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Compactness of BV Solutions of a Convex Sweeping Process of Measurable Differential Inclusion*

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A measurable sweeping process with a multivalued perturbation is considered in a separable Hilbert space. The retraction of the sweeping process is a function of bounded variation whose differential measure is absolutely continuous with respect to a positive Radon measure. The values of the multivalued perturbation are convex compact sets. By a solution of the sweeping process we mean a right continuous function of bounded variation whose differential measure is absolutely continuous with respect to some positive Radon measure and the density of the differential measure with respect to the Radon measure satisfies the corresponding differential inclusion. In order to study the existence of solutions and properties of the solution set, we introduce the topology of uniform convergence on the space of right continuous functions of bounded variation and we investigate the compactness of sets in this space. Under the most general assumptions similar to those used to deal with absolutely continuous solutions, we prove the existence of solutions and the compactness of the solution set.

Keywords: Function of bounded variation, Radon measure, differential measure, density of a measure, BV solutions, existence theorem.

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1. Introduction

Let H be a separable Hilbert space with the *norm* $\|\cdot\|$, the *metric* $d(\cdot,\cdot)$, the *scalar product* $\langle\cdot,\cdot\rangle$, and the null element Θ. We introduce the following notation:

$\mathbb{R}^+ = [0,+\infty)$, $\overline{\mathbb{R}^+} = [0,+\infty]$, $T = [0,a] \subset \mathbb{R}^+$, $a > 0$, $C\colon T \to H$ is a multivalued mapping with nonempty convex closed values, $F\colon T \times H \to H$ is a multivalued mapping with convex compact values, $\operatorname{exc}(C(s), C(t))$, $s \le t$, $s,t \in T$ is the *excess* of the set $C(s)$ over the set $C(t)$,

$$\operatorname{exc}(C(s), C(t)) = \sup\{d(x, C(t));\ x \in C(s)\}.$$

We always assume that the following inequality holds

$$\operatorname{exc}(C(s), C(t)) \le \mu(]s,t]),\ s \le t,\ s,t \in T, \tag{1.1}$$

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where μ is a positive Radon measure on T. In what follows, λ is the Lebesgue measure. Consider the following measurable sweeping process

$$-du \in \mathcal{N}(C(t); u(t)) + F(t, u(t))\, d\lambda, \quad u(0) \in C(0), \tag{1.2}$$

where $\mathcal{N}(C(t); \cdot)$ is the normal cone in the sense of convex analysis of the set $C(t)$.

Let $$\rho_F = 1, \text{ if } F \not\equiv \{\Theta\}, \text{ and } \rho_F = 0, \text{ if } F \equiv \{\Theta\}.$$

Definition 1.1. By a *solution* of inclusion (1.2) we mean a function $u\colon T \to H$ with the following properties:

(1) the function $u(\cdot)$ is a right continuous function of bounded variation on T, $u(0) = u_0,\ u(t) \in C(t),\ t \in T$;

(2) there exists a λ Bochner integrable selector $y_u(\cdot)$ of the mapping $t \to F(t, u(t))$ and a positive Radon measure ν absolutely continuously equivalent to the measure $\mu + \rho_F \lambda$ such that the differential measure du generated by the function $u(\cdot)$ is absolutely continuous with respect to the measure ν and the density $\frac{du}{d\nu}(\cdot)$ of the measure du with respect to the measure ν satisfies ν almost everywhere the inclusion

$$\frac{du}{d\nu}(t) + y_u(t)\frac{d\lambda}{d\nu} \in -\mathcal{N}(C(t); u(t)). \tag{1.3}$$

Positive Radon measures α and β are absolutely continuously equivalent if they are mutually absolutely continuous with respect to each other.

From (1.1) it follows that the retraction $\operatorname{ret}(t)$ [11] of the mapping $C(t)$ is a right continuous function of bounded variation whose differential measure $d\operatorname{ret}$ is absolutely continuous with respect to the measure μ.

The notion of a solution to a sweeping process with measure and perturbation was introduced in the works [5, 15], where the correctness of such a definition was discussed and the properties of solutions were investigated. We note that when $F(t, x) = \Theta$ the conventional notion of solution to inclusion (1.2) and that in the sense of Definition 1.1 coincide [5, 15].

Let $\overline{B}$ be the unit closed ball in H centered at the point Θ, $m(\cdot), n(\cdot) \in L^1(T, \mathbb{R}^+)$.

We always assume that the values of the mapping F are convex compact sets. For a closed bounded set $L \subset H$ we write

$$\|L\| := \{\sup \|x\|;\ x \in L\}.$$

We make the following assumptions.

Hypotheses $H_1(F)$:

(1) $d(\Theta, F(t, x)) \le m(t) + n(t)\|x\|\ \lambda$ a.e., $\ x \in H$; (1.4)

(2) the set-valued mapping $t \to F(t, x) \cap (m(t) + n(t)\|x\|)\overline{B},\ x \in H$ has a measurable selector, and the mapping $x \to F(t, x)$ has closed graph for λ almost every $t \in T$;

(3) for every bounded set $L \subset H$ the set $F(t, L) \cap (m(t) + n(t)\|L\|)\overline{B},\ t \in T$ is relatively compact for λ almost every $t \in T$.

Hypotheses $H_2(F)$:

(1) $\|F(t, x)\| \leq m(t) + n(t)\|x\|\ \lambda$ a.e., $x \in H$; (1.5)

(2) hypotheses $H_1(F)(2)$ and (3) hold.

1.1. Main results

Theorem 1.2. *Let inequality* (1.1) *and Hypotheses* $H_1(F)$ *hold. Then, inclusion* (1.2) *has a solution.*

Theorem 1.3. *Let inequality* (1.1) *and Hypotheses* $H_2(F)$ *hold. Then, the solution set* $\mathcal{R}(u_0)$ *of inclusion* (1.2) *is compact in the space of right continuous functions of bounded variation, endowed with the topology of uniform convergence on* T. *For any* $\varepsilon > 0$, *the set of all points* $t \in T$ *at which at least one of the functions* $u(\cdot) \in \mathcal{R}(u_0)$ *has a jump greater than* ε *is finite. The set of all points at which at least one of the functions* $u(\cdot) \in \mathcal{R}(u_0)$ *has a jump is at most countable.*

If the measure ν is absolutely continuous with respect to the measure λ, then the measures ν and λ are absolutely continuously equivalent. This will be the case if the measure μ is absolutely continuous with respect to the measure λ. In this case, sets of ν null measure and λ null measure coincide. Hence, from (1.3) it follows that

$$\frac{du}{d\lambda}(t) + y_u(t) \in -\mathcal{N}(C(t); u(t)) \ \ \lambda \text{ a.e.} \tag{1.6}$$

If the differential measure du of a right continuous function of bounded variation u is absolutely continuous with respect to the Lebesgue measure λ, then the function $u(t)$ is absolutely continuous and the following equality holds λ almost everywhere on T

$$\frac{du}{d\lambda}(t) = \frac{du}{dt}.$$

From this equality and (1.6) we infer that under Hypotheses $H_1(F)$ and $H_2(F)$ the differential inclusion

$$\frac{du}{dt} \in -\mathcal{N}(C(t); u(t)) + F(t, u(t)) \tag{1.7}$$

has an absolutely continuous solution, and the set of all solutions is a compact subset of the space of continuous functions of bounded variation with the topology of uniform convergence on T, and thus is a compact set in $C(T, H)$.

There is a large number of works dealing with the existence of absolutely continuous solutions to inclusion (1.2). A fairly complete survey of the relevant results can be found in reference [15]. When it comes to BV solutions of inclusion (1.2) the number of the related works is much less. Moreover, most of the latter works either do not consider perturbations at all or only single-valued perturbations.

The author is aware only of the articles [5, 13] in which the perturbations are multivalued functions. Since these two works are most closely related to our study, below we give a short account of them.

In [5] one assumes that $C(t)$ is a nonempty closed r-prox regular set satisfying the inequality

$$h(C(s),C(t)) \leq \mu(]s,t]),\ s \leq t, \tag{1.8}$$

where $h(\cdot,\cdot)$ is the Hausdorff distance between closed sets. The multivalued mapping $F(t,x)$ with convex compact values enjoys the conditions stated in Theorems 4.1 and 5.1 in [5] which are particular cases of the condition in Hypotheses $H_1(F)$, since a scalarly upper semicontinuous mapping with convex closed values has closed graph, and the assumption (iii) in Theorems 4.1 and 5.1 in [5] implies Hypothesis $H_1(F)(3)$.

In [13] one studies the same sweeping process with the perturbation being the sum of a multivalued mapping $F(t,u)$ and a single-valued mapping $f(t,u)$. The multivalued mapping $F(t,u)$ enjoys the same properties as in [5], and the single-valued function $f(t,u)$ is measurable in t and satisfies the Lipschitz condition in u. In these two works one considers only the existence of BV solutions and, as a particular case, of absolutely continuous solutions. To the best of the author's knowledge, there have been no contributions so far addressing the compactness of the set of BV solutions of inclusion (1.2), nor considering compactness criteria in the space of right continuous functions of bounded variation with the topology of uniform convergence on an interval.

Our article consists of an introduction and four sections.

In the Introduction, we state our problem and present the main results.

In Section 2, some properties of the spaces of functions of bounded variation with the topology of uniform convergence on T are studied.

In Section 3, we recall properties of positive Radon measures necessary for our investigations and prove auxiliary statements which we use later.

In Section 4, a measurable sweeping process with a single valued perturbation not depending on the phase variable is considered and some properties of the solution set which depend on the perturbation are studied.

Finally, in Section 5, we prove Theorems 1.2 and 1.3.

2. The space of right continuous function of bounded variation

Following [14], a function $x\colon T \to H$ is called *regulated* if it has either continuity points or discontinuity points of the first kind. It is known that a regulated function has at most countable number of discontinuity points [14]. Denote by $\Pi(T,H)$ the space of all regulated functions, $\Pi_+(T,X)$ the space of all right continuous regulated functions.

Let $u\colon T \to H,\ [c,d] \subset T,\ c < d$. For the partition I of the interval $[c,d]$ by the points $c = x_0 < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_n = d$ consider the sum

$$V_I(u) = \sum_{i=0}^{n-1} \|u(x_{i-1}) - u(x_i)\|.$$

If I' is a refinement of the partition I, then evidently we have $V_I(u) \leq V_{I'}(u)$.

The quantity
$$V_c^d(u) = \sup_I V_I(u),$$

where the supremum is taken over all finite partitions I of the interval $[c,d]$ is called the *variation* of u on the interval $[c,d]$. It is clear that $V_c^d(u) \leq V_0^a(u)$ for any interval $[c,d] \subset T$.

A function u is called a *function of bounded variation on* T if $V_0^a(u) < \infty$.

Monotone functions are examples of scalar functions of bounded variation defined on T.

A function $x\colon T \to H$ is called *step function from the right* if there exists a finite partition $0 = c_0 < c_1 < \ldots < c_i < \ldots < c_n = a$ of the interval T such that the function $x(t)$ is constant on each of the intervals $[c_{i-1}, c_i[,\ 1 \leq i \leq n$.

Let $V_+(T,H)$ be the space of all right continuous functions of bounded variation and $S_+(T,H)$ be the space of all right continuous step functions.

If $\mathcal{T} \subset T$, then the quantity $\sup\{\|x(t) - x(s)\|,\ t,s \in \mathcal{T}\}$ is called *the oscillation* of the function $x\colon T \to H$ on the set $\mathcal{T}$. If $t \in T$ is either a continuity point or a discontinuity point of the first kind of the function $x\colon T \to H$, then the quantity $\omega(x(t)) = \max\{\|x(t) - x(t-0)\|,\ \|x(t) - x(t+0)\|,\ \|x(t-0) - x(t+0)\|\}$ is called *the oscillation of the function x at the point $t \in T$*. Here, $x(t-0)$ and $x(t+0)$ are the left and right limits of the function x at the point t.

Let $U \subset \Pi_+(T,H)$. The set U is called *equicontinuous from the right at a point* $s \in [0,a[$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x(s) - x(t)\| \leq \varepsilon$ for all $x(\cdot) \in U$ and $t \in [s, s+\delta[$. The set $U \subset \Pi_+(T,H)$ is called *equicontinuous from the left at a point* $s \in]0,a]$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x(s-0) - x(t)\| \leq \varepsilon$ for all $x(\cdot) \in U$ and all $t \in]s-\delta, s[$.

The set $U \subset \Pi_+(T,H)$ is called *unilaterally equicontinuous* on T, if it is equicontinuous simultaneously from the left and from the right at each point $s \in]0,a[$, equicontinuous from the right at the point 0 and from the left at the point a.

We endow the spaces $\Pi(T,H)$ and $\Pi_+(T,H)$ with the topology of uniform convergence on T.

Theorem 2.1. *The space $\Pi_+(T,H)$ is a separable Banach space with the norm*

$$\|x(\cdot)\| = \sup\{\|x(t)\| :\ t \in T\}. \tag{2.1}$$

The sets $S_+(T,H)$ and $V_+(T,H)$ satisfy the inclusion $S_+(T,H) \subset V_+(T,H)$ and are dense in the space $\Pi_+(T,H)$.

Proof. According to Corollary 1.1 of [16] $\Pi(T,H)$ is a separable Banach space with the norm (2.1). Since the uniform limit of a sequence $x_n(\cdot) \in \Pi_+(T,H)$,

$n \geq 1$ is an element of the space $\Pi_+(T,H)$, $\Pi_+(T,H)$ is a closed subspace of the space $\Pi(T,H)$. Hence, $\Pi_+(T,H)$ is a separable Banach space.

Let $x(\cdot) \in \Pi_+(T,H)$. Since the point $x(\cdot)$ is a compact subset of the space $\Pi(T,H)$, in view of Theorem 1.2 of [16] for any $\varepsilon > 0$ there exists a partition $0 = c_0 < c_1 < \ldots < c_n = a$ of the interval T, such that on each of the intervals $]c_{i-1}, c_i[$ the function $x(\cdot)$ has oscillation not greater than ε. Since the function $x(\cdot)$ is right continuous, on each interval $[c_{i-1}, c_i[$, $1 \leq i \leq n$, the oscillation of $x(\cdot)$ is not greater than ε. Consider the function $x_\varepsilon\colon T \to H$ defined as follows: $x_\varepsilon(t) = x(c_{i-1})$, $t \in [c_{i-1}, c_i[$, $1 \leq i \leq n$, $x_\varepsilon(a) = x(a)$. The function $x_\varepsilon(t)$ satisfies the inequality $\|x(t) - x_\varepsilon(t)\| \leq \varepsilon$, $t \in T$ and it is a right continuous step function. Therefore, the set $S_+(T,H)$ is dense in $\Pi_+(T,H)$. Since any step function is a function of bounded variation, we have $S_+(T,H) \subset V_+(T,H) \subset \Pi_+(T,H)$ and the sets $S_+(T,H)$, $V_+(T,H)$ are dense in $\Pi_+(T,H)$. The theorem is proven. □

Since the space $\Pi_+(T,H)$ is the completion of the space $V_+(T,H)$, following [7, Ch. II, § 4, Definition 2] a set $U \subset V_+(T,H)$ is called *precompact* if its closure in $\Pi_+(T,H)$ is compact.

Theorem 2.2. *Let $U \subset V_+(T,H)$. Then, the following statements are equivalent:*

(a) *the set U is precompact;*

(b) *the set $U(t) = \{x(t);\ x(\cdot) \in U\} \subset H$, $t \in T$ is relatively compact and for any $\varepsilon > 0$ there exists a finite partition $0 = c_0 < c_i < \ldots < c_n = a$ of the interval T such that on each interval $[c_{i-1}, c_i[$, $1 \leq i \leq n$, each function $x(\cdot) \in U$ has oscillation not greater than ε.*

The following theorem is a paraphrase of Theorem 2.2.

Theorem 2.3. *Let $U \subset V_+(T,H)$. Then, the following statements are equivalent:*

(a) *the set U is precompact;*

(b) *the set $U(t) \subset H$, $t \in T$ is relatively compact and the family U is unilaterally equicontinuous.*

Theorems 2.2 and 2.3 follow from Theorems 1.2 and 1.2* of [16] since the closure of a set $U \subset V_+(T,H)$ in the topology of uniform convergence on T is a subset of the space $\Pi_+(T,H)$.

Corollary 2.4. *If a set $U \subset V_+(T,H)$ is precompact, then the set $U(T) = \{x(t);\ t \in T,\ x(\cdot) \in U\} \subset H$ is relatively compact.*

Corollary 2.5. *Let a set $U \subset V_+(T,H)$ be precompact. Then, for any $\varepsilon > 0$ the set of all points $t \in T$ at which at least one of the functions $x(\cdot) \in U$ has oscillation greater than ε is finite. The set of all points $t \in T$ at which at least one of the functions $x(\cdot) \in U$ has a jump is at most countable.*

Theorem 2.6. *A function $x\colon T \to H$ is an element of the space $\Pi_+(T,H)$ if and only if there exists a sequence of functions $x_n(\cdot) \in S_+(T,H)$ uniformly converging to $x(\cdot)$.*

Corollaries 2.4 and 2.5 and Theorem 2.6 follow from Corollaries 1.2, 1.4 and Theorem 1.5 in [16].

Theorem 2.7. *Let $U \subset V_+(T,H)$. Then, the following statements are equivalent:*

(a) *the set U is compact;*

(b) *the set U is closed in $V_+(T,H)$ and precompact.*

Since any compact set in a metric space is closed, Theorem 2.7 follows from the definition of precompactness.

We call a family U of functions of *bounded variation* from T to H uniformly bounded in variation if there exists a constant $M > 0$ such that the variation $V_0^a(x)$ on T of each function $x(\cdot) \in U$ satisfies the inequality

$$V_0^a(x) \leq M. \tag{2.2}$$

If inequality (2.2) holds for the norms $\|x(\cdot)\|$ of functions from U, then we call the family U uniformly bounded in norm.

Lemma 2.8. *If a set $U \subset V_+(T,H)$ is precompact and uniformly bounded in variation, then it is relatively compact subset of the space $V_+(T,H)$, i.e. any sequence $x_n(\cdot) \in U$, $n \geq 1$ contains a subsequence $x_{n_k}(\cdot)$, $k \geq 1$, uniformly converging to some function $x(\cdot) \in V_+(T,H)$.*

Proof. From Corollary 2.4 we infer that the set U is uniformly bounded in norm. Using Theorem 2.1 of [8, Chapter 0] we deduce that any sequence $x_n(\cdot) \in V_+(T,H)$, $n \geq 1$ contains a subsequence $y_k(\cdot) = x_{n_k}(\cdot)$, $k \geq 1$, converging pointwise in the weak topology of the space H to some function $x\colon T \to H$ of bounded variation. In view of the precompactness of U the sequence $y_k(\cdot)$, $k \geq 1$ converges in H to $x(\cdot)$ pointwise. Then, from the definition of precompactness it follows that there exists a subsequence $y_{k_l}(\cdot)$, $l \geq 1$ of the sequence $y_k(\cdot)$, $k \geq 1$ uniformly converging to the function $x(\cdot)$. Therefore, $x(\cdot) \in V_+(T,H)$. □

Corollary 2.9. *On each unilaterally equicontinuous and uniformly bounded in variation set $U \subset V_+(T,H)$ the topologies of pointwise and uniform convergences coincide.*

If a sequence $x_n(\cdot) \in U$, $n \geq 1$ converges pointwise, then for each $t \in T$ the set $\{\cup x_n(t);\ n \geq 1\}$ is relatively compact. Hence, in view of Theorem 2.3 the sequence is a precompact set. Now the corollary follows from Lemma 2.8.

Let $CV(T,H)$ be the space of continuous functions of bounded variation. It is a dense subspace of the space $C(T,H)$ of all continuous functions from T to H with the topology of uniform convergence. If we consider a set $U \subset CV(T,H)$ as a subset of the space $V_+(T,H)$, then Theorems 2.2 and 2.3 for the set U are theorems on relative compactness of sets U in the space $C(T,H)$. They are classical Arzela–Ascoli theorems for sets $U \subset CV(T,H)$.

3. Radon measures

In this section, we recall several properties of Radon measures. Some of them can be found in [5, 15]. However, for the reader's convenience and simplicity of the proofs of the main results we present these properties below.

Let $\mathcal{B}$ be the σ-algebra of all Borel sets from T. Both scalar and vector measures defined on the σ-algebra $\mathcal{B}$ are called *Stieltjes measures* [9, p. 358]. Since each positive scalar Stieltjes measure is regular [9, p. 358] it is a Radon measure [8]. In what follows, by positive Radon measures we mean scalar positive measures defined on the σ-algebra $\mathcal{B}$.

In the rest of the paper, λ is the Lebesgue measure. The variation $|m|\colon \mathcal{B} \to \mathbb{R}^+$ of a measure $m\colon \mathcal{B} \to H$ is defined as follows:

$$|m|(A) = \sup \sum_{i\in I} \|m(A_i)\|, \quad A \in \mathcal{B}, \tag{3.1}$$

where the supremum is taken over all finite families (A_i), $i \in I$ mutually disjoint sets $A_i \in \mathcal{B}$, $\bigcup\limits_{i\in I} A_i = A$ [9, proposition 2, p. 32]. If $|m|(T) < \infty$, the measure m is called a measure with bounded variation. In this case, the variation $|m|(\cdot)$ of a measure $m(\cdot)$ is a positive Radon measure. We call a positive Radon measure ν absolutely continuous with respect to a positive Radon measure μ, if from $\mu(A) = 0$, $A \in \mathcal{B}$ it follows that $\nu(A) = 0$. If the converse also holds, then the measures μ and ν are called *absolutely continuously equivalent*.

We call a Stieltjes measure m absolutely continuous with respect to a positive Radon measure $\mu(\cdot)$, if the measure $|m|$ is absolutely continuous with respect to the measure $\mu(\cdot)$. For a positive Radon measure ν on T by $L^1_\nu(T, H)$ we denote the space of equivalence classes of all ν measurable mappings $g\colon T \to H$ such that $\|g(\cdot)\|$ belongs to the space $L^1_\nu(T, \mathbb{R}^+)$.

If a Stieltjes measure m with bounded variation is absolutely continuous with respect to a positive Radon measure μ, then according to the Radon–Nikodym theorem there exists a function $\hat{m} \in L^1_\mu(T, H)$ such that

$$m(A) = \int_A \hat{m}(\tau)\, d\mu(\tau), \quad A \in \mathcal{B}.$$

the function $\hat{m}(t)$ is called the density of the measure m with respect to the Radon measure μ.

Let $T(t, r) = T \cap [t - r, t + r]$, $r > 0$, $t \in T$.

Lemma 3.1. [8] *For any positive Radon measures ν and μ we have:*

(1) *the limit*
$$\frac{d\nu}{d\mu}(t) = \lim_{r\downarrow 0} \frac{\nu(T(t,r))}{\mu(T(t,r))} \tag{3.2}$$

(*with the convention that* $\frac{0}{0} = 0$) *exists for each $t \in T$ and is finite for μ almost every $t \in T$. The function $t \to \frac{d\nu}{d\mu}(t)$ is Borel from T to $\overline{\mathbb{R}}^+$ and is called the* derivative of the measure ν *with respect to the measure μ.*

(2) *the measure ν is absolutely continuous with respect to the measure μ if and only if $\frac{d\nu}{d\mu}(t)$ is the density of the measure ν with respect to the measure μ, i.e. if and only if the following equality hold:*

$$\nu(A) = \int_A \frac{d\nu}{d\mu}(\tau)\, d\mu(\tau), \quad A \in \mathcal{B}. \tag{3.3}$$

The statement of the lemma follows from Theorem 2.12 and Corollary 2.14 in [8].

We will use the following properties of the variation (see [4]) when $c < t < d$:

$$V_c^d(u+w) \leq V_c^d(u) + V_c^d(w), \quad V_c^d(u) = V_c^t(u) + V_t^d(u). \tag{3.4}$$

For a function of bounded variation $u\colon T \to H$ let

$$\beta(u)(t) = V_0^t(u). \tag{3.5}$$

The function $\beta(u)(t)$ is increasing from T to $\mathbb{R}^+$, $\beta(0) = 0$ and it is left (right) continuous at each point $t \in T$, where the function u has the same property. For the function $\beta(u)(t)$ we have the following equality for $s < t$, $s, t \in T$:

$$\beta(u)(t) = \beta(u)(s) + V_s^t(u). \tag{3.6}$$

Theorem 3.2. *If a function of bounded variation $u\colon T \to H$ is right continuous, then there exists a unique Stieltjes measure $m\colon \mathcal{B} \to H$ such that for $0 \leq c \leq d \leq a$ the following equalities hold:*

$$m([c,d]) = u(d) - u(c-0), \quad m(]c,d]) = u(d) - u(c). \tag{3.7}$$

The statements of Theorem 3.2 are contained in Theorem 1 of [9, p. 358].

In different terminology [5, 10, 15] and others, the measure $m(\cdot)$ is called the differential measure generated by the function u and is denoted by du.

From (3.7) we infer that
$$u(t) = u(s) + \int\limits_{]s,t]} du, \tag{3.8}$$

and
$$u(t) = u(s-0) + \int\limits_{[s,t]} du, \quad s \leq t,\ s, t \in T. \tag{3.9}$$

Let $\beta(u)(t)$ be the function defined by equality (3.5). If the function $u\colon T \to H$ has bounded variation and is right continuous, then the function $\beta(u)(t)$, as an increasing function, has bounded variation and is right continuous. Hence, for the differential measure $d\beta$ of the function β the equality

$$\beta(u)(t) = \beta(u)(s) + \int\limits_{]s,t]} d\beta \tag{3.10}$$

holds. The measure $d\beta$ is a positive Radon measure.

It is known [12, Chapter V, p. 43, Theorem 1] that if a Radon measure ν is absolutely continuous with respect to a Radon measure γ, then a mapping $\hat{u}\colon T \to H$ is ν integrable if and only if the mapping $t \to \hat{u}(t)\frac{d\nu}{d\gamma}(t)$ is γ integrable. In this case, the following equality holds:

$$\int_T \hat{u}(t)\,d\nu(t) = \int_T \hat{u}(t)\frac{d\nu}{d\gamma}(t)\,d\gamma(t). \tag{3.11}$$

We denote singletons by $\{t\}$. If instead of ν the measure λ is taken, then two cases are possible: $\gamma(\{0\}) = 0$, $\gamma(\{0\}) \neq 0$. In the second case we have $\frac{d\lambda}{d\gamma}(0) = 0$. Hence, from (3.11) it follows that

$$\int_{]0,t]} \hat{u}(t)\,d\lambda(t) = \int_{]0,t]} \hat{u}(t)\frac{d\lambda}{d\gamma}(t)\,d\gamma(t), \quad t \in T. \tag{3.12}$$

If $\hat{u}(\cdot) \in L^1_\nu(T,H)$ and $u(t) = u(0) + \int_{]0,t]} \hat{u}(\tau)\,d\nu(\tau)$, $t \in T$, then the function $u(t)$ is a right continuous function of bounded variation. Then, du is absolutely continuous with respect to ν and $\hat{u}(\cdot)$ is the density of the measure du with respect to the measure ν, i.e.

$$\frac{du}{d\nu}(t) = \hat{u}(t) \quad \nu \text{ a.e.} \tag{3.13}$$

In this case [9], for the variation $|du|$ of the measure du the equality

$$|du|(]s,t]) = \int_{]s,t]} \|\hat{u}(\tau)\|\,d\nu(\tau), \quad s \le t \tag{3.14}$$

holds. Let $T^-(t,r) = [t-r,t] \cap T$, $T^+(t,r) = [t,t+r] \cap T$, $T^0(t,r) = (]t-r,t] \cap T$, $r > 0$, $t \in T$. According to [12] and [3, equality (2.12)] we have the relations

$$\hat{u}(t) = \frac{du}{d\nu}(t) = \lim_{r\downarrow 0} \frac{du(T(t,r))}{\nu(T(t,r))} = \lim_{r\downarrow 0} \frac{du(T^+(t,r))}{\nu(T^+(t,r))} =$$

$$= \lim_{r\downarrow 0} \frac{du(T^-(t,r))}{\nu(T^-(t,r))} = \lim_{r\downarrow 0} \frac{du(T^0(t,r))}{\nu(T^0(t,r))} \quad \nu \text{ a.e.} \tag{3.15}$$

Lemma 3.3. *Let $u\colon T \to H$ be a right continuous function of bounded variation, $\beta(u)\colon T \to \mathbb{R}^+$ be the function defined by equality* (3.5) *whose differential measure $d\beta$ is a positive Radon measure. Then, the following equalities hold:*

$$|du|(]s,t]) = d\beta(]s,t]), \quad s \le t, \tag{3.16}$$

$$|du|([s,t]) = d\beta([s,t]), \quad s \le t,\ s,t \in T. \tag{3.17}$$

Proof. Let $[s,t] \subset T$, $s < t$. From (3.1), (3.5), (3.6), (3.8) it follows that $\beta(u)(t) - \beta(u)(s) \le |du|(]s,t])$. From this inequality and (3.10) we obtain

$$d\beta(u)(]s,t]) \le |du|(]s,t]), \quad d\beta(u)(\{0\}) = 0, \quad s \le t. \tag{3.18}$$

According to [9, p. 361]: $|du|(A) \le d\beta(u)(A), \quad A \in \mathcal{B}$.

This inequality and (3.18) imply the equality

$$d\beta(u)(]s,t]) = |du|(]s,t]),\ |du|(\{0\}) = 0. \tag{3.19}$$

From equality (3.19) and the properties of measures we infer that

$$d\beta(u)(\{s\}) = |du|(\{s\}), \quad s \in T. \tag{3.20}$$

Hence, equality (3.17) follows from (3.20) and (3.19). The lemma is proven. □

Lemma 3.4. *Let ν, α, β be positive Radon measures and*

$$\nu(]s,t]) \le \alpha(]s,t]) + \beta(]s,t]),\ s \le t,\ s \in T, \tag{3.21}$$

$$\nu(\{0\}) \le \alpha(\{0\}) + \beta(\{0\}). \tag{3.22}$$

If α and β are absolutely continuous with respect to a positive Radon measure μ, then the measure ν is absolutely continuous with respect to the measure μ.

Proof. Since $\{t\} = \{\cap]s,t]; s \uparrow t\},\ t > 0,\ t \in T,\ [s,t] = \{s\} \cup]s,t]$,

$$]s,t[= \{\cup[s',t']; s' \downarrow s,\ t' \uparrow t,\ s' \le t',\ s',t' \in]s,t[\},$$

from (3.21) and the properties of measures it follows that

$$\nu(]s,t[) \le \alpha(]s,t[) + \beta(]s,t[),\ s \le t,\ s,t \in T. \tag{3.23}$$

Let $\mathcal{B}_0$ be the collection of all open sets $G \subset]0,a[$. Since any open set $G \subset \mathcal{B}_0$ is the union of at most countable number of mutually disjoint sets of the form $]s,t[$, from (3.23) we infer that

$$\nu(G) \le \alpha(G) + \beta(G),\ G \in \mathcal{B}_0. \tag{3.24}$$

Since the measures ν, α, β are regular [4], for any $A \in \mathcal{B}, A \subset]0,a[$ the equality [9, proposition 17, p. 313]

$$\nu(A) = \inf\{\nu(G); A \subset G,\ G \in \mathcal{B}_0\} \tag{3.25}$$

holds. Analogous equalities hold also for $\alpha(A)$ and $\beta(A)$. If the measures α and β are absolutely continuous with respect to μ, from $\mu(A) = 0, A \in \mathcal{B}, A \subset]0,a[$ we see that $\alpha(A) = 0$ and $\beta(A) = 0$. Then, from (3.24), (3.25) it follows that $\nu(A) = 0$.

If $\mu(\{0\}) = 0$, then the equality $\nu(\{0\}) = 0$ follows from (3.22). Directly from (3.21) we deduce that

$$\nu(\{a\}) \le \alpha(\{a\}) + \beta(\{a\}).$$

Hence, if $\mu(\{a\}) = 0$, then $\alpha(\{a\}) = 0, \beta(\{a\}) = 0$ and, consequently, $\nu(\{a\}) = 0$. Since any set $A \in \mathcal{B}$ can be represented in the form

$$A = (\{0\} \cap A) \cup (]0,a[\cap A) \cup (\{a\} \cap A),$$

the lemma is proven. □

Theorem 3.5. *Let $u, v\colon T \to H$ be two right continuous functions of bounded variation and their differential measures du and dv be absolutely continuous with respect to a positive Radon measure ν. Then, the function $w = u + v$ is a right continuous function of bounded variation, its differential measure dw is absolutely continuous with respect to ν and for the densities $\frac{dw}{d\nu}(t)$, $\frac{du}{d\nu}(t)$, $\frac{dv}{d\nu}(t)$ of the measures dw, du, dv with respect to ν the following equality hold:*

$$\frac{dw}{d\nu}(t) = \frac{du}{d\nu}(t) + \frac{dv}{d\nu}(t) \quad \nu \text{ a.e.} \tag{3.26}$$

Proof. The right continuity of the function w is evident. The boundedness of the variation follows from (3.4). Let $\beta(w)(t)$, $\beta(u)(t)$, $\beta(v)(t)$ be the functions defined by equality (3.5). Using (3.6) and (3.4) we obtain

$$\beta(w)(t) - \beta(w)(s) \leq \beta(u)(t) - \beta(u)(s) + \beta(v)(t) - \beta(v)(s), \quad s \leq t.$$

From this inequality and (3.10) we infer that

$$d\beta(w)(]s,t]) \leq d\beta(u)(]s,t]) + d\beta(v)(]s,t]).$$

Then, according to (3.7), (3.20) we have

$$|dw|([s,t]) \leq |du|([s,t]) + |dv|([s,t]), \quad s \leq t,\ s,t \in T.$$

Since the measures du and dv are absolutely continuous with respect to the measure ν, according to Lemma 3.4 the measure $|dw|$ is absolutely continuous with respect to the measure ν. Then, the measure dw is absolutely continuous with respect to the measure ν. From (3.15) it follows that for the density $\frac{dw}{d\nu}(t)$ of the measure dw with respect to the measure ν the equality

$$\frac{dw}{d\nu}(t) = \lim_{r\downarrow 0} \frac{dw(T^0(t,r))}{\nu(T^0(t,r))} = \lim_{r\downarrow 0} \frac{dv(T^0(t,r))}{\nu(T^0(t,r))} + \\ + \lim_{r\downarrow 0} \frac{du(T^0(t,r))}{\nu(T^0(t,r))} = \frac{dv}{d\nu}(t) + \frac{du}{d\nu}(t) \quad \nu \text{ a.e.}$$

holds – and the theorem is proven. □

4. Auxiliary results

In this section, we consider the differential inclusion

$$-dw \in \mathcal{N}(C(t); w(t)) + \hat{u}(t)\, d\lambda \tag{4.1}$$

with $\hat{u}(\cdot) \in L^1_\lambda(T,H)$, $w(0) = u_0 \in C(0)$. We prove the existence of a solution to (4.1) and we study the dependence of solutions on $\hat{u}(\cdot)$. Though these results have an auxiliary character, they are nevertheless new and are of independent interest.

Proposition 4.1. *Let $m(\cdot) \in L^1_\lambda(T, \mathbb{R}^+)$, $x\colon T \to \mathbb{R}^+$ be a right continuous function of bounded variation. If*

$$\frac{1}{2}x^2(t) \leq \frac{1}{2}c^2 + \int\limits_{]0,t]} m(s)x(s)\,d\lambda(s), \quad t \in T,\ c \geq 0, \tag{4.2}$$

then
$$x(t) \leq c + \int\limits_{]0,t]} m(s)\,d\lambda(s), \quad t \in T. \tag{4.3}$$

Proposition 4.1 is a counterpart for a continuous function $x(t)$ of Lemma A5 in [3]. The proof of this proposition repeats verbatim that of Lemma A5 [3].

Proposition 4.2. *Let ν be a positive Radon measure, $u\colon T \to H$ be a right continuous function of bounded variation whose differential measure du is absolutely continuous with respect to ν and $\frac{du}{d\nu}(t)$ be the density of the measure du with respect to the measure μ. Then, the function $t \to \|u(t)\|^2$ is a right continuous function of bounded variation and*

$$\frac{\|u(t)\|^2}{2} \leq \frac{\|u(0)\|}{2} + \int\limits_{]0,t]} \langle u(\tau), \frac{du}{d\nu}(\tau)\rangle d\nu, \quad t \in T. \tag{4.4}$$

Proposition 4.2 follows from Proposition 3.3 in [15].

Let $\hat{r}(\cdot) \in L^1_\lambda(T, \mathbb{R}^+)$, $r(t) = r_0 + \int_{]0,t]} \hat{r}(\tau)\,d\lambda(\tau),\ t \in T,\ r_0 \geq \|u_0\|,\ u_0 \in C(0)$, and dr be the differential measure generated by the function $r(t)$. The measure dr is a positive Radon measure which is absolutely continuous with respect to the measure λ, $\hat{u}(\cdot) \in L^1_\lambda(T, H)$,

$$\|\hat{u}(t)\| \leq \hat{r}(t) \quad \lambda\,\text{a.e.} \tag{4.5}$$

Let
$$u(t) = \int\limits_{]0,t]} \hat{u}(\tau)\,d\lambda(\tau),\ t \in T. \tag{4.6}$$

Then, $u(t)$ is a right continuous function of bounded variation, the differential measure du of this function is absolutely continuous with respect to the measure λ and according to (3.8), (3.14) and (4.5) the inequality

$$\|u(t) - u(s)\| \leq |du|(]s,t]) \leq dr(]s,t]), \quad s \leq t,\ s,t \in T \tag{4.7}$$

holds. In what follows, ν is the positive Radon measure defined by the equality

$$\nu = \mu + dr + \lambda. \tag{4.8}$$

The measure λ is absolutely continuous with respect to ν, and the measures ν and $\mu + \lambda$ are absolutely continuously equivalent. Hence, the measure du is absolutely continuous with respect to ν. Using (4.6) and (3.12) we obtain

$$u(t) = \int\limits_{]0,t]} \hat{u}(\tau)\,d\lambda(\tau) = \int\limits_{]0,t]} \hat{u}(\tau)\frac{d\lambda}{d\nu}(\tau)\,d\nu(\tau).$$

From this equality it follows that the derivative $\frac{du}{d\nu}(t)$ of the measure du with respect to the measure ν satisfies the equality

$$\frac{du}{d\nu}(t) = \hat{u}(\tau)\frac{d\lambda}{d\nu}(t) \ \nu \text{ a.e.} \tag{4.9}$$

Let

$$C_u(t) = C(t) + u(t). \tag{4.10}$$

From (1.1), (4.7), and (4.8) we infer that

$$\operatorname{exc}(C_u(s), C_u(t)) \leq \nu(]s,t]), \quad s \leq t. \tag{4.11}$$

Note that $\mathcal{N}(C(t), x)$ is the subdifferential of the indicator function of the set $C(t)$ at the point $x \in C(t)$. Therefore [17],

$$y \in \mathcal{N}(C_u(t), x) \Longrightarrow y \in \mathcal{N}(C(t), x - u(t)). \tag{4.12}$$

Theorem 4.3. *Let inequality* (1.1) *hold. Then, for any $\hat{u}(t)$ satisfying inequality* (4.5) *there exists a unique solution $w(\hat{u})(t)$ of inclusion* (4.1) *satisfying the inequality*

$$\|w(\hat{u})(t) - w(\hat{u})(s)\| \leq \nu(]s,t]) + dr(]s,t]), \ s \leq t. \tag{4.13}$$

Proof. Let $\hat{u}(t)$ satisfy inequality (4.5) and the function $u(t)$ be defined by equality (4.6). From (4.10), (4.11) and Corollary 4.4 [15] (see also [11]) it follows that there exists a unique function $v\colon T \to H$ with the properties:

$$(1) \ \ v(0) = u_0 \in C(0), \quad v(t) \in C_u(t), \ \ t \in T; \tag{4.14}$$

(2) function $v(t)$ is a right continuous function of bounded variation;

(3) the differential measure dv generated by v is absolutely continuous with respect to ν and the derivative $\frac{dv}{d\nu}(t)$ of the measure dv with respect to the measure ν satisfies the inclusion

$$-\frac{dv}{d\nu}(t) \in N(C_u(t); v(t)) \quad \nu \text{ a.e.} \tag{4.15}$$

(4) the inequality

$$\|v(t) - v(s)\| \leq \nu(]s,t]), \quad s \leq t, \ s,t \in T \tag{4.16}$$

holds. Let

$$w(\hat{u})(t) = v(t) - u(t). \tag{4.17}$$

Then,

$$w(\hat{u})(0) = u_0 \in C(0), \ w(\hat{u})(t) \in C(t). \tag{4.18}$$

From Theorem 3.5 and (4.9) it follows that the function $w(\hat{u})(t)$ is a right continuous function of bounded variation, the differential measure $dw(\hat{u})$ of the function $w(\hat{u})$ is absolutely continuous with respect to the measure ν and for the density $\frac{dw(\hat{u})}{d\nu}(t)$ of the differential measure $dw(\hat{u})$ with respect to the measure ν the following equality holds:

$$\frac{dw(\hat{u})}{d\nu}(t) = \frac{dv}{d\nu}(t) - \frac{du}{d\nu}(t) = \frac{dv}{d\nu}(t) - \hat{u}(t)\frac{d\lambda}{d\nu}(t) \quad \nu \text{ a.e.} \tag{4.19}$$

Using (4.12), (4.15), and (4.19) we obtain

$$-\frac{dw(\hat{u})}{d\nu}(t) - \hat{u}(t)\frac{d\lambda}{d\nu} \in N(C(t), w(\hat{u})(t) \quad \nu \text{ a.e.} \tag{4.20}$$

Recall that the measure $\nu(\cdot)$ is absolutely uniformly equivalent to the measure $\mu + \lambda$. Consequently, from (4.18), (4.20) it follows that $w(\hat{u})$ is a solution of inclusion (4.1).

The uniqueness of the solution $w(\hat{u})$ for a fixed $\hat{u}$ directly follows from (4.20) and Proposition 4.2. We obtain inequality (4.13) from (4.16), (4.17), (4.7). The theorem is proven. □

Denote by $S(\hat{r})$ the set of all $\hat{u}(\cdot) \in L^1_\lambda(T, H)$ satisfying inequality (4.5). Let $\mathcal{R}_S(u_0)$ be the collection of all solutions $w(\hat{u})$ of inclusion (4.1) corresponding to $\hat{u}(\cdot) \in S(\hat{r})$.

Theorem 4.4. *Let inequality* (1.1) *hold. Then, the following statements are true:*

(a) *the set $\mathcal{R}_S(u_0)$ is not empty;*

(b) *the inequality* $$\|w(\hat{u})(t) - w(\hat{u})(s)\| \leq \nu(]s,t]) + dr(]s,t]); \tag{4.21}$$

$s \leq t,\ s,t \in T,\ \hat{u} \in S(\hat{r})$ *holds;*

(c) *the solution $w(\hat{u})$ is unique and for any $\hat{u}_1, \hat{u}_2 \in S(\hat{r})$ the inequality*

$$\|w(\hat{u}_1)(t) - w(\hat{u}_2)(t)\| \leq \int_{]0,t]} \|\hat{u}_1(\tau) - \hat{u}_2(\tau)\| d\lambda(\tau),\ t \in T \tag{4.22}$$

holds;

(d) *the set $\mathcal{R}_S(u_0)$ is uniformly bounded in norm and in variation;*

(e) *the set $\mathcal{R}_S(u_0)$ is unilaterally equicontinuous.*

Proof. For $\hat{u}(\cdot) \neq \theta$ the existence of a solution to inclusion (4.1) was proved in Theorem 4.1. According to (1.1) and (4.8)

$$\operatorname{exc}(C(s), C(t) \leq \mu(]s,t]) \leq \nu(]s,t]), \quad s \leq t,\ s,t \in T.$$

Then, from Corollary 4.4 [3] it follows that there exists a unique function $w(\theta)$, $w(\theta)(0) = u_0$, $w(\theta)(t) \in C(t)$ such that $w(\theta)$ is a right continuous function of bounded variation, the differential measure $dw(\theta)$ is absolutely continuous with respect to ν and for the density $\frac{dw(\theta)}{d\nu}(t)$ of the measure $dw(\theta)$ with respect to ν the inclusion

$$-\frac{dw(\theta)}{d\nu}(t) \in \mathcal{N}(C(t);\ w(\theta)(t))\ \nu \text{ a.e.} \tag{4.23}$$

holds. In this case, we have

$$\|w(\theta)(t) - w(\theta(s)\| \leq \nu(]s,t]),\ s \leq t,\ s,t \in T. \tag{4.24}$$

Hence, according to (4.20) and (4.23) the function $w(\theta)$ is a solution of inclusion (4.1) for $\hat{u}(\cdot) = \theta$. Consequently, $w(\theta) \in \mathcal{R}_S(u_0)$ and the statement a) is proven.

Statement (b) follows from (4.13) and (4.24).

Let $w(\hat{u}_1)$ and $w(\hat{u}_2)$ be two solutions corresponding to $\hat{u}_1$ and $\hat{u}_2$, respectively. Making use of (4.20) and Proposition 4.2 we obtain

$$\frac{1}{2}\|w(\hat{u}_1)(t) - w(\hat{u}_2)(t)\|^2 \leq \tag{4.25}$$

$$\leq \int_{]0,t]} \langle w(\hat{u}_1)(\tau) - w(\hat{u}_2)(\tau), \hat{u}_2(\tau) - \hat{u}_1(\tau)\rangle \frac{d\lambda}{d\nu}(\tau) d\nu(\tau),\ t \in T. \tag{4.26}$$

Since the function $t \to \langle w(\hat{u}_1)(\tau) - w(\hat{u}_2)(\tau), \hat{u}_2(\tau) - \hat{u}_1(\tau)\rangle$ is λ-integrable, from (3.12) and (4.26) we infer that

$$\frac{1}{2}\|w(\hat{u}_1)(t) - w(\hat{u}_2)(t)\|^2 \leq$$

$$\leq \int_{]0,t]} \|w(\hat{u}_1)(\tau) - w(\hat{u}_2)(\tau)\| \cdot \|\hat{u}_1(\tau) - \hat{u}_2(\tau)\| d\lambda(\tau),\ t \in T.$$

This inequality and Proposition 4.1 imply inequality (4.22).

The uniqueness of the solution $w(\hat{u})$ for each $\hat{u} \in S(\hat{r})$ is established in Theorem 4.3. The statement (c) is proven. Let

$$M = \sup\{\|w(\theta)(t)\|,\ t \in T\}. \tag{4.27}$$

Then, from (4.22) and (4.5) we obtain

$$\|w(\hat{u})(t)\| \leq M + \int_{]0,t]} \hat{r}(\tau) d\lambda(\tau),\ t \in T,\ \hat{u}(\cdot) \in S(\hat{r}). \tag{4.28}$$

Now the uniform boundedness of the family $\mathcal{R}_S(u_0)$ in norm and in variation follows from (4.24), (4.28).

The equicontinuity of the set $\mathcal{R}_{S'}(u_0)$ from the right at each point $t \in [0, a[$ follows directly from (4.24). The equicontinuity of the set $\mathcal{R}_S(u_0)$ from the left at each point $t \in (0, a]$ follows from the inequality

$$\|w(\hat{u})(t-0) - w(\hat{u})(s)\| \leq \nu(]s,t[) + dr(]s,t[),\ s < t,\ s,t \in T.$$

This inequality follows from (4.24) and the properties of measures. The theorem is proven. □

5. Main results

In this section, we prove the main results.

In the rest of the paper, $\omega - L^1_\lambda(T,H)$ is the space of λ-integrable functions from T to H endowed with the weak topology. By the symbol $\omega - S,\ S \subset L^1_\lambda(T,H)$ we denote the set S endowed with the topology induced from $\omega - L^1_\lambda(T,H)$.

Let $r(t)$ be the solution of the differential equation

$$\dot{r}(t) = m(t) + n(t)r(t),\ r(0) = M,\ t \in T, \tag{5.1}$$

where M is the constant (4.27), which exists and is unique.

Lemma 5.1. *If a function $w \in V_+(T,H)$ satisfies the inequality*

$$\|w(t)\| \leq M + \int_{]0,t]} (m(\tau) + n(\tau)\|w(\tau)\|) d\lambda(\tau), \; t \in T, \tag{5.2}$$

then
$$\|w(t)\| \leq r(t), t \in T. \tag{5.3}$$

Proof. Let
$$l(t) = M + \int_{]0,t]} (m(\tau) + n(\tau)\|w(\tau)\|) d\lambda(\tau), \; t \in T. \tag{5.4}$$

Since the Lebesgue measure λ of singletons is equal to zero, equality (5.4) is valid also for $t = 0$. Hence, $l(t)$ is an absolutely continuous function whose derivative $\dot{l}(t)$ satisfies the following equality almost everywhere

$$\dot{l}(t) = m(t) + n(t)\|w(t)\|.$$

From this equality and (5.2), (5.4) it follows that

$$\dot{l}(t) - n(t)l(t) \leq m(t).$$

Multiply the left-hand side of the last inequality by $e^{-\int_0^t n(s)ds}$. Then, we obtain

$$\frac{d}{dt}\left(l(t) \cdot e^{-\int_0^t n(s)ds}\right) \leq e^{-\int_0^t n(s)ds} m(t).$$

Integrating this inequality from 0 to t we see that

$$l(t) \leq e^{\int_0^t n(s)ds}(M + \int_0^t e^{-\int_0^\tau n(s)ds} m(\tau)d\tau). \tag{5.5}$$

It is well-known that the right-hand side of inequality (5.5) is the solution $r(t)$ of equality (5.1). Therefore,
$$l(t) \leq r(t), \; t \in T.$$

From this inequality and (5.2), (5.4) we deduce inequality (5.3). The lemma is proven. □

Lemma 5.2. *Let Hypotheses $H(F_1)$ hold. Then, for any function $w \in V_+(T,H)$ the mapping*
$$t \to F(t, w(t)) \cap (m(t) + n(t)\|w(t)\|)\bar{B}$$
has a λ-integrable selector.

Proof. Let
$$U(t,x) = F(t,x) \cap (m(t) + n(t)\|x\|)\bar{B}. \tag{5.6}$$

From Hypotheses $H_1(F)$ 1), 2) it follows that the values of the mapping $U(t,x)$ are nonempty convex compact sets, the mapping $x \to U(t,x)$ has a measurable selector for λ almost every $t \in T$.

Let $w(\cdot) \in V_+(T,H)$. According to Theorem 2.6 there exists a sequence $w_n(\cdot) \in S_+(T,H), n \geq 1$, converging uniformly to $w(\cdot)$. Then, for each function $w_n(\cdot)$,

$n \geq 1$, there exists a measurable function $v_n(\cdot)$ satisfying the following inclusion almost everywhere

$$v_n(t) \in U(t, w_n(t)),\ n \geq 1. \tag{5.7}$$

Since the set $w(\cdot) \cup (\cup w_n(\cdot); n \geq 1)$ is compact in the space $V_+(T, H)$, from Corollary 2.4 we conclude that there exists a compact set $C \subset H$ such that

$$w(t) \cup (\cup w_n(t); n \geq 1) \subset C,\ t \in T.$$

From Hypothesis $H_1(F)$ 3) we deduce that the values of the multivalued mapping $t \to \overline{\text{co}} U(t, C)$ are convex compact sets and

$$U(t, x) \subset \overline{\text{co}} U(t, C),\ x \in C,\ t \in T, \tag{5.8}$$

$$v_n(t) \in \overline{\text{co}} U(t, C) \subset (m(t) + n(t)\|C\|)\bar{B} \tag{5.9}$$

λ almost everywhere, $n \geq 1$, where $\overline{\text{co}}$ stands for the closed convex hull of a set.

From (5.8) it follows that the restriction of the mapping $x \to U(t, x), t \in T$ to the set C is upper semicontinuous. From inclusion (5.9) we infer that the sequence $v_n(\cdot), n \geq 1$ is a relatively compact set in the space $\omega - L^1_\lambda(T, H)$. Since the space $L^1_\lambda(T, H)$ is separable, any compact set in the space $\omega - L^1_\lambda(T, H)$ is metrizable in the topology of this space.

Therefore, from the sequence $v_n(\cdot), n \geq 1$ one can extract a subsequence $v_{n_k}(\cdot), k \geq 1$ weakly converging to some function $v(\cdot) \in L^1_\lambda(T, H)$. To avoid introducing new notations, without loss of generality, we can assume that the sequence $v_n(\cdot), n \geq 1$ itself weakly converges to $v(\cdot)$. Then, from (5.7), the upper semicontinuity of the mapping $x \to U(t, x)$ on C and Mazur's lemma on weakly convergent sequences we obtain λ a.e.

$$v(t) \in \cap_{n \geq 1} \overline{\text{co}} \cup_{k \geq n} U(t, w_k(t)) \subset U(t, w(t)). \tag{5.10}$$

Therefore, $v(t)$ is a λ-integrable selector of the mapping $t \to F(t, w(t)) \cap (m(t) + n(t)\|w(t)\|)\bar{B}$, and the lemma is proven. □

Remark 5.3. The mapping $t \to F(t, x) \cap (m(t) + n(t)\|x\|)\overline{B}$, $x \in H$ has a measurable selector if for every $x \in H$ the mapping $t \to F(t, x)$ is measurable [6].

Let $r(t)$ be the solution of equation (5.1) and

$$S = \{\hat{u} \in L^1_\lambda(T; H);\ \|\hat{u}(t)\| \leq \dot{r}(t)\ \ \lambda\ \text{ a.e.}\}. \tag{5.11}$$

Setting $\hat{r}(t) = \dot{r}(t)$ and using (4.28) we deduce that the inequality

$$\|w(\hat{u})(t)\| \leq M + \int_{]0,t]} \dot{r}(\tau) ds(\tau) = r(t),\ \ t \in T, \tag{5.12}$$

holds for any solution $w(\hat{u})$ of inclusion (4.1) with $\hat{u} \in S$.

The set S is convex compact and metrizable in the space $\omega - L^1_\lambda(T, H)$.

Let $\mathcal{T}\colon S \to V_+(T,H)$ be the operator which with each $\hat{u} \in S$ associates the unique solution of inclusion (4.1), i.e. $w(\hat{u}) = \mathcal{T}(\hat{u})$, $\hat{u} \in S$. Let

$$\mathcal{T}(S) = \{\mathcal{T}(\hat{u});\ \hat{u} \in S\}, \quad \mathcal{T}(S)(t) = \{\mathcal{T}(\hat{u})(t);\ \hat{u} \in S\},\ \ t \in T. \tag{5.13}$$

From (5.12) we obtain

$$\|\mathcal{T}(S)(t)\| \le r(t),\ t \in T. \tag{5.14}$$

According to (5.14) and Hypothesis $H_1(F)$ 3) the set

$$F(t, \mathcal{T}(S)(t)) \cap (m(t) + n(t)\|\mathcal{T}(S)(t)\|)\bar{B}$$

is relatively compact for λ a.e. $t \in T$. Let

$$F^*(\mathcal{T}(S)(t)) = \overline{\mathrm{co}}(F(t, \mathcal{T}(S)(t)) \cap (m(t) + n(t)\|\mathcal{T}(S)(t)\|)\bar{B}. \tag{5.15}$$

Then, $F^*(\mathcal{T}(S)(t))$ is a multivalued mapping with convex compact values for λ a.e. $t \in T$. From Lemma 5.2 it follows that for any $\mathcal{T}(\hat{u})$, $\hat{u} \in S$ the mapping

$$t \to F(t, \mathcal{T}(\hat{u})(t) \cap (m(t) + n(t)\|\mathcal{T}(\hat{u})(t)\|)\bar{B}$$

has a λ-integrable selector. From (5.14), (5.15) we infer that

$$\|F^*(\mathcal{T}(S)(t))\| \le m(t) + n(t)r(t) = \dot{r}(t)\ \lambda \text{ a.e.} \tag{5.16}$$

Hence, the set

$$S_{F^*} = \{\hat{u} \in L^1_\lambda(T,H),\ \hat{u}(t) \in F^*(\mathcal{T}(S)(t))\ \lambda \text{ a.e.}\} \tag{5.17}$$

is a nonempty convex metrizable compact subset of the space $\omega - L^1_\lambda(T,H)$.

Lemma 5.4. *Let Hypotheses $H_1(F)$ hold. Then, the operator $\mathcal{T}$ is continuous from $\omega - S_{F^*}$ to $V_+(T,H)$.*

Proof. From (5.11), (5.16), (5.17) it follows that

$$S_{F^*} \subset S. \tag{5.18}$$

Let a sequence $\hat{u}_n \in S_{F^*}$, $n \ge 1$ weakly converges to $\hat{u} \in S_{F^*}$. From (5.18) and Theorem 4.4 we infer that the set $\{\cup\mathcal{T}(\hat{u}_n); n \ge 1\} \cup \mathcal{T}(\hat{u})$ is uniformly bounded in norm and in variation and it is unilaterally equicontinuous. According to Theorem 2.1 in [8, Chapter 0] there exists a subsequence $\mathcal{T}(\hat{u}_{n_k}), k \ge 1$ of the sequence $\mathcal{T}(\hat{u}_n), n \ge 1$, converging pointwise in the weak topology of the space H to some function w of bounded variation. From Proposition 4.2 and (4.20) it follows that

$$\frac{1}{2}\|\mathcal{T}(\hat{u}_{n_k})(t) - \mathcal{T}(\hat{u})(t)\|^2 \le$$
$$\le \int_{]0,t]} \langle \mathcal{T}(\hat{u}_{n_k})(\tau) - \mathcal{T}(\hat{u})(\tau), \hat{u}(\tau) - \hat{u}_{n_k}(\tau)\rangle \frac{d\lambda}{d\nu}(\tau) d\nu(\tau).$$

Since the functions $\langle \mathcal{T}(\hat{u}_{n_k})(\tau) - \mathcal{T}(\hat{u})(\tau), \hat{u}(\tau) - \hat{u}_{n_k}(\tau)\rangle, k \geq 1$ are λ-integrable, similarly to (4.26) we obtain

$$\frac{1}{2}\left\|\mathcal{T}(\hat{u}_{n_k})(t) - w(\hat{u})(t)\right\|^2 \leq$$
$$\leq \int_{]0,t]} \langle \mathcal{T}(\hat{u}_{n_k})(\tau) - \mathcal{T}(\hat{u})(\tau), \hat{u}(\tau) - \hat{u}_{n_k}(\tau)\rangle d\lambda(\tau).$$

Since the function w is λ-integrable, we rewrite the last inequality in the form

$$\frac{1}{2}\left\|\mathcal{T}(\hat{u}_{n_k})(t) - \mathcal{T}(\hat{u})(t)\right\|^2 \leq \tag{5.19}$$
$$\leq \int_{]0,t]} \langle \mathcal{T}(\hat{u}_{n_k})(\tau) - w(\tau), \hat{u}(\tau) - \hat{u}_{n_k}(\tau)\rangle d\lambda(\tau) + \int_{]0,t]} \langle w(\tau), \hat{u}(\tau) - \hat{u}_{n_k}(\tau)\rangle d\lambda(\tau).$$

The fact that $\hat{u}_{n_k} \in S_{F^*}, k \geq 1$, implies that for λ almost every $t \in T$ the set $\cup(\hat{u}_{n_k}(t) - \hat{u}(t); r \geq 1) \in H$ is relatively compact.

Since the sequence $\mathcal{T}(\hat{u}_{n_k})(\tau) - w(\tau), k \geq 1, \tau \in T$ is bounded, the sequence of functions $h \to \langle \mathcal{T}(\hat{u}_{n_k}(\tau) - w(\tau), h\rangle, k \geq 1$ is equicontinuous. It is known that on each equicontinuous set the topology of pointwise convergence coincides with that of uniform convergence on compact sets. Hence,

$$\lim_{k\to\infty} \langle \mathcal{T}(\hat{u}_{n_k})(\tau) - w(\tau), \hat{u}(\tau) - \hat{u}_{n_k}(\tau)\rangle = 0 \ \lambda \text{ a.e.}$$

Therefore, from (5.14), (5.18) and Lebesgue's dominated convergence theorem it follows that

$$\lim_{k\to\infty} \int_{]0,t]} \langle \mathcal{T}(\hat{u}_{n_k})(\tau) - w(\tau), \hat{u}(\tau) - \hat{u}_{n_k}(\tau)\rangle d\lambda(\tau) = 0, \ t \in T.$$

The second integral in (5.19) converges to zero since the sequence $\hat{u}_{n_k}, k \geq 1$ weakly converges to $\hat{u}_0$. Consequently, from (5.19) we infer that the sequence $\mathcal{T}(\hat{u}_{n_k})(t), k \geq 1$ for each $t \in T$ converges to $\mathcal{T}(\hat{u})(t)$ in the space H. Then, from Corollary 2.9 we see that the sequence $\mathcal{T}(\hat{u}_{n_k}), k \geq 1$ converges to $\mathcal{T}(\hat{u})$ in the space $V_+(T, H)$.

We have thus proved than if a sequence $\hat{u}_n \in S_{F^*}, n \geq 1$ weakly converges to $\hat{u} \in S_{F^*}$, then there exists a subsequence $\hat{u}_{n_k}$, $k \geq 1$ of the sequence $\hat{u}_n$, $n \geq 1$ such that the sequence $\mathcal{T}(\hat{u}_{n_k}), k \geq 1$ converges to $\mathcal{T}(\hat{u})$ in $V_+(T, H)$.

Suppose that the sequence $\mathcal{T}(\hat{u}_n)$, $n \geq 1$ does not converge to $\mathcal{T}(\hat{u})$ in the space $V_+(T, H)$. Then, there exists a subsequence $\mathcal{T}(\hat{u}_{n_k})$, $k \geq 1$ of the sequence $\mathcal{T}(\hat{u}_n), n \geq 1$ such that any subsequence of the sequence $\mathcal{T}(\hat{u}_{n_k}), k \geq 1$ does not converge to $\mathcal{T}(\hat{u})$ in the space $V_+(T, H)$. Repeating the reasoning above to the sequence $\mathcal{T}(\hat{u}_{n_k}), k \geq 1$ and taking into account the fact that inclusion (4.1) with u_0 has a unique solution, we arrive at a contradiction.

Therefore, if a sequence $\hat{u}_n \in S_{F^*}, n \geq 1$ weakly converges to $\hat{u} \in S_F$, then the sequence $\mathcal{T}(\hat{u}_n), n \geq 1$ converges to $\mathcal{T}(\hat{u})$ in the space $V_+(T, H)$. The lemma is proven. □

Proof of Theorem 1.2.

Let $\hat{u} \in S_{F^*}$ and $U(t, x)$ be the mapping defined by equality (5.6). According to Lemma 5.2 the mapping $t \to U(t, \mathcal{T}(\hat{u})(t))$ has a λ-integrable selector. Denote by $S_U(\mathcal{T}(\hat{u}))$ the multivalued mapping from S_{F^*} to $L^1_\lambda(T, H)$ defined by the rule

$$S_U(\mathcal{T}(\hat{u})) = \{v \in L^1_\lambda(T, H);\ v(t) \in U(t, \mathcal{T}(\hat{u})(t))\ \ \lambda \text{ a.e.}\},\quad \hat{u} \in S_{F^*}.$$

From (5.16) we see that the values of the mapping $S_U(\mathcal{T}(\hat{u}))$ are nonempty convex compact sets of the space $\omega - L^1_\lambda(T, H)$. From (5.18) and (5.13) it follows that $\mathcal{T}(S_{F^*}) \subset \mathcal{T}(S)$. Hence,

$$S_U(\mathcal{T}(\hat{u})) \subset S_{F^*},\quad \hat{u} \in S_{F^*}. \tag{5.20}$$

Let the sequence $\hat{u}_n \in S_{F^*}, n \geq 1$ weakly converges to $\hat{u} \in S_{F^*}$. Then, in view of Lemma 5.4 the sequence $\mathcal{T}(\hat{u}_n), n \geq 1$ converges to $\mathcal{T}(\hat{u})$ in the space $V_+(T, H)$. Since

$$U(t, \mathcal{T}(\hat{u}_n)(t) \in F^*(t)\ \ \lambda\ \text{ a.e.}$$

and the values of the mapping $F^*(t)$ are compact sets, similarly to the proof of Lemma 5.2 we obtain

$$\bigcap_{n\geq 1} \overline{\text{co}} \bigcup_{k\geq n} U(t, \mathcal{T}(\hat{u}_k)(t)) \subset U(t, \mathcal{T}(\hat{u})(t))\ \ \lambda\ \text{ a.e.} \tag{5.21}$$

Let a sequence $v_n \in S_U(\mathcal{T}(\hat{u}_n)), n \geq 1$ weakly converge to v. Then, from (5.21) we infer that $v \in S_U(\mathcal{T}(\hat{u}))$. Consequently, the mapping $\hat{u} \to S_U(\mathcal{T}(\hat{u}))$, $\hat{u} \in S_{F^*}$ has a weakly closed graph. Then, in accordance with (5.20) from the convexity and compactness of the set S_{F^*} in the space $\omega - L^1_\lambda(T, H)$ it follows that the mapping $\hat{u} \to S_U(\mathcal{T}(\hat{u})), \hat{u} \in S_F$ is an upper semicontinuous mapping from $\omega - S_{F^*}$ to $\omega - S_{F^*}$ with convex compact values. According to Theorem 1 of [7], if K is a convex compact set of a linear topological space and a multivalued mapping $W\colon K \to K$ with convex compact values is upper semicontinuous, then there exists a fixed point of this mapping. Therefore, the mapping $\hat{u} \to S_u(\mathcal{T}(\hat{u}))$, $\hat{u} \in S_F$ has a fixed point $\hat{u}_*$, i.e.

$$\hat{u}_* \in S_U(\mathcal{T}(\hat{u}_*)).$$

Hence, for $w(\hat{u}_*) = \mathcal{T}(\hat{u}_*)$ we have

$$\hat{u}_*(t) \in U(t, \mathcal{T}(\hat{u}_*)(t) = F(t, w(\hat{u}_*)(t)) \cap (m(t) + n(t)\|w(\hat{u}_*)(t)\|)\bar{B}. \tag{5.22}$$

Since $w(\hat{u}_*) = \mathcal{T}(\hat{u}_*)$ is a solution of inclusion (4.1), from (5.22) we see that $w(\hat{u}_*)$ is a solution of inclusion (1.2), and the proof is complete. □

Proof of Theorem 1.3.

Let hypotheses $H_2(F)$ hold. Then, from Theorem 1.2 it follows that the set $\mathcal{R}(u_0)$ of solutions of inclusion (1.2) is not empty. Let $w(\hat{u}) \in \mathcal{R}(u_0)$. Then,

$$\hat{u}(t) \in F(T, w(\hat{u})(t))\ \lambda \text{ a.e.} \tag{5.23}$$

Using (4.22) and (1.5) we obtain

$$\|w(\hat{u})(t) - w(\theta)(t)\| \leq \int_{]0,t]} (m(\tau) + n(\tau)\|w(\hat{u})(\tau)\|)d\lambda(\tau).$$

Then,

$$\|w(\hat{u})(t)\| \leq M + \int_{]0,t]} (m(\tau) + n(\tau)\|w(\hat{u})(\tau)\|)d\lambda(\tau),$$

where the constant M is defined by equality (4.27).

From Lemma 5.1 and inequality (1.5) we infer that

$$\|w(\hat{u})(t)\| \leq r(t), \ t \in T, \tag{5.24}$$

$$\|\hat{u}(t)\| \leq m(t) + n(t)r(t) = \dot{r}(t). \tag{5.25}$$

From (5.24) it follows that the set $\mathcal{R}(u_0)(t) = \{w(\hat{u})(t); \ w(\hat{u}) \in \mathcal{R}(u_0)\}$ is bounded for each $t \in T$. Then, from (1.5) and Hypothesis $H_2(F)$ 2) we see that the values of the mapping $t \to F^*(t) = \overline{\text{co}}F(t, \mathcal{R}(u_0)(t))$ are convex compact sets and according to (1.5), (5.25)

$$\|F^*(t, \mathcal{R}(u_0)(t))\| \leq m(t) + n(t)r(t) = \dot{r}(t) \ \lambda \text{ a.e.}$$

From this inequality it follows that the set

$$S_{F^*} = \{\hat{u} \in L^1_\lambda(T, H); \ \hat{u}(t) \in F^*(t, \mathcal{R}(u_0)(t)) \ \lambda \text{ a.e.}\}$$

is a convex metrizable compact subset of the space $\omega - L^1_\lambda(T, H)$. Then, according to Lemma 5.4 the mapping $\hat{u} \to \mathcal{T}(\hat{u}) = w(\hat{u})$ is continuous from $\omega - S_{F^*}$ to $V_+(T, H)$. Since for any $w(\hat{u}) \in \mathcal{R}(u_0)$ the inclusion $\hat{u} \in S_{F^*}$ holds, the set $\mathcal{R}(u_0)$ is a relatively compact subset of the space $V_+(T, H)$. Show that the set $\mathcal{R}(u_0)$ is closed in $V_+(T, H)$.

Let
$$S_F(\mathcal{T}(\hat{u})) = \{v \in L^1_\lambda(T, H); \ v(t) \in F(t, \mathcal{T}(\hat{u})(t)) \ \lambda \text{ a.e.}\} \tag{5.26}$$

As it was shown, the mapping $\hat{u} \to S_F(\mathcal{T}(\hat{u}))$, $\hat{u} \in S_{F^*}$ considered as a mapping from $\omega - S_{F^*}$ to $\omega - L^1_\lambda(T, H)$ has a weakly closed graph.

Let the sequence $w(\hat{u}_n) \in \mathcal{R}(u_0), n \geq 1$ converge uniformly on T to a function w. Then, $w \in V_+(T, H)$. Since $\hat{u}_n \in S_{F^*}, n \geq 1$, there exists a subsequence $\hat{u}_{n_k}, k \geq 1$ of the sequence $\hat{u}_n, n \geq 1$, weakly converging to $\hat{u} \in S_{F^*}$. Taking into account the continuity of the mapping $\mathcal{T}$ on $\omega - S_{F^*}$ we obtain

$$w = \mathcal{T}(\hat{u}). \tag{5.27}$$

On the other hand, $\hat{u}_{n_k} \in S_F(\mathcal{T}(\hat{u}_{n_k}))$ for $k \geq 1$.

Passing to the limit in this inclusion we obtain $\hat{u} \in S_F(\mathcal{T}(\hat{u}))$.

From (5.26) it follows that $\hat{u}(t) \in F(t, w(t))$ λ a.e., and equality (5.27) says that w is a solution of inclusion (1.2). Hence, $w(\hat{u}) \in \mathcal{R}(u_0)$ and the set $\mathcal{R}(u_0)$

is compact. The properties of the set $\mathcal{R}(u_0)$ stated in the theorem follow from Corollary 2.5. The theorem is proven. □

When the measure μ is absolutely continuous with respect to the measure λ, the measures λ and ν are absolutely equivalent. In this case, we arrive at inclusion (1.7) whose solutions are absolutely continuous functions. We do not state paraphrases of Theorem 1.2 and 1.3 applied to solutions of inclusion (1.7) as they are evident.

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