-1\}$.*

As we can see from Proposition 4.9, each critical angle can be identified with a corresponding semiaxis length, and viceversa. In particular, any statement concerning semiaxes lengths can be rephrased in terms of critical angles. By combining Theorem 4.2 and Proposition 4.9, we see that any orthogonally invariant function $\mathfrak{S}\colon \Xi_n \to \mathbb{R}$ can be represented as a composition $\mathfrak{S} = h \circ \theta$ involving a certain permutation invariant function $h\colon\ ]0,\pi[^{\,n-1} \to \mathbb{R}$. This observation opens the way to alternative measures of size for an ellipsoidal cone. For instance, we may consider the angular average

$$\mathfrak{S}_{\mathrm{ang}}(K) := \frac{1}{n-1}\sum_{i=1}^{n-1} \theta_i(K)$$

or any other reasonable combination of the critical angles of K. We close this section with an interesting application of the extremal angles $\theta_{\min}(K)$ and $\theta_{\max}(K)$. Any proper cone K can be approximated from inside by its inscribed revolution cone $R^\uparrow(K)$ and from outside by its circumscribed revolution cone $R^\downarrow(K)$, i.e.,

$$R^\uparrow(K) \subseteq K \subseteq R^\downarrow(K). \tag{38}$$

By definition, $R^\uparrow(K)$ is the largest revolution cone contained in K and $R^\downarrow(K)$ is the smallest revolution cone enclosing K. Such optimal revolutions cones exist, are unique, and can be computed by using various calculus rules available in the literature, cf. Goffin [19] and Henrion and Seeger [28, 29].

Proposition 4.10. *Let K be an ellipsoidal cone in $\mathbb{R}^n$. Then*

$$\begin{aligned} R^{\uparrow}(K) &= \operatorname{rev}\left(\cos\left[(1/2)\theta_{\min}(K)\right], \sigma_K\right), \\ R^{\downarrow}(K) &= \operatorname{rev}\left(\cos\left[(1/2)\theta_{\max}(K)\right], \sigma_K\right). \end{aligned}$$

In view of Proposition 4.9, the revolution cones $R^{\uparrow}(K)$ and $R^{\downarrow}(K)$ can be expressed also in terms of the extremal semiaxes lengths:

$$\begin{aligned} R^{\uparrow}(K) &= \operatorname{rev}\left(\left(1+[\gamma_{\min}(K)]^2\right)^{-1/2}, \sigma_K\right), \\ R^{\downarrow}(K) &= \operatorname{rev}\left(\left(1+[\gamma_{\max}(K)]^2\right)^{-1/2}, \sigma_K\right). \end{aligned}$$

5. Volume and other geometric properties

5.1. Ball-truncated volume

A natural way of measuring the size of a proper cone K in $\mathbb{R}^n$ is by evaluating its volume. Since a proper cone is an unbounded set, the concept of volume must be defined with care. The first idea that comes to mind is to intersect K with a ball centered at the origin and then to evaluate the n-dimensional Lebesgue measure of that intersection. We get in this way the expression

$$\operatorname{btv}(K) := \operatorname{vol}_n(K \cap \mathbb{B}_n), \tag{39}$$

which is called the ball-truncated volume of K. Up to a multiplicative factor, the ball-truncated volume of K is equal to the solid angle of K. The computation of ball-truncated volumes of proper cones in spaces of dimension beyond three is a theme discussed in Gourion and Seeger [20, 21] and Ribando [43]. A mayor drawback of the concept of ball-truncated volume is that the evaluation of the expression (39) is computationally expensive in general, even for relatively simple cones in small dimensional spaces. In principle we can use Monte-Carlo simulation on the probabilistic formula

$$\frac{\operatorname{btv}(K)}{\operatorname{vol}_n(\mathbb{B}_n)} = \mathbb{P}[\,\mathbf{g} \in K] = \frac{1}{(2\pi)^{n/2}} \int_K e^{-\frac{1}{2}\|x\|^2} dx, \tag{40}$$

where $\mathbf{g} \sim \textsc{Normal}\,(0, I_n)$ is an n-dimensional Gaussian vector. The basic idea of the Monte-Carlo method is to produce a huge sample $\{\mathbf{g}^{(1)}, \mathbf{g}^{(2)}, \ldots, \mathbf{g}^{(N)}\}$ of stochastically independent n-dimensional Gaussian vectors and to count the number of times that the target K is being hit. The following proposition specifies formula (40) to the case of an ellipsoidal cone in pre-Lorentzian format.

Proposition 5.1. *Let $B \in \mathbb{GL}(n)$ and $\mathbf{g} \sim \textsc{Normal}\,(0, I_n)$. Then*

$$\frac{\operatorname{btv}(\Phi(B))}{\operatorname{vol}_n(\mathbb{B}_n)} = \mathbb{P}[\,B\mathbf{g} \in \mathbb{L}_n] = \frac{1}{(2\pi)^{n/2}|\det B|} \int_{\mathbb{L}_n} e^{-\frac{1}{2}\|B^{-1}u\|^2} du. \tag{41}$$

The random vector $B\mathbf{g}$ follows a normal distribution centered at the origin and covariance matrix $BB^{\top}$. We have not found in the literature a close form expression for evaluating the multiple integral in (41), except when $\Phi(B)$ happens to be a revolution cone. As shown in Shannon [57], evaluating the ball-truncated volume of a revolution cone reduces to the numerical computation of a one-dimensional integral over a bounded interval. Recall that the half-aperture angle of a revolution cone is defined as the half of its maximal angle.

Proposition 5.2. (Shannon [57]) *Let K be an n-dimensional revolution cone with half-aperture angle equal to ϑ. Then*

$$\frac{\mathrm{btv}(K)}{\mathrm{vol}_n(\mathbb{B}_n)} = \frac{1}{2\kappa_n}\int_0^{\vartheta}(\sin t)^{n-2}dt$$

with $\kappa_n := \int_0^{\pi/2}(\sin t)^{n-2}dt$.

Since $\mathrm{btv}\colon \Xi_n \to \mathbb{R}$ is orthogonally invariant, the ball-truncated volume of an ellipsoidal cone depends only on its collection of semiaxes lengths. So, without loss of generality, we may concentrate on the computation of

$$g_{\mathrm{btv}}(a) := \mathrm{vol}_n(\mathcal{E}_a \cap \mathbb{B}_n) \tag{42}$$

for $a \in \mathrm{int}(\mathbb{R}^{n-1}_+)$. Handling the particular case (42) remains unfortunately a difficult numerical task. We do not push further the discussion on the exact evaluation of the ball-truncated volume of an ellipsoidal cone, but we provide easily computable lower and upper bounds. The first result in this vein is obtained by using the sandwich (38).

Proposition 5.3. *Let K be an ellipsoidal cone in $\mathbb{R}^n$. Then*

$$\frac{1}{2\kappa_n}\int_0^{(1/2)\theta_{\min}(K)}(\sin t)^{n-2}dt \le \frac{\mathrm{btv}(K)}{\mathrm{vol}_n(\mathbb{B}_n)} \le \frac{1}{2\kappa_n}\int_0^{(1/2)\theta_{\max}(K)}(\sin t)^{n-2}dt.$$

The quality of these bounds depends on whether the extremal angles $\theta_{\min}(K)$ and $\theta_{\max}(K)$ are close or far from each other. We saw in Proposition 4.9 that

$$\begin{aligned}\theta_{\min}(K) &= 2\arctan[\gamma_{\min}(K)],\\ \theta_{\max}(K) &= 2\arctan[\gamma_{\max}(K)].\end{aligned}$$

Hence, the bounds in Proposition 5.3 can also be written in terms of the extremal semiaxes lengths. The next result is rather counter-intuitive: it says that $\mathrm{btv}(K)$ is near zero if $\gamma_{\min}(K)$ is near zero, even if the remaining semiaxes lengths are all huge! This curious phenomenon puts into question mark the usefulness of the concept of ball-truncated volume.

Proposition 5.4. *Let K be an ellipsoidal cone in $\mathbb{R}^n$. Let m be a positive integer smaller than or equal to the multiplicity of $\gamma_{\min}(K)$. Then*

$$\frac{\mathrm{btv}(K)}{\mathrm{vol}_n(\mathbb{B}_n)} \le \frac{1}{2\,\kappa_{m+1}}\int_0^{\arctan[\gamma_{\min}(K)]}(\sin t)^{m-1}dt. \tag{43}$$

In particular,

$$\frac{\mathrm{btv}(K)}{\mathrm{vol}_n(\mathbb{B}_n)} \le \frac{1}{\pi}\arctan[\gamma_{\min}(K)]. \tag{44}$$

Proof. We give a proof that uses interesting probabilistic arguments. Without loss of generality we may assume that $K = \mathcal{E}_a$ with

$$\gamma_{\min}(K) = a_1 = \ldots = a_m \leq a_{m+1} \leq \ldots \leq a_{n-1}.$$

Let $\mathbf{g} \sim \textsc{Normal}\,(0, I_n)$. Then

$$\begin{aligned}\frac{\mathrm{btv}(\mathcal{E}_a)}{\mathrm{vol}_n(\mathbb{B}_n)} \;=\; \mathbb{P}[\,\mathbf{g} \in \mathcal{E}_a] \;&\leq\; \mathbb{P}\left[(1/a_1)\,\left(\mathbf{g}_1^2 + \ldots + \mathbf{g}_m^2\right)^{1/2} \leq \mathbf{g}_n\right]\\ &=\; \mathbb{P}\left[(1+a_1^2)^{-1/2}\,\left(\mathbf{g}_1^2 + \ldots + \mathbf{g}_m^2 + \mathbf{g}_n^2\right)^{1/2} \leq \mathbf{g}_n\right].\end{aligned}$$

The last line corresponds to the probability that a Gaussian vector

$$(\mathbf{g}_1, \ldots, \mathbf{g}_m, \mathbf{g}_n)^\top \sim \textsc{Normal}\,(0, I_{m+1})$$

hits a revolution cone with half-aperture angle

$$\vartheta = \arccos\left[(1+a_1^2)^{-1/2}\right] = \arctan a_1.$$

Shannon's result implies then the announced formula (43). The particular choice $m = 1$ yields formula (44). $\square$

5.2. Canonical volume

The concept of ball-truncated volume has yet another inconvenience: it is badly suited to deal with duality issues. For instance, it is not possible to express $\mathrm{btv}(K^*)$ in terms of $\mathrm{btv}(K)$, even if we restrict the attention to ellipsoidal cones. All in all, the ball truncation strategy does not seem a good option for handling volumetric issues. The half-space truncation strategy is more tractable from a numerical point of view and, in addition, it leads to a concept of volume that enjoys a number of interesting theoretical properties. A half-space truncation of $K \in \Pi_n$ is obtained by choosing a vector $y \in \mathrm{int}(K^*)$ and forming

$$T(K, y) := \{x \in K : \langle y, x\rangle \leq 1\}.$$

In contrast with a cross section of K, the set $T(K, y)$ is a convex body in the underlying space $\mathbb{R}^n$. The canonical volume of K is defined as the positive number

$$\mathrm{Vol}(K) := \min_{\substack{y\in \mathrm{int}(K^*)\\ \|y\|=1}} \mathrm{vol}_n\left[T(K, y)\right]. \tag{45}$$

The coefficient (45) was introduced recently in Seeger and Torki [52, 53] under the name of least partial volume of K, but the function

$$y \in \mathrm{int}(K^*) \mapsto v_K(y) := \mathrm{vol}_n\left[T(K, y)\right] \tag{46}$$

has been in use for a long time. Gigena [18] refers to $v_K \colon \mathrm{int}(K^*) \to \mathbb{R}$ as the volume function associated to the proper cone K. By an elementary integration argument we see that (46) can be written in the form

$$v_K(y) = \frac{\mathrm{vol}_{n-1}\left[S(K, y)\right]}{n\,\|y\|}. \tag{47}$$

So, we can forget about half-space truncations and concentrate on cross sections.

Example 5.5. If $K \in \Pi_n$ is axially symmetric, then the minimum in (45) is attained at the axial symmetry center σ_K and we get

$$\mathrm{Vol}(K) \;=\; \frac{\mathrm{vol}_{n-1}\,[S(K,\sigma_K)]}{n}\,.$$

In particular, the canonical volume of the Lorentz cone is

$$\mathrm{Vol}(\mathbb{L}_n) = \frac{\mathrm{vol}_{n-1}\,(\mathbb{B}_{n-1})}{n}\,. \qquad \square$$

Güler [26] proves that the volume function (46) admits the integral representation

$$v_K(y) = \frac{1}{n!}\int_K e^{-\langle y,x\rangle}dx.$$

Such integral is easy to evaluate when K is an ellipsoidal cone given in Lorentzian format.

Proposition 5.6. *Let $A \in \mathbb{GL}(n)$. Then, for all y in the interior of $[A(\mathbb{L}_n)]^*$, we have*

$$v_{A(\mathbb{L}_n)}(y) = \left|\langle y, AJ_nA^\top y\rangle\right|^{-n/2} |\mathrm{det}A|\,\mathrm{Vol}(\mathbb{L}_n) \tag{48}$$

and, in particular,

$$\mathrm{vol}_{n-1}\,[S(A(\mathbb{L}_n),y)] = n\|y\|\,\left|\langle y, AJ_nA^\top y\rangle\right|^{-n/2} |\mathrm{det}A|\,\mathrm{Vol}(\mathbb{L}_n). \tag{49}$$

Proof. The classical formula of change of variables in an integral leads to

$$v_{A(\mathbb{L}_n)}(y) = |\mathrm{det}A|\,v_{\mathbb{L}_n}(A^\top y).$$

It suffices then to recall the well known formula

$$v_{\mathbb{L}_n}(u) = |\langle u, J_n u\rangle|^{-n/2}\,\mathrm{Vol}(\mathbb{L}_n)$$

of the volume function associated to $\mathbb{L}_n$. Formula (49) follows from (47)–(48). $\square$

The canonical volume of $A(\mathbb{L}_n)$ is obtained by evaluating (48) at $y = \sigma_{A(\mathbb{L}_n)}$. We recover in this way the formula

$$\mathrm{Vol}(A(\mathbb{L}_n)) = |\mathrm{det}A|\,|\lambda_{\min}(A^{-\top}J_nA^{-1})|^{n/2}\,\mathrm{Vol}(\mathbb{L}_n) \tag{50}$$

established in Seeger and Torki [52, Proposition 4.6]. How to compute the canonical volume of an ellipsoidal cone when a Lorentzian representation is not readily available? As shown in [52, Proposition 3.2], the function $\mathrm{Vol}\colon \Pi_n \to \mathbb{R}$ is orthogonally invariant. Hence, the canonical volume of an ellipsoidal cone depends only on its collection of semiaxes lengths.

Proposition 5.7. (Seeger and Torki [54]) *Let K be an ellipsoidal cone in $\mathbb{R}^n$. Then*

$$\mathrm{Vol}(K) = \mathrm{Vol}(\mathbb{L}_n)\,\Pi_{j=1}^{n-1}\,\gamma_j(K)\,. \tag{51}$$

Except for a scaling operation, formula (51) is consistent with the definition of volume of an ellipsoidal cone proposed by Lemmon [39, Definition 2]. In fact, Lemmon omits the multiplicative factor $\mathrm{Vol}(\mathbb{L}_n)$ and takes the square root of the product of all semiaxes lengths. Formula (51) takes various particular forms depending on the way the ellipsoidal cone K is parametrized. Besides the example of a Lorentzian representation considered in (50), other examples are displayed in the next proposition.

Proposition 5.8. *The canonical volume of an ellipsoidal cone can be computed as follows:*

(a) *If* $W \in \mathbb{P}(n-1)$, *then* $\mathrm{Vol}(\mathcal{E}(W)) = (\det W)^{-1/2}\, \mathrm{Vol}(\mathbb{L}_n)$.

(b) *If* $B \in \mathbb{GL}(n)$, *then* $\mathrm{Vol}(\Phi(B)) = |\det B|^{-1} |\lambda_{\min}(B^\sharp)|^{n/2}\, \mathrm{Vol}(\mathbb{L}_n)$.

(c) *If* (R, q) *is as in* (12), *then* $\mathrm{Vol}(\Gamma(R, q)) = [\det_+(R)]^{-1/2}\, \mathrm{Vol}(\mathbb{L}_n)$, *with* $\det_+(R)$ *denoting the product of the positive eigenvalues of* R.

(d) *If* (G, c) *is as in* (21), *then* $\mathrm{Vol}(\Theta(G, c)) = |\det G^c|^{1/2} |\lambda_{\min}(G^c)|^{-n/2}\, \mathrm{Vol}(\mathbb{L}_n)$. *As in Proposition* 3.13, *we use here the notation with* $G^c := GG^\top - cc^\top$.

The combination of Proposition 4.4 and Proposition 5.7 yields the following remarkable duality formula.

Proposition 5.9. (Seeger and Torki [52]) *Let K be an ellipsoidal cone in $\mathbb{R}^n$. Then*

$$\mathrm{Vol}(K)\,\mathrm{Vol}(K^*) = [\mathrm{Vol}(\mathbb{L}_n)]^2 . \tag{52}$$

What is even more striking is that (52) serves in fact to characterize the ellipsoidal cones. The next theorem, a more complete version of Proposition 5.9, can be seen as a conic formulation of a celebrated result of Blaschke [6] and Santaló [45] relating the volume of an origin-symmetric convex body Ω in $\mathbb{R}^n$ and the volume of its polar convex body

$$\Omega^\circ := \{y \in \mathbb{R}^n : \langle y, x\rangle \le 1 \text{ for all } x \in \Omega\}.$$

Ellipsoids enjoy some extremality properties within the class of convex bodies. In the same vein, ellipsoidal cones enjoy some extremality properties within the class of proper cones.

Theorem 5.10. (Seeger and Torki [53]) *Let K be a proper cone in $\mathbb{R}^n$. Then*

$$\mathrm{Vol}(K)\,\mathrm{Vol}(K^*) \le [\mathrm{Vol}(\mathbb{L}_n)]^2 , \tag{53}$$

with equality if and only if K is an ellipsoidal cone.

The so-called Mahler volume of an origin-symmetric convex body Ω in $\mathbb{R}^n$ is defined as the product of $\mathrm{vol}_n(\Omega)$ and $\mathrm{vol}_n(\Omega^\circ)$. By analogy with this definition we say that

$$\mathrm{MCV}(K) := \mathrm{Vol}(K)\,\mathrm{Vol}(K^*)$$

is the Mahler canonical volume of a proper cone K. Observe that the function $\mathrm{MCV}\colon \Pi_n \to \mathbb{R}$ is invariant under orthogonal transformations, but not under general invertible linear transformations. Theorem 5.10 asserts that the set of maximizers of $\mathrm{MCV}\colon \Pi_n \to \mathbb{R}$ is exactly the set of ellipsoidal cones in $\mathbb{R}^n$.

5.3. Pointedness and solidity coefficients

The role of a solidity coefficient is to measure the degree of solidity of a proper cone K, cf. [21, 32]. The most popular example of solidity coefficient is

$$\mathfrak{s}(K) := \max_{x \in K \cap \mathbb{S}_n} \operatorname{dist}(x, \operatorname{bd}(K)) \tag{54}$$

$$= \max\{r : x + r\mathbb{B}_n \subseteq K, \, \|x\| = 1\}, \tag{55}$$

where $\operatorname{dist}(x, \Omega)$ stands for the distance from a point x to a set Ω. From (55) we see that $\mathfrak{s}(K)$ corresponds to the radius of the largest ball contained in K and centered at a unit vector. Such a ball exists and it is unique: its center is the unique solution to the maximization problem (54) and its radius is equal to $\mathfrak{s}(K)$. A proper cone has low degree of solidity if $\mathfrak{s}(K)$ is near 0 and high degree of solidity if $\mathfrak{s}(K)$ is near 1. Calculus rules for computing $\mathfrak{s}(K)$ have been developed in a number of papers, cf. [3, 29, 32, 47]. Similarly, the role of a pointedness coefficient is to measure the degree of pointedness of a proper cone K. The most frequently mentioned example of pointedness coefficient is

$$\mathfrak{p}(K) := \min_{x \in \operatorname{co}(K \cap \mathbb{S}_n)} \|x\| , \tag{56}$$

where "co" stands for the convex hull operation. A proper cone has low degree of pointedness if $\mathfrak{p}(K)$ is near 0 and high degree of pointedness if $\mathfrak{p}(K)$ is near 1. As shown in Iusem and Seeger [32, Proposition 6.3] and Freund and Vera [17, Proposition 2.1], the solidity coefficient (54) and the pointedness coefficient (56) are related through the duality formula

$$\mathfrak{p}(K) = \mathfrak{s}(K^*).$$

It is also known that $\mathfrak{s}\colon \Pi_n \to \mathbb{R}$ and $\mathfrak{p}\colon \Pi_n \to \mathbb{R}$ are orthogonally invariant functions. Hence, for the case of an ellipsoidal cone K, the coefficients $\mathfrak{s}(K)$ and $\mathfrak{p}(K)$ are entirely determined by the semiaxes lengths of K.

Proposition 5.11. *Let K be an ellipsoidal cone. Then*

$$\mathfrak{s}(K) = \frac{\gamma_{\min}(K)}{\sqrt{1 + [\gamma_{\min}(K)]^2}} \qquad \text{and} \qquad \mathfrak{p}(K) = \frac{1}{\sqrt{1 + [\gamma_{\max}(K)]^2}}.$$

Note that $\mathfrak{s}(K)$ depends only on $\gamma_{\min}(K)$ and $\mathfrak{p}(K)$ depends only on $\gamma_{\max}(K)$. These formulas take various forms depending on the way the ellipsoidal cone K is parametrized. Some examples are displayed in Table 5.1. Since $\mathfrak{p}(K) = \mathfrak{s}(K^*)$, we give only the corresponding formulas for the solidity coefficient.

6. Miscellaneous results involving ellipsoidal cones

6.1. Projection onto an ellipsoidal cone

The projection of a point w onto a proper cone K is denoted by $\operatorname{Proj}(w, K)$ and it is defined as the unique solution to the distance minimization problem

$$\operatorname{dist}(w, K) := \min_{x \in K} \|w - x\|.$$

Data	K	$\mathfrak{s}(K)$	Reference
$W \in \mathbb{P}(n-1)$	$\mathcal{E}(W)$	$[1+\lambda_{\max}(W)]^{-1/2}$	[32, Prop. 6.4]
$B \in \mathbb{GL}(n)$	$\Phi(B)$	$\left[\frac{-\lambda_{\min}(B^\sharp)}{\lambda_{\max}(B^\sharp)-\lambda_{\min}(B^\sharp)}\right]^{1/2}$	–
(R,q) as in (11) $R^q := R - qq^\top$	$\Gamma(R,q)$	$\left[\frac{-\lambda_{\min}(R^q)}{\lambda_{\max}(R^q)-\lambda_{\min}(R^q)}\right]^{1/2}$	[3, p. 1075]

Table 5.1: Solidity coefficient of an ellipsoidal cone.

Projecting onto a proper cone is a difficult numerical task in general. However, projecting onto an ellipsoidal cone is fairly easy if its semiaxial representation is already available. Since

$$\mathrm{Proj}[w, U(\mathcal{E}_a)] = U\,\mathrm{Proj}(U^\top w, \mathcal{E}_a)$$

for any orthogonal matrix U, everything boils down to project a point v onto the particular ellipsoidal cone $\mathcal{E}_a$. Such operation can be carried out as explained in the next proposition. For avoiding trivialities we assume that v is neither in $\mathcal{E}_a$ nor in the polar cone of $\mathcal{E}_a$. The polar cone of $\mathcal{E}_a$ is denoted by $\mathcal{E}_a^\circ$ and it is defined as the negative of the dual cone $\mathcal{E}_a^*$.

Proposition 6.1. *Let $a \in \mathrm{int}(\mathbb{R}^{n-1}_+)$. Suppose that $v \in \mathbb{R}^n$ is not in $\mathcal{E}_a \cup \mathcal{E}_a^\circ$. Then the components of the projection*

$$\bar{x} := \mathrm{Proj}(v, \mathcal{E}_a) = \mathrm{argmin}_{x\in\mathcal{E}_a}\|v - x\|$$

are given by

$$\bar{x}_i = \frac{a_i^2 v_i}{a_i^2 + \bar{\mu}} \text{ for } i \in \{1, ..., n-1\}, \quad \bar{x}_n = \left[\sum_{i=1}^{n-1}\left(\frac{a_i v_i}{a_i^2 + \bar{\mu}}\right)^2\right]^{1/2},$$

where $\bar{\mu}$ is the unique root of the function $h\colon \mathbb{R}_+ \to \mathbb{R}$ defined by

$$h(\mu) := (1-\mu)\left[\sum_{i=1}^{n-1}\left(\frac{a_i v_i}{a_i^2 + \mu}\right)^2\right]^{1/2} - \ v_n\,.$$

Proof. It suffices to work out the optimality conditions for the optimization problem

$$\mathrm{minimize}_{x\in\mathbb{R}^n} \sum_{i=1}^{n}(v_i - x_i)^2$$

$$\sum_{i=1}^{n-1}(x_i/a_i)^2 - x_n^2 \leq 0 \tag{57}$$

$$-x_n \leq 0. \tag{58}$$

The nonnegative variable μ plays the role of a Karush-Kuhn-Tucker multiplier associated to the constraint (57). Evaluated at the optimum, the inequality (58) is strict because $v \notin \mathcal{E}_a^\circ$, whereas (57) holds as an equality because $v \notin \mathcal{E}_a$. Note that the function h is continuous and decreasing. Since it decreases

from a positive term $$h(0) = \left[\sum_{i=1}^{n-1} (v_i/a_i)^2\right]^{1/2} - v_n$$

to a negative term $$\lim_{\mu\to\infty} h(\mu) = -\left[\sum_{i=1}^{n-1} (a_i v_i)^2\right]^{1/2} - v_n\,,$$

it follows that h vanishes at a unique positive real $\bar{\mu}$. □

To the best of our knowledge, Proposition 6.1 is a new result. The numerical computation of the root of h offers no difficulty. A standard bisection algorithm does the job. Parenthetically, Proposition 6.1 yields in particular an explicit formula for the projection onto a revolution cone.

6.2. Copositivity and exponential nonnegativity

A symmetric matrix is copositive if its associated quadratic form is nonnegative on the nonnegative orthant. More generally, a symmetric matrix S is copositive on a closed convex cone K if the quadratic form $x \mapsto \langle x, Sx\rangle$ is nonnegative on K. In general, checking copositivity on ellipsoidal cones is much simpler than checking copositivity on polyhedral cones. It is important however to parametrize in a convenient way the ellipsoidal cone under consideration. Below we consider four different options.

Proposition 6.2. *For $S \in \mathrm{Sym}(n)$ and $A \in \mathbb{GL}(n)$, the following conditions are equivalent:*

(a) *S is copositive on $A(\mathbb{L}_n)$.*

(b) *There exists a nonnegative μ such that $A^\top SA + \mu J_n \succeq 0$.*

Proposition 6.2 is stated in Loewy and Schneider [40] for the case $A = I_n$, but the extension to a general invertible matrix A is straightforward. The change of variables $B = A^{-1}$ leads to the following reformulation of Proposition 6.2.

Proposition 6.3. *For $S \in \mathrm{Sym}(n)$ and $B \in \mathbb{GL}(n)$, the following conditions are equivalent:*

(a) *S is copositive on $\Phi(B)$.*

(b) *There exists a nonnegative μ such that $S + \mu B^\sharp \succeq 0$.*

The next proposition considers the case of an ellipsoidal cone in Stern-Wolkowicz format. The equivalence (a) ⇔ (b) in Proposition 6.4 is obvious because any quadratic form is an even function. The equivalence (b) ⇔ (c) is a consequence of the S-lemma of Yacubovich.

Proposition 6.4. *Let $S \in \mathrm{Sym}(n)$ and (C, b) be as in Theorem* 3.3. *Then the following conditions are equivalent:*

(a) *S is copositive on* $\mathrm{sw}(C, b)$.

(b) $\langle x, Sx \rangle \geq 0$ *whenever* $\langle x, Cx \rangle \leq 0$.

(c) *There exists a nonnegative μ such that $S + \mu C \succeq 0$.*

The next result considers the case of an ellipsoidal cone in Bishop-Phelps format. Proposition 6.5 has been proven recently in Tian et al. [61, Theorem 2.12], but in fact it follows directly from Proposition 6.4.

Proposition 6.5. *Let $S \in \mathrm{Sym}(n)$ and (R, q) be as in* (11). *Then the following conditions are equivalent:*

(a) *S is copositive on $\Gamma(R, q)$.*

(b) *There exists a nonnegative μ such that $S + \mu(R - qq^\top) \succeq 0$.*

Copositivity is a notion of positivity specially tailored for symmetric matrices. Exponential nonnegativity is a notion of positivity that applies to general square matrices and that is motivated by the analysis of linear differential systems. A matrix $M \in \mathbb{M}_n$ is called exponentially nonnegative on $K \in \Pi_n$ if

$$e^{tM}(K) \subseteq K \quad \text{for all } t \geq 0. \tag{59}$$

The invariance condition (59) means that for an arbitrary starting point $x \in K$, the solution $\varphi(t) = e^{tM}x$ to the linear Cauchy problem

$$\dot{\varphi}(t) = M\varphi(t), \quad \varphi(0) = x \tag{60}$$

remains in K for all future time. The concept of exponential nonnegativity has been widely studied in the literature. The book of Berman et al. [4] is a good introduction to this topic. It has been shown in Schneider and Vidyasagar [46] that M is exponentially nonnegative on K if and only if

$$K \ni x \perp y \in K^* \quad \text{implies} \quad \langle y, Mx \rangle \geq 0, \tag{61}$$

where $\perp$ stands for orthogonality. By using (61) we see that M is exponentially nonnegative on $\mathbb{R}^n_+$ if and only if the off-diagonal entries of M are nonnegative. More generally, M is exponentially nonnegative on a simplicial cone $A(\mathbb{R}^n_+)$ if and only if the off-diagonal entries of $A^{-1}MA$ are nonnegative. Exponential nonnegativity on ellipsoidal cones is characterized as follows.

Proposition 6.6. (Stern and Wolkowicz [59]) *Let $A \in \mathbb{GL}(n)$. Then the following conditions are equivalent:*

(a) *M is exponentially nonnegative on $A(\mathbb{L}_n)$.*

(b) *$A^{-1}MA$ is exponentially nonnegative on $\mathbb{L}_n$.*

(c) *There exists $\mu \geq 0$ such that $\mu J_n \succeq J_n A^{-1}MA + (J_n A^{-1}MA)^\top$.*

6.3. Lyapunov rank of an ellipsoidal cone

The concept of Lyapunov-like matrix is related to exponential nonnegativity but its motivation comes from a different source. Instead of a differential system like

(60), we now consider a complementarity problem

$$y = F(x), \quad K \ni x \perp y \in K^* \tag{62}$$

involving a proper cone K in $\mathbb{R}^n$ and a given function $F\colon \mathbb{R}^n \to \mathbb{R}^n$. For theoretical and algorithmic reasons it is important to have a good understanding of the structure of the complementarity set

$$\mathcal{C}_K := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : K \ni x \perp y \in K^*\}$$

associated to K. Following Gowda and Tao [22], we say that $M \in \mathbb{M}_n$ is a Lyapunov-like matrix on K if

$$K \ni x \perp y \in K^* \quad \text{implies} \quad \langle y, Mx\rangle = 0. \tag{63}$$

Conditions (63) and (61) are not quite the same: any Lyapunov-like matrix on K is exponentially nonnegative on K, but not conversely. The set of Lyapunov-like matrices on K is a linear subspace of $\mathbb{M}_n$. The dimension of that linear subspace is called the Lyapunov rank of K and it is denoted by $\beta(K)$. In short,

$$\beta(K) := \dim \{M \in \mathbb{M}_n : \langle y, Mx\rangle = 0 \text{ for all } (x,y) \in \mathcal{C}_K\}. \tag{64}$$

The term (64) was originally introduced in Rudolf et al. [44] under the name of bilinearity rank of K. Such parameter plays a useful role in the analysis of the linear complementarity problem (62). Calculus rules for computing (64) have been developed in Gowda and Tao [22], Gowda and Trott [23], and Rudolf et al. [44].

Proposition 6.7. *Let K be an ellipsoidal cone in $\mathbb{R}^n$. Then*

$$\beta(K) = \frac{n(n-1)}{2} + 1. \tag{65}$$

As shown in Gowda and Tao [22, Section 4], the Lyapunov rank of $\mathbb{L}_n$ is equal to the term on the right-hand side of (65). So, for proving Proposition 6.7 it suffices to recall that the Lyapunov rank of a proper cone is stable under invertible linear transformations. The next proposition characterizes the Lyapunov-like matrices on an ellipsoidal cone given in Lorentzian format. The notation

$$\mathrm{Lie}\,[\mathrm{Aut}(\mathbb{L}_n)] := \{N \in \mathbb{M}_n : e^{tN} \in \mathrm{Aut}(\mathbb{L}_n) \text{ for all } t \in \mathbb{R}\}$$

stands for the Lie algebra on $\mathrm{Aut}(\mathbb{L}_n)$.

Proposition 6.8. *Let $A \in \mathbb{GL}(n)$. Then the following conditions are equivalent:*

(a) *M is Lyapunov-like on $A(\mathbb{L}_n)$.*

(b) *$A^{-1}MA$ is Lyapunov-like on $\mathbb{L}_n$.*

(c) *$A^{-1}MA \in \mathrm{Lie}\,[\mathrm{Aut}(\mathbb{L}_n)]$.*

The equivalence (b) $\Leftrightarrow$ (c) in Proposition 6.8 is a particular instance of [22, Theorem 3]. Observe that formula (65) is consistent with the well known fact that

$$\dim\left(\mathrm{Lie}\,[\mathrm{Aut}(\mathbb{L}_n)]\right) = \frac{n(n-1)}{2} + 1.$$

Let us come back to the complementarity problem (62). If we wish to solve this problem for a given ellipsoidal cone K, then the simplest way of proceeding is to represent K in a Lorentzian format $A(\mathbb{L}_n)$ for a suitable $A \in \mathbb{GL}(n)$. In such a case, (62) can be converted into a complementarity problem

$$v = A^\top F(Au), \quad \mathbb{L}_n \ni u \perp v \in \mathbb{L}_n \tag{66}$$

relative to the Lorentz cone. The model (66) has been studied by a number of authors when F is an affine function or, more generally, a smooth function.

6.4. Inscribed and circumscribed ellipsoidal cones

The classical problem of finding an ellipsoidal approximation of a convex body Ω admits two variants: the inner approximation problem where we seek the maximum volume ellipsoid contained in Ω and the outer approximation problem where we seek the minimum volume ellipsoid enclosing Ω. By solving this pair of optimization problems we get

$$e^{\uparrow}(\Omega) \subseteq \Omega \subseteq e^{\downarrow}(\Omega),$$

where $e^{\uparrow}(\Omega)$ is the ellipsoid inscribed in Ω and $e^{\downarrow}(\Omega)$ is the ellipsoid circumscribed around Ω. The existence and uniqueness of such optimal ellipsoids is guaranteed by the celebrated Loewner-John theorem. Some authors refer to the inscribed ellipsoid as the John ellipsoid of Ω and the circumscribed ellipsoid as the Loewner ellipsoid of Ω. The technique of ellipsoidal approximation has been extended in Seeger and Torki [54] from convex bodies to proper cones. The conic version of the Loewner-John theorem asserts that a proper cone K in $\mathbb{R}^n$ can be sandwiched between a pair of optimal ellipsoidal cones:

$$E^{\uparrow}(K) \subseteq K \subseteq E^{\downarrow}(K).$$

The inscribed ellipsoidal cone $E^{\uparrow}(K)$ is the unique solution to the maximization problem

$$\begin{cases} \text{maximize } \ \mathrm{Vol}(Q) \\ Q \in \Xi_n \\ Q \subseteq K, \end{cases} \tag{67}$$

whereas the circumscribed ellipsoidal cone $E^{\downarrow}(K)$ is the unique solution to the minimization problem

$$\begin{cases} \text{minimize } \ \mathrm{Vol}(Q) \\ Q \in \Xi_n \\ Q \supseteq K. \end{cases} \tag{68}$$

The optimization problems (67) and (68) are not dual to each other in the sense of convex optimization theory, but the inscribed and circumscribed ellipsoidal cones obey to the duality formulas stated in the next proposition.

Proposition 6.9. (Seeger and Torki [54]) *Let $K \in \Pi_n$. Then*

$$E^{\uparrow}(K) = \left[E^{\downarrow}(K^*)\right]^*, \quad E^{\downarrow}(K) = \left[E^{\uparrow}(K^*)\right]^*.$$

So, we just need to compute one of the two optimal ellipsoidal cones and get the other by dualization.

6.4.1. Characterizing the inscribed ellipsoidal cone

The next proposition gives a necessary and sufficient condition for an ellipsoidal cone in the Seeger-Torki format to be the inscribed ellipsoidal cone in a given proper cone.

Proposition 6.10. (Seeger and Torki [54]) *Let $K \in \Pi_n$ and (R, q) be as in* (12)*. Then $\Gamma(R, q) = E^{\uparrow}(K)$ if and only if the following two conditions hold:*

(i) $\Gamma(R, q) \subseteq K$.

(ii) *There exist $m \in \mathbb{N}$, positive reals $t_1, \ldots, t_m$, and vectors $w_1, \ldots, w_m \in \mathrm{bd}(K)$ such that*

$$\langle w_k, Rw_k\rangle^{1/2} = \langle q, w_k\rangle = 1, \qquad \sum_{k=1}^{m} t_k\,(w_k - q) = 0,$$
$$\sum_{k=1}^{m} t_k\,(w_k - q)\,(w_k - q)^{\top} = R^{\dagger}.$$

The w_k's in Proposition 6.10 have an interesting geometric interpretation. We refer to them as inner contact points of K because the inscribed ellipsoidal cone $E^{\uparrow}(K)$ touches the boundary of K at these points. By relying on Proposition 6.10 it is possible for instance to derive an explicit formula for $E^{\uparrow}(K)$ when K is a simplicial cone, cf. [55].

6.4.2. Numerical computation of the circumscribed ellipsoidal cone

Seeger and Vidal-Nuñez [56] address the problem of computing the ellipsoidal cone circumscribed around a polyhedral cone

$$H(\mathbb{R}^p_+) := \{H\alpha : \alpha \in \mathbb{R}^p_+\},$$

where $H := [h_1, \ldots, h_p]$ is a matrix whose columns are nonzero vectors of $\mathbb{R}^n$. In such work, the minimization problem (68) is formulated with the help of a semi-axial parametrization and also with the help of a pre-Lorentzian parametrization. The first choice leads to a problem with the following structure:

$$\left\{\begin{array}{l} \text{minimize } \Pi_{i=1}^{n-1} a_i \\ a \in \mathrm{int}(\mathbb{R}^{n-1}_+) \\ U^{\top}U = I_n \\ U^{\top}h_j \in \mathcal{E}_a \ \text{ for } j = 1, \ldots, p. \end{array}\right. \tag{69}$$

The second choice leads to a problem of a seemingly simpler form:

$$\left\{\begin{array}{l} \text{minimize } \ \lambda_{\max}(B^{\sharp}\,) \\ \det B = 1 \\ Bh_j \in \mathbb{L}_n \ \text{ for } j = 1, \ldots, p. \end{array}\right. \tag{70}$$

Without loss of generality, the matrix B in the pre-Lorentzian representation can be chosen so that $\det B = 1$. Such a normalization condition simplifies the expression of $\mathrm{Vol}(\Phi(B))$, cf. Proposition 5.8.

Proposition 6.11. (Seeger and Vidal-Nuñez [56]) *Let* $K := H(\mathbb{R}^p_+)$ *be a proper polyhedral cone. Then*

(a) $E^{\downarrow}(K) = U_*(E_{a_*})$, *where* (U_*, a_*) *is any solution to* (69).

(b) $E^{\downarrow}(K) = \Phi(B_*)$, *where* B_* *is any solution to* (70).

Problems (69) and (70) were solved numerically in [56] for various sizes of n and p. The pre-Lorentzian approach performed always better than the semiaxial approach, be it in terms of number of iterations or in terms of time needed to reach a solution.

References

[1] A. M. Aliliuko, O. H. Mazko: *Invariant cones and stability of linear dynamical systems*, Ukranian Math. J. 58 (2006) 1635–1655.

[2] D. Amelunxen, M. Lotz, M. B. Mccoy, J. A. Tropp: *Living on the edge: Phase transitions in convex programs with random data*, Inform. Inference 3 (2015) 224–294.

[3] A. Belloni, R. M. Freund: *On the second-order feasibility cone: primal-dual representation and efficient projection*, SIAM J. Optim. 19 (2008) 1073–1092.

[4] A. Berman, M. Neuman, R. J. Stern: *Nonnegative Matrices in Dynamics Systems*, Wiley, New York (1989).

[5] R. Bhattacharya, J. Fung, R. M. Murray, A. Tiwari: *Ellipsoidal cones and rendezvous of multiple agents*, in: 43rd IEEE Conf. on Decision and Control, Nassau, Bahamas (2004).

[6] W. Blaschke: *Vorlesungen über Differentialgeometrie II*, Springer, Berlin (1923).

[7] G. C. Calafiori: *Approximation of n-dimensional data using spherical and ellipsoidal primitives*, IEEE Trans. Systems, Man, Cybernetics 32 (2002) 269–278.

[8] B. C. Carlson: *Some inequalities for hypergeometric functions*, Proc. Amer. Math. Soc. 17 (1966) 32–39.

[9] C. B. Chua: *The primal-dual second-order cone approximations algorithm for symmetric cone programming*, Found. Comput. Math. 7 (2007) 271–302.

[10] F. H. Clarke, Y. S. Ledyaev, R. J. Stern: *Complements, approximations, smoothings and invariance properties*, J. Convex Analysis 4 (1997) 189–219.

[11] F. H. Clarke, Y. S. Ledyaev, R. J. Stern: *Invariance, monotonicity, and applications*, in: *Nonlinear Analysis, Differential Equations and Control*, NATO Sci. Ser. C Math. Phys. Sci. 528, Kluwer Academic Publishers, Dordrecht (1999) 207–305.

[12] G. Eichfelder, J. Povh: *On the set-semidefinite representation of nonconvex quadratic programs with cone constraints*, Croat. Oper. Res. Rev. CRORR 1 (2010) 26–39.

[13] F. Facchinei, J. S. Pang: *Finite Dimensional Variational Inequalities I*, Springer, New York (2003).

[14] J. Faraut, A. Korányi: *Analysis on Symmetric Cones*, Oxford University Press, New York (1994).

[15] M. Fiedler, E. Haynsworth: *Cones which are topheavy with respect to a norm*, Linear Multilinear Algebra 1 (1973) 203–211.

[16] D. D. Fischer: *Minimum ellipsoids*, Math. Comput. 18 (1964) 669–673.

[17] R. M. Freund, J. R. Vera: *Condition-based complexity of convex optimization in conic linear form via the ellipsoid algorithm*, SIAM J. Optim. 10 (1999) 155–176.

[18] S. Gigena: *Integral invariants of convex cones*, J. Differential Geometry 13 (1978) 191–222.

[19] J.-L. Goffin: *The relaxation method for solving systems of linear inequalities*, Math. Oper. Res. 5 (1980) 388–414.

[20] D. Gourion, A. Seeger: *Deterministic and stochastic methods for computing volumetric moduli of convex cones*, Comput. Appl. Math. 29 (2010) 215–246.

[21] D. Gourion, A. Seeger: *Solidity indices for convex cones*, Positivity 16 (2012) 685–705.

[22] M. S. Gowda, J. Tao: *On the bilinearity rank of a proper cone and Lyapunov-like transformations*, Math. Program. 147 (2014) 155–170.

[23] M. S. Gowda, D. Trott: *On the irreducibility, Lyapunov rank, and automorphisms of special Bishop-Phelps cones*, J. Math. Anal. Appl. 419 (2014) 172–184.

[24] P. M. Gruber, T. Ódor: *Ellipsoids are the most symmetric convex bodies*, Arch. Math. (Basel) 73 (1999) 394–400.

[25] C. Grussler, A. Rantzig: *Modified balanced truncation preserving ellipsoidal cone-invariance*, in: 53rd IEEE Conference on Decision and Control, Los Angeles (2014) 2365–2370.

[26] O. Güler: *Barrier functions in interior point methods*, Math. Oper. Res. 21 (1996) 860–885.

[27] X. L. Guo, Z. B. Deng, S. C. Fang, Z. B. Wang, W. X. Xing: *Quadratic optimization over a second-order cone with linear equality constraints*, J. Oper. Res. Soc. China 2 (2014) 17–38.

[28] R. Henrion, A. Seeger: *On properties of different notions of centers for convex cones*, Set-Valued Variat. Analysis 18 (2010) 205–231.

[29] R. Henrion, A. Seeger: *Inradius and circumradius of various convex cones arising in applications*, Set-Valued Variat. Analysis 18 (2010) 483–511.

[30] R. Henrion, A. Seeger: *Condition number and eccentricity of a closed convex cone*, Math. Scandinavica 109 (2011) 285–308.

[31] M. Ito, B. F. Lourenço: *The p-cones in dimension $n \geq 3$ are not homogeneous when $p \neq 2$*, Linear Algebra Appl. 533 (2017) 326–335.

[32] A. Iusem, A. Seeger: *Axiomatization of the index of pointedness for closed convex cones*, Comput. Appl. Math. 24 (2005) 245–283.

[33] A. Iusem, A. Seeger: *On pairs of vectors achieving the maximal angle of a convex cone*, Math. Progr. 104 (2005) 501–523.

[34] A. Iusem, A. Seeger: *Measuring the degree of pointedness of a closed convex cone: a metric approach*, Math. Nachrichten 279 (2006) 599–618.

[35] A. Iusem, A. Seeger: *Angular analysis of two classes of non-polyhedral convex cones: the point of view of optimization theory*, Comput. Appl. Math. 26 (2007) 191–214.

[36] A. Iusem, A. Seeger: *Searching for critical angles in a convex cone*, Math. Progr. 120 (2009) 3–25.

[37] J. Jahn: *Bishop-Phelps cones in optimization*, Int. J. Optim. Theory Methods Appl. 1 (2009) 123–139.

[38] J. Jerónimo-Castro, T. B. Mcallister: *Two characterizations of ellipsoidal cones*, J. Convex Analysis 20 (2013) 1181–1187.

[39] M. Lemmon: *Inductive inference of invariant subspaces*, in: Proc. American Control Conf., San Francisco (1993) 1219–1223.

[40] R. Loewy, H. Schneider: *Positive operators on the n-dimensional ice cream cone*, J. Math. Anal. Appl. 49 (1975) 375–392.

[41] S. Olovjanischnikoff: *Über eine kennzeichnende Eigenschaft des Ellipsoides*, Uchenye Zapiski Leningrad State Univ., Math. Series 83 (1941) 114–128.

[42] J. Peña, J. Renegar: *Computing approximate solutions for convex conic systems of constraints*, Math. Program. 87 (2000) 351–383.

[43] J. Ribando: *Measuring solid angles beyond dimension three*, Discrete Comput. Geom. 36 (2006) 479–487.

[44] G. Rudolf, N. Noyan, D. Papp, F. Alizadeh: *Bilinear optimality constraints for the cone of positive polynomials*, Math. Program. 129 (2011) 5–31.

[45] L. A. Santaló: *Un invariante afin para los cuerpos convexos del espacio de n dimensiones*, Portugal Math. 8 (1949) 155–161.

[46] H. Schneider, M. Vidyasagar: *Cross-positive matrices*, SIAM J. Numer. Anal. 7 (1970) 508–519.

[47] A. Seeger: *Epigraphical cones I*, J. Convex Analysis 18 (2011) 1171–1196.

[48] A. Seeger: *Epigraphical cones II*, J. Convex Analysis 19 (2012) 1–21.

[49] A. Seeger: *Aperture angle analysis for ellipsoids*, Electronic J. Linear Algebra 26 (2013) 732–741.

[50] A. Seeger: *On measures of size for convex cones*, J. Convex Analysis 23 (2016) 947–980.

[51] A. Seeger, D. Sossa: *Critical angles between two convex cones. II: Special cases*, TOP 24 (2016) 66–87.

[52] A. Seeger, M. Torki: *Centers and partial volumes of convex cones. I: Basic theory*, Beitr. Algebra Geom. 56 (2015) 227–248.

[53] A. Seeger, M. Torki: *Centers and partial volumes of convex cones. II: Advanced topics*, Beitr. Algebra Geom. 56 (2015) 491–514.

[54] A. Seeger, M. Torki: *Conic version of Loewner-John ellipsoid theorem*, Math. Program. 155 (2016) 403–433.

[55] A. Seeger, M. Torki: *Inscribed and circumscribed ellipsoidal cones: volume ratio analysis*, Beitr. Algebra Geom. 59 (2018) 717–737.

[56] A. Seeger, J. Vidal-Nuñez: *Measuring centrality and dispersion in directional datasets: the ellipsoidal cone covering approach*, J. Global Optim. 68 (2017) 279–306.

[57] C. Shannon: *Probability of error for optimal codes in a Gaussian channel*, Bell System Tech. 38 (1959) 611–656.

[58] R. Stern, H. Wolkowicz: *Invariant ellipsoidal cones*, Linear Algebra Appl. 150 (1991) 81–106.

[59] R. Stern, H. Wolkowicz: *Exponential nonnegativity on the ice cream cone*, SIAM J. Matrix Anal. Appl. 12 (1991) 160–165.

[60] G. J. Tee: *Surface area and capacity of ellipsoids in n dimensions*, New Zealand J. Math. 34 (2005) 165–198.

[61] Y. Tian, S. C. Fang, Z. B. Deng, W. X. Xing: *Computable representation of the cone of nonnegative quadratic forms over a general second-order cone and its applications to completely positive programming*, J. Indust. Manag. Optim. 9 (2013) 703–721.