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The Split Common Fixed Point Problem by the Hybrid Method for Families of New Demimetric Mappings in Banach Spaces

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We consider the split common fixed point problem for families of mappings in Banach spaces. Using the hybrid method, we prove two strong convergence theorems of finding a solution of the split common fixed point problem for families of generalized demimetric mappings in Banach spaces. We also apply these results to obtain well-known and new results for the split common fixed point problem in Banach spaces.

Keywords: Split common fixed point problem, metric projection, metric resolvent, generalized demimetric mapping, hybrid method.

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1. Introduction

Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-*demimetric* [30] if

$$2\langle x - q, J(x - Ux)\rangle \geq (1 - \eta)\|x - Ux\|^2$$

for all $x \in C$ and $q \in F(U)$, where $F(U)$ is the set of fixed points of U and J is the duality mapping on E. We have from [30] that $F(U)$ is closed and convex.

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This property is important. Using this, we proved weak and strong convergence theorems for demimetric mappings in Hilbert spaces and Banach spaces; see [15, 28, 30, 31, 34]. Very recently, Kawasaki and Takahashi [11] generalized the concept of demimetric mappings as follows: Let θ be a real number with $\theta \neq 0$. A mapping $U\colon C \to E$ with $F(U) \neq \emptyset$ is called *generalized demimetric* [11] if

$$\theta\langle x - q, J(x - Ux)\rangle \geq \|x - Ux\|^2 \tag{1}$$

for all $x \in C$ and $q \in F(U)$. This mapping U is called θ *- generalized demimetric*. The set $F(U)$ is also closed and convex; see [11].

Let H_1 and H_2 be two real Hilbert spaces. Given mappings $S\colon H_1 \to H_1$ and $T\colon H_2 \to H_2$, respectively, and a bounded linear operator $A\colon H_1 \to H_2$, the *split common fixed point problem* [8, 17] is to find a point $z \in H_1$ such that $z \in F(S) \cap A^{-1}F(T)$, where $F(S)$ and $F(T)$ are fixed point sets of S and T, respectively. The split common fixed point problem generalizes the split feasibility problem [7] and the split common null point problem [6]. Many authors have studied the split feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces; see, for instance, [1, 6, 8, 17, 36]. However, it is difficult to prove such results outside Hilbert spaces. Recently, using the hybrid method [18, 19, 22], Takahashi [29] proved the following strong convergence theorem for demimetric mappings. In its formulation condition (I) is used, which is defined exlicitly in Definition 3.1.

Theorem 1.1. ([29]) *Let H be a Hilbert space and let V be a smooth, strictly convex and reflexive Banach space. Let J_V be the duality mapping on V and let $\{\eta_n\}$ be a sequence of real numbers with $\eta_n \in (-\infty, 1)$. Let $\{T_n\}$ be a sequence of nonexpansive mappings of H to H satisfying condition* (I) *and let $\{U_n\}$ be a sequence of η_n-demimetric mappings of V to V satisfying condition* (I)*. Let $A\colon H \to V$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A.*

Suppose that $\mathbf{G} := \cap_{n=1}^{\infty} F(T_n) \cap A^{-1} \cap_{n=1}^{\infty} F(U_n) \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = T_n\Big(x_n - \lambda_n A^* J_V(Ax_n - U_n Ax_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\eta_n\} \subset (-\infty, 1)$, $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy conditions:

$$0 \leq \alpha_n \leq a < 1 \quad \text{and} \quad 0 < b \leq \lambda_n\|A\|^2 \leq c < d \leq 1 - \eta_n$$

for some $a, b, c, d \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in \mathbf{G}$, where $z_0 = P_{\mathbf{G}} x_1$.

In Theorem 1.1, one of the Banach spaces was a Hilbert space. Takahashi, Wen and Yao [33] also obtained the following strong convergence theorem for demimetric mappings in two Banach spaces.

Theorem 1.2. ([33]) *Let E and V be uniformly convex and smooth Banach spaces and let J_E and J_V be the duality mappings on E and V, respectively. Let $\{\tau_n\}$ and $\{\eta_n\}$ be sequences of real numbers with $\tau_n, \eta_n \in (-\infty, 1)$. Let $\{T_n\}$ be a family of τ_n-demimetric mappings of E into itself with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ satisfying condition* (I) *and let $\{U_n\}$ be a family of η_n-demimetric mappings of V into itself with $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$ satisfying condition* (I). *Let $A\colon E \to V$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $\cap_{n=1}^{\infty} F(T_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(U_n)) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n J_E^{-1} A^* J_V(Ax_n - U_n A x_n), \\ y_n = T_n z_n, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ D_n = \{z \in E : 2\langle z_n - z, J_E(z_n - y_n)\rangle \geq (1 - \tau_n)\|z_n - y_n\|^2\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $\{\tau_n\}, \{\eta_n\} \subset (-\infty, 1)$ satisfy conditions such that for some $a, b \in \mathbb{R}$,

$$0 < a \leq 2r_n\|A\|^2 \leq 1 - \eta_n \quad \textit{and} \quad 0 < b \leq 1 - \tau_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to $z_0 \in \cap_{n=1}^{\infty} F(T_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(U_n))$, where $z_0 = P_{\cap_{n=1}^{\infty} F(T_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(U_n))} x_1$.

In this paper, motivated by these problems and results, we consider the split common fixed point problem for families of mappings in Banach spaces. Using the hybrid method [18, 19, 22], we prove two strong convergence theorems for finding a solution of the split common fixed point problem for families of generalized demimetric mappings in Banach spaces. We also apply these results to obtain well-known and new results which are connected with the split common fixed point problem in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot\ ,\ \cdot \rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [25] that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle; \tag{2}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{3}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{4}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The *nearest point projection* of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such a mapping P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{5}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [23].

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the *strong convergence* of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the *weak convergence* by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the *Kadec-Klee property*, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [9, 21].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{6}$$

exists. In the case, E is called *smooth*. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [23, 24]. We know the following result.

Lemma 2.1. ([23]) *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call such a mapping P_C the metric projection of E onto C.

Lemma 2.2. ([23]) *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x$;

(2) $\langle z - y, J(x - z)\rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let B be a mapping of E into 2^{E^*}. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u^* \in Bx$, and $v^* \in By$. A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [4]; see also [24, Theorem 3.5.4].

Theorem 2.3. ([4]) *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if for any $r > 0$,*

$$R(J + rB) = E^*,$$

where $R(J + rB)$ is the range of $J + rB$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rBx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of B. The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [24].

Let B be a maximal monotone operator on a Hilbert space H. In a Hilbert space H, the metric resolvent J_r of B is simply called the resolvent of B. It is known that the resolvent J_r of B for $r > 0$ is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H.$$

Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$. Then a mapping $U\colon C \to E$ with $F(U) \neq \emptyset$ is called generalized demimetric [11] if it satisfies (1), that is, for all $x \in C$ and $q \in F(U)$,

$$\theta\langle x - q, J(x - Ux)\rangle \geq \|x - Ux\|^2.$$

Example 2.4. Here are some examples of generalized demimetric mappings.

(1) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let t be a real number with $0 \le t < 1$. A mapping $U\colon C \to H$ is called a *t-strict pseudo-contraction* [5] if

$$\|Ux - Uy\|^2 \le \|x - y\|^2 + t\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a t-strict pseudo-contraction and $F(U) \neq \emptyset$, then U is $\frac{2}{1-t}$ - generalized demimetric; see [11].

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U\colon C \to H$ is called *generalized hybrid* [12] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \le \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2 \qquad (7)$$

for all $x, y \in C$. Such a mapping U is called *(α, β) - generalized hybrid*. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is 2 - generalized demimetric. In fact, setting $x = u \in F(U)$ and $y = x \in C$ in (7), we have that

$$\alpha\|u - Ux\|^2 + (1-\alpha)\|u - Ux\|^2 \le \beta\|u - x\|^2 + (1-\beta)\|u - x\|^2$$

and hence $\|Ux - u\|^2 \le \|x - u\|^2$. From

$$\|Ux - u\|^2 = \|Ux - x\|^2 + \|x - u\|^2 + 2\langle Ux - x, x - u\rangle,$$

we have

$$2\langle x - u, x - Ux\rangle \ge \|x - Ux\|^2$$

for all $x \in C$ and $u \in F(U)$. This means that U is 2 - generalized demimetric. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0) - generalized hybrid mapping is nonexpansive. It is nonspreading [13, 14] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [26] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [10].

(3) Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty, closed and convex subset of E. Let P_D be the metric projection of E onto D. Then P_D is 1 - generalized demimetric; see [11].

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is 1 - generalized demimetric; see [11].

(5) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let T be a mapping from C into H. Suppose that T is Lipschitzian, that is, there exists $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$. Let $S = (L+1)I - T$ and assume $F(\frac{T}{L}) \neq \emptyset$. Then $F(S) = F(\frac{T}{L})$ and S is $(-2L)$ - generalized demimetric; see [11, 32].

(6) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $\alpha > 0$. If B be an α-inverse strongly monotone mapping from C into H with $B^{-1}0 \neq \emptyset$. Define $T = I + B$. Then $F(T) = B^{-1}0$ and T is $\left(-\frac{1}{\alpha}\right)$ - generalized demimetric; see [11, 32]. □

The following lemmas are important and crucial in the proofs of our main results.

Lemma 2.5. ([11]) *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. If a mapping $U\colon C \to E$ is θ - generalized demimetric and $\theta > 0$, then U is $\left(1 - \frac{2}{\theta}\right)$-demimetric.*

Lemma 2.6. ([11]) *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let θ be a real number with $\theta \neq 0$. Let T be a θ - generalized demimetric mapping of C into E. Then $F(T)$ is closed and convex.*

Lemma 2.7. ([11]) *Let E be a smooth Banach space, let C be a nonempty subset of E and let θ be a real number with $\theta \neq 0$. Let T be a θ - generalized demimetric mapping from C into E and let $k \in \mathbb{R}$ with $k \neq 0$. Then $(1-k)I + kT$ is θk - generalized demimetric from C into E.*

We also know the following lemma from [34]:

Lemma 2.8. ([34]) *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 1)$ and let T be a k-demimetric mapping of C into H such that $F(T)$ is nonempty. Let λ be a real number with $0 < \lambda \leq 1 - k$ and define $S = (1-\lambda)I + \lambda T$. Then S is a quasi-nonexpansive mapping of C into H.*

3. Main results

In this section, using the hybrid method by Nakajo and Takahashi [18], we first prove a strong convergence theorem for finding a solution of the split common fixed point problem with families of generalized demimetric mappings in Banach spaces.

Definition 3.1. ([2]) Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let $\{U_n\}$ be a sequence of mappings of C into E such that $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$. The sequence $\{U_n\}$ is said to satisfy condition

(I) if for any bounded sequence $\{z_n\}$ of C with $\lim_{n\to\infty} \|z_n - U_n z_n\| = 0$ every weak cluster point of $\{z_n\}$ belongs to $\cap_{n=1}^{\infty} F(U_n)$.

Theorem 3.2. *Let H be a Hilbert space and let V be a smooth, strictly convex and reflexive Banach space. Let J_V be the duality mapping on V. Let $\{\theta_n\}$ and $\{\tau_n\}$ be sequences of real numbers with $\theta_n, \tau_n \neq 0$, let $\{S_n\}$ be a sequence of θ_n - generalized demimetric mappings of H to H with $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ satisfying condition* (I) *and* *let* $\{T_n\}$ *be a sequence of* τ_n*-generalized demimetric mappings of* V *to* V *with* $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ *satisfying condition (I). Let* $\{k_n\}$ *and* $\{h_n\}$ *be sequences of real numbers with* $\theta_n k_n > 0$ *and* $\tau_n h_n > 0$*, respectively. Let* $A\colon H \to V$ *be a bounded linear operator such that* $A \neq 0$*. Suppose that*

$$\mathbf{G} := \cap_{n=1}^{\infty} F(S_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(T_n)) \neq \emptyset.$$

Let $x_1 \in H$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = ((1-\lambda_n)I + \lambda_n S_n)\Big(x_n - r_n h_n A^* J_V(Ax_n - T_n Ax_n)\Big), \\ y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c, d, e, f, \lambda_0 \in \mathbb{R}$*,* $\{\alpha_n\} \subset [0,1]$*,* $\{r_n\} \subset (0,\infty)$ *and* $\{\lambda_n\}, \{k_n\}, \{h_n\} \subset \mathbb{R}$ *satisfy the following:* $0 < a \leq \alpha_n \leq 1, \quad 0 < b \leq |h_n| \leq c,$

$$0 < d \leq r_n \leq e < f \leq \frac{2}{\tau_n h_n \|A\|^2}, \quad 0 < \frac{\lambda_n}{k_n} \leq \frac{2}{\theta_n k_n} \quad \text{and} \quad 0 < \lambda_0 \leq |\lambda_n|$$

for all $n \in \mathbb{N}$*. Then* $\{x_n\}$ *converges strongly to* $z_0 \in \mathbf{G}$*, where* $z_0 = P_{\mathbf{G}} x_1$*.*

Proof. Since
$$\begin{aligned} \|y_n - z\| \leq \|x_n - z\| &\iff \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ &\iff \|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z\rangle \leq 0, \end{aligned}$$

it follows that C_n is closed and convex for all $n \in \mathbb{N}$. It is obvious that D_n is closed and convex. Then $C_n \cap D_n$ is closed and convex for all $n \in \mathbb{N}$. Let us show that $\mathbf{G} \subset C_n$ for all $n \in \mathbb{N}$. Let $z \in \mathbf{G}$. Then $z = S_n z$ and $Az = T_n Az$. Since $T_n\colon V \to V$ is τ_n-generalized demimetric, we have from Lemma 2.7 that $(1-h_n)I + h_n T_n$ is $\tau_n h_n$-generalized demimetric. Since $S_n\colon H \to H$ is θ_n-generalized demimetric, we also have from Lemma 2.7 that $(1-k_n)I + k_n S_n$ is $\theta_n k_n$-generalized demimetric. Furthermore, from Lemma 2.5 and $\theta_n k_n > 0$, we have that $(1-k_n)I + k_n S_n$ is $\big(1 - \frac{2}{\theta_n k_n}\big)$-demimetric in the sense of [30]. Since

$$0 < \frac{\lambda_n}{k_n} \leq \frac{2}{\theta_n k_n} = 1 - \Big(1 - \frac{2}{\theta_n k_n}\Big)$$

and
$$(1-\lambda_n)I + \lambda_n S_n = \left(1 - \frac{\lambda_n}{k_n}\right) I + \frac{\lambda_n}{k_n}((1-k_n)I + k_n S_n),$$

we have from Lemma 2.8 that $(1-\lambda_n)I + \lambda_n S_n$ is quasi-nonexpansive. Putting $s_n = x_n - r_n h_n A^* J_V(Ax_n - T_n Ax_n)$, from $0 < d \leq r_n \leq e < f \leq \frac{2}{\tau_n h_n \|A\|^2}$, we have that for $z \in \mathbf{G}$,

$$
\begin{aligned}
\|z_n - z\|^2 &= \|((1-\lambda_n)I + \lambda_n S_n)s_n - ((1-\lambda_n)I + \lambda_n S_n)z\|^2 \\
&\leq \|x_n - r_n h_n A^* J_V(Ax_n - T_n Ax_n) - z\|^2 \\
&= \|x_n - z\|^2 - 2\langle x_n - z, r_n h_n A^* J_V(Ax_n - T_n Ax_n)\rangle \\
&\qquad + \|r_n h_n A^* J_V(Ax_n - T_n Ax_n)\|^2 \\
&\leq \|x_n - z\|^2 - 2r_n\langle Ax_n - Az, J_V(Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\rangle \\
&\qquad + r_n^2 h_n^2 \|A\|^2 \|J_V(Ax_n - T_n Ax_n)\|^2 \\
&\leq \|x_n - z\|^2 - \frac{2r_n}{\tau_n h_n}\|Ax_n - ((1-h_n)I + h_n T_n)Ax_n\|^2 \qquad (8) \\
&\qquad + r_n^2 h_n^2 \|A\|^2 \|Ax_n - T_n Ax_n\|^2 \\
&= \|x_n - z\|^2 - \frac{2r_n}{\tau_n h_n} h_n^2 \|Ax_n - T_n Ax_n\|^2 + r_n^2 h_n^2 \|A\|^2 \|Ax_n - T_n Ax_n\|^2 \\
&= \|x_n - z\|^2 + r_n h_n^2 (r_n \|A\|^2 - \frac{2}{\tau_n h_n})\|Ax_n - T_n Ax_n\|^2 \leq \|x_n - z\|^2
\end{aligned}
$$

and hence

$$
\begin{aligned}
\|y_n - z\| &= \|(1-\alpha_n)x_n + \alpha_n z_n - z\| \leq (1-\alpha_n)\|x_n - z\| + \alpha_n \|z_n - z\| \\
&\leq (1-\alpha_n)\|x_n - z\| + \alpha_n \|x_n - z\| = \|x_n - z\|.
\end{aligned}
$$

Therefore, $\mathbf{G} \subset C_n$ for all $n \in \mathbb{N}$. Let us show that $\mathbf{G} \subset D_n$ for all $n \in \mathbb{N}$. It is obvious that $\mathbf{G} \subset D_1$. Suppose that $\mathbf{G} \subset D_j$ for some $j \in \mathbb{N}$. Then $\mathbf{G} \subset C_j \cap D_j$. From $x_{j+1} = P_{C_j \cap D_j} x_1$, we have that

$$\langle x_{j+1} - z, x_1 - x_{j+1}\rangle \geq 0, \quad \forall z \in C_j \cap D_j$$

and hence

$$\langle x_{j+1} - z, x_1 - x_{j+1}\rangle \geq 0, \quad \forall z \in \mathbf{G}.$$

Then $\mathbf{G} \subset D_{j+1}$. We have by induction that $\mathbf{G} \subset D_n$ for all $n \in \mathbb{N}$. Thus, we have that $\mathbf{G} \subset C_n \cap D_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $F(S_n)$ and $F(T_n)$ are closed and convex from Lemma 2.6 and A is linear and continuous, $\mathbf{G}$ is also nonempty, closed and convex. Thus, there exists $z_0 \in \mathbf{G}$ such that $z_0 = P_{\mathbf{G}} x_1$. From $x_{n+1} = P_{C_n \cap D_n} x_1$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - y\|, \quad \forall y \in C_n \cap D_n.$$

Since $z_0 \in \mathbf{G} \subset C_n \cap D_n$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - z_0\|. \qquad (9)$$

This means that $\{x_n\}$ is bounded.

Next we show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From the definition of D_n, we have that $x_n = P_{D_n} x_1$. From $x_{n+1} = P_{C_n \cap D_n} x_1$ we have $x_{n+1} \in D_n$. Thus

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$$

for all $n \in \mathbb{N}$. This implies that $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. From $x_{n+1} \in D_n$ we have that

$$\langle x_n - x_{n+1}, x_1 - x_n \rangle \geq 0.$$

This implies from (4) that

$$0 \leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - \|x_{n+1} - x_n\|^2$$

and hence
$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.$$

Since there exists the limit of $\{\|x_1 - x_n\|\}$, we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{10}$$

From $x_{n+1} \in C_n$, we also have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. Then we get from (10) that $\|y_n - x_{n+1}\| \to 0$. Using this, we have that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{11}$$

We have from (8) that for any $z \in \mathbf{G}$,

$$\begin{aligned}
\|y_n - z\|^2 = \|(1-\alpha_n)x_n + \alpha_n z_n - z\|^2 &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n \|z_n - z\|^2 \\
&\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n \|x_n - z\|^2 \\
&\qquad + \alpha_n r_n h_n^2 (r_n \|A\|^2 - \frac{2}{\tau_n h_n}) \|Ax_n - T_n A x_n\|^2 \\
&= \|x_n - z\|^2 + \alpha_n r_n h_n^2 (r_n \|A\|^2 - \frac{2}{\tau_n h_n}) \|Ax_n - T_n A x_n\|^2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\alpha_n r_n h_n^2 (\frac{2}{\tau_n h_n} - r_n \|A\|^2) \|Ax_n - T_n A x_n\|^2 \leq \|x_n - z\|^2 - \|y_n - z\|^2 \\
&= (\|x_n - z\| + \|y_n - z\|)(\|x_n - z\| - \|y_n - z\|) \leq (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|.
\end{aligned}$$

From $\|y_n - x_n\| \to 0$, $0 < a \leq \alpha_n \leq 1$ and $0 < d \leq r_n \leq e < f \leq \frac{2}{\tau_n h_n \|A\|^2}$, we have

$$\lim_{n\to\infty} \|Ax_n - T_n A x_n\|^2 = 0. \tag{12}$$

We also have that $\|y_n - x_n\| = \|(1-\alpha_n)x_n + \alpha_n z_n - x_n\| = \alpha_n \|z_n - x_n\|$. From $\|y_n - x_n\| \to 0$ and $0 < a \leq \alpha_n \leq 1$, we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{13}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. Since A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Since $\lim_{n\to\infty} \|Ax_n - T_n A x_n\| = 0$ and the family $\{T_n\}$ satisfies condition

(I), we have that $Aw \in \cap_{n=1}^{\infty} F(T_n)$ and hence $w \in A^{-1} \cap_{n=1}^{\infty} F(T_n)$. We show that $w \in \cap_{n=1}^{\infty} F(S_n)$. Putting $s_n = x_n - r_n h_n A^* J_V(Ax_n - T_n Ax_n)$, we have that

$$\|s_n - z_n\| = \|s_n - ((1-\lambda_n)I + \lambda_n S_n)s_n\| = \|\lambda_n(s_n - S_n s_n)\| \geq \lambda_0 \|s_n - Ss_n\|.$$

Furthermore, we have $\|s_n - x_n\| = \|r_n h_n A^* J_V(Ax_n - T_n Ax_n)\| \to 0$. We have from $\|s_n - z_n\| \leq \|s_n - x_n\| + \|x_n - z_n\|$ and (13) that $\|s_n - z_n\| \to 0$. This implies

$$\lim_{n\to\infty} \|s_n - S_n s_n\| = 0. \tag{14}$$

Since $\|s_n - x_n\| \to 0$, we also have that $\{s_n\}$ converges weakly to w. Since $\{S_n\}$ satisfies condition (I), we have $w \in \cap_{n=1}^{\infty} F(S_n)$. This implies that $w \in \cap_{n=1}^{\infty} F(S_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(T_n)) = \mathbf{G}$. From $z_0 = P_{\mathbf{G}} x_1$, $w \in \mathbf{G}$ and (9), we have

$$\begin{aligned}\|x_1 - z_0\| \leq \|x_1 - w\| &\leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \leq \|x_1 - z_0\|.\end{aligned}$$

Then we get that $\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - z_0\|$.

Since H satisfies the Kadec-Klee property, we have $x_1 - x_{n_i} \to x_1 - w$ and hence $x_{n_i} \to w = z_0$. Therefore, we have $x_n \to w = z_0$. This completes the proof. □

Next, using the hybrid method [18], we prove a strong convergence theorem of finding a solution of the split common fixed point problem for families of generalized demimetric mappings in two Banach spaces.

Theorem 3.3. *Let E and V be uniformly convex and smooth Banach spaces and let J_E and J_V be the duality mappings on E and V, respectively. Let $\{\theta_n\}$ and $\{\tau_n\}$ be sequences of real numbers with $\theta_n, \tau_n \neq 0$. Let $\{S_n\}$ be a sequence of θ_n-generalized demimetric mappings of E into E satisfying condition (I) and $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and let $\{T_n\}$ be a sequence of τ_n-generalized demimetric mappings of V into V satisfying condition (I) and $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{k_n\}$ and $\{h_n\}$ be sequences of real numbers with $\theta_n k_n > 0$ and $\tau_n h_n > 0$, respectively. Let $A\colon E \to V$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that*

$$\mathbf{G} := \cap_{n=1}^{\infty} F(S_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(T_n)) \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - r_n h_n J_E^{-1} A^* J_V(Ax_n - T_n Ax_n), \\ y_n = ((1-k_n)I + k_n S_n)z_n, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ D_n = \{z \in E : \theta_n k_n \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c, d \in \mathbb{R}$, $\{r_n\} \subset (0, \infty)$ and $\{k_n\}, \{h_n\} \subset \mathbb{R}$ satisfy the following:

$$0 < a \leq |h_n|, \quad 0 < b \leq |k_n|, \quad \theta_n k_n \leq c, \quad \text{and} \quad 0 < d \leq r_n \leq \frac{1}{\tau_n h_n \|A\|^2}$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \mathbf{G}$, where $z_0 = P_{\mathbf{G}} x_1$.

Proof. It is obvious that $C_n \cap D_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. To show that $\mathbf{G} \subset C_n$ for all $n \in \mathbb{N}$, let us show that $\langle z_n - z, J_E(x_n - z_n)\rangle \geq 0$ for all $z \in A^{-1}(\cap_{n=1}^{\infty} F(T_n))$ and $n \in \mathbb{N}$. In fact, since $(1-h_n)I + h_n T_n$ is $\tau_n h_n$ - generalized demimetric from Lemma 2.7, we have that for any $z \in A^{-1}(\cap_{n=1}^{\infty} F(T_n))$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\langle z_n - z, &J_E(x_n - z_n)\rangle = \langle z_n - x_n + x_n - z, J_E(x_n - z_n)\rangle \\
&= \langle -r_n h_n J_E^{-1} A^* J_V(Ax_n - T_n Ax_n) \\
&\qquad + x_n - z, J_E(r_n h_n J_E^{-1} A^* J_V(Ax_n - T_n Ax_n))\rangle \\
&= \langle -r_n J_E^{-1} A^* J_V(h_n Ax_n - h_n T_n Ax_n) \\
&\qquad + x_n - z, J_E(r_n J_E^{-1} A^* J_V(h_n Ax_n - h_n T_n Ax_n))\rangle \\
&= \langle -r_n J_E^{-1} A^* J_V(Ax_n - ((1-h_n)I + h_n T_n)Ax_n) \\
&\qquad + x_n - z, r_n A^* J_V(Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\rangle \qquad (15) \\
&\geq -r_n^2 \|A\|^2 \|Ax_n - ((1-h_n)I + h_n T_n)Ax_n\|^2 \\
&\qquad + \langle Ax_n - Az, r_n J_V(Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\rangle \\
&\geq -r_n^2 \|A\|^2 \|Ax_n - ((1-h_n)I + h_n T_n)Ax_n\|^2 \\
&\qquad + \frac{r_n}{\tau_j h_n} \|Ax_n - ((1-h_n)I + h_n T_n)Ax_n\|^2 \\
&= r_n \Big(\frac{1}{\tau_n h_n} - r_n \|A\|^2\Big) \|Ax_n - ((1-h_n)I + h_n T_n)Ax_n\|^2 \geq 0.
\end{aligned}$$

Then we have that $\mathbf{G} \subset A^{-1}(\cap_{n=1}^{\infty} F(T_n)) \subset C_n$ for all $n \in \mathbb{N}$. Next, to show that $\mathbf{G} \subset D_n$ for all $n \in \mathbb{N}$, let us show that $\theta_n k_n \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2$ for all $z \in \cap_{n=1}^{\infty} F(T_n)$ and $n \in \mathbb{N}$. In fact, since $(1-k_n)I + k_n S_n$ is $\theta_n k_n$ - generalized demimetric from Lemma 2.7, we have that for any $z \in \cap_{n=1}^{\infty} F(S_n)$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\theta_n k_n \langle z_n - z, J_E(z_n - y_n)\rangle - \|z_n - y_n\|^2 \\
&= \theta_n k_n \langle z_n - z, J_E(z_n - ((1-k_n)I + k_n S_n)z_n)\rangle \qquad (16) \\
&\qquad - \|z_n - ((1-k_n)I + k_n S_n)z_n\|^2 \\
&\geq \|z_n - ((1-k_n)I + k_n S_n)z_n\|^2 - \|z_n - ((1-k_n)I + k_n S_n)z_n\|^2 = 0.
\end{aligned}$$

Then we have that $\mathbf{G} \subset \cap_{n=1}^{\infty} F(S_n) \subset D_n$. We show that $\mathbf{G} \subset Q_n$ for all $n \in \mathbb{N}$. It is obvious from $Q_1 = E$ that $\mathbf{G} \subset Q_1$. Suppose that $\mathbf{G} \subset Q_j$ for some $j \in \mathbb{N}$. Then $\mathbf{G} \subset C_k \cap D_j \cap Q_j$. From $x_{j+1} = P_{C_j \cap D_j \cap Q_j} x_1$, we have that

$$\langle x_{j+1} - z, J_E(x_1 - x_{j+1})\rangle \geq 0, \quad \forall z \in C_j \cap D_j \cap Q_j$$

and hence $$\langle x_{j+1} - z, J_E(x_1 - x_{j+1})\rangle \geq 0, \quad \forall z \in \mathbf{G}.$$

Then $\mathbf{G} \subset Q_{j+1}$. By mathematical induction, we have $\mathbf{G} \subset Q_n$ for all $n \in \mathbb{N}$. Thus we have $\mathbf{G} \subset C_n \cap D_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $F(S_n)$ and $F(T_n)$ are closed and convex from Lemma 2.6 and A is linear and continuous, we have that $\mathbf{G} = \cap_{n=1}^{\infty} F(S_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(T_n))$ is nonempty, closed and convex. Then there exists $z_0 \in \mathbf{G}$ such that $z_0 = P_{\mathbf{G}} x_1$. From $x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1$, we have that $\|x_1 - x_{n+1}\| \leq \|x_1 - y\|$ for all $y \in C_n \cap D_n \cap Q_n$. Since $z_0 \in \mathbf{G} \subset C_n \cap D_n \cap Q_n$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - z_0\|. \tag{17}$$

This means that $\{x_n\}$ is bounded.

Next we show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From the definition of Q_n, we have that $x_n = P_{Q_n} x_1$. From $x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1$ we have that $x_{n+1} \in Q_n$. Thus

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$$

for all $n \in \mathbb{N}$. This implies that $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. Put $\lim_{n\to\infty} \|x_n - x_1\| = l$. If $l = 0$, then $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Assume that $l > 0$. Since $x_n = P_{Q_n} x_1$, $x_{n+1} \in Q_n$ and $\frac{x_n + x_{n+1}}{2} \in Q_n$, we have

$$\|x_1 - x_n\| \leq \|x_1 - \frac{x_n + x_{n+1}}{2}\| \leq \frac{1}{2}(\|x_1 - x_n\| + \|x_1 - x_{n+1}\|)$$

and hence
$$\lim_{n\to\infty} \|x_1 - \frac{x_n + x_{n+1}}{2}\| = l.$$

Since E is uniformly convex, we get that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

From the definition of C_n, we have that $z_n = P_{C_n} x_n$. From $x_{n+1} \in C_n$ we have

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\|.$$

From $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$, we have that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Furthermore, from $x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1$, we have that

$$\theta_n k_n \langle z_n - x_{n+1}, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2$$

and hence
$$\theta_n k_n \langle z_n - x_n + x_n - x_{n+1}, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2.$$

From $\|z_n - x_n\| \to 0$ and $\|x_n - x_{n+1}\| \to 0$, we get that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{18}$$

Since $\|y_n - z_n\| = |k_n| \|z_n - S_n z_n\| \geq b\|z_n - S_n z_n\|$, we get that

$$\lim_{n\to\infty} \|z_n - S_n z_n\| = 0. \tag{19}$$

On the other hand, we know that

$$\|x_n - z_n\| = \|J_E(x_n - z_n)\| = \|r_n A^* J_V(Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\|$$

and

$$\begin{aligned}
\tau_n h_n &\langle x_n - z, A^* J_V(Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\rangle \\
&= \tau_n h_n \langle Ax_n - Az, J_V(Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\rangle \\
&\geq \|Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\|^2 = h_n^2 \|Ax_n - T_n Ax_n)\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - z_n\| = 0$, we have that

$$\lim_{n\to\infty} \|A^* J_V(Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\| = 0.$$

Then we get from $h_n^2 \geq a^2 > 0$ that

$$\lim_{n\to\infty} \|Ax_n - T_n Ax_n\| = 0. \tag{20}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. From $\|z_n - x_n\| \to 0$, we have that $\{z_{n_i}\}$ converges weakly to w. Since $\lim_{n\to\infty} \|S_n z_n - z_n\| = 0$ from (19) and the family $\{S_n\}$ satisfies condition (I), we have that $w \in \cap_{n=1}^{\infty} F(S_n)$. Furthermore, since A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Since $\lim_{n\to\infty} \|Ax_n - T_n Ax_n\| = 0$ from (20)and $\{T_n\}$ satisfies condition (I), we have that $Aw \in \cap_{n=1}^{\infty} F(T_n)$. Then we have $w \in \cap_{n=1}^{\infty} F(S_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(T_n)) = \mathbf{G}$. From $z_0 = P_{\mathbf{G}} x_1$, $w \in G$ and (17), we have that

$$\|x_1 - z_0\| \leq \|x_1 - w\| \leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \leq \|x_1 - z_0\|.$$

Thus we get that $\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - z_0\|$. From the Kadec-Klee property of E, we have that $x_1 - x_{n_i} \to x_1 - w$ and hence $x_{n_i} \to w = z_0$. Therefore, we have $x_n \to w = z_0$. This completes the proof. □

4. Applications

In this section, using Theorems 3.2 and 3.3, we get well-known and new strong convergence theorems which are connected with the split common fixed point problem with families of generalized demimetric mappings in Banach spaces.

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U\colon C \to H$ is called *demiclosed* if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup w$ and $x_n - Ux_n \to 0$, $w = Uw$ holds. For example, if C is a nonempty, closed and convex subset of H and T is a nonexpansive mapping of C of H, then T is demiclosed; see [4]. In fact, let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. Since C is weakly closed, we have that $u \in C$. Furthermore, we have from $x_n \rightharpoonup u$ that $\{x_n\}$ is bounded and then $\{Tx_n\}$ is bounded. Thus, we have that

$$
\begin{aligned}
\|u-Tu\|^2 &= \|u-x_n+x_n-Tu\|^2 = \|u-x_n\|^2 + \|x_n-Tu\|^2 + 2\langle u-x_n, x_n-Tu\rangle \\
&= \|u-x_n\|^2 + \|x_n-Tx_n+Tx_n-Tu\|^2 + 2\langle u-x_n, x_n-u+u-Tu\rangle \\
&= \|u-x_n\|^2 + \|x_n-Tx_n\|^2 + \|Tx_n-Tu\|^2 + 2\langle x_n-Tx_n, Tx_n-Tu\rangle \\
&\qquad - 2\|u-x_n\|^2 + 2\langle u-x_n, u-Tu\rangle \qquad (21)\\
&\le \|u-x_n\|^2 + \|x_n-Tx_n\|^2 + \|x_n-u\|^2 + 2\langle x_n-Tx_n, Tx_n-Tu\rangle \\
&\qquad - 2\|u-x_n\|^2 + 2\langle u-x_n, u-Tu\rangle \\
&= \|x_n-Tx_n\|^2 + 2\langle x_n-Tx_n, Tx_n-Tu\rangle + 2\langle u-x_n, u-Tu\rangle \to 0.
\end{aligned}
$$

Therefore, we have that $u = Tu$. Furthermore, we know the following result obtained by Marino and Xu [16]; see also [35].

Lemma 4.1. ([16, 35]) *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let t be a real number with $0 \le t < 1$. Let $U\colon C \to H$ be a t-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Using (21), we prove the following result.

Lemma 4.2. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T\colon C \to H$ be nonexpansive mappings such that $F(S) \cap F(T) \neq \emptyset$ and let $\{\gamma_n\}$ be a sequence of real numbers. Assume that there exist $a, b \in \mathbb{R}$ such that $0 < a \le \gamma_n \le b < 1$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, put $U_n = \gamma_n S + (1-\gamma_n)T$ and $T_n = 2I - U_n$. Then $\{T_n\}$ is a sequence of (-2) - generalized demimetric mappings of C into H satisfying condition (I) and $\cap_{n=1}^{\infty} F(T_n) = \cap_{n=1}^{\infty} F(U_n) = F(S) \cap F(T)$.*

Proof. Since S and T are nonexpansive mappings, $U_n = \gamma_n S + (1-\gamma_n)T$ is nonexpansive. From (5) in Examples, $T_n = 2I - U_n$ is (-2) - generalized demimetric. Furthermore, since S and T are nonexpansive mappings and $F(S) \cap F(T) \neq \emptyset$, S and T are quasi-nonexpansive mappings. Using this, we have from (3) that, for $z_0 \in F(S) \cap F(T)$, $z \in \cap_{n=1}^{\infty} F(U_n)$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
\|z-z_0\|^2 &= \|U_n z - z_0\|^2 = \|(\gamma_n S + (1-\gamma_n)T)z - z_0\|^2 \\
&= \|\gamma_n(Sz-z_0) + (1-\gamma_n)(Tz-z_0)\|^2 \\
&= \gamma_n\|Sz-z_0\|^2 + (1-\gamma_n)\|Tz-z_0\|^2 - \gamma_n(1-\gamma_n)\|Sz-Tz\|^2 \\
&\le \gamma_n\|z-z_0\|^2 + (1-\gamma_n)\|z-z_0\|^2 - \gamma_n(1-\gamma_n)\|Sz-Tz\|^2 \\
&= \|z-z_0\|^2 - \gamma_n(1-\gamma_n)\|Sz-Tz\|^2.
\end{aligned}
$$

This means that $\gamma_n(1-\gamma_n)\|Sz-Tz\|^2 \le 0$. Since $0 < a \le \gamma_n \le b < 1$ for all $n \in \mathbb{N}$, we have $Sz = Tz$. Since

$$
\begin{aligned}
\|Sz-z\| &= \|\gamma_n Sz + (1-\gamma_n)Sz - z\| = \|\gamma_n Sz + (1-\gamma_n)Tz - z\| \\
&= \|(\gamma_n S + (1-\gamma_n)T)z - z\| = \|z-z\| = 0,
\end{aligned}
$$

we get $Sz = z$. Similarly, we obtain $Tz = z$. This implies the inclusion $\cap_{n=1}^{\infty} F(U_n) \subset F(S) \cap F(T)$. It is obvious that $F(S) \cap F(T) \subset \cap_{n=1}^{\infty} F(U_n)$. Thus $\cap_{n=1}^{\infty} F(U_n) = F(S) \cap F(T)$. Since

$$z \in F(T_n) \Longleftrightarrow z = 2z - U_n z \Longleftrightarrow U_n z = z \Longleftrightarrow z \in F(U_n),$$

we also have that $\cap_{n=1}^{\infty} F(U_n) = \cap_{n=1}^{\infty} F(T_n)$

Suppose that $\{z_n\}$ is a bounded sequence such that $z_n - T_n z_n \to 0$. From

$$T_n z_n - z_n = 2z_n - U_n z_n - z_n = z_n - U_n z_n$$

we get $z_n - U_n z_n \to 0$. Then (2) and (3) imply for $z \in \cap_{n=1}^{\infty} F(U_n)$

$$\begin{aligned}
\|z_n - z\|^2 &= \|z_n - U_n z_n + U_n z_n - z\|^2 \leq \|U_n z_n - z\|^2 + 2\langle z_n - U_n z_n, z_n - z\rangle \\
&= \|\gamma_n S z_n + (1-\gamma_n) T z_n - z\|^2 + 2\langle z_n - U_n z_n, z_n - z\rangle \\
&= \gamma_n \|S z_n - z\|^2 + (1-\gamma_n)\|T z_n - z\|^2 \\
&\qquad - \gamma_n(1-\gamma_n)\|S z_n - T z_n\|^2 + 2\langle z_n - U_n z_n, z_n - z\rangle \\
&\leq \gamma_n \|z_n - z\|^2 + (1-\gamma_n)\|z_n - z\|^2 \\
&\qquad - \gamma_n(1-\gamma_n)\|S z_n - T z_n\|^2 + 2\langle z_n - U_n z_n, z_n - z\rangle \\
&= \|z_n - z\|^2 - \gamma_n(1-\gamma_n)\|S z_n - T z_n\|^2 + 2\langle z_n - U_n z_n, z_n - z\rangle
\end{aligned}$$

and hence $\gamma_n(1-\gamma_n)\|S z_n - T z_n\|^2 \leq 2\langle z_n - U_n z_n, z_n - z\rangle$.

Since $z_n - U_n z_n \to 0$ and $\{z_n\}$ is bounded, we have that $S z_n - T z_n \to 0$. Using this, we obtain

$$\begin{aligned}
\|z_n - S z_n\| &= \|z_n - U_n z_n + U_n z_n - S z_n\| \leq \|z_n - U_n z_n\| + \|U_n z_n - S z_n\| \\
&= \|z_n - U_n z_n\| + (1-\gamma_n)\|T z_n - S z_n\| \to 0.
\end{aligned}$$

If a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ converges weakly to w, then we can conclude from $z_n - S z_n \to 0$ that $w \in F(S)$. Similarly, $w \in F(T)$. Thus every weak cluster point $\{z_n\}$ belongs to $F(S) \cap F(T) = \cap_{n=1}^{\infty} F(U_n) = \cap_{n=1}^{\infty} F(T_n)$. This completes the proof. □

Using Lemma 4.2, we obtain the following theorem.

Theorem 4.3. *Let H_1 and H_2 be Hilbert spaces. Let $s \in [0,1)$. Let $U\colon H_1 \to H_1$ be an s-strict pseudo-contraction such that $F(U) \neq \emptyset$ and let $S, T\colon H_2 \to H_2$ be nonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$. Let $\{\delta_n\}$ and $\{\gamma_n\}$ be sequences of real numbers. Assume that there exist $u, v, w \in \mathbb{R}$ such that $0 \leq \delta_n \leq u < 1$ and $0 < v \leq \gamma_n \leq w < 1$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, put $S_n = \delta_n I + (1-\delta_n)U$. Furthermore, put $U_n = \gamma_n S + (1-\gamma_n)T$ and $T_n = 2I - U_n$ for all $n \in \mathbb{N}$. Let $A\colon H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Suppose that*

$$\mathbf{G} := F(U) \cap A^{-1}(F(S) \cap F(T)) \neq \emptyset.$$

Let $x_1 \in H_1$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = ((1-\lambda_n)I + \lambda_n S_n)\Big(x_n + r_n A^*(Ax_n - T_n Ax_n)\Big), \\ y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ C_n = \{z \in H_1 : \|y_n - z\| \le \|x_n - z\|\}, \\ D_n = \{z \in H_1 : \langle x_n - z, x_1 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, d, e, f, \lambda_0 \in \mathbb{R}$, $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and $\{\lambda_n\} \subset \mathbb{R}$ satisfy the following: $0 < a \le \alpha_n \le 1$,

$$0 < d \le r_n \le e < f \le \frac{1}{\|A\|^2}, \quad 0 < \lambda_n \le 1-s \quad \textit{and} \quad 0 < \lambda_0 \le \lambda_n$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $z_0 \in \mathbf{G}$, where $z_0 = P_{\mathbf{G}} x_1$.

Proof. Since U is an s-strict pseudo-contraction of H_1 into itself with $F(U) \neq \emptyset$, we know from Example 2.4(1) that U is $\frac{2}{1-s}$ - generalized demimetric. We have for $x \in H_1$ and $p \in F(T_n) = F(U)$,

$$\begin{aligned} \langle x-p, x-S_n x\rangle &= \langle x-p, x-(\delta_n x + (1-\delta_n)Ux)\rangle = (1-\delta_n)\langle x-p, x-Ux\rangle \\ &\ge (1-\delta_n)\frac{1-s}{2}\|x-Ux\|^2 = \frac{1-s}{2(1-\delta_n)}\|x-(\delta_n x + (1-\delta_n)Ux)\|^2 \\ &\ge \frac{1-s}{2}\|x-S_n x\|^2 \end{aligned}$$

and hence $\{S_n\}$ is a family of $\frac{2}{1-s}$ - generalized demimetric mappings of H_1 into H_1 such that $F(U) = \cap_{n=1}^\infty F(S_n)$. Furthermore, let $\{u_n\}$ be a bounded sequence of H_1 such that $u_n - S_n u_n \to 0$. Then we have

$$(1-\delta_n)(u_n - Uu_n) = u_n - S_n u_n \to 0$$

and hence $u_n - Uu_n \to 0$ from $0 \le \delta_n \le u < 1$. It follows from Lemma 4.1 that every weak cluster point of $\{u_n\}$ belongs to $F(U) = \cap_{n=1}^\infty F(S_n)$. This means that the family $\{S_n\}$ satisfies condition (I). On the other hand, we have from Lemma 4.2 that $\{T_n\}$ is a sequence of (-2) - generalized demimetric mappings and satisfies condition (I). Put $\tau_n = -2$ and $h_n = -1$ in Theorem 3.2. Then we have that

$$0 < d \le r_n \le e < f \le \frac{1}{\|A\|^2}, \quad 0 < \lambda_n \le 1-s \quad \text{and} \quad 0 < \lambda_0 \le \lambda_n$$

for all $n \in \mathbb{N}$. We have the desired result from Theorem 3.2. □

We also know the following result from Plubtieng and Takahashi [20].

Lemma 4.4. ([20]) *Let H_1, H_2 be Hilbert spaces and let $\alpha > 0$. Let $G\colon H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda G)^{-1}$ be the resolvent of G for $\lambda > 0$. Let $T\colon H_2 \to H_2$ be an α-inverse strongly monotone mapping and let $A\colon H_1 \to H_2$ be a bounded linear operator. Suppose that $G^{-1}0 \cap A^{-1}(T^{-1}0) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H_1$. Then the following conditions are equivalent:*

(i) $z = J_\lambda(I - rA^*TA)z$; (ii) $0 \in A^*TAz + Gz$; (iii) $z \in G^{-1}0 \cap A^{-1}(T^{-1}0)$.

Theorem 4.5. *Let H be a Hilbert space and let V be a uniformly convex and smooth Banach space. Let J_V be the duality mapping on V. Let G and B be maximal monotone operators of H and V, respectively. Let J_s and Q_t be the metric resolvents of G for $s > 0$ and B for $t > 0$, respectively. Let $U\colon H \to H$ be an α-inverse strongly monotone mapping with $U^{-1}0 \neq \emptyset$. Let $A\colon H \to V$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $G^{-1}0 \cap U^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_{s_n}(I - s_nU)\big(x_n - r_nA^*J_V(Ax_n - Q_{t_n}Ax_n)\big), \\ y_n = (1 - \alpha_n)x_n + \alpha_n z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c, d, e \in \mathbb{R}$, $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and $\{s_n\}, \{t_n\} \subset (0, \infty)$ satisfy the following:

$$0 < a \leq \alpha_n \leq 1, \quad 2\alpha \geq s_n \geq b > 0, \quad t_n \geq c > 0 \quad \text{and} \quad 0 < d \leq r_n\|A\|^2 \leq e < 2$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a point

$$z_0 \in G^{-1}0 \cap U^{-1}0 \cap A^{-1}(B^{-1}0),$$

where $z_0 = P_{G^{-1}0 \cap U^{-1}0 \cap A^{-1}(B^{-1}0)} x_1$.

Proof. Put $S_n = J_{s_n}(I - s_nU)$ for all $n \in \mathbb{N}$. Since $2\alpha \geq s_n \geq b > 0$, S_n is nonexpansive and hence 2 - generalized demimetric. We also have from Lemma 4.4 that $G^{-1}0 \cap U^{-1}0 = \cap_{n=1}^\infty F(S_n)$. Furthermore, the family $\{S_n\}$ satisfies condition (I). In fact, let $\{z_n\}$ be a bounded sequence in H such that $z_n - S_nz_n \to 0$. Without loss of generality, we may assume that $z_n \rightharpoonup w$ and $z_n - S_nz_n \to 0$, Let $v_n = S_nz_n = J_{s_n}(I - s_nU)z_n$. Then

$$Uv_n + \frac{z_n - s_nUz_n - v_n}{s_n} \in Uv_n + Gv_n.$$

Since $U + G$ is monotone, we have that, for any $(u, v) \in U + G$,

$$\langle v_n - u,\ Uv_n + \frac{z_n - s_nUz_n - v_n}{s_n} - v\rangle \geq 0, \quad \text{and hence}$$

$$\langle v_n - u, -v\rangle \geq \langle v_n - u, -Uv_n - \frac{z_n - s_nUz_n - v_n}{s_n}\rangle$$
$$= \langle v_n - u, Uz_n - Uv_n + \frac{1}{s_n}(v_n - z_n)\rangle = \frac{1}{s_n}\langle v_n - u, (I - s_nU)v_n - (I - s_nU)z_n\|$$
$$\geq -\frac{1}{s_n}\|v_n - u\| \cdot \|(I - s_nU)v_n - (I - s_nU)z_n\| \geq -\frac{1}{s_n}\|v_n - u\| \cdot \|v_n - z_n\|.$$

From $z_n \rightharpoonup w$ and $z_n - v_n \to 0$, we have $\langle w - u, -v\rangle \geq 0$. Since $U + G$ is maximal monotone, $w \in (U + G)^{-1}0$. The family $\{T_n\}$ satisfies condition (I). On the other hand, since Q_{t_n} is the metric resolvent of B for $t_n > 0$, from (4) in Examples, Q_{t_n} is 1 - generalized demimetric. We also have that the family $\{Q_{t_n}\}$ satisfies condition (I). In fact, let $\{u_n\}$ be a bounded sequence in V such that $u_n - Q_{t_n}u_n \to 0$. Without loss of generality, we may assume that $u_n \rightharpoonup p$ and $u_n - Q_{t_n}u_n \to 0$. Since Q_{t_n} is the metric resolvent of B, we have that

$$J_V(u_n - Q_{t_n}u_n)/t_n \in BQ_{t_n}u_n$$

for all $n \in \mathbb{N}$; see [3, 24]. From the monotonicity of B, we have

$$0 \leq \left\langle u - Q_{t_n}u_n, v^* - \frac{J_V(u_n - Q_{t_n}u_n)}{t_n}\right\rangle$$

for all $(u, v^*) \in B$ and $n \in \mathbb{N}$. Taking $n \to \infty$, we get that $\langle u - p, v^*\rangle \geq 0$ for all $(u, v^*) \in B$. Since B is a maximal monotone operator, we have

$$p \in B^{-1}0 = \cap_{n=1}^{\infty} F(Q_{t_n}).$$

This means that the family $\{Q_{t_n}\}$ satisfies condition (I). Since $\theta_n = 2$ and $\tau_n = 1$ in Theorem 3.2, we may take $k_n = 1$ and $h_n = 1$. Then we have that

$$z_n = J_{s_n}(I - s_nU)\big(x_n - r_nA^*J_V(Ax_n - Q_{t_n}Ax_n)\big).$$

Furthermore, we have that

$$0 < d \leq r_n\|A\|^2 \leq e < 2$$

for all $n \in \mathbb{N}$. Therefore, we have the desired result from Theorem 3.2. $\square$

Putting $U = 0$ in Theorem 4.5, we have the following theorem by Takahashi and Yao [37].

Theorem 4.6 ([37]). *Let H be a Hilbert space and let V be a uniformly convex and smooth Banach space. Let J_V be the duality mapping on V. Let G and B be maximal monotone operators of H and V, respectively. Let J_s and Q_t be the metric resolvents of G for $s > 0$ and B for $t > 0$, respectively. Let $A\colon H \to V$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_{s_n}\big(x_n - r_n A^* J_V(Ax_n - Q_{t_n}Ax_n)\big), \\ y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ C_n = \{z \in H : \|y_n - z\| \le \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c, d, e \in \mathbb{R}$, $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and $\{s_n\}, \{t_n\} \subset (0,\infty)$ satisfy the following for all $n \in \mathbb{N}$:

$$0 < a \le \alpha_n \le 1, \quad s_n \ge b > 0, \quad t_n \ge c > 0 \quad \text{and} \quad 0 < d \le r_n\|A\|^2 \le e < 2$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $z_0 = P_{G^{-1}0\cap A^{-1}(B^{-1}0)}x_1$.

Using Theorem 3.3, as in the proof of Theorem 4.5, we get the following theorem for the split common null point problem in two Banach spaces by Takahashi [27].

Theorem 4.7. ([27]) *Let E and V be uniformly convex and smooth Banach spaces and let J_E and J_V be the duality mappings on E and V, respectively. Let G and B be maximal monotone operators of E into E^* and V into V^*, respectively. Let J_s and Q_t be the metric resolvents of G for $s > 0$ and B for $t > 0$, respectively. Let $A\colon E \to V$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Define $\mathbf{G} := G^{-1}0 \cap A^{-1}(B^{-1}0)$ and suppose $\mathbf{G} \neq \emptyset$. For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n J_E^{-1} A^* J_V(Ax_n - Q_{t_n}Ax_n), \\ y_n = J_{s_n} z_n, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n)\rangle \ge 0\}, \\ D_n = \{z \in E : \langle z_n - z, J_E(z_n - y_n)\rangle \ge \|z_n - y_n\|^2\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n)\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where where $b, c, d \in \mathbb{R}$ and $\{r_n\} \subset (0,\infty)$ satisfy the following:

$$s_n \ge b > 0, \quad t_n \ge c > 0 \quad \text{and} \quad 0 < d \le r_n \le \frac{2}{\|A\|^2}$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \mathbf{G}$, where $z_0 = P_{\mathbf{G}}x_1$.

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