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Existence and Relaxation of BV Solutions for a Sweeping Process with a Nonconvex-Valued Perturbation*

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We study a measurable sweeping process with a multivalued perturbation in a separable Hilbert space. The values of the perturbation are closed, not necessarily convex sets. The retraction of the sweeping process is bounded by a positive Radon measure. A solution of the sweeping process is a pair consisting of a right continuous function of bounded variation whose differential measure is absolutely continuous with respect to some positive Radon measure and an integrable selector of the perturbation considered along this function. The density of the differential measure with respect to the Radon measure above satisfies the corresponding inclusion.

Along with the original perturbation, we consider the perturbation with the convexified values. We prove theorems on the existence and relaxation of solutions. The latter means the density of the solution set of the sweeping process with the original perturbation in the solution set of the sweeping process with the convexified perturbation. These solution sets are considered as subsets of the Cartesian product of the space of right continuous functions of bounded variation and the space of integrable functions. These spaces are endowed with the topology of uniform convergence on an interval and the weak topology, respectively.

Keywords: Function of bounded variation, Radon measure, differential measure, density of a measure, BV solutions, existence theorem, relaxation theorem.

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1. Introduction

Let H be a separable Hilbert space with the norm $\|\cdot\|$, the metric $d(\cdot,\cdot)$, the scalar product $\langle\cdot,\cdot\rangle$, and the null element Θ. We introduce the following notation: $\mathbb{R}^+ = [0,+\infty)$, $\overline{\mathbb{R}^+} = [0,+\infty]$, $T = [0,a] \subset \mathbb{R}^+$, $a > 0$, $B \subset H$ is the *unit open ball* centered at Θ, $\overline{B}$ is its *closure*, $C\colon T \rightrightarrows H$ is a multivalued mapping with nonempty convex closed values, $F\colon T \times H \rightrightarrows H$ is a multivalued mapping with nonempty closed values.

The *excess* $\operatorname{exc}(C(s),C(t))$, $s \le t$, $s,t \in T$, of the set $C(s)$ over the set $C(t)$ is defined as follows

$$\operatorname{exc}(C(s),C(t)) = \sup\{d(x,C(t));\ x \in C(s)\}.$$

For a positive Radon measure ν on T denote by $L^1_\nu(T,H)$ the space of all equivalence classes of ν-measurable mappings $u\colon T \to H$ such that the function $t \to \|u(t)\|$ is ν-integrable with the standard norm. The space $L^1_\nu(T,H)$ endowed with the weak topology is denoted by ω-$L^1_\nu(T,H)$.

Positive Radon measures α and β on T are called *absolutely continuously equivalent* if they are mutually absolutely continuous with respect to each other.

Let $BV_+(T,H)$ be the space of all right continuous functions of bounded variation $u\colon T \to H$ with the topology of uniform convergence on T and $cl(H)$ be the collection of all nonempty closed sets from H with the *Hausdorff metric*

$$\operatorname{haus}(A,B) = \sup\{\operatorname{exc}(A,B), \operatorname{exc}(B,A)\}.$$

Since the metric $\operatorname{haus}(\cdot,\cdot)$ may admit infinite values on the space $cl(H)$, it is called the *generalized Hausdorff metric*.

In the rest of the paper, μ is a positive Radon measure on T, λ is the Lebesgue measure and we always assume that the following inequality holds

$$\operatorname{exc}(C(s),C(t)) \le \mu(]s,t]),\ s \le t,\ s,t \in T. \tag{1.1}$$

Consider the measurable sweeping process [5, 12]:

$$-du \in \mathcal{N}(C(t);u(t)) + F(t,u(t))\,d\lambda, \quad t \in T, \quad u(0) = u_0 \in C(0), \tag{1.2}$$

where $\mathcal{N}(C(t);x)$ is the normal cone in the sense of the convex analysis of the set $C(t)$ at a point $x \in C(t)$ and $F\colon T \times H \to cl(H)$ is a multivalued perturbation.

Along with inclusion (1.2) we consider the following sweeping process

$$-du \in \mathcal{N}(C(t);u(t)) + \overline{\operatorname{co}}\,F(t,u(t))\,d\lambda, \quad t \in T, \tag{1.3}$$

where $\overline{\operatorname{co}}$ stands for the closed convex hull of a set.

Definition 1.1. By a *solution* of inclusion (1.2) we mean a pair $(u(\hat g),\hat g)$ such that:

(1) $u(\hat g) \in BV_+(T,H)$, $u(\hat g)(0) = u_0$, $\hat g \in L^1_\lambda(T,H)$, $u(\hat g)(t) \in C(t)$, $t \in T$;

(2) there exists a positive Radon measure ν absolutely continuously equivalent to the measure $\lambda + \mu$ such that the differential measure $du(\hat{g})$ generated by the function $u(\hat{g})$ is absolutely continuous with respect to the measure ν;

(3) the density $\frac{du(\hat{g})}{d\nu} \in L^1_\lambda(T, H)$ of the measure $du(\hat{g})$ with respect to the measure ν and the function $\hat{g}$ satisfy the following inclusions

$$\frac{du(\hat{g})}{d\nu}(t) + \hat{g}(t)\frac{d\lambda}{d\nu}(t) \in -\mathcal{N}(C(t); u(\hat{g})(t)) \quad \nu \text{ a.e.}, \tag{1.4}$$

$$\hat{g}(t) \in F(t, u(\hat{g})(t)) \quad \lambda \text{ a.e.} \tag{1.5}$$

Here, $\frac{d\lambda}{d\nu}(t)$ is the density of the measure λ with respect to the measure ν.

A solution of inclusion (1.3) is defined similarly replacing in the above definition inclusion (1.5) by the inclusion

$$\hat{g}(t) \in \overline{\text{co}}\, F(t, u(\hat{g})(t)) \quad \lambda \text{ a.e.} \tag{1.6}$$

Definition 1.1 is a reformulation of Definitions 2.1 [5] and 2.2 [12] adapted to inclusion (1.2). This definition provides us with a simple control system (1.4) with a trajectory $u(\hat{g})$, a control $\hat{g}$, and the control constraint (1.5). Hence, in what follows, if $(u(\hat{g}), \hat{g})$ is a solution to inclusion (1.2), by analogy with control systems, the function $u(\hat{g})$ is called a trajectory corresponding to the perturbation $\hat{g}$. The set of all solutions to inclusion (1.2) is denoted by $\mathcal{R}_F(u_0)$ and the set of all trajectories by $\mathcal{T}r_F(u_0)$. Similarly, $\mathcal{R}_{\overline{\text{co}}\,F}(u_0)$ and $\mathcal{T}r_{\overline{\text{co}}\,F}(u_0)$ denote the sets of all solutions and trajectories, respectively, of inclusion (1.3).

We assume that the mapping $F\colon T \times H \to cl(H)$ enjoys the following properties.

Hypotheses $H(F)$:

(1) the mapping $t \rightrightarrows F(t, x)$, $x \in H$, is measurable [7];

(2)
$$\text{haus}\,(F(t, x), F(t, y)) < k(t)\|x - y\| \quad \lambda \text{ a.e.}, \tag{1.7}$$
$$x, y \in H,\ x \neq y,\ k(\cdot) \in L^1_\lambda(T, \mathbb{R}^+),\ k(t) > 0,\ t \in T;$$

(3)
$$d(\Theta, F(t, \Theta)) < c(t) \ \ \lambda \text{ a.e.},\ c(\cdot) \in L^1_\lambda(T, \mathbb{R}^+),\ c(t) > 0,\ t \in T; \tag{1.8}$$

(3*)
$$\|F(t, \Theta)\| = \sup\{\|v\|;\ v \in F(t, \Theta)\} < c(t) \quad \lambda \text{ a.e.}; \tag{1.9}$$

(4) for all $\rho > d(\Theta, C(t))$, $t \in T$, the set $C_\rho(t) = C(t) \cap \rho \cdot \overline{B} \subset H$ is compact.

1.1. Main results

Theorem 1.2. *Let inequality* (1.1) *hold. Then:*

(1) *if Hypotheses* H(F)(1)–(3) *are valid, then the set* $\mathcal{R}_F(u_0)$ *is not empty;*

(2) *if Hypotheses* H(F)(1),(2),(3*) *are valid, then we have*

$$\overline{\mathcal{R}_{\overline{\text{co}}\,F}(u_0)} = \overline{\mathcal{R}_F(u_0)}, \tag{1.10}$$

where the bar stands for the closure in the space $BV_+(T, H) \times \omega\text{-}\ L^1_\lambda(T, H)$.

Theorem 1.3. *Let inequality* (1.1) *and the Hypotheses* $H(F)(1), (2), (3^*), (4)$ *hold. Then the set* $\mathcal{R}_{\overline{\text{co}}\, F}(u_0)$ *is a compact subset of the space* $BV_+(T, H) \times \omega\text{-}L^1_\lambda(T, H)$ *and we have* $\mathcal{R}_{\overline{\text{co}}\, F}(u_0) = \overline{\mathcal{R}_F(u_0)}$.

The statement (1) of Theorem 1.2 is an existence theorem for a solution of inclusion (1.2) with a perturbation whose values are closed, nonconvex, unbounded sets while the statement of Theorem 1.2(2) is a relaxation theorem.

In the case when Hypotheses $H(F)(1)$–(3) hold, equality (1.10) is valid if the space $L^1_\lambda(T, H)$ is endowed with the so-called weak norm (see (2.11) and Corollary 5.4).

A survey of results on the existence of solutions to inclusion (1.2) in the sense of Definition 1.1 is given in the work [12]. Only one of the references in this survey (namely, the article [5]) deals with a multivalued perturbation which admits convex compact values. All the other works, in particular [12], consider either a single valued perturbation or no perturbation at all. The present paper continues the research initiated in [14], where one studied the existence of solutions and the compactness of the solution set for inclusion (1.2) with a convex and compact valued perturbation. Unlike [5] treating only the existence of solutions, in [14] the compactness of the solution set was proved. In addition, the assumptions on the perturbation $F(t, x)$ in [14] are more general than those in [5].

To the best of the authors' knowledge, inclusion (1.2) with a perturbation $F(t, x)$ having closed nonconvex values has never been considered in the literature before.

Our article consists of five sections. In the first section, Introduction, we state our problem and present the main results. In Section 2, we give main notations, definitions and recall some facts necessary for our investigations. In Section 3, we prove several auxiliary results. Section 4 explores the existence of a solution to inclusion (1.2). In the last Section 5, we prove the relaxation theorem establishing a connection between the solutions of inclusions (1.2) and (1.3).

2. Main notations, definitions and preliminaries

Let $\mathcal{B}$ be the σ-algebra of all Borel sets of the interval $T = [0, a]$. Both scalar and vector measures defined on $\mathcal{B}$ are called Borel or Stieltjes measures [4, p. 358]. Since every positive scalar Borel measure is regular [4, p. 358], it is a Radon measure in the sense of Definition 1.5 in [8]. Hence, in what follows, by positive Radon measures we mean positive scalar Borel measures. For convenience, one says that Borel measures are defined on T instead of the σ-algebra $\mathcal{B}$.

A positive Radon measure ν is absolutely continuous with respect to a positive Radon measure μ, if from $\mu(B) = 0$, $B \in \mathcal{B}$, it follows that $\nu(B) = 0$. If the converse also holds, then the measures μ and ν are called absolutely continuously equivalent.

In the rest of the paper, λ is the Lebesgue measure on T. The variation $|m| \colon \mathcal{B} \to \mathbb{R}^+$ of a measure $m \colon \mathcal{B} \to H$ is defined as follows:

$$|m|(A) = \sup \sum_{i \in I} \|m(A_i)\|, \quad A \in \mathcal{B},$$

where the supremum is taken over all finite families $\{A_i\}$, $i \in I$, of mutually disjoined sets $A_i \in \mathcal{B}$, $\bigcup_{i\in I} A_i = A$ [4, Proposition 2, p. 32]. If $|m|(T) < \infty$, the measure m is called a measure with bounded variation. In this case, the variation $|m|$ of the measure m is a positive Radon measure.

A measure $m\colon \mathcal{B} \to H$ with bounded variation is called absolutely continuous with respect to a positive Radon measure μ, if the measure $|m|$ is absolutely continuous with respect to the measure μ. If a Stieltjes measure $m\colon \mathcal{B} \to H$ with bounded variation is absolutely continuous with respect to a positive Radon measure μ, then according to the Radon–Nikodym theorem [3] there exists a function $\hat{m} \in L^1_\mu(T, H)$ such that

$$m(A) = \int_A \hat{m}(\tau)\, d\mu(\tau), \quad A \in \mathcal{B}.$$

The function $\hat{m}(t)$ is called the density of the measure m with respect to the positive Radon measure μ and is denoted by $\frac{dm}{d\mu}(t)$, $t \in T$.

Theorem 2.1. *Let $u\colon T \to H$ be a right continuous function of bounded variation. Then, there exists a unique Stieltjes measure of bounded variation $m\colon \mathcal{B} \to H$ such that for $0 \le c \le d \le a$ the following equalities hold:*

$$m(]c, d]) = u(d) - u(c), \quad m([c, d]) = u(d) - u(c - 0). \tag{2.1}$$

Theorem 2.1 can be found in [4, p. 358, Theorem 1].

In different terminology (see, e.g., [5, 12, 10]), the measure $m(\cdot)$ is called the differential measure generated by the function u and is denoted by du.

From (2.1) it follows that
$$u(t) = u(s) + \int\limits_{]s,t]} du, \tag{2.2}$$

and
$$u(t) = u(s - 0) + \int\limits_{[s,t]} du, \quad s \le t,\ s, t \in T. \tag{2.3}$$

If $\hat{u} \in L^1_\nu(T, H)$ and
$$u(t) = u(0) + \int\limits_{]0,t]} \hat{u}(\tau)\, d\nu(\tau), \ \ t \in T, \tag{2.4}$$

then the function $u(t)$ is a right continuous function of bounded variation, the differential measure du generated by u is absolutely continuous with respect to the positive measure ν, $\hat{u}(\cdot)$ is the density of the measure du with respect to the measure ν, i.e.
$$\frac{du}{d\nu}(t) = \hat{u}(t) \quad \nu \ \text{a.e.}, \tag{2.5}$$

and for the variation $|du|$ of the measure du we have the equality

$$|du|(]s, t]) = \int\limits_{]s,t]} \|\hat{u}(\tau)\|\, d\nu(\tau), \quad s \le t,\ s, t \in T. \tag{2.6}$$

Let the Lebesgue measure λ be absolutely continuous with respect to a positive Radon measure ν, $\frac{d\lambda}{d\nu}(t)$ be the density of the measure λ with respect to the measure ν, $\hat{u} \in L^1_\lambda(T, H)$, and

$$u(t) = \int\limits_{]0,t]} \hat{u}(\tau)\, d\lambda(\tau), \quad t \in T.$$

Then,
$$u(t) = \int\limits_{]0,t]} \hat{u}(\tau)\, d\lambda(\tau) = \int\limits_{]0,t]} \hat{u}(\tau) \frac{d\lambda}{d\nu}(\tau)\, d\nu(\tau), \quad t \in T, \tag{2.7}$$

$u(t)$ is a right continuous function of bounded variation, the differential measure du generated by u is absolutely continuous with respect to the measure ν and we have ν a.e. the equality

$$\frac{du}{d\nu}(t) = \hat{u}(\tau) \frac{d\lambda}{d\nu}(t)\,. \tag{2.8}$$

Let $T(t,r) = [t-r, t+r] \cap T$, $T^+(t,r) = [t, t+r] \cap T$, $T^-(t,r) = [t-r, t] \cap T$, $r > 0$, $t \in T$. If a function $u\colon T \to H$ of bounded variation is right continuous and the differential measure du generated by u is absolutely continuous with respect to a positive Radon measure ν, then for the density $\frac{du}{d\nu}(t)$ of the measure du with respect to ν we have [11]:

$$\frac{du}{d\nu}(t) = \lim_{r \downarrow 0} \frac{du(T(t,r))}{\nu(T(t,r))} = \lim_{r \downarrow 0} \frac{du(T^+(t,r))}{\nu(T^+(t,r))} = \lim_{r \downarrow 0} \frac{du(T^-(t,r))}{\nu(T^-(t,r))} \quad \nu \text{ a.e.} \tag{2.9}$$

The space of all right continuous functions of bounded variation $u\colon T \to H$ with the topology of uniform convergence on T is denoted by $BV_+(T,H)$. The topology of the space $BV_+(T,H)$ is induced by the norm [14]:

$$\|u(\cdot)\|_{BV_+} = \sup\{\|u(t)\|;\ t \in T\}.$$

Let $\mathcal{U} \subset BV_+(T,H)$. The set $\mathcal{U}$ is called right equicontinuous at a point $s \in [0,a[$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x(s) - x(t)\| \le \varepsilon$ for all $x(\cdot) \in \mathcal{U}$ and $t \in]s, s+\delta[$. The set $\mathcal{U}$ is called left equicontinuous at a point $s \in]0,a]$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x(s-0) - x(t)\| \le \varepsilon$ for all $x(\cdot) \in \mathcal{U}$ and $t \in]s-\delta, s[$.

The set $\mathcal{U}$ is called *unilaterally equicontinuous* on T, if it is simultaneously right and left equicontinuous at every point $s \in]0,a[$, right equicontinuous at the point 0 and left equicontinuous at the point a.

A family $\mathcal{U}$ of functions of bounded variation $u\colon T \to H$ is called *uniformly bounded in variation* if there exists a constant $M > 0$ such that the variation $\mathrm{Var}(u)$ on T of every function $u(\cdot) \in \mathcal{U}$ satisfies the inequality

$$\mathrm{Var}(u) \le M. \tag{2.10}$$

If inequality (2.10) holds for the norms $\|u(\cdot)\|_{BV_+}$, $u(\cdot) \in \mathcal{U}$, then we call the family $\mathcal{U}$ uniformly bounded in norm.

In the space $L^1_\lambda(T,H)$ along with the standard norm

$$\|f\|_{L^1} = \int_T \|f(\tau)\|\,d\lambda(\tau)$$

we consider the so-called "weak" norm:

$$\big|\|f\|\big| = \sup_{0\le s\le t\le a} \left\| \int_s^t f(\tau)\,d\lambda(\tau) \right\|. \tag{2.11}$$

The space $L^1_\lambda(T,H)$ with the norm (2.11) is denoted by $|\omega|$-$L^1_\lambda(T,H)$.

It is evident that the norm (2.11) is equivalent to the norm

$$\|f\|_{|\omega|} = \sup_{t\in T} \left\| \int_{[0,t]} f(\tau)\,d\lambda(\tau) \right\|.$$

Recall that the space $L^1_\lambda(T,H)$ endowed with the weak topology is denoted by ω-$L^1_\lambda(T,H)$.

3. Auxiliary results

In this section, we state several auxiliary results which we will use to prove the main results of the paper.

Lemma 3.1. [13] *Let $V \subset \omega$-$L^1_\lambda(T,H)$ be a compact set. If a sequence $f_n \in V$, $n \ge 1$, converges to f in the space $L^1_\lambda(T,H)$ endowed with the "weak" norm* (2.11), *then it converges to f in the space ω-$L^1_\lambda(T,H)$.*

Theorem 3.2. [14] *Let $\mathcal{U} \subset BV_+(T,H)$ and the following conditions be fulfilled:*

(1) *the set $\mathcal{U}(t) = \{u(t);\ u(\cdot) \in \mathcal{U}\} \subset H,\ t \in T$, is relatively compact;*

(2) *the set $\mathcal{U}$ is unilaterally equicontinuous;*

(3) *the set $\mathcal{U}$ is uniformly bounded in variation.*

Then, the set $\mathcal{U}$ is a relatively compact subset of the space $BV_+(T,H)$, i.e. any sequence $u_n(\cdot) \in \mathcal{U}$, $n \ge 1$, contains a subsequence $u_{n_k}(\cdot)$, $k \ge 1$, uniformly converging on T to some function $u(\cdot) \in BV_+(T,H)$.

Corollary 3.3. *On every unilaterally equicontinuous and uniformly bounded in variation set $\mathcal{U} \subset BV_+(T,H)$ the topologies of pointwise and uniform convergences on T coincide.*

Theorem 3.4. [14] *Let $u, v\colon T \to H$ be two right continuous functions of bounded variation and their differential measures du and dv be absolutely continuous with respect to a positive Radon measure ν. Then, the function $w = u + v$ is a right continuous function of bounded variation, its differential measure dw is absolutely continuous with respect to the measure ν and for the densities $\frac{dw}{d\nu}(t)$, $\frac{du}{d\nu}(t)$, $\frac{dv}{d\nu}(t)$ of the measures dw, du, dv with respect to the measure ν we have*

$$\frac{dw}{d\nu}(t) = \frac{du}{d\nu}(t) + \frac{dv}{d\nu}(t) \quad \nu\ a.e.$$

Define
$$r(t) = r_0 + \int\limits_{]0,t]} \hat{r}(\tau)\, d\lambda(\tau), \quad t \in T, \quad \hat{r} \in L^1_\lambda(T, \mathbb{R}^+), \tag{3.1}$$

where $r_0 \geq \|u_0\|$, $u_0 \in C(0)$. The function $r(t)$ is a function of bounded variation, its differential measure dr is a positive Radon measure absolutely continuous with respect to the measure λ.

Let μ be a positive Radon measure and

$$\nu(r) = \mu + dr + \lambda. \tag{3.2}$$

The measure λ is absolutely continuous with respect to the measure $\nu(r)$ and the measures $\nu(r)$ and $\mu + \lambda$ are absolutely continuously equivalent.

Consider the multivalued mapping $\mathcal{U}\colon T \rightrightarrows H$,

$$\mathcal{U}(t) = \hat{r}(t) \cdot \overline{B}, \quad t \in T, \tag{3.3}$$

which is measurable with closed bounded values.

Consider the differential inclusion

$$-du \in \mathcal{N}(C(t); u(t)) + \mathcal{U}(t)\, d\lambda, \quad u(0) = u_0 \in C(0). \tag{3.4}$$

A solution to inclusion (3.4) is defined similarly to that for inclusion (1.2) changing $F(t, u(t))$ by $\mathcal{U}(t)$. The set of all solutions to inclusion (3.4) is denoted by $\mathcal{R}_{\mathcal{U}}(u_0)$ and the set of all trajectories by $\mathcal{T}r_{\mathcal{U}}(u_0)$. Let

$$S(\hat{r}) = \{\hat{g}(\cdot) \in L^1_\lambda(T, H);\ \|\hat{g}(t)\| \leq \hat{r}(t) \ \ \lambda \text{ a.e.}\}.$$

It is evident that if $(u(\hat{g}), \hat{g}) \in \mathcal{R}_{\mathcal{U}}(u_0)$, then $u(\hat{g})$ is a solution to the differential inclusion

$$-du \in \mathcal{N}(C(t); u(t)) + \hat{g}(t)\, d\lambda, \quad u(0) = u_0 \in C(0), \quad \hat{g} \in S(\hat{r}). \tag{3.5}$$

The inverse is also true, i.e. if $u(\hat{g})$ is a solution to inclusion (3.5) with $\hat{g} \in S(\hat{r})$, then $(u(\hat{g}), \hat{g})$ is a solution to inclusion (3.4).

Denote by $\mathcal{R}_S(u_0)$ the set of all solutions to inclusion (3.5) with $\hat{g} \in S(\hat{r})$.

Theorem 3.5. *Let inequality* (1.1) *hold. Then:*

(1) *the sets $\mathcal{R}_{\mathcal{U}}(u_0)$ and $\mathcal{R}_S(u_0)$ are not empty;*

(2) *the equality $\mathcal{T}r_{\mathcal{U}}(u_0) = \mathcal{R}_S(u_0)$ holds and for any $\hat{g} \in S(\hat{r})$ the function $u(\hat{g}) \in \mathcal{T}r_{\mathcal{U}}(u_0)$ is a unique solution to inclusion* (3.5) *such that*

$$-\frac{du(\hat{g})}{d\nu(r)}(t) - \hat{g}(t)\frac{d\lambda}{d\nu(r)} \in \mathcal{N}(C(t), u(\hat{g})(t) \quad \nu(r) \ a.e.; \tag{3.6}$$

(3) *for any $u(\hat{g}) \in \mathcal{T}r_{\mathcal{U}}(u_0)$ the following inequality holds for $s \leq t,\ s, t \in T$:*

$$\|u(\hat{g})(t) - u(\hat{g})(s)\| \leq \nu(r)(]s,t]) + dr(]s,t]);$$

(4) *for any* $(u(\hat{g}_i), \hat{g}_i) \in \mathcal{R}_{\mathcal{U}}(u_0)$, $i = 1, 2$, *we have the inequality*

$$\|u(\hat{g}_1)(t) - u(\hat{g}_2)(t)\| \leq \int_{]0,t]} \|\hat{g}_1(\tau) - \hat{g}_2(\tau)\| d\lambda(\tau),\ t \in T; \tag{3.7}$$

(5) *the set* $\mathcal{T}r_{\mathcal{U}}(u_0)$ *is uniformly bounded in norm and in variation;*

(6) *the set* $\mathcal{T}r_{\mathcal{U}}(u_0)$ *is unilaterally equicontinuous.*

Theorem 3.5 directly follows from Theorem 4.2 in [14].

Let $\hat{g}_i \in L^1_\lambda(T, H)$, $i = 1, 2$,

$$g_i(t) = \int_{]0,t]} \hat{g}_i(\tau) d\lambda(\tau),\ \ t \in T,\ i = 1, 2, \tag{3.8}$$

$|dg_i|$, $i = 1, 2$, be the variations of the differential measures dg_i generated by the functions g_i.

Corollary 3.6. *Let inequality* (1.1) *hold. Then, for any* $\hat{g}_i \in L^1_\lambda(T, H)$, $i = 1, 2$, *inclusion* (3.5) *has a unique solution* $u(\hat{g}_i)$ *satisfying the inclusion*

$$-\frac{du(\hat{g}_i)}{d\nu_i}(t) - \hat{g}_i(t)\frac{d\lambda}{d\nu_i} \in \mathcal{N}(C(t), u(\hat{g}_i)(t)) \quad \nu_i \text{ a.e.},\ i = 1, 2, \tag{3.9}$$

where

$$\nu_i = \mu + |dg_i| + \lambda,\ \ i = 1, 2, \tag{3.10}$$

and inequality (3.7) *holds.*

Fix i and let $\hat{r}(t) = \|\hat{g}_i(t)\|$ λ a.e. Then, from (2.6), (3.1), (3.8) it follows that

$$dr = |dg_i|. \tag{3.11}$$

Using (3.10), (3.11), (3.2), (3.3) and Theorem 3.5 we see that inclusion (3.5) has a unique solution satisfying (3.9).

To obtain inequality (3.7) for any $\hat{g}_i \in L^1_\lambda(T, H)$, $i = 1, 2$, it is enough to take

$$\hat{r}(t) = \|\hat{g}_1(t)\| + \|\hat{g}_2(t)\|\ \ \lambda \text{ a.e.}$$

in (3.3) and apply Theorem 3.5.

Lemma 3.7. *Let*

$$g_*(t) = \int_{]0,t]} \hat{g}_*(\tau) d\lambda(\tau),\ \ \hat{g}_* \in L^1_\lambda(T, H), \tag{3.12}$$

$$Cg_*(t) = C(t) + g_*(t),\ \ t \in T. \tag{3.13}$$

Then,

(1) $$\operatorname{exc}(Cg_*(s), Cg_*(t)) \leq \mu(]s,t]) + |dg_*|(]s,t]),\quad s \leq t,\ s, t \in T; \tag{3.14}$$

(2) $w \in \mathcal{N}(Cg_*(t), z)$, $t \in T$, $z \in Cg_*(t)$, *if and only if* $w \in \mathcal{N}(C(t), x)$ *and* $x = z - g_*(t)$.

Proof. From (3.13) we derive

$$\operatorname{exc}(Cg_*(s), Cg_*(t)) \leq \operatorname{exc}(C(s), C(t)) + \|g_*(t) - g_*(s)\|, \quad s \leq t,\ s, t \in T.$$

Now, inequality (3.14) follows from the last inequality and (1.1), (3.12), (2.5), (2.6).

Let $w \in \mathcal{N}(Cg_*(t), z)$, $z \in Cg_*(t)$. Then, according to (3.13) for an arbitrary $y \in C(t)$ there exists $v \in Cg_*(t)$ such that $y = v - g_*(t)$. From the definition of the normal cone $\mathcal{N}(Cg_*(t), \cdot)$ it follows that $\langle w, v - z\rangle \leq 0$.

From this inequality and (3.13) we obtain

$$\langle w, y - x\rangle \leq 0, \quad x = z - g_*(t), \quad x \in C(t).$$

Since $y \in C(t)$ is arbitrary, we have $w \in \mathcal{N}(C(t), x)$.

If $w \in \mathcal{N}(C(t), x)$, $x \in C(t)$, then the inclusion

$$w \in \mathcal{N}(Cg_*(t), z), \quad z = x + g_*(t), \quad z \in Cg_*(t),$$

is proved similarly. □

Let
$$\nu(g_*) = \mu + |dg_*| + \lambda, \tag{3.15}$$
where dg_* is the differential measure of the function $g_*(t)$ defined by equality (3.12). In view of Corollary 3 inclusion (3.5) has a unique solution $u(\hat{g}_*)$ for which the inclusion
$$-\frac{du(\hat{g}_*)}{d\nu(g_*)}(t) - \hat{g}_*(t)\frac{d\lambda}{d\nu(g_*)} \in \mathcal{N}(C(t); u(\hat{g}_*)(t)) \tag{3.16}$$
is valid $\nu(g_*)$ a.e. Consider the inclusion

$$-dz \in \mathcal{N}(Cg_*(t); z(t)) + \hat{f}(t)\, d\lambda, \quad z(0) = u_0 \in C(0), \quad \hat{f} \in L^1_\lambda(T, H). \tag{3.17}$$

From inequality (3.14) it follows that the measure $\mu(g_*) = \mu + |dg_*|$ for inclusion (3.17) is a counterpart of the measure μ in inequality (1.1). Let

$$f(t) = \int_{]0,t]} \hat{f}(\tau) d\lambda(\tau), \ \ t \in T,$$

df be the differential measure generated by the function f and

$$\nu(f) = \mu(g_*) + |df| + \lambda = \mu + |dg_*| + |df| + \lambda. \tag{3.18}$$

From inequality (3.14) and Corollary we see that inclusion (3.17) has a unique solution $z(\hat{f})$, $z(\hat{f})(0) = u_0$, the differential measure $dz(\hat{f})$ generated by the function $z(\hat{f})$ is absolutely continuous with respect to the measure $\nu(f)$ and for the density $\frac{dz(\hat{f})}{d\nu(f)}(t)$ of the measure $dz(\hat{f})$ with respect to the measure $\nu(f)$ we have

$$-\frac{dz(\hat{f})}{d\nu(f)}(t) - \hat{f}(t)\frac{d\lambda}{d\nu(f)} \in \mathcal{N}(Cg_*(t); z(\hat{f})(t)) \quad \nu(f) \text{ a.e.} \tag{3.19}$$

Lemma 3.8. *A function $z(\hat{f})$, $z(\hat{f})(0) = u_0$, is a solution to inclusion* (3.17) *if and only if the function*

$$u(t) = z(\hat{f})(t) - g_*(t) \tag{3.20}$$

is a solution to inclusion (3.5) *with*

$$\hat{g}(t) = \hat{f}(t) + \hat{g}_*(t) \quad \lambda \text{ a.e.}, \tag{3.21}$$

i.e.

$$u(\hat{g})(t) = z(\hat{f})(t) - g_*(t). \tag{3.22}$$

Proof. Let $z(\hat{f})$, $z(\hat{f})(0) = u_0$, be a solution to inclusion (3.17). Then, inclusion (3.19) holds true. According to (3.18) the measure λ is absolutely continuous with respect to the measure $\nu(f)$. Then, from (3.12) it follows that the measure dg_* is absolutely continuous with respect to the positive Radon measure $\nu(f)$. With the help of (2.8) we obtain

$$\frac{dg_*}{d\nu(f)}(t) = \hat{g}_*(t)\frac{d\lambda}{d\nu(f)} \quad \nu(f) \text{ a.e.} \tag{3.23}$$

From Theorem 3.4 and (3.19), (3.23) we have

$$-\frac{d\big(z(\hat{f}) - g_*\big)}{d\nu(f)}(t) - \big(\hat{g}_*(t) + \hat{f}(t)\big)\frac{d\lambda}{d\nu(f)} \in \mathcal{N}\big(Cg_*(t); z(t)\big) \quad \nu(f) \text{ a.e.}$$

Employing this inclusion and Lemma 3.7 we obtain

$$-\frac{d\big(z(\hat{f}) - g_*\big)}{d\nu(f)}(t) - \big(\hat{g}_*(t) + \hat{f}(t)\big)\frac{d\lambda}{d\nu(f)} \in \mathcal{N}\big(C(t); z(\hat{f})(t) - g_*(t)\big) \tag{3.24}$$

$\nu(f)$ a.e. Since the function $z(\hat{f})(t) - g_*(t)$, $z(\hat{f})(0) - g_*(0) = u_0$, is a right continuous function of bounded variation, the differential measure $d(z(\hat{f}) - g_*)$ generated by this function is absolutely continuous with respect to the measure $\nu(f)$, and the measures $\mu + \lambda$ and $\nu(f)$ are absolutely continuously equivalent, from (3.24) it follows that the function $u(t)$ defined by (3.20) is a solution to inclusion (3.5) with $\hat{g}(t)$ satisfying equality (3.21), i.e. equality (3.22) holds.

Let now a function $u(\hat{g})$, $u(\hat{g})(0) = u_0$, be a solution to inclusion (3.5) with $\hat{g}(t)$ defined by (3.21). According to Corollary 3 for

$$\nu(g) = \mu + |dg| + \lambda \text{ and } g(t) = \int_{]0,t]} \hat{g}(\tau)d\lambda(\tau) = \int_{]0,t]} \big(\hat{g}_*(\tau) + \hat{f}(\tau)\big)d\lambda(\tau)$$

we have the following inclusion $\nu(g)$ a.e.

$$-\frac{du(\hat{g})}{d\nu(g)}(t) - \big(\hat{g}_*(t) + \hat{f}(t)\big)\frac{d\lambda}{d\nu(g)} \in \mathcal{N}(C(t); u(\hat{g})(t)). \tag{3.25}$$

Since the measure λ is absolutely continuous with respect to the measure $\nu(g)$, from (3.12) we infer that the measure dg_* is absolutely continuous with respect to the Radon measure $\nu(g)$. Hence, in view of (2.8) we have

$$\frac{dg_*}{d\nu(g)}(t) = \hat{g}_*(t)\frac{d\lambda}{d\nu(g)} \quad \nu(g) \text{ a.e.} \tag{3.26}$$

Next, from Theorem 3.4 and (3.25), (3.26) we obtain

$$-\frac{d\big(u(\hat{g}) + g_*\big)}{d\nu(g)}(t) - \hat{f}(t)\frac{d\lambda}{d\nu(g)} \in \mathcal{N}\big(C(t); u(\hat{g})(t)\big) \quad \nu(g) \text{ a.e.}$$

Then, from Lemma 3.7 we deduce that

$$-\frac{d\big(u(\hat{g}) + g_*\big)}{d\nu(g)}(t) - \hat{f}(t)\frac{d\lambda}{d\nu(g)} \in \mathcal{N}\big(Cg_*(t); u(\hat{g})(t) + g_*(t)\big) \quad \nu(g) \text{ a.e.} \tag{3.27}$$

Since the function $z(\hat{f})(t) = u(\hat{g})(t) + g_*(t)$, $t \in T$, $z(\hat{f})(0) = u_0$, is a right continuous function of bounded variation the differential measure $dz(\hat{f})$ of which is absolutely continuous with respect to the measure $\nu(g)$, and the measures $\mu + |dg_*| + \lambda$ and $\nu(g) = \mu + |dg| + \lambda$ are absolutely continuously equivalent, from (3.27) it follows that the function $z(f)(t) = u(\hat{g})(t) + g_*(t)$, $t \in T$, is a solution to inclusion (3.17). $\square$

From Lemma 3.8 it follows that for a solution $u(\hat{g}_*)(t)$, $u(\hat{g}_*)(0) = u_0$, of inclusion (3.5) with $\hat{g}(t) = \hat{g}_*(t)$, $t \in T$, we have

$$u(\hat{g}_*)(t) = z(\Theta)(t) - \hat{g}_*(t), \tag{3.28}$$

where $z(\Theta)(t)$, $z(\Theta)(0) = u_0$, is a solution of inclusion (3.17) with $\hat{f}(t) = \Theta$, $t \in T$.

4. Existence of solutions

In this section, we prove the existence of solutions to inclusion (1.2).

According to Corollary 3 for any $\hat{g} \in L^1_\lambda(T, H)$ inclusion (3.5) has a unique solution $u(\hat{g})$, $u(\hat{g})(0) = u_0$. If $\hat{g}_i \in L^1_\lambda(T, H)$, $i = 1, 2$, then the solutions $u(\hat{g}_i)$, $u(\hat{g}_i)(0) = u_0$, $i = 1, 2$, of inclusion (3.5) satisfy inequality (3.7).

Let $u(\Theta), u(\Theta)(0) = u_0$, be the solution to inclusion (3.5) with $\hat{g}(t) = \Theta$, $t \in T$. Then, in view of (3.7) for any $\hat{g} \in L^1_\lambda(T, H)$ we have

$$\|u(\hat{g})(t) - u(\Theta)(t)\| \le \int_{]0,t]} \|\hat{g}(\tau)\| d\lambda(\tau). \tag{4.1}$$

Assume that
$$d(\Theta, F(t, u(\Theta)(t)) < a(t) \quad \lambda \text{ a.e.}, \tag{4.2}$$

$a(\cdot) \in L^1_\lambda(T, \mathbb{R}^+),\ a(t) > 0,\ t \in T$. Consider the differential equation

$$\dot{r}(t) = a(t) + k(t)r(t), \quad t \in T, \quad r(0) = r_0 \ge 0, \tag{4.3}$$

which has the unique solution

$$r(t) = r_0 e^{m(t)} + \int_{]0,t]} e^{m(t)-m(s)} a(s)\, d\lambda(s), \quad m(t) = \int_{]0,t]} k(\tau)\, d\lambda(\tau),\ t \in T, \tag{4.4}$$

where $k(\cdot)$ is the function appearing in inequality (1.7).

Since λ is the Lebesgue measure, in the rest of the paper, in place of $d\lambda(\tau)$ and λ a.e. we write as usual $d\tau$ and a.e., respectively.

Theorem 4.1. *Let Hypotheses $H(F)(1),(2)$ and inequality (4.2) hold. Then, for any $u_0 \in C(0)$ inclusion (1.2) has a solution $(u(\hat{g}), \hat{g})$, $u(\hat{g})(0) = u_0$, satisfying the inequalities*

$$\|u(\hat{g})(t) - u(\Theta)(t)\| \le r(t),\ t \in T, \qquad \|\hat{g}(t)\| \le \dot{r}(t) \ a.e. \tag{4.5}$$

Proof. Let
$$U(t,u) = F(t, u + u(\Theta)(t)), \quad u \in H,\ t \in T. \tag{4.6}$$

From Hypotheses $H(F)(1),(2)$ and (4.2) it follows that the map $U\colon T \times H \to cl(H)$ enjoys the following properties:

(a) the mapping $t \rightrightarrows U(t,x)$ is measurable;

(b) $\operatorname{haus}(U(t,u), U(t,v)) < k(t)\|u - v\|$ a.e. $u, v \in H,\ u \ne v;$ (4.7)

(c) $d(\Theta, U(t,u)) < a(t) + k(t)\|u\|$ a.e., $u \in H.$ (4.8)

From this inequality, (4.7) and (4.3) it directly follows that

$$U(t,u) \bigcap \dot{r}(t)\overline{B} \subset U(t,v) + k(t)\|u - v\|B \text{ a.e.}, \tag{4.9}$$

for $\|u\| \le r(t)$, $\|v\| \le r(t)$, $t \in T$, where $r(t)$ is the solution of the differential equation (4.3). Construct by induction a sequence $y_{i+1}(t), \hat{g}_i(t),\ i = 0,1,2,\dots,$ with the properties

$$y_0(t) = \Theta, \quad y_{i+1}(t) = u(\hat{g}_i)(t) - u(\Theta)(t), \quad t \in T, \tag{4.10}$$

where $u(\hat{g}_i)(t)$ is the solution of inclusion (3.5) with $\hat{g} = \hat{g}_i \in L^1_\lambda(T,H)$,

$$\hat{g}_i(t) \in U(t, y_i(t)) \ \text{ a.e.}, \tag{4.11}$$

$$\|\hat{g}_i(t)\| \le \dot{r}(t) \ \text{ a.e.}, \quad \|y_{i+1}(t)\| \le r(t),\ t \in T, \tag{4.12}$$

$$\|\hat{g}_i(t) - \hat{g}_{i-1}(t)\| \le k(t)\|y_i(t) - y_{i-1}(t)\| \ \text{ a.e.},\ i \ge 1, \tag{4.13}$$

$$\begin{aligned} &\|\hat{g}_i(t) - \hat{g}_{i-1}(t)\| \le \\ &\le k(t)\left\{ r_0 \frac{[m(t)]^{i-1}}{(i-1)!} + \int_{]0,t]} \frac{[m(t) - m(s)]^{i-1}}{(i-1)!} a(s)\, ds \right\} \ \text{ a.e.},\ i \ge 1, \end{aligned} \tag{4.14}$$

$$\|y_{i+1}(t) - y_i(t)\| \le r_0 \frac{[m(t)]^i}{i!} + \int_{]0,t]} \frac{[m(t) - m(s)]^i}{i!} a(s)\, ds. \tag{4.15}$$

From (4.8), (4.10) we deduce that

$$\alpha_0(t) := d(\Theta, U(t, y_0(t)) < a(t) \text{ a.e.} \tag{4.16}$$

Since the function $\alpha_0(t)$ is measurable and

$$\alpha_0(t) < \frac{\alpha_0(t) + a(t)}{2} < a(t) \text{ a.e.},$$

from (4.16) it follows that the mapping

$$t \rightrightarrows U(t, y_0(t)) \bigcap \frac{a(t) + \alpha_0(t)}{2} \overline{B}$$

is measurable with nonempty closed values. Then [7], there exists a measurable selector $\hat{g}_0(t)$ of this mapping satisfying the inequality

$$\|\hat{g}_0(t)\| \leq a(t) \text{ a.e.}, \quad \hat{g}_0(t) \in U(t, y_0(t)) \text{ a.e.} \tag{4.17}$$

From (4.1), (4.3), (4.10), and (4.17) it follows that

$$\|\hat{g}_0(t)\| \leq \dot{r}(t) \text{ a.e.}, \quad \|y_0(t)\| \leq r(t),\ t \in T, \tag{4.18}$$

$$\|y_1(t)\| = \|u(\hat{g}_0)(t) - u(\Theta)(t)\| \leq \int_{]0,t]} a(s)\, ds \leq r(t), \quad t \in T. \tag{4.19}$$

Using (4.9), (4.17)–(4.19) we obtain

$$d(\hat{g}_0(t), U(t, y_1(t))) < k(t)\|y_1(t) - y_0(t)\| \text{ a.e.} \tag{4.20}$$

when $\|y_1(t) - y_0(t)\| \neq 0$.

Let $\mathcal{T}_0 = \{t \in T;\ \|y_1(t)\| = 0\}$. As the function $t \to y_1(t)$ is Borel, the set $\mathcal{T}_0$ is measurable. Since the function $\alpha_1(t) := d(\hat{g}_0(t), U(t, y_1(t))$ is measurable and

$$\alpha_1(t) < \frac{\alpha_1(t) + k(t)\|y_1(t)\|}{2} < k(t)\|y_1(t)\|,\ t \notin \mathcal{T}_0,$$

the multivalued mapping

$$t \rightrightarrows U(t, y_1(t)) \bigcap \left(\hat{g}_0(t) + \frac{\alpha_1(t) + k(t)\|y_1(t)\|}{2} \overline{B} \right), \quad t \in T \backslash \mathcal{T}_0$$

is measurable with nonempty closed values. Hence, there exists a measurable selector $\hat{g}_1^*(t)$ of this mapping defined on $T \backslash \mathcal{T}_0$. Let $\hat{g}_1(t) = \hat{g}_1^*(t)$, $t \in T \backslash \mathcal{T}_0$, $\hat{g}_1(t) = \hat{g}_0(t)$, $t \in \mathcal{T}_0$. Then,

$$\hat{g}_1(t) \in U(t, y_1(t)) \text{ a.e. and,} \tag{4.21}$$

$$\|\hat{g}_1(t) - \hat{g}_0(t)\| \leq k(t)\|y_1(t) - y_0(t)\| \text{ a.e.} \tag{4.22}$$

From (4.17), (4.3), (4.19), (4.22) it follows that

$$\|\hat{g}_1(t)\| \leq a(t) + k(t)\left(r_0 + \int_{]0,t]} a(s)\,ds\right) \leq a(t) + k(t)r(t) \quad \text{a.e.} \tag{4.23}$$

From this inequality, (4.1) and the equality

$$\|y_2(t)\| = \|u(\hat{g}_1)(t) - u(\Theta)(t)\| \tag{4.24}$$

we obtain

$$\|\hat{g}_1(t)\| \leq \dot{r}(t) \;\text{ a.e.}, \quad \|y_2(t)\| \leq r(t), \;\; t \in T. \tag{4.25}$$

Using (3.7), (4.3), (4.19), (4.22) we see that

$$\|\hat{g}_1(t) - \hat{g}_0(t)\| \leq k(t)\left(r_0 + \int_{]0,t]} a(s)\,ds\right) \quad \text{a.e.}, \tag{4.26}$$

$$\|y_2(t) - y_1(t)\| = \|u(\hat{g}_1)(t) - u(\hat{g}_0)(t)\| \leq$$

$$\leq \int_{]0,t]} k(\tau)\left(r_0 + \int_{]0,\tau]} a(s)\,ds\right) d\tau = r_0 m(t) + \int_{]0,t]} k(\tau)\left(\int_{]0,\tau]} a(s)\,ds\right) d\tau. \tag{4.27}$$

Since

$$\int_{]0,t]} k(\tau)\left(\int_{]0,\tau]} a(s)\,ds\right) d\tau = \int_{]0,t]}\int_{]0,t]} k(\tau)a(s)\,d\tau\,ds - \tag{4.28}$$

$$- \int_{]0,t]} k(\tau)\left(\int_{]\tau,t]} a(s)\,ds\right) d\tau = \int_{]0,t]} m(t)a(s)\,ds - \tag{4.29}$$

$$- \int_{]0,t]} a(s)\left(\int_{]0,s]} k(\tau)\,d\tau\right) ds = \int_{]0,t]} [m(t) - m(s)]a(s)\,ds, \tag{4.30}$$

from (4.27) it follows that

$$\|y_2(t) - y_1(t)\| \leq r_0 m(t) + \int_{]0,t]} [m(t) - m(s)]a(s)\,ds. \tag{4.31}$$

From (4.21), (4.22), (4.25), (4.26), (4.31) we deduce that relations (4.11)–(4.15) hold for $i = 1$.

Suppose that we have constructed $y_i(t) = u(\hat{g}_{i-1})(t) - u(\Theta)(t), \hat{g}_{i-1}(t)$ satisfying (4.11)–(4.15). In particular,

$$\hat{g}_{i-1}(t) \in U(t, y_{i-1}(t)) \;\text{ a.e.},$$

$$\|\hat{g}_{i-1}(t)\| \leq \dot{r}(t) \;\text{ a.e.}, \quad \|y_i(t)\| \leq r(t), \quad \|y_{i-1}(t)\| \leq r(t), \;\; t \in T.$$

From these inequalities and (4.9) it follows that

$$\hat{g}_{i-1}(t) \in U(t, y_{i-1}(t)) \bigcap \dot{r}(t)\overline{B} \subset U(t, y_i(t)) + k(t)\|y_i(t) - y_{i-1}(t)\|B \quad \text{a.e.}$$

According to this inclusion we have

$$d(\hat{g}_{i-1}(t), U(t, y_i(t)) < k(t)\|y_i(t) - y_{i-1}(t)\| \quad \text{a.e.},$$

when $\|y_i(t) - y_{i-1}(t)\| \neq 0$. Reasoning as above, we conclude that there exists a measurable function $\hat{g}_i(t)$ such that

$$\hat{g}_i(t) \in U(t, y_i(t)) \text{ a.e.,} \tag{4.32}$$

$$\|\hat{g}_i(t) - \hat{g}_{i-1}(t)\| \leq k(t)\|y_i(t) - y_{i-1}(t)\| \quad \text{a.e.} \tag{4.33}$$

From (4.33) we see that

$$\|\hat{g}_i(t)\| \leq k(t) \sum_{j=1}^{i} \|y_j(t) - y_{j-1}(t)\| + \|\hat{g}_0(t)\|.$$

Using the last inequality, (4.15), (4.17) we obtain

$$\|\hat{g}_i(t)\| \leq k(t) \left[r_0 \sum_{j=0}^{i-1} \frac{[m(t)]^j}{j} + \sum_{j=0}^{i-1} \int_{]0,t]} \frac{[m(t) - m(s)]^j}{j!} a(s)\, ds \right] + a(t) \text{ a.e.} \tag{4.34}$$

Since
$$1 + \frac{z}{1!} + \ldots + \frac{z^j}{j!} \leq e^z, \quad z \geq 0,$$

from (4.4) we deduce that

$$\|\hat{g}_i(t)\| \leq a(t) + k(t)r(t) = \dot{r}(t) \quad \text{a.e.} \tag{4.35}$$

Then, according to (4.1), (4.10) we have

$$\|y_{i+1}(t)\| = \|u(\hat{g}_i)(t) - u(\Theta)(t)\| \leq r(t), \quad t \in T. \tag{4.36}$$

Using (4.33), (4.15) we obtain

$$\|\hat{g}_i(t) - \hat{g}_{i-1}(t)\| \leq$$
$$\leq k(t) \left[r_0 \frac{[m(t)]^{i-1}}{(i-1)!} + \int_{]0,t]} \frac{[m(t) - m(s)]^{i-1}}{(i-1)!} a(s)\, ds \right] \text{ a.e.} \tag{4.37}$$

From (4.37), (3.7) and the equality

$$\|y_{i+1}(t) - y_i(t)\| = \|u(\hat{g}_i)(t) - u(\hat{g}_{i-1})(t)\|, \quad t \in T,$$

it follows that inequality (4.15) holds for i. Indeed, it is enough to compare the derivative of the right-hand side of inequality (4.15) with the right-hand side of inequality (4.14) and take into account the fact that the left- and right-hand sides of the inequality (4.15) equal zero when $t = 0$.

From (4.32)–(4.37) it follows that relations (4.11)–(4.15) hold for $y_{i+1}(t), \hat{g}_i(t)$ and the sequence $y_{i+1}(t), \hat{g}_i(t)$, $i \geq 0$, is thus constructed.

From (4.37) we infer that the series $\sum\limits_{i=0}^{\infty} \|\hat{g}_{i+1}(t) - \hat{g}_i(t)\|$ converges almost everywhere. Therefore, $\hat{g}_i(t)$, $i \geq 1$, is a Cauchy sequence for a.e. $t \in T$. Consequently, the sequence $\hat{g}_i(t)$, $i \geq 1$, converges almost everywhere to a measurable function $\hat{g}(t)$.

From (4.12) it follows that $\quad \|\hat{g}(t)\| \leq \dot{r}(t) \quad \text{a.e.} \qquad (4.38)$

Hence, the sequence $\hat{g}_i(t),\ i \geq 1$, converges to $\hat{g}$ in the space $L^1_\lambda(T,H)$. Let $u(\hat{g})$ be the solution to inclusion (3.5). Then, according to Corollary 3 from inequality (3.7) it follows that

$$\|u(\hat{g})(t) - u(\hat{g}_i)(t)\| \leq \int_{]0,t]} \|\hat{g}(s) - \hat{g}_i(s)\|\, ds, \quad i \geq 0.$$

From this inequality we infer that the sequence $u(\hat{g}_i),\ i \geq 1$, converges to $u(\hat{g})$ in the space $BV_+(T,H)$. Consequently, the sequence $y_{i+1}(t) = u(\hat{g}_i)(t) - u(\Theta)(t)$, $i \geq 0$, converges to $y(t) = u(\hat{g})(t) - u(\Theta)(t)$ in the space $BV_+(T,H)$. According to (4.12) we have

$$\|y(t)\| = \|u(\hat{g})(t) - u(\Theta)(t)\| \leq r(t), \ \ t \in T. \tag{4.39}$$

Using (4.9), (4.11), (4.12), (4.39) we obtain

$$d(\hat{g}_i(t), U(t, y(t)) \leq k(t)\|y_i(t) - y(t)\| \ \text{ a.e.}$$

Passing in this inequality to the limit as $i \to \infty$ and taking account of (4.6) we deduce that

$$\hat{g}(t) \in U(t, y(t)) = U(t, u(\hat{g})(t) - u(\Theta)(t)) = F(t, u(\hat{g})(t)) \ \text{ a.e.} \tag{4.40}$$

From (4.38)–(4.40) it follows that $(u(\hat{g}), \hat{g})$ is a solution to inclusion (1.2) satisfying inequalities (4.5). □

Corollary 4.2. *Let Hypotheses* $H(F)(1) - (3)$ *hold. Then, for any* $u_0 \in C(0)$ *inclusion* (1.2) *has a solution.*

From $H(F)(2),(3)$ it follows that $d(\Theta, F(t, u(\Theta)(t)) < c(t) + k(t)\|u(\Theta)(t)\|$ a.e. Hence, in inequality (4.2) the function $c(t) + k(t)\|u(\Theta)(t)\|$ can be taken as the function $a(t)$.

Remark 4.3. A solution $(u(\hat{g}), \hat{g})$, $u(\hat{g})(0) = u_0$, of inclusion (1.2) which exists according to Theorem 4.1 satisfies inclusion (1.4) with the measure $\nu = \mu + dr + \lambda$, where $r(t)$ is the solution of Eq. (4.3) and dr is the differential measure generated by $r(t)$.

Remark 4.4. The inequalities (4.5) are also valid for the solution $r(t)$ of Eq. (4.3) with $r(0) = 0$.

5. Relaxation

In this section, we prove the relaxation theorem.

Let $\hat{r} \in L^1_\lambda(T;\mathbb{R}^+)$, $r(t)$, $t \in T$, be the function defined by (3.1), $\nu(r)$ be the measure defined by (3.2), and $\mathcal{U}\colon T \rightrightarrows H$ be a measurable multivalued mapping with convex compact values satisfying the inequality

$$\|\mathcal{U}\| = \sup\{\|v\|;\ v \in \mathcal{U}(t)\} \leq \hat{r}(t) \quad \text{a.e.} \tag{5.1}$$

Denote by $S_{\mathcal{U}}$ the collection of all measurable selectors of the mapping $t \rightrightarrows \mathcal{U}(t)$. The set $S_{\mathcal{U}}$ is nonempty, compact, and metrizable in the topology of the space ω-$L^1_\lambda(T, H)$.

Theorem 5.1. *Let inequality* (1.1) *hold. Then, for any sequence $\hat{g}_n \in S_{\mathcal{U}}$, $n \geq 1$, converging in the space ω-$L^1_\lambda(T, H)$ to $\hat{g}_0$, the sequence $u(\hat{g}_n)$, $n \geq 1$, of the solutions of inclusion* (3.5) *with $\hat{g} = \hat{g}_n$ converges in the space $BV_+(T, H)$ to the solution of inclusion* (3.5) *with $\hat{g} = \hat{g}_0$.*

Proof. From Theorem 3.5 it follows that the sequence $u(\hat{g}_n)$, $n \geq 1$, is a unilaterally equicontinuous, uniformly bounded in norm and in variation subset of the space $BV_+(T, H)$. From Theorem 2.1 [9, Chapter 0] it follows that there exists a subsequence $u(\hat{g}_{n_k})$, $k \geq 1$, of the sequence $u(\hat{g}_n)$, $n \geq 1$, converging pointwise in the weak topology of the space H to some function of bounded variation w. Using (3.6) and [12, Proposition 3.3] we obtainfor $t \in T$

$$\frac{1}{2}\|u(\hat{g}_{n_k})(t) - u(\hat{g}_0)(t)\|^2 \leq$$
$$\leq \int_{]0,t]} \langle u(\hat{g}_{n_k})(\tau) - u(\hat{g}_0)(\tau), \hat{g}_0(\tau) - \hat{g}_{n_k}(\tau)\rangle \frac{d\lambda}{d\nu}(\tau) d\nu(\tau), \qquad (5.2)$$

Since the function w is integrable, similarly to inequality (5.19) in [14] from inequality (5.2) we obtain

$$\frac{1}{2}\|u(\hat{g}_{n_k})(t) - u(\hat{g}_0)(t)\|^2 \leq$$
$$\leq \int_{]0,t]} \langle u(\hat{g}_{n_k})(\tau) - w(\tau), \hat{g}_0(\tau) - \hat{g}_{n_k}(\tau)\rangle d\tau +$$
$$+ \int_{]0,t]} \langle w(\tau), \hat{g}_0(\tau) - \hat{g}_{n_k}(\tau)\rangle d\tau, \quad t \in T. \qquad (5.3)$$

The fact that $\hat{g}_{n_k}(\tau) \in \mathcal{U}(\tau)$, $k \geq 1$, implies that the set $\{\hat{g}_0(\tau) - \hat{g}_{n_k}(\tau); k \geq 1\} \subset H$ is relatively compact for almost every $\tau \in T$.

From the boundedness of the sequence $u(\hat{g}_{n_k})(\tau) - w(\tau)$, $k \geq 1$, $\tau \in T$, in H it follows that the sequence of functions $h \to \langle u(\hat{g}_{n_k})(\tau) - w(\tau), h\rangle$, $k \geq 1$, $\tau \in T$, is equicontinuous. It is known that on every equicontinuous set the topology of pointwise convergence coincides with that of uniform convergence on compact sets. Hence,

$$\lim_{k\to\infty} \langle u(\hat{g}_{n_k})(\tau) - w(\tau), \hat{g}_0(\tau) - \hat{g}_{n_k}(\tau)\rangle = 0 \quad \text{a.e.}$$

Now, from (5.1) and Lebesgue's dominated convergence theorem we conclude that

$$\lim_{k\to\infty} \int_{]0,t]} \langle u(\hat{g}_{n_k})(\tau) - w(\tau), \hat{g}_0(\tau) - \hat{g}_{n_k}(\tau)\rangle d\tau = 0, \ t \in T.$$

The second integral in (5.3) converges to zero since the sequence $\hat{g}_{n_k}$, $k \geq 1$, converges in the space ω-$L^1_\lambda(T, H)$ to $\hat{g}_0$, and the function $w(\tau)$ is measurable and bounded. Therefore, from (5.3) we infer that the sequence $u(\hat{g}_{n_k})(t)$, $k \geq 1$, converges pointwise to $u(\hat{g}_0)(t)$ in the space H. Then, from Corollary 3.3 we see that the sequence $u(\hat{g}_{n_k})$, $k \geq 1$, converges to $u(\hat{g}_0)$ in the space $BV_+(T, H)$.

We have proved that if a sequence $\hat{g}_n \in S_{\mathcal{U}}$, $n \geq 1$, converges in the space ω-$L^1_\lambda(T, H)$ to $\hat{g}_0$, then there exists a subsequence $\hat{g}_{n_k}$, $k \geq 1$, of the sequence $\hat{g}_n$, $n \geq 1$, such that the sequence $u(\hat{g}_{n_k})$, $k \geq 1$, converges in the space $BV_+(T, H)$ to $u(\hat{g}_0)$. Suppose that the sequence $u(\hat{g}_n)$, $n \geq 1$, does not converge in the space $BV_+(T, H)$ to $u(\hat{g}_0)$. Then, there exists a subsequence $u(\hat{g}_{n_k})$, $k \geq 1$, of the sequence $u(\hat{g}_n)$, $n \geq 1$, such that any subsequence of the sequence $u(\hat{g}_{n_k})$, $k \geq 1$, does not converge to $u(\hat{g}_0)$ in the space $BV_+(T, H)$. Repeating the reasoning above to the sequence $u(\hat{g}_{n_k})$, $k \geq 1$, and taking into account the fact that inclusion (3.5) with $\hat{g} = \hat{g}_0$ has a unique solution $u(\hat{g}_0)$, we arrive at a contradiction. □

Let $\hat{g}_* \in L^1_\lambda(T, H)$, $g_*(t)$ be the function defined by (3.12), and $Cg_*(t)$ be the multivalued mapping defined by (3.13).

Lemma 5.2. *Let Hypotheses $H(F)(1), (2)$ hold and assume that for the solution $u(\hat{g}_*)$, $u(\hat{g}_*)(0) = u_0$, of inclusion (3.5) with $\hat{g} = \hat{g}_*$ the inequality*

$$d(\hat{g}_*(t), F(t, \hat{g}(u_*)(t)) < a(t), \tag{5.4}$$

$a \in L^1_\lambda(T, \mathbb{R}^+)$, $a(t) > 0$, $t \in T$ holds a.e. Then, there exists a solution $(u(\hat{g}), \hat{g})$, $u(\hat{g})(0) = u_0$, of inclusion (1.2) such that

$$\|u(\hat{g})(t) - u(\hat{g}_*)(t)\| \leq r(t), \quad t \in T, \tag{5.5}$$

$$\|\hat{g}(t) - \hat{g}_*(t)\| \leq \dot{r}(t) \quad a.e., \tag{5.6}$$

where $r(t)$ is the solution of Eq. (4.3) with $r(0) = r_0 \geq 0$.

Proof. Let
$$U(t, z) = -\hat{g}_*(t) + F(t, z - g_*(t)). \tag{5.7}$$

Consider the inclusion

$$-dz \in \mathcal{N}(Cg_*(t), z(t)) + U(t, z(t))\, d\lambda, \quad z(0) = u_0 \in Cg_*(0) = C(0), \tag{5.8}$$

which we rewrite in the form

$$-dz \in \mathcal{N}(Cg_*(t), z(t)) + \hat{f}(t)\, d\lambda, \tag{5.9}$$

$$z(0) = u_0 \in Cg_*(0), \quad \hat{f}(t) \in U(t, z(t)). \tag{5.10}$$

From Hypotheses $H(F)(1), (2)$ it follows that the mapping $U(t, z)$ satisfies the same hypotheses with $F(t, u)$ changed by $U(t, z)$. According to (3.28), (5.4), (5.7) we have

$$\begin{aligned} d(\Theta, U(t, z(\Theta)(t)) &= d(\hat{g}_*(t), F(t, z(\Theta)(t)) - g_*(t)) \\ &= d(\hat{g}_*(t), F(t, u(\hat{g}_*)(t)) < a(t) \quad \text{a.e.} \end{aligned}$$

Therefore, all the assumptions of Theorem 4.1 are valid for the mapping $U(t,z)$. According to this theorem inclusion (5.8) has a solution $(z(\hat{f}),\hat{f})$, $z(\hat{f})(0)=u_0$,

$$\hat{f}(t)\in U(t,z(\hat{f})(t)) \quad \text{a.e.}, \tag{5.11}$$

satisfying the inequalities

$$\|z(\hat{f})(t)-z(\Theta)(t)\|\le r(t),\;\; t\in T, \quad \|\hat{f}(t)\|\le \dot{r}(t) \quad \text{a.e.} \tag{5.12}$$

Using Lemma 3.8 we see that the pair $(u(\hat{g}),\hat{g})$, where $u(\hat{g})$ is defined by (3.22) and $\hat{g}(t)$ by (3.21) is the solution of inclusion (3.5) with $g=\hat{g}$. Since

$$\|u(\hat{g})(t)-u(\hat{g}_*)(t)\|=\|z(\hat{f})(t)-z(\Theta)(t)\|,\; t\in T,\; \|\hat{g}(t)-\hat{g}_*(t)\|=\|\hat{f}(t)\| \text{ a.e.},$$

inequalities (5.5), (5.6) follow from inequalities (5.12).

To finish the proof we need to show the validity of the inclusion

$$\hat{g}(t)\in F(t,u(\hat{g})(t)) \quad \text{a.e.}$$

This inclusion follows from (3.21), (3.22), (5.7), and (5.11). □

Theorem 5.3. *Let Hypotheses $H(F)(1)-(3)$ hold. Then, for any solution $(u(\hat{g}_*),\hat{g}_*)\in\mathcal{R}_{\overline{\mathrm{co}}\,F}(u_0)$ there exists a sequence of solutions $(u(\hat{g}_k),\hat{g}_k)\in\mathcal{R}_F(u_0)$, $k\ge 1$, such that*

$$u(\hat{g}_k)\to u(\hat{g}_*) \quad \textit{in } BV_+(T,H), \tag{5.13}$$

$$\hat{g}_k\to\hat{g}_* \quad \textit{in } |\omega|\text{-}L^1_\lambda(T,H). \tag{5.14}$$

If Hypothesis $H(F)(3^*)$ holds instead of $H(F)(3)$, then

$$\hat{g}_k\to\hat{g}_* \quad \textit{in } \omega\text{-}L^1_\lambda(T,H). \tag{5.15}$$

Proof. From Corollary 4.2 it follows that the sets $\mathcal{R}_F(u_0)$ and $\mathcal{R}_{\overline{\mathrm{co}}\,F}(u_0)$ are not empty. Let $(u(\hat{g}_*),\hat{g}_*)\in\mathcal{R}_{\overline{\mathrm{co}}\,F}(u_0)$. Then,

$$\hat{g}_*(t)\in\overline{\mathrm{co}}\,F(t,u(\hat{g}_*)(t)) \quad \text{a.e.} \tag{5.16}$$

Denote by S_F and $S_{\overline{\mathrm{co}}\,F}$ the sets of the measurable selectors of the mappings $t\rightrightarrows F(t,u(\hat{g}_*)(t))$ and $t\rightrightarrows\overline{\mathrm{co}}\,F(t,u(\hat{g}_*)(t))$, respectively, which are elements of the space $L^1_\lambda(T,H)$. According to inequalities (1.7), (1.8) we have

$$d(\Theta,F(t,u(\hat{g}_*)(t))<c(t)+k(t)\|u(\hat{g}_*)(t)\| \quad \text{a.e.}$$

From this inequality it follows that the set S_F is not empty. Then, from [6, Theorem 1.5] we infer that

$$S_{\overline{\mathrm{co}}\,F}=\overline{\mathrm{co}}\,S_F,$$

where the closed convex hull $\overline{\mathrm{co}}$ on the right-hand side of this equality is taken in the space $L^1_\lambda(T,H)$. From this equality and (5.16) we see that $\hat{g}_*\in\overline{\mathrm{co}}\,S_F$. Therefore, for any $\varepsilon>0$ there exists an element $\hat{g}_\varepsilon\in L^1_\lambda(T,H)$ such that

$$\hat{g}_\varepsilon\in\mathrm{co}\,S_F \;\text{ and }\; \|\hat{g}_*-\hat{g}_\varepsilon\|_{L^1_\lambda(T,H)}\le\varepsilon. \tag{5.17}$$

Let $u(\hat{g}_\varepsilon)$ be the solution of inclusion (3.5) with $\hat{g} = \hat{g}_\varepsilon$. Then, from Corollary 3 it follows that

$$\|u(\hat{g}_\varepsilon)(t) - u(\hat{g}_*)(t)\| \leq \int_{]0,t]} \|\hat{g}_\varepsilon(\tau) - \hat{g}_*(\tau)\| d\tau, \ t \in T.$$

From this inequality we infer that

$$u(\hat{g}_\varepsilon) \to u(\hat{g}_*) \quad \text{in} \ \ BV_+(T, H) \ \ \text{as} \ \ \varepsilon \to 0, \tag{5.18}$$

$$\hat{g}_\varepsilon \to \hat{g}_* \quad \text{in} \ \ L^1_\lambda(T, H) \ \ \text{as} \ \ \varepsilon \to 0. \tag{5.19}$$

Fix $\hat{g}_\varepsilon$. According to (5.17) there exists a finite collection of elements

$$\hat{g}_i(\cdot) \in S_F, \quad i = 1, \dots, n, \tag{5.20}$$

and numbers $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$ such that $\hat{g}_\varepsilon = \sum\limits_{i=1}^{n} \alpha_i \hat{g}_i$.

Let $U \colon T \rightrightarrows H$ be the multivalued mapping

$$U(t) = \{\hat{g}_i(t); \ 1 \leq i \leq n\}, \ \ t \in T,$$

which has compact values. Then,

$$\hat{g}_\varepsilon(t) \in \operatorname{co} U(t) \quad \text{a.e.} \tag{5.21}$$

From (5.20) we see that the function

$$t \to \|U(t)\| = \sup\{\|v\|; \ v \in U(t)\}$$

is an element of the space $L^1_\lambda(T, \mathbb{R}^+)$. Hence, from (5.21) and [15, Theorem 2.2] it follows that there exists a sequence $\hat{v}_n \in L^1_\lambda(T, H)$, $n \geq 1$,

$$\hat{v}_n(t) \in U(t) \subset F(t, u(\hat{g}_*)(t)) \quad \text{a.e.} \tag{5.22}$$

such that
$$\hat{v}_n \to \hat{g}_\varepsilon \quad \text{in} \ \ |\omega|\text{-}L^1_\lambda(T, H) \ \ \text{as } n \to \infty. \tag{5.23}$$

Let $u(\hat{v}_n)$, $u(\hat{v}_n)(0) = u_0$, $n \geq 1$, be the solution of inclusion (3.5) with $\hat{g} = \hat{v}_n$. Since the set $V = \{w \in L^1_\lambda(T, H); \ \|w(t)\| \leq \|U(t)\| \ \text{a.e.}\}$ is convex and compact in the space ω–$L^1_\lambda(T, H)$, from (5.22), (5.23) and Lemma 3.1 we deduce that

$$\hat{v}_n \to \hat{g}_\varepsilon \quad \text{in} \ \ \omega\text{-}L^1_\lambda(T, H) \ \ \text{as } n \to \infty.$$

Since the values of the mapping co $U(t)$ are convex compact sets in the space H, from (5.22) and Theorem 5.1 we infer that

$$u(\hat{v}_n) \to u(\hat{g}_\varepsilon) \quad \text{in} \ \ BV_+(T, H) \ \ \text{as } n \to \infty. \tag{5.24}$$

From (5.18), (5.19), (5.23) and (5.24) it follows that there exists a sequence $\hat{v}_k \in L^1_\lambda(T,H)$, $k \geq 1$,

$$\hat{v}_k(t) \in F(t, u(\hat{g}_*)(t)) \quad \text{a.e.,} \tag{5.25}$$

such that

$$\hat{v}_k \to \hat{g}_* \quad \text{in } \ |\omega|\text{-}L^1_\lambda(T,H) \quad \text{as } k \to \infty, \tag{5.26}$$

$$u(\hat{v}_k) \to u(\hat{g}_*) \quad \text{in } \ BV_+C(T,H) \quad \text{as } k \to \infty. \tag{5.27}$$

If Hypothesis $H(F)(3^*)$ holds instead of $H(F)(3)$, the set $\overline{\text{co}}\, S_F$ is convex and compact in the space ω-$L^1_\lambda(T,H)$. Since $\hat{v}_k \in \overline{\text{co}}\, S_F$, $k \geq 1$, from (5.27) and Lemma 3.1 it follows that

$$\hat{v}_k \to \hat{g}_* \quad \text{in } \ \omega\text{-}L^1_\lambda(T,H) \quad \text{as } k \to \infty. \tag{5.28}$$

Let

$$a_k(t) = k(t)\|u(\hat{g}_*)(t) - u(\hat{v}_k)(t)\| + \frac{1}{k}. \tag{5.29}$$

From (5.27), (5.29) it follows that $a_k \in L^1_\lambda(T,\mathbb{R}^+)$, $a_k(t) > 0$, $k \geq 1$, $t \in T$, and

$$a_k \to \Theta \quad \text{in } \ L^1_\lambda(T,\mathbb{R}^+) \quad \text{as } k \to \infty. \tag{5.30}$$

According to (1.7), (5.25), (5.29) we have

$$d(\hat{v}_k(t), F(t, u(\hat{v}_k)(t)) < a_k(t) \quad \text{a.e.}$$

From this inequality and Lemma 5.2 we conclude that there exists a solution $(u(\hat{g}_k), \hat{g}_k) \in \mathcal{R}_F(u_0)$ such that

$$\|u(\hat{g}_k)(t) - u(\hat{v}_k)(t)\| \leq r_k(t), \quad t \in T,\ k \geq 1, \tag{5.31}$$

$$\|\hat{g}_k(t) - \hat{v}_k(t)\| \leq \dot{r}_k(t) \quad \text{a.e.},\ k \geq 1, \tag{5.32}$$

where $r_k(t)$, $r_k(0) = 0$, is the solution of equation (4.3) with $a(t) = a_k(t)$. From (5.30), (4.3), (4.4) it follows that

$$r_k \to 0 \quad \text{in } \ C(T,\mathbb{R}^+) \quad \text{as } k \to \infty, \tag{5.33}$$

and

$$\dot{r}_k \to 0 \quad \text{in } \ L^1_\lambda(T,\mathbb{R}^+) \quad \text{as } k \to \infty. \tag{5.34}$$

Using (5.31), (5.32), (5.26), (5.27), (5.33), (5.34) we arrive at (5.13), (5.14).

If Hypothesis $H(F)(3^*)$ holds, then instead of (5.26) we need to use (5.28) and (5.31), (5.32), (5.27), (5.33), (5.34) to arrive at (5.13), (5.15). □

Corollary 5.4. *Let Hypotheses $H(F)(1)-(3)$ and inequality (1.1) hold. Then, we have*

$$\overline{\mathcal{R}_{\overline{\text{co}}\,F}(u_0)}^{|\omega|} = \overline{\mathcal{R}_F(u_0)}^{|\omega|}, \tag{5.35}$$

where the bar above stands for the closure in the space $BV_+(T,H) \times |\omega|$-$L^1_\lambda(T,H)$, and

$$\overline{\mathcal{T}r_{\overline{\text{co}}\,F}(u_0)} = \overline{\mathcal{T}r_F(u_0)}, \tag{5.36}$$

where the bar above stands for the closure in the space $BV_+(T,H)$.

In the case when Hypotheses $H(F)(1),(2),(3^*)$ hold, equalities (1.10) and (5.36) are valid.

Theorem 1.2 now follows from Corollaries 4.2 and 5.4. The equalities (1.10), (5.35), and (5.36) are usually called *relaxation*.

Proof of Theorem 1.3. From the inequalities (1.7), (1.8) it follows that

$$\|\overline{\mathrm{co}}F(t,u)\| = \sup\{\|v\|;\ v \in \overline{\mathrm{co}}F(t,u)\} \le c(t) + k(t)\|u\|.$$

From this inequality and Corollary 3 we infer that for any $(u(\hat{g}),\hat{g}) \in \mathcal{R}_{\overline{\mathrm{co}}\,F}(u_0)$ we have

$$\|u(\hat{g})(t)\| \le M + \int_{]0,t]} (c(\tau) + k(\tau)\|u(\hat{g})(\tau)\|)\,d\tau, \quad t \in T, \tag{5.37}$$

where $M = \sup\{\|u(\Theta)(t)\|;\ t \in T\}$. Using Lemma 5.1 in [14] and (5.37) we see that for any $(u(\hat{g}),\hat{g}) \in \mathcal{R}_{\overline{\mathrm{co}}\,F}(u_0)$ the following inequalities hold:

$$\|u(\hat{g})(t)\| \le r(t), \quad t \in T, \tag{5.38}$$

$$\|\hat{g}(t)\| \le \dot{r}(t) \quad \text{a.e.}, \tag{5.39}$$

where $r(t)$ is the solution of the equation

$$\dot{r}(t) = c(t) + k(t)r(t), \quad r(0) = M.$$

From inequalities (5.38), (5.39), Hypothesis $H(F)(4)$ and Theorem 3.5 we deduce that the following properties are valid:

(1) the set $\mathcal{T}r_{\overline{\mathrm{co}}\,F}(u_0)$ is uniformly bounded in norm and variation;

(2) the set $\mathcal{T}r_{\overline{\mathrm{co}}\,F}(u_0)$ is unilaterally equicontinuous;

(3) the set $\mathcal{U}(t) = \{u(\hat{g})(t);\ (u(\hat{g}),\hat{g}) \in \mathcal{R}_{\overline{\mathrm{co}}\,F}(u_0)\}$, $t \in T$, is relatively compact in H.

Hence, from (5.39) and Theorem 3.2 it follows that the set $\mathcal{R}_{\overline{\mathrm{co}}\,F}(u_0)$ is relatively compact in the space $BV_+(T,H) \times \omega\text{-}L^1_\lambda(T,H)$.

To finish the proof of Theorem 1.3 it is enough to show that the set $\mathcal{R}_{\overline{\mathrm{co}}\,F}(u_0)$ is closed in the space $BV_+(T,H) \times \omega\text{-}L^1_\lambda(T,H)$. From inequality (5.39) and the metrizability of compact subsets of the space $\omega\text{-}L^1_\lambda(T,H)$ we infer that it is enough to show the sequential closedness of the set $\mathcal{R}_{\overline{\mathrm{co}}\,F}(u_0)$.

Let $(u(\hat{g}_n),\hat{g}_n) \in \mathcal{R}_{\overline{\mathrm{co}}\,F}(u_0)$, $n \ge 1$, converge to $(w,\hat{g}_0)$ in the space $BV_+(T,H) \times \omega\text{-}L^1_\lambda(T,H)$. Then, invoking Hypothesis $H(F)(2)$ and Mazur's lemma on weakly convergent sequences we obtain

$$\hat{g}_0(t) \in \bigcap_{n=1}^{\infty} \overline{\mathrm{co}} \bigcup_{k \ge n} \mathrm{co}\,F(t, u(\hat{g}_k)(t)) \subset \overline{\mathrm{co}}F(t,w(t)) \quad \text{a.e.} \tag{5.40}$$

Let $u(\hat{g}_0)$ be the solution of inclusion (3.5) with $\hat{g} = \hat{g}_0$. Then, similarly to (5.3) we have

$$\frac{1}{2}\|u(\hat{g}_n)(t)-u(\hat{g}_0)(t)\|^2 \leq$$
$$\leq \int_{]0,t]}\langle u(\hat{g}_n)(\tau)-w(\tau),\hat{g}_0(\tau)-\hat{g}_n(\tau)\rangle d\tau + \int_{]0,t]}\langle w(\tau),\hat{g}_0(\tau)-\hat{g}_n(\tau)\rangle d\tau. \quad (5.41)$$

Since

$$\int_{]0,t]}\langle u(\hat{g}_n)(\tau)-w(\tau),\hat{g}_0(\tau)-\hat{g}_n(\tau)\rangle\, d\tau \leq \sup_{t\in T}\{\|u(\hat{g}_n)(t)-w(t)\|\}\cdot 2\int_T \dot{r}(\tau)\, d\tau,$$

from the convergence of $u(\hat{g}_n)$, $n\geq 1$, to w in the space $BV_+(T,H)$, the convergence of $\hat{g}_n$, $n\geq 1$, in the space ω-$L^1_\lambda(T,H)$, and (5.41) it follows that

$$u(\hat{g}_n)(t)\to u(\hat{g}_0)(t) \quad \text{in } H,\ t\in T, \quad \text{as } n\to\infty.$$

Now, from properties (1), (2) of the set $\mathcal{T}r_{\overline{\text{co}}\,F}(u_0)$ and Corollary 3.3 we conclude that

$$u(\hat{g}_n)\to u(\hat{g}_0) \quad \text{in } BV_+(T,H) \ \text{ as } n\to\infty.$$

Consequently, $u(\hat{g}_0)(t)=w(t)$, $t\in T$. Finally, (5.40) implies that $(u(\hat{g}_0),\hat{g}_0)\in \mathcal{R}_{\overline{\text{co}}\,F}(u_0)$. And, the theorem thus follows.

Corollary 5.5. *Let Hypotheses $H(F)(1),(2),(3^*),(4)$ hold. Then, $\mathcal{T}r_{\overline{\text{co}}\,F}(u_0)$ is a compact subset of the space $BV_+(T,H)$ and we have*

$$\mathcal{T}r_{\overline{\text{co}}\,F}(u_0)=\overline{\mathcal{T}r_F(u_0)},$$

where the bar above stands for the closure in the space $BV_+(T,H)$.

In conclusion, we remark that in the proofs of both the existence theorem and the relaxation theorem (inequality (5.35)) we do not assume the boundedness of values of the mapping $F(t,u)$.

In [1, Theorem 4.1] the authors proved the existence of a unique solution to inclusion

$$-du\in\mathcal{N}(C(t);u(t))+f(t,u(t))\,d\lambda, \quad t\in T, \quad u(0)=u_0\in C(0), \qquad (5.42)$$

under the following assumptions:

$$|d(y,C(t))-d(y,C(s))|\leq \mu(]s,t]), \quad y\in H,\ s\leq t,\ s,t\in T, \qquad (5.43)$$

and the mapping $f\colon T\times H\to H$ is such that

(i) there exists a function $\beta(\cdot)\in L^1_\lambda(T,\mathbb{R}^+)$ such that

$$\|f(t,x)\|\leq\beta(t)(1+\|x\|), \quad x\in\bigcup_{t\in T}C(t); \qquad (5.44)$$

(ii) for every $r>0$ the family of functions $\{f(\cdot,x);\ x\in r\overline{B}\}$ is equicontinuous and there exists a function $L_r(\cdot)\in L^1_\lambda(T,\mathbb{R}^+)$ such that

$$\|f(t,x)-f(t,y)\|\leq L_r(t)\|x-y\|, \quad t\in T,\ x,y\in r\overline{B}. \qquad (5.45)$$

We now show that the existence of a solution for inclusion (5.42) follows from the statement 1) of our Theorem 1.2. In particular, the existence result of [2], where the equicontinuity of the above family of functions is not assumed, is recovered by our theorem for the case of a convex moving set $C(t)$.

Given a point $x \in H$, let $\mathrm{pr}(C(t);x)$ be the projection of x onto the set $C(t)$. The function $(t,x) \to \mathrm{pr}(C(t);x)$ enjoys the following properties:

(1) the function $t \to \mathrm{pr}(C(t);x)$ is Lebesgue measurable for any $x \in H$;

(2) $\| \mathrm{pr}(C(t);x) - \mathrm{pr}(C(t);y)\| \leq \|x-y\| \quad x,y \in H,\ t \in T;$ (5.46)

(3) $\| \mathrm{pr}(C(t);x)\| \leq d(\Theta, C(0)) + \mu(T) + 2\|x\|, \quad t \in T.$ (5.47)

Indeed, from (5.43) it follows that for every $x \in H$ the function $t \to d(x, C(t))$ has bounded variation and, thus, is Lebesgue measurable. Invoking [7, Theorem 3.5] we see that the mapping $t \rightrightarrows C(t) \cap (x + d(x,C(t))\overline{B})$, $t \in T$, has nonempty values and is measurable as an intersection of measurable mappings. Since

$$\mathrm{pr}(C(t);x) = C(t) \cap (x + d(x,C(t))\overline{B}),\ t \in T,$$

statement (1) is proved.

Inequality (5.46) follows from the properties of projections. Inequality (5.47) follows from the inequality

$$d(x,C(t)) \leq d(x,C(0)) + \mu(T) \leq d(\Theta, C(0)) + \mu(T) + \|x\|, \quad t \in T.$$

which is a consequence of (5.43) and the inequality

$$\| \mathrm{pr}(C(t);x)\| \leq d(x,C(t)) + \|x\|.$$

Let
$$M(r) = d(\Theta, C(0)) + \mu(T) + 2r, \quad r \geq 0. \tag{5.48}$$

$$K_r(t) = L_{M(r)}(t) + 1, \tag{5.49}$$

$K_r(t) > 0$, $t \in T$, $K_r(\cdot) \in L^1_\lambda(T, \mathbb{R}^+)$, where $L_{M(r)}(t)$ is the function appearing in inequality (5.45), and

$$\hat{f}(t,x) = f(t, \mathrm{pr}(C(t);x)), \quad x \in H,\ t \in T. \tag{5.50}$$

From (5.44), (5.47) we deduce that

$$\|\hat{f}(t,x)\| < \alpha(t)(1 + \|x\|), \quad x \in H,\ t \in T, \tag{5.51}$$

$$\alpha(t) = 2(\beta(t)+1)(1 + d(\Theta, C(t)) + \mu(T)), \quad \alpha(t) > 0,\ \alpha(\cdot) \in L^1_\lambda(T, \mathbb{R}^+).$$

Using (ii), (5.45), (5.46), (5.48) (5.49), (5.50) we obtain

$$\|\hat{f}(t,x) - \hat{f}(t,y)\| \leq K_r(t)\|x-y\|, \quad x,y \in r\overline{B}, \tag{5.52}$$

and the function $t \to \hat{f}(t,x)$, $x \in H$, is Lebesgue measurable. From (5.43) it follows that inequality (1.1) is valid. Then, there exists a solution $u(\Theta)(t)$, $u(\Theta)(0) = u_0$, of inclusion (5.42) without perturbation. Let

$$M = \sup\{u(\Theta)(t);\ t \in T\}.$$

Denote by $r(t)$, $r(0) = M$, the solution of the differential equation

$$\dot{r} = \alpha(t)(1 + r(t)), \quad r(0) = M. \tag{5.53}$$

Set $R = \max\{r(t);\ t \in T\}$ and

$$\phi(t, x) = \hat{f}(t, \mathrm{pr}(R\overline{B}; x)). \tag{5.54}$$

Then, from (5.51), (5.52), (5.54) we infer that the function $\phi(t, x)$ enjoys the following properties $H(\phi)$:

(1) the function $t \to \phi(t, x)$, $x \in H$, is Lebesgue measurable;

(2) $\|\phi(t, x) - \phi(t, y)\| \leq K_R(t)\|x - y\|, \quad x, y \in H;$

(3) $\|\phi(t, x)\| < \alpha(t)(1 + \|x\|), \quad x \in H.$

Consider the inclusion

$$-du \in \mathcal{N}(C(t); u(t)) + \phi(t, u(t))\, d\lambda. \tag{5.55}$$

From the properties $H(\phi)\, 1) - 3)$ and the statement (1) of Theorem 1.2 it follows that inclusion (5.55) has a solution

$$(u(\hat{g}), \hat{g}), \quad \hat{g}(t) = \phi(t, u(\hat{g})(t)), \quad t \in T. \tag{5.56}$$

Using Corollary 3 and (5.56) we see that for $u(\hat{g})$, $u(\Theta)$ inequality (3.7) holds which in turn implies that

$$\|u(\hat{g})(t)\| \leq M + \int_{]0,t]} \alpha(\tau)(1 + \|u(\hat{g})(\tau)\|)\, d\lambda(\tau),\ t \in T. \tag{5.57}$$

Let
$$\gamma(t) = M + \int_{]0,t]} \alpha(\tau)(1 + \|u(\hat{g})(\tau)\|)\, d\lambda(\tau),\ t \in T. \tag{5.58}$$

Then, the function $\gamma(t)$, $\gamma(0) = M$, is absolutely continuous and

$$\dot{\gamma}(t) \leq \alpha(t)(1 + \gamma(t)) \quad \text{a.e.}$$

From (5.57), (5.58) and theorems on differential inequalities it follows that

$$\|u(\hat{g})(t)\| \leq \gamma(t) \leq r(t), \quad t \in T. \tag{5.59}$$

From this inequality we deduce that $\|u(\hat{g})(t)\| \leq R$, $t \in T$. Since $u(\hat{g})(t) \in C(t)$, employing (5.50), (5.54) we obtain

$$f(t, u(\hat{g})(t)) = \phi(t, u(\hat{g})(t)), \quad t \in T.$$

Consequently, the function $u(\hat{g})(t)$, $t \in T$, $\hat{g}(t) = f(t, u(\hat{g})(t))$ is a solution to inclusion (5.42).

The uniqueness of a solution to inclusion (5.42) can be proved, using the results of our work, as follows. If $u(\hat{g}_i)(t)$, $\hat{g}_i(t) = f(t, u(\hat{g}_i)(t))$, $i = 1, 2$, are two solutions of inclusion (5.42), with the help of Corollary 3 and (3.7), (5.45) we obtain

$$\|u(\hat{g}_1)(t) - u(\hat{g}_2)(t)\| \leq \int_{]0,t]} L_m(\tau)\|u(\hat{g}_1)(\tau) - u(\hat{g}_2)(\tau)\|\, d\tau,\ t \in T,$$

where $m = \sup_{t\in T} \max\{\|u(\hat{g}_1)(t)\|, \|u(\hat{g}_2)(t)\|\}$. From this inequality similarly to (5.59) we have

$$\|u(\hat{g}_1)(t) - u(\hat{g}_2)(t)\| \leq r_0(t), \quad t \in T,$$

where $r_0(t)$, $r_0(0) = 0$, is the solution of the differential equation

$$\dot{r}_0(t) = L_m(t)r(t), \quad t \in T.$$

Since $r(t) = 0$, $t \in T$, we see that

$$u(\hat{g}_1)(t) = u(\hat{g}_2)(t), \quad t \in T.$$

We note that Theorem 4.1 in [1] is valid if in the assumption (ii) we suppose that the function $t \to f(t, x)$ is Lebesgue measurable for every $x \in H$ as, in this case, the function $t \to \hat{f}(t, x)$ is Lebesgue measurable for every $x \in H$ as well.

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