

Does Deconvolution Exist?

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Blurring of a photographic image by a wrong focus can be modeled by convolution. This paper discusses some points for the inverse operation with particular interest on the set of integers $\mathbb{Z}$.

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1. Introduction.

When I was a beginner I read many papers by A.D. Ioffe, for example [4, 5], this about fifty years ago; since then he went on beautifully and it is honor and pleasure dedicating this work to Alexander Ioffe.

Briefly a problem is *ill-posed* if there is a “bad” transformation A (“signal” $\mapsto$ “blurred signal” for example) and one tries recovering the preimage of any y , expecting to find an x such that $Ax = y$. Difficulties could lie in: A is not one-to-one, or very different initial points may have very closed image (see Section 4), or (frequently this happens simultaneously) the map A is not onto.

Photographic images often present blurring, for example due to a wrong focus setting. Several other defects due to different causes are possible (cf. [1]). Defect of focus is roughly equivalent to convolution of the image source with the brightness of the image of one point light in 0^1 . Numerous papers use the word *deconvolution*. Is it more than a word? Surely this belongs to the class of ill-posed problems (see [10, Ch.IV pp.91–115]).

Several authors add a stochastic component. There is a clear reason: when the map “signal” $\mapsto$ “blurred signal” is not onto (this may highly depends on the functional space under consideration), finding a preimage to any point in the target space of A needs some stochastic adjustment. Many papers speak of *bayesian*. The

¹ The density could be $k\mathbf{1}_{B(0,r)}$ ($\mathbf{1}_A$ denotes the *indicator function* of the subset A) where $r > 0$ is a radius and $k = (\pi r^2)^{-1}$. Question in dimension 2 and with the Euclidean norm: is it a zero divisor for convolution? We will see (Section 2) that in dimension 1 we do have a zero divisor.

signal itself can be seen as a trajectory of a stochastic process². *Robustness* may also be referred to because the perturbation is not exactly known. A lot of recent papers use *wavelets*³.

The literature is prolific and difficult to understand. The words *mask*, *sharpening* are keys on the Net, as also *filtration* and *denoising*. An astonishing algorithm is due to Van Cittert: see Wikipedia (German) at “Van-Cittert-Deconvolution”, or [6], where the algorithm is also explained. A paper with a heralding title, which quotes Van Cittert and illustrates the interest to the question outside the purely mathematical world is [2].

We will give some calculi with the space $\mathbb{Z}$ (dimension 1) and refine the threshold $\frac{1}{2}$ highlighted by Duval [3] (see Section 6).

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2. Convolution. Notations. Zero divisors

Convolution and Fourier transform have as framework either $\mathbb{R}^d$ (which is its own dual group), or $\mathbb{Z}^d$ and its dual group⁴ $\mathbb{T}^d$ (maybe the groups $\mathbb{Z}/n\mathbb{Z}$?). When μ and ν are bounded measures on $\mathbb{R}^d$, their convolution product, denoted by $\mu * \nu$, is the image (pushforward⁵) by $(x, y) \mapsto x + y$ of their product $\mu \otimes \nu$ (cf. the sum of independent random variables in probability). For Lebesgue integrable functions on $\mathbb{R}^d$, their convolution product is classically

$$f * g : x \mapsto \int_{\mathbb{R}^d} f(x - y) g(y) dy.$$

The excellent paper by Ross [8] examines mainly convolution of L^1 functions.

Concerning convolution of distributions, Schwartz begins with the case of two when one of them has compact support. Then he proves [9, vol. 2, Th. VII, p. 14] that the convolution of a finite number of distributions which have all, except one, compact supports is associative and commutative and then [9, vol. 2, ch. 6, §5, p. 26] moves to more general situations. He proves that, in dimension $d = 1$, the *convolution algebra* $\mathcal{D}'_+$ (the set of distributions with supports limited on left) has no zero divisors [9, vol. 2, ch. 6, Th. XIV, p. 29]. The same result holds for $\mathcal{D}'_-$ (the set of distributions with supports limited on right).

The unit mass in 0, δ_0 , is always a neutral element and convolution by δ_x ($x \in \mathbb{R}^d$) amounts to translation by the vector x . Uniqueness of a possible inverse holds when one works in a subspace where associativity holds (see (1) hereafter). We will see cases where several inverses do coexist (Theorem 3.3).

² Cf. the Wiener filter, I learned in R. Pallu de La Barrière [7].

³ Using the Fourier transform is tempting but disappointing.

⁴ Cf. Fourier series.

⁵ Cf. the writing $\mu * \nu = S_{\#}(\mu \otimes \nu)$ where S denotes the sum.

Let us show that δ_x has as unique inverse δ_{-x} (who doubts it?). Suppose H is another distribution inverse of δ_x . Among the three distributions δ_x , δ_{-x} and H , two have compact support, hence associativity holds and

$$H = H * \delta_0 = H * (\delta_x * \delta_{-x}) = (H * \delta_x) * \delta_{-x} = \delta_0 * \delta_{-x} = \delta_{-x}. \quad (1)$$

When dealing with $\mathbb{Z}$ and measures such as $\mu = \sum_{n \in \mathbb{Z}} x_n \delta_n$, or $\nu = \sum_{n \in \mathbb{Z}} y_n \delta_n$, the point of view of convolution is to consider the function $n \mapsto x_n$ defined on $\mathbb{Z}$ (resp. $n \mapsto y_n$). The convolution of μ and ν returns to the convolution $z := x * y$ where $z_n = \sum_{k \in \mathbb{Z}} x_{n-k} y_k$. Next h will equivalently denote a measure or a function on $\mathbb{Z}$.

Example 2.1. (Examples of zero divisors)

(1) With $\mathbb{R}$ let consider the *gate function* $h = \mathbf{1}_{[-1,1]}$. Then

$$h * \mathbf{1}_{\cup_{n \in \mathbb{Z}} [2n, 2n+1]} = h * \left(\frac{1}{2} \mathbf{1}_{\mathbb{R}} \right)$$

hence $f \mapsto h * f$ is not injective on $L^\infty(\mathbb{R})$, and h is a zero divisor:

$$h * \left(\mathbf{1}_{\cup_{n \in \mathbb{Z}} [2n, 2n+1]} - \frac{1}{2} \mathbf{1}_{\mathbb{R}} \right) = 0.$$

(2) With $\mathbb{Z}$, take $h := \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1$ or $h := \frac{1}{4} \delta_{-1} + \frac{1}{2} \delta_0 + \frac{1}{4} \delta_1$. Then we have

$$h * \mathbf{1}_{2\mathbb{Z}} = h * \left(\frac{1}{2} \mathbf{1}_{\mathbb{Z}} \right), \quad (2)$$

hence $f \mapsto h * f$ is not injective on $\ell^\infty(\mathbb{Z})$, and h is a zero divisor:

$$h * \left(\mathbf{1}_{2\mathbb{Z}} - \frac{1}{2} \mathbf{1}_{\mathbb{Z}} \right) = 0.$$

3. Convolution and inverse, particular cases

Let us begin with $h = a \delta_0 + (1-a) \delta_1$ ($a \in]0, 1[$) (a kind of “gate function”).

Lemma 3.1. *Let $a \in]0, 1[$ and $h = a \delta_0 + (1-a) \delta_1$. Then an inverse of h in $\mathcal{D}'_+(\mathbb{R})$ is⁶*

$$J = \frac{1}{a} \delta_0 - \frac{1-a}{a^2} \delta_1 + \frac{(1-a)^2}{a^3} \delta_2 + \dots \quad (3)$$

(the limit is for the weak topology $\sigma(\mathcal{D}', \mathcal{D})$)

Proof. Indeed $h * \frac{1}{a} \sum_{n=0}^k \left[-\frac{1-a}{a} \right]^n \delta_n = \delta_0 + (-1)^k \left[\frac{1-a}{a} \right]^{k+1} \delta_{k+1} \rightarrow \delta_0$

because for any α_n , $\alpha_n \delta_n \rightarrow 0$ in the topology $\sigma(\mathcal{D}', \mathcal{D})$ when $n \rightarrow \infty$. □

⁶ Cf. the known formula $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$ for $x \in \mathbb{R}$.

Lemma 3.2. *An inverse of h in $\mathcal{D}'(\mathbb{R})$ is*

$$\frac{1}{1-a} \delta_{-1} - \frac{a}{(1-a)^2} \delta_{-2} + \frac{a^2}{(1-a)^3} \delta_{-3} - \frac{a^3}{(1-a)^4} \delta_{-4} + \dots \quad (4)$$

(the limit still for $\sigma(\mathcal{D}', \mathcal{D})$)

Proof. One can write $h = (1-a) \delta_1 * (\delta_0 + \frac{a}{1-a} \delta_{-1})$.

Then $(1-a) \delta_1$ admits the inverse $\frac{1}{1-a} \delta_{-1}$ and for the second factor one can develop “on left” as in the preceding lemma. $\square$

Theorem 3.3. *The distribution $\frac{1}{2}(\delta_0 + \delta_1)$ on $\mathbb{R}$ admits several inverses in $\mathcal{D}'$ with respect to convolution (the limits are for $\sigma(\mathcal{D}', \mathcal{D})$):*

$$J_1 = 2 \lim_{k \rightarrow \infty} \sum_{n=0}^k (-1)^n \delta_n = 2(\delta_0 - \delta_1 + \delta_2 - \delta_3 + \dots), \quad (5)$$

$$J_2 = 2 \lim_{k \rightarrow \infty} \sum_{n=1}^k (-1)^{n-1} \delta_{-n} = 2(\delta_{-1} - \delta_{-2} + \delta_{-3} - \delta_{-4} + \dots)$$

and especially $H = \frac{1}{2}(J_1 + J_2)$ i.e.

$$H = \dots - \delta_{-4} + \delta_{-3} - \delta_{-2} + \delta_{-1} + \delta_0 - \delta_1 + \delta_2 - \delta_3 + \dots \quad (6)$$

Moreover for any $f \in \mathbb{R}^{(\mathbb{Z})}$ (the space of real sequences on $\mathbb{Z}$ with compact supports)

$$(f * h) * H = f.$$

Remark 3.4. For any $\lambda \in \mathbb{R}$, $\lambda J_1 + (1-\lambda) J_2$ is also an inverse of h . And $J_1 - J_2$ forms with h a couple of zero divisors.

Proof. The lemmas imply the assertions about inverses. The last formula follows from the fact that f and h have compact supports, hence $(f * h) * H = f * (h * H) = f * \delta_0$. $\square$

Now we turn to a measure carried by $\{-1, 0, 1\}$, still positive with total mass 1. With the parameter $a \in]\frac{1}{2}, 1[$

$$h := \frac{1-a}{2} \delta_{-1} + a \delta_0 + \frac{1-a}{2} \delta_1 \quad (7)$$

or, with the parameter $b = \frac{1-a}{2} \in]0, \frac{1}{4}[$ which will be often better suited,

$$h := b \delta_{-1} + (1-2b) \delta_0 + b \delta_1. \quad \square$$

Lemma 3.5. *Let $b \in]0, \frac{1}{4}[$. Then $\lambda = \frac{1}{2b}[2b-1 + \sqrt{1-4b}]$ belongs to $] -1, 0[$, tends to 0 if $b \rightarrow 0$, and tends to -1 if $b \rightarrow 1/4$.*

Proof. Elementarily λ is a root of the equation $\lambda^2 + \frac{1-2b}{b}\lambda + 1 = 0$. One has $\lambda \leq 0$ because

$$\begin{aligned} 2b - 1 + \sqrt{1 - 4b} \leq 0 &\iff \sqrt{1 - 4b} \leq 1 - 2b \iff 1 - 4b \leq (1 - 2b)^2 \\ &\iff 1 - 4b \leq 1 - 4b + 4b^2 \end{aligned}$$

and $\lambda > -1$ because $2b - 1 + \sqrt{1 - 4b} > -2b \iff \sqrt{1 - 4b} > 1 - 4b$ which holds, since on $]0, 1[$, $\sqrt{x}$ is $> x$. The convergences are easy. $\square$

Theorem 3.6. Let h be given by (7), i.e. $h := \frac{1-a}{2}\delta_{-1} + a\delta_0 + \frac{1-a}{2}\delta_1$.

Let c be defined for all $n \in \mathbb{Z}$ by $c_n = \lambda^{|n|} (\sqrt{1 - 4b})^{-1}$.

The c_n are alternatively > 0 and < 0 and $\sum_{n \in \mathbb{Z}} c_n = 1$. The measure (or sequence) c is an inverse of h , that is $h * c = \delta_0$. Moreover for any $f \in \ell^\infty(\mathbb{Z})$

$$(f * h) * c = f.$$

Remark 3.7. Since $-1 < \lambda < 0$, c considered as a function oscillates as the famous *cardinal sine function*: $\text{sinc } x = \frac{\sin x}{x}$ (cf. also the *mexican hat*). This seems quite general. For another comment see Section 6.

Proof. (1) One has

$$\begin{aligned} \sum_{n \in \mathbb{Z}} c_n &= c_0 + 2 \sum_{n \geq 1} c_n = c_0 \left(1 + 2 \sum_{n \geq 1} \lambda^n \right) = c_0 \left(1 + 2 \frac{\lambda}{1 - \lambda} \right) \\ &= \frac{1}{\sqrt{1 - 4b}} \frac{1 + \lambda}{1 - \lambda} = \frac{1}{\sqrt{1 - 4b}} \frac{4b - 1 + \sqrt{1 - 4b}}{1 - \sqrt{1 - 4b}} = 1. \end{aligned}$$

$$(2) \text{ Firstly } (h * c)_n = \sum_{i \in \mathbb{Z}} h(n - i) c_i = \sum_{i \in \mathbb{Z}} h(i) c_{n-i}.$$

For $n = 0$ this gives

$$\begin{aligned} (h * c)_0 &= h(-1) c_1 + h(0) c_0 + h(1) c_{-1} \\ &= b \frac{\lambda}{\sqrt{1 - 4b}} + (1 - 2b) \frac{1}{\sqrt{1 - 4b}} + b \frac{\lambda}{\sqrt{1 - 4b}} = \frac{1}{\sqrt{1 - 4b}} [2b\lambda + 1 - 2b] \\ &= \frac{1}{\sqrt{1 - 4b}} [2b - 1 + \sqrt{1 - 4b} + 1 - 2b] = 1. \end{aligned}$$

For $n \geq 1$ this gives

$$\begin{aligned} (h * c)_n &= h(-1) c_{n+1} + h(0) c_n + h(1) c_{n-1} = b(c_{n-1} + c_{n+1}) + (1 - 2b) c_n \\ &= c_0 [b\lambda^{n-1} + b\lambda^{n+1} + (1 - 2b)\lambda^n] = \frac{\lambda^{n-1}}{\sqrt{1 - 4b}} [b + (1 - 2b)\lambda + b\lambda^2] = 0 \end{aligned}$$

because $\lambda^2 + \frac{1-2b}{b}\lambda + 1 = 0$.

(3) As for $(f * h) * c$, the functions are respectively, bounded for f , with compact support for h , integrable for c (convergent sum). So associativity holds. $\square$

4. Illustration (pictures on $\mathbb{Z}$)

A monochrome photographic image can be modeled by a (measurable) function $f: \mathbb{R}^2 \rightarrow [0, 1]$, f measuring the brightness.

We will expose some examples with $f: \mathbb{Z} \rightarrow [0, 1]$, that is a one dimensional picture formed from pixels. So the basic space is $\ell^\infty(\mathbb{Z})$. Another natural space is $\mathbb{R}^{(\mathbb{Z})}$, that is the space of real sequences on $\mathbb{Z}$ with compact supports (this is Bourbaki's notation); it is a natural space since pictures do have compact supports. Other vector spaces could be considered in abstract studies ($p \in]1, \infty[$):

$$\mathbb{R}^{(\mathbb{Z})} \subset \ell^1(\mathbb{Z}) \subset \ell^p(\mathbb{Z}) \subset c_0(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}) \subset \mathbb{R}^{\mathbb{Z}}.$$

As measure spaces, $\mathbb{R}^{\mathbb{Z}} \sim \mathcal{M}(\mathbb{Z})$ (the space of all measures on $\mathbb{Z}$) and $\ell^1(\mathbb{Z}) \sim \mathcal{M}^b(\mathbb{Z})$ (the space of all bounded measures on $\mathbb{Z}$).

Let us consider the linear map

$$A := \begin{cases} \mathbb{R}^{\mathbb{Z}} \longrightarrow \mathbb{R}^{\mathbb{Z}} \\ (x_n)_{n \in \mathbb{Z}} \mapsto (y_n)_{n \in \mathbb{Z}} \quad \text{where} \quad y_n = \frac{1}{2}(x_{n-1} + x_n). \end{cases}$$

Applying A is the same thing as convolution by the “gate function” $h = \frac{1}{2}(\delta_0 + \delta_1)$. It is not one-to-one, its kernel being (elementary verification)

$$\ker A = \{\lambda((-1)^n)_{n \in \mathbb{Z}}; \lambda \in \mathbb{R}\} = \{\lambda(\mathbf{1}_{2\mathbb{Z}} - \mathbf{1}_{2\mathbb{Z}+1}); \lambda \in \mathbb{R}\}.$$

This kernel expression holds too with the space $\ell^\infty(\mathbb{Z})$. But by restricting the linear transformation A to $c_0(\mathbb{Z})$ or to a smaller subspace, the kernel becomes $\{0\}$ and the map $x \mapsto Ax$ is then one-to-one.

Here come our main observations:

- (a) $x = \frac{1}{2} \mathbf{1}_{\mathbb{Z}}$ is *perfect grey*;
- (b) $x_n = 1$ if n is even, 0 otherwise is *macroscopically grey*;
- (c) the same ones on, for example $\{0, \dots, 999\}$, will reveal to have quite different properties.

Pictures belonging to $\ell^\infty(\mathbb{Z})$. Let the convolution by h be the blurring action. Then $\mathbf{1}_{2\mathbb{Z}} * h$ and $\frac{1}{2} \mathbf{1}_{\mathbb{Z}}$ ($= [\frac{1}{2} \mathbf{1}_{\mathbb{Z}}] * h$) are identical. Inversion of A and deconvolution are impossible.

Pictures belonging to $\mathbb{R}^{(\mathbb{Z})}$. Then A is one-to-one (its kernel, $\ker A$, vanishes). If $x \in \mathbb{R}^{(\mathbb{Z})}$ the blurred picture $h * x$ has also compact support and convolution with H defined in (6) is possible. Thus

$$(x * h) * H = x * (h * H) = x * \delta_0 = x. \quad (8)$$

But some different x can give very closed blurred pictures. More precisely take

$$x_n = \begin{cases} 1 & \text{if } n \text{ is even and } 0 \leq n \leq 998 \\ 0 & \text{otherwise} \end{cases}$$

(there are 500 pixels with value 1). The blurring gives the picture $y = h * x$ with

$$y_n = \frac{1}{2} x_n + \frac{1}{2} x_{n-1} = \frac{1}{2} \text{ for } 0 \leq n \leq 999 \text{ and } 0 \text{ otherwise} \quad (9)$$

(there are 1000 pixels with value $1/2$). But the almost perfect grey picture $\tilde{x} = \frac{1}{2} \mathbf{1}_{\{0,999\}}$ (it is grey on a large interval) is blurred into $\tilde{y}$ where

$$\tilde{y}_n = \begin{cases} \frac{1}{2} & \text{if } 1 \leq n \leq 999 \\ \frac{1}{4} & \text{if } n = 0 \text{ or } 1000 \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

which is very closed to y obtained in (9). This illustrates the ill-posedness of the inversion problem⁷. Note also that despite the possibility of deconvolution (8), H is an unbounded measure with unbounded support. This inversion is in some sense *academical*.

Practitioners use high-pass filters under the form of convolution with a small supported *mask* (look on the Net at “sharpening”), for example in dimension 2 a measure supported by $\{-1, 0, 1\} \times \{-1, 0, 1\}$ as maybe

$$\begin{array}{|c|c|c|} \hline 0 & -1 & 0 \\ \hline -1 & 5 & -1 \\ \hline 0 & -1 & 0 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|c|} \hline -1 & -1 & -1 \\ \hline -1 & 9 & -1 \\ \hline -1 & -1 & -1 \\ \hline \end{array}$$

the sum of all coefficients being 1.

5. Exercices

When the picture x or $\tilde{x}$ belong to $\mathbb{R}^{\mathbb{Z}}$, deconvolution works theoretically perfectly.

Case of macroscopic grey. As for $y = A(x)$ given in (9) the formula

$$\sum_{k \in \mathbb{Z}} y_k H_{n-k}$$

(H_m is the m -th term of H defined in (6)) gives exactly x_n . This could be an exercise. The inverse J_1 (cf. (5)) can equally do the job, with

$$\sum_{k \in \mathbb{Z}} y_k J_{1,n-k} \quad \text{where} \quad J_{1,m} = 2(-1)^m \text{ for } m \geq 0.$$

Case of almost perfect grey. As for $\tilde{y} = A(\tilde{x})$ given in (10) the formulas

$$\sum_{k \in \mathbb{Z}} \tilde{y}_k H_{n-k} \quad \text{or} \quad \sum_{k \in \mathbb{Z}} \tilde{y}_k J_{1,n-k}$$

give exactly $\tilde{x}_n$.

⁷ In this example there is a bad behavior as analysed in *sampling theory*.

6. About the threshold $1/2$

In [3] Duval studies convolution by $a\delta_0 + \alpha g(x) dx$ imposing $a > \frac{1}{2}$.

We refound this in Lemma 3.1 where the multiplicative factor $-\frac{1-a}{a}$ has absolute value < 1 if and only if $a > \frac{1}{2}$.

We refound again this in Theorem 3.6 where the multiplicative factor λ belongs to $] -1, 0[$ and badly tends to -1 when $a \searrow \frac{1}{2}$ (equivalently $b \nearrow \frac{1}{4}$).

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