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Lipschitz-like Property of Subdifferential Mapping of Convex Continuous Functions on Hilbert Space

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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A Lipschitz-like property of subdifferential mappings of convex continuous functions on Hilbert spaces is investigated. It is shown that the subdifferential mapping of such a function admits a Lipschitz-like property at all points from a dense subset of its domain. In particular, Lipschitz-like properties of subdifferentials of distance functions from sets with convex complements are discovered.

Keywords: Convex functions, Aubin Lipschitz-like properties, best approximation, metric projection mapping, Borel sets.

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1. Introduction

We review briefly the state of the art on second-order differentiability of convex continuous functions. It is known, see [5, Section 5.1] for example, that there are convex continuous functions on any nonseparable Hilbert space which are nowhere second-order differentiable. Moreover, even in standard Euclidean space, there are convex continuous functions such that the set of points where a second order Fréchet derivative exists, cannot be residual; see [6, Exercise 2.6.13] for example. In the light of this information we should not expect second-order properties of convex continuous functions in the general cases. Instead of guaranteeing second-order expansions we should look for results on local Lipschitz behavior of subdifferentials. This is the aim of this paper. For this reason fibers of metric projection are explored and their properties are used as a tool to discover Lipschitz properties of subdifferentials. The basic tool is the diameter of fiber of metric projection mapping. It turns out that it forms an upper semicontinuous function, say F_S; see Proposition 2.4. Properties of this function allow us to get several results on Lipschitz-like properties of subdifferentials; see for example Theorem 3.1 or Proposition 3.4 below. It is also interesting to know, how large the set of points, at which subdifferential admits a local Lipschitz-like property, is. In an

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infinite dimensional space it is a dense subset, see Corollary 3.9 and Corollary 4.1; it has full measure, whenever the Euclidean space and distance functions from sets with convex complements are considered, see Proposition 4.3. In the last result in the paper, see Proposition 4.4, it is shown that the lower semicontinuity of F_S at points of its is rather a rare property. In the case S is the hypograph of a convex continuous function, to get the continuity of F_S, at a point from the support of F_S, there is a necessity to impose strong assumptions on the convex function or vice versa the continuity of F_S at a support point implies that the convex function is locally $C^{1,1}$, we refer to [12] and references therein for more on functions from smooth classes of functions and conditions preserving a proper level of smoothness.

All properties of functions are presented in a Hilbert space setting. It seems that it could also be possible to extend them to more general settings but, at this moment, it is an open problem.

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2. Preliminaries

Let $\mathbb{H}$ be a real Hilbert space and $\mathbb{R}^n$ stands for the n-dimensional Euclidean space; see for example [4, Example 2.3]. For every real $r > 0$ and every $x \in \mathbb{H}$ we denote by $B_{\mathbb{H}}(x, r)$ (resp. $B_{\mathbb{H}}[x, r]$) the open (resp. closed) ball centred at x and of radius r, the unit sphere of $\mathbb{H}$ is denoted by $S_{\mathbb{H}}[0, 1] := \{x \in \mathbb{H} \mid \|x\| = 1\}$, the *boundary* of a subset $D \subset \mathbb{H}$ is denoted by $\operatorname{fr} D$, where $\operatorname{fr} D := \operatorname{cl} D \setminus \operatorname{int} D$, and "int" stands for the *interior* of D as well as "cl" stands for the *strong topological closure* (the closure with respect to the norm). For given $x, y \in \mathbb{H}$ we put $[x, y] := \{tx + (1 - t)y \mid t \in [0, 1]\}$ and $]x, y[:= \{tx + (1 - t)y \mid t \in]0, 1[\}$. For every nonempty subset $S \subset \mathbb{H}$ the *distance function* from S is denoted by $d_S(\cdot)$, that is,

$$\forall x \in \mathbb{H}, \quad d_S(x) = \inf_{u \in S} \|u - x\|,$$

where the norm, $\|x\| := \sqrt{\langle x, x\rangle}$ for all $x \in \mathbb{H}$, is induced by the scalar product (inner product, dot product). The *metric projection mapping* on S is defined by

$$P_S(x) := \{s \in \operatorname{cl} S \mid d_S(x) = \|x - s\|\}. \tag{1}$$

Define the *fiber* over $s \in \mathbb{H}$ as

$$D_s(S) := \{x \in \mathbb{H} \mid d_S(x) = \|x - s\|\} \tag{2}$$

and relate to the fibers the following function $F_S : \operatorname{cl}(\mathbb{H} \setminus S) \longrightarrow [0, \infty]$ by

$$F_S(u) := \operatorname{diam} D_u(S_u), \tag{3}$$

where $S_u := \{z \in \mathbb{H} \mid d_S(z) \le d_S(u)\}$, $D_u(S_u) := \{x \in \mathbb{H} \mid d_{S_u}(x) = \|x - u\|\}$ and

$$\operatorname{diam} D_u(S_u) := \sup_{a \in D_u(S_u),\, b \in D_u(S_u)} \|a - b\|.$$

Negative infinity is defined as the supremum of the empty set, i.e., $\sup \emptyset := -\infty$.

Several properties of fibers as defined in (2) can be found in [22]. For example, fibers are closed and convex subsets; see [22, Lemma 2.3]. Let $\bar{s} \in \operatorname{cl} S$, $u \in D_{\bar{s}}(S)$, and $d_S(u) > 0$. Then each point $u_\rho := \rho u + (1-\rho)\bar{s}$, with $\rho \in [0,1[$, has the following properties $u_\rho \in D_{\bar{s}}(S)$ and

$$\|u_\rho - s_i\| \longrightarrow d_S(u_\rho) \Longrightarrow s_i \longrightarrow \bar{s}, \tag{4}$$

where $\{s_i\}_{i\in\mathbb{N}}$ is a sequence of elements from S. This result was given in [25, Lemma 1.5], see also [22, Lemma 2.4]. In other words, we have the continuity of the metric projection mapping at each $u_\rho \in D_{\bar{s}}(S)$, and thus the Fréchet differentiability of the distance function at each $u_\rho \in D_{\bar{s}}(S) \setminus (\operatorname{cl} S \cup \{u\})$, see [8, Theorem 3.1]. This property can also be found in the proof of Proposition 4.1 in [13].

Lemma 2.1. *Let $S \subset \mathbb{H}$ be a closed nonempty subset of a Hilbert space $\mathbb{H}$ such that $\mathbb{H} \setminus S \neq \emptyset$ and let $u \in \mathbb{H} \setminus S$ be given. If $F_S(u) > 0$, then the Fréchet derivative of the distance function d_S exists at u, that is $D_F d_S(u)$ exists, and for $v := u - d_S(u)\, D_F d_S(u)$ we have $d_S(u) = \|u - v\|$, $v \in S$.*

Proof. Take any $z \in D_u(S_u) \setminus \{u\}$ and define $v := u + \frac{d_S(u)}{\|u-z\|}(u-z)$. We have $\|u-v\| = d_S(u)$ and $d_S(z) > d_S(u)$. Suppose that $d_S(z) < \|z-v\|$ and take $s' \in S$ such that $d_S(z) \leq \|s'-z\| < \|z-v\|$, and $y \in [s', z] \cap \{w \in \mathbb{H} \,|\, d_S(w) = d_S(u)\}$. Observe that $y \in S_u$ and $\|y - z\| \geq \|u - z\| = d_{S_u}(z)$. Thus

$$\begin{aligned} d_S(u) = d_S(y) &\leq \|s' - y\| = \|s' - z\| - \|y - z\| < \|z - v\| - \|y - z\| \\ &\leq \|z - u\| + \|u - v\| - \|y - z\| = d_{S_u}(z) + d_S(u) - \|y - z\| \\ &\leq d_{S_u}(z) + d_S(u) - d_{S_u}(z) = d_S(u), \end{aligned}$$

a contradiction, so $d_S(z) \geq \|z - v\|$. In this case we get

$$d_S(z) \geq \|z - v\| = \|u - z\| + d_S(u) \geq d_S(z),$$

which implies $d_S(z) = \|z - v\|$. For every $n \in \mathbb{N}$ let us choose $s_n \in S$ such that $\lim_{n\to\infty} \|u - s_n\| = d_S(u)$. The set $S' := S \cup \{v\}$ is closed and $d_{S'}(z) = \|z - v\|$, $d_S(u) = \|u - v\| = d_{S'}(u)$. It follows from (4) (take v, z, S' instead of $\bar{s}$, u and S, respectively, and $\rho := \|u - v\|/\|v - z\|$) that $\lim_{n\to\infty} \|v - s_n\| = 0$ and $D_F d_S(u)$ exists with $D_F d_S(u) = (z - u)/\|z - u\|$, so $v \in S$ and

$$v := u + \frac{d_S(u)}{\|u - z\|}(u - z) = u - \frac{d_S(u)}{\|z - v\|}(z - v) = u - d_S(u)\, D_F d_S(u). \qquad \square$$

A straightforward consequence of Lemma 2.1 is the following

Corollary 2.2. *Let $S \subset \mathbb{H}$ be a closed nonempty subset of a Hilbert space $\mathbb{H}$ such that $U := (\mathbb{H} \setminus S) \neq \emptyset$ and let $z \in U$ and $u \in \operatorname{cl} U$ be given. If $z \in (D_u(S_u) \setminus \{u\})$, then for $s := u + d_S(u)(u - z)/\|u - z\|$ we have $d_S(u) = \|u - s\|$, $s \in S$.*

A nested sequence of closed sets with convex complements, say $\{S_i\}_{i\in\mathbb{N}}$, generates a countable family of functions $\{F_{S_i}\}_{i\in\mathbb{N}}$. It is of interest if this family of functions has a converging property. Below, a kind of upper semicontinuity, with respect to the strong topology, is established. However, let us point out that there are decreasing nested families of sets $\{S_i\}_{i\in\mathbb{N}}$ such that

$$\forall u \in \mathbb{H}, \limsup_{i\to\infty, u_i\to u} F_{S_i}(u_i) = 0,$$

see (21) in Example 3.3 below.

Lemma 2.3. *Let $S_i \subset \mathbb{H}$ be closed nonempty subsets of a Hilbert space $\mathbb{H}$ such that $U_i := (\mathbb{H} \setminus S_i) \neq \emptyset$ are convex and $S_{i+1} \subset S_i$ for all $i \in \mathbb{N}$, $S := \bigcap_{i\in\mathbb{N}} S_i \neq \emptyset$, and $\mu > 0$ be given. If $F_{S_i}(u_i) > \mu$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \|u_i - u\| = 0$, then $F_S(u) \geq \mu$. Moreover, if additionally $\min\{d_S(u), d_{S_i}(u_i)\} > 0$ for all $i \in \mathbb{N}$, then the Fréchet derivative of the distance function d_S exists at u, that is $D_F d_S(u)$ exists, and $\lim_{i\to\infty} \|D_F d_{S_i}(u_i) - D_F d_S(u)\| = 0$.*

Proof. We have for all $j > i$, $d_S(u) \geq d_{S_j}(u) \geq d_{S_i}(u)$, thus

$$\zeta := \lim_{i\to\infty} d_{S_i}(u)$$

is well defined. Take $s_i \in S_i$ and $z_i \in D_{s_i}(S_i)$ such that $u_i \in [s_i, z_i]$; see Corollary 2.2, and $1 + \zeta + \mu > \|z_i - s_i\| \geq \|z_i - u_i\| > \mu$ for all $i \in \mathbb{N}$, and $\lim_{i\to\infty} \|z_i - u_i\|$ exists. For all $i, j \in \mathbb{N}$ put

$$\delta_{ij} := \|z_i - u_i\|^{-1}\|z_j - u_j\|^{-1}\langle z_i - u_i, z_j - u_j\rangle,$$

$$a_{ij} := \delta_{ij}\|z_j - u_j\|^{-1}(z_j - u_j), \quad \text{and} \quad b_{ij} := \|z_i - u_i\|^{-1}(z_i - u_i) - a_{ij}.$$

Fix $t > 0$ and notice that $\|z_j - s_j - tb_{ij}\|^2 = \|z_j - s_j\|^2 + t^2\|b_{ij}\|^2$, hence

$$\forall j > i, \quad s_j + tb_{ij} \in S_j \subset S_i. \tag{5}$$

We also have for all $j > i$,

$$\begin{aligned}
&\|z_i - u_j - tb_{ij}\|^2 = \|(z_i - u_i) + (u_i - u_j) - tb_{ij}\|^2 \\
&= \|z_i - u_i\|^2 + \|u_i - u_j\|^2 + t^2\|b_{ij}\|^2 + 2\langle(z_i - u_i), (u_i - u_j)\rangle - 2t\langle(z_i - u_j), b_{ij}\rangle \\
&= \|z_i - u_i\|^2 + \|u_i - u_j\|^2 + t^2\|b_{ij}\|^2 \\
&\qquad\qquad + 2\langle(z_i - u_i) - tb_{ij}, (u_i - u_j)\rangle - 2t\langle(z_i - u_i), b_{ij}\rangle \\
&= \|z_i - u_i\|^2 + \|u_i - u_j\|^2 + t^2\|b_{ij}\|^2 \\
&\qquad\qquad + 2\langle(z_i - u_i) - tb_{ij}, (u_i - u_j)\rangle - 2t\|z_i - u_i\|(1 - \delta_{ij}^2) \\
&\leq \|z_i - u_i\|^2 + t^2 + 2\|u_i - u_j\| \left(\|u_i - u_j\| + 1 + \zeta + \mu + t\right) - 2t\mu(1 - \delta_{ij}^2).
\end{aligned}$$

We exclude the case $\kappa := \liminf_{i\to\infty} \inf_{j>i} \delta_{ij} < 1$.
Suppose $\kappa < 1$. It follows from (5) that for $t > 0$ small enough we get

$$\begin{aligned}
\liminf_{i\to\infty} \|z_i - s_i\| &\leq \liminf_{i\to\infty} \inf_{j>i} \|z_i - s_j - tb_{ij}\| \\
&\leq \liminf_{i\to\infty} \inf_{j>i} \left(\|z_i - u_j - tb_{ij}\| + d_{S_j}(u_j)\right) \\
&\leq \liminf_{i\to\infty} \inf_{j>i} \Big(\|z_i - s_i\| + d_{S_j}(u_j) - d_{S_i}(u_i) \\
&\qquad + \frac{t^2 + 2\|u_i - u_j\|(\|u_i - u_j\| + 1 + \zeta + \mu + t) - 2t\mu(1 - \delta_{ij}^2)}{2\|z_i - u_i\|}\Big)
\end{aligned}$$

$$\le \liminf_{i\to\infty} \inf_{j>i} \Big(\|z_i - s_i\| + d_{S_j}(u) + \|u_j - u\| + \|u_i - u\| - d_{S_i}(u) \Big) + \frac{t^2 - t\mu(1-\kappa)}{2\|z_i - u_i\|}$$

$$\le \liminf_{i\to\infty} \inf_{j>i} \Big(\|z_i - s_i\| + \frac{t^2 - t\mu(1-\kappa)}{2(1+\zeta+\mu)} \Big) < \liminf_{i\to\infty} \|z_i - s_i\|,$$

a contradiction, whenever $\kappa < 1$ and $t < \mu(1-\kappa)$; thus $\kappa = 1$.

Hence there are $s, z \in \mathbb{H}$ such that $\lim_{i\to\infty} \|z_i - z\| = 0$, $\lim_{i\to\infty} \|s_i - s\| = 0$ and $\lim_{i\to\infty} \|z_i - u_i\|^{-1}(z_i - u_i) = \|z - u\|^{-1}(z - u)$ as well as $s \in \bigcap_{i\in\mathbb{N}} S_i = S$, and

$$d_S(u) \ge \zeta \ge \liminf_{i\to\infty}(d_{S_i}(u_i) - \|u - u_i\|) = \lim_{i\to\infty} \|u_i - s_i\| = \|u - s\| \ge d_S(u).$$

Hence $u \in D_s(S)$, $F_S(u) \ge \liminf_{i\to\infty} \|z_i - u_i\| \ge \mu$. Lemma 2.1 implies directly that $D_F d_S(u)$ exists whenever $d_S(u) > 0$. By simple algebra we have

$$D_F d_S(u) = \|z - u\|^{-1}(z - u) = \lim_{i\to\infty} \|z_i - u_i\|^{-1}(z_i - u_i) = \lim_{i\to\infty} D_F d_{S_i}(u_i). \qquad \square$$

As a simple consequence of Lemma 2.3, the upper semicontinuity of the function defined in (3) is established, whenever U is a nonempty convex and open subset of $\mathbb{H}$ with $\mathbb{H} \setminus U \neq \emptyset$.

Proposition 2.4. *Let $U \subset \mathbb{H}$ be a nonempty open and convex set, $S := \mathbb{H} \setminus U$ be nonempty. Then the function F_S is upper semicontinuous on* $\mathrm{cl}\, U$.

Proof. To get the upper semicontinuity of F_S it is enough to use Lemma 2.3 with $S_i := S$ for all $i \in \mathbb{N}$. $\square$

Subdifferentials are natural tools to explore convex functions. Formally, for a lower semicontinuous convex function, say $f\colon \mathbb{H} \longrightarrow \mathbb{R} \cup \{+\infty\}$, which is finite at $x \in \mathbb{H}$, the *subdifferential* (in the sense of Convex Analysis) of f at $x \in \mathbb{H}$ is defined as follows

$$\partial f(x) := \{x^* \in \mathbb{H} \mid \forall h \in \mathbb{H}, \langle x^*, h\rangle \le f(x+h) - f(x)\}.$$

For information on subdifferentials and their properties we refer to [4, 6, 11], for example. To define the subdifferential of a convex function $f\colon \mathbb{H} \longrightarrow \mathbb{R} \cup \{+\infty\}$, in the sense of Convex Analysis, we use global properties of the function. However there is a possibility to define subdifferentials for functions being convex on smaller sets. It can be done according to a general theory of abstract subdifferentials, see for example [12, (P2),(P3), page 192]. Namely, if $U \subset \mathbb{H}$ is a convex open and nonempty subset such that $f\colon U \longrightarrow R$ is a convex continuous function and $u \in U$ is given, then

$$\partial f(u) := \partial f_U^\infty(u), \tag{6}$$

where $f_U^\infty\colon \mathbb{H} \longrightarrow \mathbb{R} \cup \{+\infty\}$, $f_U^\infty(v) := f(v)$ for all $v \in U$ and $f_U^\infty(v) := \infty$ for all $v \notin U$. Let us also recall that the distance function from a set S with convex complement is a concave function on the complement; see [10, Proposition 1, p. 66] and the comments following the proposition. If $u \in U$ is fixed then we can use (6) in order to define $\partial(-d_S)(u)$, with $f := -d_S$.

Lemma 2.5. *Let $U \subset \mathbb{H}$ be a convex open subset of a Hilbert space $\mathbb{H}$, $S := (\mathbb{H} \setminus U) \neq \emptyset$, and $z \in U$. Then for every $x, y \in \mathbb{U}$, $\alpha > 0, \beta > 0$ such that $\alpha + \beta = 1$, $z = \alpha x + \beta y$ we have*

$$\alpha d_S(x) + \beta d_S(y) \leq d_S(z). \tag{7}$$

A straightforward consequence of the concavity of distance function is the following equivalence for all points $u \in U$

$$P_S(u) \neq \emptyset \Longleftrightarrow \partial(-d_S)(u) \cap S_{\mathbb{H}}[0,1] \neq \emptyset, \tag{8}$$

whenever $U \subset \mathbb{H}$ is a convex open nonempty subset of a Hilbert space $\mathbb{H}$ with $S := (\mathbb{H} \backslash U) \neq \emptyset$. Since the equivalence can be checked by simple algebra, we leave it without proof. In a finite dimensional space, by local compactness, $P_S(u) \neq \emptyset$ for all $u \in U$, thus $\partial(-d_S)(u) \setminus \mathbb{B}_{\mathbb{H}}(0,1) \neq \emptyset$ and $\partial(-d_S)(u) \cap S_{\mathbb{H}}[0,1] \neq \emptyset$ for all $u \in U$ (we recall that $\mathbb{B}_{\mathbb{H}}(0,1)$ stands for the open ball at the origin with radius 1). This mean that a potential usefulness of the equivalence in (8) can be explored in infinite dimensional cases. For example, knowing that $\partial(-d_S)(u) \cap S_{\mathbb{H}}[0,1] = \emptyset$ we conclude that $P_S(u) = \emptyset$. Below an example of a distance function such that $\partial(-d_S)(0) \subset \mathbb{B}_{\mathbb{H}}(0,1)$ is provided.

Example 2.6. Let $\mathbb{H}$ be a Hilbert space, Q be a closed bounded convex and nonempty subset, and $\alpha > 0$ be given. For all $x \in \mathbb{H}$ put

$$\Delta_Q(x) := \sup_{q \in Q} \|x - q\| \quad \text{and} \quad S_\alpha := \{u \in \mathbb{H} \mid \Delta_Q(u) \geq \alpha + \inf_{x \in \mathbb{H}} \Delta_Q(x)\}. \tag{9}$$

Suppose that for all $q \in Q$: $\Delta_Q(0) > \|q\|$ and $\alpha + \inf_{x \in \mathbb{H}} \Delta_Q(x) > \Delta_Q(0)$.

For all $u \in (\mathbb{H} \setminus S_\alpha)$ we have $\alpha + \inf_{x \in \mathbb{H}} \Delta_Q(x) = d_{S_\alpha}(u) + \Delta_Q(u);$ (10)

see [7, Corollary 6.2], we also refer to [14, Theorem 5].

Take any $u \in \mathbb{H}$ such that $\alpha + \inf_{x \in \mathbb{H}} \Delta_Q(x) \geq \Delta_Q(u)$. If $F_{S_\alpha}(u) = 0$, then it follows from Lemma (2.3) that $\lim_{v \to u} F_{S_\beta}(v) = 0$ for all $\beta > \alpha$.

Obviously the following relations hold true

$$\partial(-d_{S_\alpha})(0) = \partial(\Delta_Q)(0) \subset \mathbb{B}_{\mathbb{H}}(0,1). \tag{11}$$

Thus $P_{S_\alpha}(0) = \emptyset$, by (8). In particular $\partial(-d_{S_\alpha})(0) = \{0\}$ and $P_{S_\alpha}(0) = \emptyset$ for $\mathbb{H} := l_2$, $Q := \{\frac{1}{2}e_1, \frac{2}{3}e_2, \ldots, \frac{i}{i+1}e_i, \ldots\}$, where $e_i = (e_{i1}, e_{i2}, \ldots)$, and

for all $i, j \in \mathbb{N}$, $\quad e_{ij} := \begin{cases} 0, & \text{whenever } i \neq j; \\ 1, & \text{whenever } i = j. \end{cases}$

It is interesting to know which subdifferential properties are specific for distance functions. If f is a concave distance function, then its subdifferential is included in the unit ball for all $u \in \mathbb{H}$, $\partial(-f)(u) \subset \mathbb{B}_{\mathbb{H}}[0,1]$ (we recall that $\mathbb{B}_{\mathbb{H}}[0,1]$ stands for the closed ball at the origin of radius 1), also the monotonicity of the subdifferential mapping $\partial(-f)(\cdot)$ is preserved. However the equality $\partial(-d_{S_\alpha})(0) = \{0\}$, in the example above, shows that we should not expect that $\sup_{u^* \in \partial(-f)(u)} \|u^*\| = 1$ is a property of the distance function. Of course the use of subdifferentials in order to characterize distance functions should not be limited to Convex Analysis. For this purpose some abstract subdifferentials can be utilized, see for example [12]. At this moment such a characterization, to the best of the author's knowledge, is unknown. This is an open problem, which is stated explicitly below.

Question 2.7.
How can distance functions be characterized by means of subdifferentials?

In view of Example 2.6, considering (11) with any x replacing the origin, it is also of interest to know: *how large is the set* $\{x \in \mathbb{H} \mid \partial(\Delta_Q)(x) \subset \mathbb{B}_{\mathbb{H}}(0,1)\}$?

Implicitly (keep in mind that $\mathbb{B}_{\mathbb{H}}(0,1)$ stands for the open ball at the origin with radius 1 and (8) holds true) we ask how large is the set $\{x \in \mathbb{H} \mid P_{S_\alpha}(x) = \emptyset\}$?

We address this question below. It is known that the set

$$\{x \in \mathbb{H} \mid \partial(\Delta_Q)(x) \subset \mathbb{B}_{\mathbb{H}}(0,1)\}$$

can be represented as a countable union of nowhere dense sets, that is, this set is of first category; see [16, page 40]; we refer to [14, Lemma 2] for details how to get such results in a more general setting. We present it in a simple case. First let us notice that we have the following inclusion

$$\forall x \in \mathbb{H}, \Big(\{x \in \mathbb{H} \mid \partial(\Delta_Q)(x) \subset \mathbb{B}_{\mathbb{H}}(0,1)\} \\ \subset \{x \in \mathbb{H} \mid \max_{x^* \in \partial(\Delta_Q)(x)} \min_{q \in Q} (\Delta_Q(x) - \langle x^*, x-q\rangle) > 0\}\Big). \quad (12)$$

To justify this inclusion we use some results on the existence of saddle points; we refer to [3, Chapter II] for necessary information, for example, [3, Corollary 2.127], also comments following (2.132) and inequalities in (2.133). Fix any $x \in \mathbb{H}$. There is $(\tilde{x}^*, \tilde{q}) \in \partial(\Delta_Q)(x) \times Q$ such that

$$\Delta_Q(x) - \langle \tilde{x}^*, x - \tilde{q}\rangle = \max_{x^* \in \partial(\Delta_Q)(x)} \min_{q \in Q} (\Delta_Q(x) - \langle x^*, x-q\rangle).$$

Observe that if $\partial(\Delta_Q)(x) \subset \mathbb{B}_{\mathbb{H}}(0,1)$, then $\|\tilde{x}^*\| < 1$ and

$$\Delta_Q(x) - \langle \tilde{x}^*, x - \tilde{q}\rangle \geq \|x - \tilde{q}\| - \langle \tilde{x}^*, x - \tilde{q}\rangle > 0,$$

which establishes the inclusion. For each integer $i \in \mathbb{N}$ let

$$A_i := \{x \in \mathbb{H} \mid \exists x^* \in \partial(\Delta_Q)(x), \max_{q \in Q} \langle x^*, x-q\rangle + \frac{1}{i} \leq \Delta_Q(x)\}.$$

It is known that these sets are closed and nowhere dense; see [14, Lemma 2] for details. Thus $\bigcup_{i \in \mathbb{N}} A_i$ is of first category. The formula

$$\bigcup_{i\in\mathbb{N}} A_i = \{x \in \mathbb{H} \mid \max_{x^*\in\partial(\Delta_Q)(x)} \min_{q\in Q}(\Delta_Q(x) - \langle x^*, x-q\rangle) > 0\}$$

is easy to justify, thus the set $\{x \in \mathbb{H} \mid \partial(\Delta_Q)(x) \subset \mathbb{B}_{\mathbb{H}}(0,1)\}$ is of first category.

A next observation is that if $F_S(u) = \infty$, then S is a half space. In other words 0-Lipschitz property of Fréchet derivative of d_S is preserved on a half space. Thus the Fréchet derivative of the distance function is a constant on some open half space (it is a fixed element from the unit sphere), whereas it is zero on the other open half space; there is nothing to explore in this case.

Lemma 2.8. *Let $U \subset \mathbb{H}$ be an open nonempty and convex set, $S := (\mathbb{H} \setminus U) \neq \emptyset$ and $u \in \operatorname{cl} U$ be given. If $F_S(u) = \infty$, then there is $x^* \in S_{\mathbb{H}}[0,1]$ such that*

$$S_u = \{z \in \mathbb{H} \mid \langle x^*, z-u\rangle \leq 0\}.$$

Proof. Let us choose $a_n, b_n \in D_u(S_u)$ such that $\lim_{n\to\infty} \|a_n - b_n\| = \infty$. For all $n \in \mathbb{N}$ take $c_n \in \{a_n, b_n\}$ such that $\lim_{n\to\infty} \|c_n - u\| = \infty$. We may assume that $0 < \|c_n - u\| < \|c_{n+1} - u\|$ for all $n \in \mathbb{N}$; if not, then a proper subsequence can be selected. We have

$$S_u \cap \left(\bigcup_{c\in D_u(S_u)} B_{\mathbb{H}}(c, \|c-u\|) \right) = \emptyset. \tag{13}$$

Case 1: Assume that $B_{\mathbb{H}}[c_n, \|c_n - u\|] \subset B_{\mathbb{H}}[c_{n+1}, \|c_{n+1} - u\|]$ for all $n \in \mathbb{N}$; we recall that $B_{\mathbb{H}}(c,r)$ stands for the open ball centred at c and of radius r, whereas $B_{\mathbb{H}}[c,r]$ for the closed one. In this case the set

$$\bigcup_{n\in\mathbb{N}} B_{\mathbb{H}}(c_n, \|c_n - u\|)$$

becomes a half space, it can be checked directly or [21, Theorem 3.7] can be used for this purpose. It is also easy to notice; see [21, Theorem 3.7] for more details if needed, that the limit $\lim_{n\to\infty} \|c_n - u\|^{-1}(c_n - u))$ exists. Thus for

$$x^* := \lim_{n\to\infty} \|c_n - u\|^{-1}(c_n - u)$$

we are done. In fact, if $\langle x^*, \bar{z} - u\rangle > 0$ for some $\bar{z} \in \mathbb{H}$, then

$$\begin{aligned}
&\limsup_{n\to\infty}(\|c_n - \bar{z}\|^2 - \|c_n - u\|^2) \\
&\qquad = \limsup_{n\to\infty}(2\|c_n - u\| \, \langle \|c_n - u\|^{-1}(c_n - u), u - \bar{z}\rangle + \|u - \bar{z}\|^2) < 0,
\end{aligned}$$

hence $\bar{z} \in \bigcup_{c\in D_u(S_u)} B_{\mathbb{H}}(c, \|c-u\|)$ and $\bar{z} \notin S_u$, by (13), which implies

$$\forall z \in S_u,\ \langle x^*, z-u\rangle \leq 0.$$

On the other hand, if $\tilde{z} \notin S_u$, then for some $r > 0$, by the convexity of $\mathbb{H} \setminus S_u$, we get

$$\forall n \in \mathbb{N},\ \forall y \in \mathbb{B}_{\mathbb{H}}[\tilde{z}, r],\ \langle \|c_n - u\|^{-1}(c_n - u), y-u\rangle > 0,$$

thus
$$\forall y \in \mathbb{B}_{\mathbb{H}}(\tilde{z}, r),\ \langle x^*, y-u\rangle > 0,$$

and $\langle x^*, \tilde{z} - u\rangle > 0$ in particular, which, in this case, yields

$$S_u = \{z \in \mathbb{H} \ \mid \ \langle x^*, z - u\rangle \leq 0\}.$$

Case 2: Assume that $B_{\mathbb{H}}[c_n, \|c_n - u\|] \setminus B_{\mathbb{H}}[c_m, \|c_m - u\|] \neq \emptyset$ for some $n < m$. Let us exclude this case. For this reason let us observe that by the convexity of $D_u(S_u)$ we get $[c_n, c_m] \subset D_u(S_u)$ and consequently we get the convexity of the following set

$$\bigcup_{t\in[0,1]} B_{\mathbb{H}}(((1-t)c_n + tc_m), \|((1-t)c_n + tc_m) - u\|).$$

Indeed, take $x, y \in \bigcup_{t\in[0,1]} B_{\mathbb{H}}(((1-t)c_n + tc_m), \|((1-t)c_n + tc_m) - u\|)$.
For some $t_x, t_y \in [0,1]$ we have

$$\|x - ((1-t_x)c_n + t_xc_m))\| < \|u - ((1-t_x)c_n + t_xc_m)\| = d_{S_u}(((1-t_x)c_n + t_xc_m)),$$

$$\|y - ((1-t_y)c_n + t_yc_m))\| < \|u - ((1-t_y)c_n + t_yc_m)\| = d_{S_u}(((1-t_y)c_n + t_yc_m)).$$

Fix $t \in [0,1]$. By (7) we obtain

$$\begin{aligned}
&\|(1-t)x + ty - ((1-t)((1-t_x)c_n + t_xc_m) + t((1-t_y)c_n + t_yc_m))\| \\
&\quad < (1-t)d_{S_u}(((1-t_x)c_n + t_xc_m)) + td_{S_u}(((1-t_y)c_n + t_yc_m)) \\
&\quad \leq d_{S_u}((1-t)((1-t_x)c_n + t_xc_m) + t((1-t_y)c_n + t_yc_m)) \\
&\quad = \|u - ((1-t)((1-t_x)c_n + t_xc_m) + t((1-t_y)c_n + t_yc_m))\|,
\end{aligned}$$

thus $$[x, y] \subset \bigcup_{t\in[0,1]} B_{\mathbb{H}}(((1-t)c_n + tc_m), \|((1-t)c_n + tc_m) - u\|)$$

and the convexity is established. Now choose $w \in B_{\mathbb{H}}(c_n, \|c_n - u\|)$ such that $\langle u - c_m, u - w\rangle < 0$ and take $\alpha > 0$, $\beta > 0$ such that

$$u + \alpha(u - w) + \alpha\beta(c_n - c_m) \in B_{\mathbb{H}}(c_m, \|c_m - u\|)$$

and $$w - \beta(c_n - c_m) \in B_{\mathbb{H}}(c_n, \|c_n - u\|).$$

Notice that for $\bar{t} := 1/(1+\alpha)$ we get

$$\begin{aligned}
u &= \bar{t}(u + \alpha(u - w) + \alpha\beta(c_n - c_m)) + (1-\bar{t})(w - \beta(c_n - c_m)) \\
&\in \bigcup_{t\in[0,1]} B_{\mathbb{H}}(((1-t)c_n + tc_m), \|((1-t)c_n + tc_m) - u\|),
\end{aligned}$$

which contradicts (13). Thus Case 2 is excluded and we are done. □

The concavity of the distance function can be also used to show that parts of the boundary of an open convex set can be regarded as parts of graphs of Lipschitz functions around some points. First, let us recall that the sets $\{u \in U \mid d_S(u) \geq \gamma\}$ are convex, whenever $\gamma > 0$, U is a nonempty convex and open subset of $\mathbb{H}$, where $S := (\mathbb{H} \setminus U) \neq \emptyset$.

Lemma 2.9. *Let $\mathbb{H}$ be a Hilbert space, $U \subset \mathbb{H}$ be a convex nonempty and open subset such $S := (\mathbb{H} \setminus U) \neq \emptyset$, and $c \in U$ be given such that $F_S(c) \in]0, \infty[$ (see Lemmas 2.8, 2.1) and $r := F_S(c)/4$, $c^* := -D_F d_S(c)$, $Y := \{w \in \mathbb{H} \,|\, \langle w, c^* \rangle = 0\}$. Then for all $\gamma \in]\max\{0, d_S(c) - 2^{-1}r\}, d_S(c) + 2^{-1}r]$ the function $f_{c,\gamma} \colon Y \longrightarrow \mathbb{R} \cup \{\infty\}$ is lower semicontinuous and convex, and $f_{c,\gamma}(y) \in [0, 2r]$, whenever $y \in \mathbb{B}_Y[0, 3r]$, where for all $y \in \mathbb{B}_Y[0, 3r]$,*

$$f_{c,\gamma}(y) := \inf \left\{ t \geq 0 \;\middle|\; \begin{array}{l} c + (d_S(c) - \gamma)c^* + y - tc^* \\ \in \{u \in U \,|\, d_S(u) \geq \gamma\} \end{array} \right\}$$

and $f_{c,\gamma}(y) := \infty$, whenever $y \notin \mathbb{B}_Y[0, 3r]$. Moreover, $f_{c,\gamma}$ restricted to $\mathbb{B}_Y[0, r]$ is 1-Lipschitz, that is, $|f_{c,\gamma}(a) - f_{c,\gamma}(b)| \leq \|a - b\|$, whenever $a, b \in \mathbb{B}_Y[0, r]$, and $f_{c,\gamma}(0) = 0$, $D_F f_{c,\gamma}(0) = 0$, and, for all $d \in (U \cap \mathbb{B}_{\mathbb{H}}[c, 3r])$, $\gamma = d_S(d)$ implies

$$d = c + (d_S(c) - \gamma)c^* + \big((d - c) - \langle d - c, c^* \rangle c^*\big) - f_{c,\gamma}(d - c - \langle d - c, c^* \rangle c^*)c^*.$$

Proof. Let us fix $\gamma \in]\max\{0, d_S(c) - 2^{-1}r\}, d_S(c) + 2^{-1}r]$ and take $z \in D_c(S_c)$ such that $\|z - c\| = 4r$. Notice that $c^* := \|c - z\|(c - z)$ and $d_S(c + (d_S(c) - \gamma)c^*) = \gamma$ (see Corollary 2.2), thus $f_{c,\gamma}(0) = 0$ and $D_F f_{c,\gamma}(0) = 0$ (see also Lemma 2.1), moreover, $\{u \in U \,|\, d_S(u) \geq \gamma\}$ is convex and nonempty. Since

$$\|z - (c + (d_S(c) - \gamma)c^*)\| = d_{S_{c+(d_S(c)-\gamma)c^*}}(z),$$

we have for all $u \in c + (d_S(c) - \gamma)c^* + Y$, $\ d_S(u) \leq \gamma$.

Now it is easy to check that function $f_{c,\gamma}$ is convex lower semicontinuous and $f_{c,\gamma}(y) \in [0, 2r]$, whenever $y \in \mathbb{B}_Y[0, 3r]$. Let $a, b \in \mathbb{B}_Y[0, r]$ and $h \in S_Y[0, 3r]$ be such that $b \in [a, h]$. For some $\mu \in [0, 1]$ we have

$$b = (1 - \mu)h + \mu a, \quad \|b - a\| = (1 - \mu)\|h - a\| \ \text{ and}$$

$$\begin{aligned} f_{c,\gamma}(b) - f_{c,\gamma}(a) &\leq (1 - \mu)(f_{c,\gamma}(h) - f_{c,\gamma}(a)) \leq (1 - \mu)f_{c,\gamma}(h) \leq (1 - \mu)2r \\ &= (1 - \mu)2r\frac{\|h - a\|}{\|h - a\|} = 2r\frac{\|b - a\|}{\|h - a\|} \leq 2r\frac{\|b - a\|}{2r} = \|b - a\|. \end{aligned}$$

Similarly we can justify the following inequality $f_{c,\gamma}(a) - f_{c,\gamma}(b) \leq \|b - a\|$. Thus $f_{c,\gamma}$ restricted to $\mathbb{B}_Y[0, r]$ is 1-Lipschitz with $D_F f_{c,\gamma}(0) = 0$.

Let us fix $d \in \mathbb{B}_{\mathbb{H}}[c, 3r] \cap U$ such that $d_S(d) = \gamma$. By a simple algebra we get $w := (d - c - \langle d - c, c^* \rangle c^*) \in \mathbb{B}_Y[0, 3r]$, and

$$d = c + (d_S(c) - d_S(d))c^* + w - f_{c,\gamma}(w)c^*. \qquad \square$$

In the next property it is recalled that the cone generated by the subdifferential at a given point of continuity of a convex function coincides with the normal cone to the level set at this point; see [11, Proposition 2, p. 206].

Proposition 2.10. *Let $f\colon \mathbb{H} \longrightarrow \mathbb{R}\cup\{\infty\}$ be a convex function, which is finite and continuous at a given point $u \in \mathbb{H}$ such that $\{z \in \mathbb{H} \mid f(z) < f(u)\} \neq \emptyset$. Then*

$$\{z^* \in \mathbb{H} \mid \forall z \in \mathbb{H}, f(z) \leq f(u) \Longrightarrow \langle z^*, z-u\rangle \leq 0\} = \bigcup_{\lambda\geq 0} \lambda\partial f(u).$$

Moreover, see Lemma 2.5, if $U \subset \mathbb{H}$ is a convex open nonempty subset of a Hilbert space $\mathbb{H}$, $S := (\mathbb{H}\setminus U) \neq \emptyset$, then for all $u \in U$ such that $d_S(v) > d_S(u)$ for some $v \in U$ we get

$$\{z^* \in \mathbb{H} \mid \forall z \in \mathbb{H}, d_S(z) \geq d_S(u) \Longrightarrow \langle z^*, z-u\rangle \leq 0\} = \bigcup_{\lambda\geq 0} \lambda\partial(-d_S)(u).$$

3. Lipschitz-like properties of subdifferential mapping

It is known that the local boundedness of a convex function is equivalent to the local Lipschitz continuity of the function; see for example [6, Lemma 4.1.3]. Thus the continuity at some point implies the Lipschitz continuity around this point. It is of interest to know whether local boundedness of convex function can be translated into a local Lipschitz-like property of subdifferential mapping, for example in the sense of the Aubin Lipschitz-like property; see [1]. In view of [12, Corollary 4.7] the answer is negative in a general case, even in Hilbert space. However, if the demand is restricted (of Lipschitz-like property of subdifferential mapping at points) to some dense subset, then the answer is positive. This is investigated below. Also, it should be added, that such a property can be obtained on residual subsets only for special classes of convex functions. This fact is to be investigated in more details in the next section.

Theorem 3.1. *Let $\mathbb{H}$ be a Hilbert space and $f\colon \mathbb{H} \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a convex function. Assume that $x, y \in \mathbb{H}$, $\beta \in R$ are such that $\max\{f(x), f(y)\} < \beta$ and*

$$\forall (u,\gamma) \in \mathbb{H}\times\mathbb{R}, \|u-y\|^2 + (\gamma-\beta)^2 \leq \|x-y\|^2 + (f(x)-\beta)^2 \Longrightarrow f(u) \leq \gamma, \quad (14)$$

$$\forall u \in \mathbb{H}, \|x-y\|^2 + (f(x)-\beta)^2 \leq \|u-y\|^2 + (f(u)-\beta)^2. \quad (15)$$

Then for $\alpha := (2(\beta - f(x)))^{-1}$ the following statements hold true:

(i) *the Fréchet derivative of f exists at x, say $D_F f(x)$, and*

$$x^* := D_F f(x) = 2\alpha(x-y);$$

(ii) *for all $u \in \mathbb{H}$ and all $u^* \in \partial f(u)$ we have for all $h \in \mathbb{H}$,*

$$\langle u^* - x^*, h\rangle \leq \alpha((f(u+h) - f(x))^2 + \|u+h-x\|^2) + (\langle x^*, u-x\rangle + f(x) - f(u));$$

(iii) *for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $u \in \mathbb{B}_{\mathbb{H}}[x,\delta]$*

$$\partial f(u) \subset x^* + 4\alpha((\|x^*\| + \epsilon)^2 + 1)\|u-x\|\mathbb{B}_{\mathbb{H}}[0,1];$$

(iv) $\displaystyle\limsup_{u\to x} \frac{\sup_{u^*\in\partial f(u)} \|u^* - x^*\|}{\|u-x\|} \leq 4\alpha(\|x^*\|^2 + 1).$

Moreover, if for some $M \geq 0$ the function f is M-Lipschitz continuous on $\mathbb{H}$, that is $|f(b) - f(a)| \leq M\|b - a\|$ for all $a, b \in \mathbb{H}$, then

(v) *for all $u \in \mathbb{H}$, $\partial f(u) \subset x^* + 4\alpha(M^2 + 1)\|u - x\|\mathbb{B}_{\mathbb{H}}[0, 1]$.*

Proof. The continuity of f on $\mathbb{B}_{\mathbb{H}}(y, \sqrt{\|x - y\|^2 + (f(x) - \beta)^2})$ is a consequence of the local boundedness of f, which follows from (14). Moreover, this property implies the local Lipschitz property of f, see for example [6, Proposition 4.1.4].

It follows from (15) that for all $h \in \mathbb{H}$,

$$\|x - y\|^2 + (f(x) - \beta)^2 \leq \|x + h - y\|^2 + (f(x + h) - \beta)^2.$$

Thus, for all $h \in \mathbb{H}$,

$$2(\beta - f(x))(f(x + h) - f(x)) \leq 2\langle x - y, h\rangle + \|h\|^2 + (f(x + h) - f(x))^2$$

and consequently, for all $h \in \mathbb{H}$,

$$f(x + h) - f(x) \leq \langle x^*, h\rangle + \alpha(\|h\|^2 + (f(x + h) - f(x))^2). \tag{16}$$

Since f is Lipschitz continuous on some open neighborhood of x, it follows from (16) that

$$\lim_{h\to 0} [f(x + h) - f(x) - \langle x^*, h\rangle]/\|h\| = 0,$$

thus $D_F f(x) = x^*$. Let us take $u, u^* \in \mathbb{H}$ such that $u^* \in \partial f(u)$. Again using (16) for all $h \in \mathbb{H}$ we get

$$\begin{aligned}\langle u^*, h\rangle \leq f(u + h) - f(u) = f(u + h) - f(x) + f(x) - f(u) \leq \langle x^*, h\rangle \\ + \alpha(\|u + h - x\|^2 + (f(u + h) - f(x))^2) + (\langle x^*, u - x\rangle + f(x) - f(u))\end{aligned}$$

and statement (ii) is justified. It is known that if a lower semicontinuous convex function is Fréchet differentiable at a point from the interior of its domain, then the function is strictly Fréchet differentiable at this point; see for example [6, Proposition 6.1.4] or [13, Proposition 4.4]. Thus we have

$$\forall \epsilon > 0, \exists \delta > 0 : \forall u \in \mathbb{B}_{\mathbb{H}}[x, 2\delta],\ \partial f(u) \subset (x^* + \epsilon\mathbb{B}_{\mathbb{H}}[0, 1]).$$

Statement (iii) easily follows from (ii) (consider $h \in S_{\mathbb{H}}[0, \|u - x\|]$) and (iv) follows immediately from (iii). In order to complete the proof we observe that whenever f is M-Lipschitz function, then (v) is a simple consequence of (ii). □

It remains a problem how to preserve the assumption $f(x) < \beta$ in the theorem above. It turns out that the continuity of f is enough for that purpose. We recall

$$f'(x; y - x) = \inf_{t>0}[f(x + t(y - x)) - f(x)]/t \ \text{ for all } y \in \mathbb{H}.$$

Lemma 3.2. *Let $\mathbb{H}$ be a Hilbert space and $f : \mathbb{H} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function such that $f(x) < \infty$ at a given point $x \in \mathbb{H}$. Assume that $y \in \mathbb{H}$, $\beta \in R$ and $M \geq 0$ are such that $f(y) < \beta$, $|f'(x; y - x)| \leq M\|x - y\|$, and*

$$\forall (u, \gamma) \in \mathbb{H} \times \mathbb{R},\ \|u - y\|^2 + (\gamma - \beta)^2 \leq \|x - y\|^2 + (f(x) - \beta)^2 \Longrightarrow f(u) \leq \gamma, \tag{17}$$

$$\forall u \in \mathbb{H},\ \|x-y\|^2 + (f(x)-\beta)^2 \leq \|u-y\|^2 + (f(u)-\beta)^2. \tag{18}$$

Then $f(x) < \beta$ *and all assumptions of Theorem* 3.1 *are satisfied, so* (i), (ii), (iii) *and* (iv) *hold true. Additionally we have* $\|x-y\|^2 \leq -(\beta - f(x))f'(x, y-x)$ *and*

$$M(\beta - f(x)) \geq \|y-x\| \text{ and } (1+M^2)(\beta - f(x)) \geq \beta - f(y). \tag{19}$$

Proof. Let us notice that $\beta \geq f(x)$. Indeed, if $\beta < f(x)$ then

$$\|y-x\|^2 + (\beta - (f(x)-\sigma))^2 < \|y-x\|^2 + (\beta - f(x))^2,$$

for some $\sigma > 0$, so it follows from (17) that $f(x) < f(x) - \sigma$, which is impossible. Using (18) we have for all $t \in]0,1[$

$$\begin{aligned}\|x-y\|^2 + (\beta - f(x))^2 &\leq \|(x+t(y-x)-y)\|^2 + (\beta - (f(x+t(y-x)))^2 \\ = (1-2t+t^2)\|x-y\|^2 &+ (\beta-f(x))^2 - 2(\beta - f(x))(f(x+t(y-x)) - f(x)) \\ &+ (f(x+t(y-x)) - f(x))^2,\end{aligned}$$

thus for all $t \in]0,1[$, $\quad 2\|x-y\|^2 \leq$

$$\leq t\|x-y\|^2 - 2(\beta - f(x))\frac{(f(x+t(y-x)) - f(x))}{t} + \frac{(f(x+t(y-x)) - f(x))^2}{t}.$$

Hence
$$\|x-y\|^2 \leq -(\beta - f(x))f'(x; y-x).$$

Notice that if $\beta = f(x)$ then $f(y) < \beta = f(x)$, so $\|y-x\| > 0$ and

$$0 < \|y-x\|^2 \leq 0,$$

a contradiction. Hence we get $f(x) < \beta$ and $\|x-y\|^2 \leq -(\beta - f(x))f'(x, y-x)$. Using the inequality $-f'(x; y-x) \leq M\|x-y\|$ we infer that $\|y-x\| \leq M(\beta - f(x))$ and

$$\begin{aligned}\beta - f(x) = \beta - f(y) + f(y) - f(x) &\geq \beta - f(y) + f'(x; y-x) \\ &\geq \beta - f(y) - M\|y-x\| \geq \beta - f(y) - M^2(\beta - f(x)),\end{aligned}$$

which justifies (19). □

Example 3.3. (Continuation of Example 2.6) Let Q be as in Example 2.6. We know that $\partial(\Delta_Q)(0) \subset \mathbb{B}_{\mathbb{H}}(0,1)$; see (11). An obvious consequence of this inclusion is the lack of "good" directions of convergence of subdifferentials of Δ_Q at the origin; we refer to [24] for a discussion on the role of "good" directions of convergence in exploring subdifferentials, in [23] sufficient conditions for a direction to be a "good" direction for the directional convergence of subdifferentials are proposed. Namely by (11) we have

$$S_{\mathbb{H}}[0,1] \cap \limsup_{i\to\infty} \partial\Delta_Q(t_i h_i) \subset S_{\mathbb{H}}[0,1] \cap \partial\Delta_Q(0) = \emptyset, \tag{20}$$

for all $h \in \mathbb{H}$ and all sequences $\{h_i\}_{i\in\mathbb{N}}$, $\{t_i\}_{i\in\mathbb{N}}$, which converge to h and 0, respectively, with $t_i > 0$ for all $i \in \mathbb{N}$, where

$$\limsup_{i\to\infty} \partial\Delta_Q(t_i h_i) := \{x^* \in \mathbb{H} \mid \liminf_{i\to\infty} d_{\partial\Delta_Q(t_i h_i)}(x^*) = 0\} \subset \partial\Delta_Q(0) \subset \mathbb{B}_{\mathbb{H}}(0,1).$$

Let us define convex continuous functions $f_i\colon \mathbb{H} \longrightarrow \mathbb{R}$ as follows

$$\forall i \in \mathbb{N},\, \forall h \in \mathbb{H},\;\; f_i(h) := i(\Delta_Q(i^{-1}h) - \Delta_Q(0))$$

and subsets $S_i \subset \mathbb{H} \times \mathbb{R}$ as their hypographs, namely

$$\forall i \in \mathbb{N},\;\; S_i := \{(h,\gamma) \in \mathbb{H} \times \mathbb{R} \mid \gamma \leq f_i(h)\}.$$

Put $\quad \forall h \in \mathbb{H},\;\; f(h) := \inf_{i\in\mathbb{N}} f_i(h)$ and $S := \{(h,\gamma) \in \mathbb{H} \times \mathbb{R} \mid \gamma \leq f(h)\}$

and observe that $S_1 \supset S_2 \supset \ldots$, and $S = \bigcap_{i\in\mathbb{N}} S_i$. We show that

$$\forall (u,\theta) \in \mathbb{H} \times \mathbb{R} \setminus S, \quad \limsup_{i\to\infty,(u_i,\theta_i)\to(u,\theta)} F_{S_i}((u_i,\theta_i)) = 0. \tag{21}$$

Suppose that its contrary holds true, that is, there are $(y,\beta) \in \mathbb{H} \times \mathbb{R} \setminus S$ and a sequence $\{y_i,\beta_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty}(\|y_i - y\|^2 + (\beta_i - \beta)^2) = 0$ and

$$\limsup_{i\to\infty} F_{S_i}((y_i,\beta_i)) > 0.$$

That is, there are $\mu > 0$ and a subsequence $\{i_j\}_{j\in\mathbb{N}}$ for which $F_{S_{i_j}}((y_{i_j},\beta_{i_j})) > \mu$ for all $j \in \mathbb{N}$, and by Lemma 2.3 the Fréchet derivative of the distance function d_S exists at (y,β) ($D_F d_S((y,\beta))$ exists), and

$$\lim_{j\to\infty} \|D_F d_{S_{i_j}}((y_{i_j},\beta_{i_j})) - D_F d_S((y,\beta))\| = 0. \tag{22}$$

Take $x_j \in \mathbb{H}$, $x \in \mathbb{H}$ such that

$$d^2_{S_{i_j}}((y_{i_j},\beta_{i_j})) = \|y_{i_j} - x_j\|^2 + (\beta_{i_j} - f_{i_j}(x_j))^2, \quad d^2_S((y,\beta)) = \|y - x\|^2 + (\beta - f(x))^2$$

and $\quad D_F d_{S_{i_j}}((y_{i_j},\beta_{i_j})) = \dfrac{1}{d_{S_{i_j}}((y_{i_j},\beta_{i_j}))}(y_{i_j} - x_j, \beta_{i_j} - f_{i_j}(x_j))$

for all $j \in \mathbb{N}$ and

$$D_F d_S((y,\beta)) = \frac{1}{d_S((y,\beta))}(y - x, \beta - f(x)).$$

We may assume that the limit

$$d := \lim_{j\to\infty} d_{S_{i_j}}((y_{i_j},\beta_{i_j})) = \lim_{j\to\infty} \sqrt{\|y_{i_j} - x_j\|^2 + (\beta_{i_j} - f_{i_j}(x_j))^2}$$

exists, otherwise we take a proper subsequence. It is a simple algebra to show that $d \in [0, d_S((y,\beta))]$. If $d = 0$, then

$$\lim_{j\to\infty} \|y - x_j\|^2 + (\beta - f_{i_j}(x_j))^2 = 0$$

and $(y,\beta) \in S$, a contradiction. By (22) we obtain $\lim_{j\to\infty} \|x_j - x\| = 0$, $\lim_{j\to\infty} \|f_{i_j}(x_j) - f(x)\| = 0$. Hence, it follows from Theorem 3.1, see also Lemma 3.2, that

$$\lim_{j\to\infty} \|D_F f_{i_j}(x_j) - D_F f(x)\| = 0.$$

Thus $$1 = \lim_{j\to\infty} \|D_F \Delta_Q({i_j}^{-1} x_j)\| = \lim_{j\to\infty} \|D_F f_{i_j}(x_j)\| = \|D_F f(x)\|.$$

It is easy to verify that $D_F f(x) \in \partial(\Delta_Q)(0) \subset \mathbb{B}_{\mathbb{H}}(0,1)$ (see (20)), which yields $\|D_F f(x)\| < 1$, a contradiction. Thus (21) holds true.

It is known, see for example [12, Theorem 4.2], that continuity assumptions of some selection of subdifferential of a convex function can be translated into smoothness properties of the function. However, assuming continuity or Lipschitz continuity on some dense subset, we may only infer properties of subdifferentials on this set; see [13, Corollary 4.3 or 4.5]. Little is known about the continuity of subdifferentials around points from that set. Below, we provide some additional facts, whenever a distance function is explored.

Proposition 3.4. *Let $\mathbb{H}$ be a Hilbert space, $U \subset \mathbb{H}$ be a convex nonempty and open subset such that $S := (\mathbb{H} \setminus U) \neq \emptyset$, and $u \in U$ be given with $F_S(u) \in]0,\infty[$ (see Lemma 2.8), $r := F_S(u)/4$. Then for all $v \in \mathbb{B}_{\mathbb{H}}(u, 2^{-1}r) \cap U$ and all $v^* \in \partial(-d_S)(v)$ we have $0 > \langle v^*, D_F d_S(u)\rangle$ and*

$$\| - \frac{1}{\langle v^*, D_F d_S(u)\rangle} v^* + D_F d_S(u)\| \le \frac{5}{F_S(u)} \|v-u\|. \tag{23}$$

Moreover, if $v \in \mathbb{B}_{\mathbb{H}}(u, 2^{-1}r) \cap U$, $v^* \in \partial(-d_S)(v)$ and

$$\left(-\frac{1}{\langle v^*, D_F d_S(u)\rangle} + 1\right) \|v^*\|^2 + 2\langle v^*, D_F d_S(u)\rangle \ge 0, \tag{24}$$

then

$$\|v^* + D_F d_S(u)\| \le \| - \frac{1}{\langle v^*, D_F d_S(u)\rangle} v^* + D_F d_S(u)\| \le \frac{5}{F_S(u)} \|v-u\|.$$

Proof. First, let us recall that $D_F d_S(u)$ exists; see Lemma 2.1. Let us fix $v \in \mathbb{B}_{\mathbb{H}}(u, 2^{-1}r) \cap U$ and $v^* \in \partial(-d_S)(v)$. Put $\gamma := d_S(v)$, $c := u$, $c^* := -D_F d_S(u)$ and take $z \in D_c(S_c)$ such that $\|z - c\| = 4r$. It follows from Lemma 2.9 that the function $f_{c,\gamma}$ (defined in Lemma 2.9) is 1-Lipschitz on $\mathbb{B}_Y[0,r]$ with

$$Y := \{w \in \mathbb{H} \,|\, \langle w, c^*\rangle = 0\}.$$

Moreover $D_F f_{c,\gamma}(0) = 0$ and

$$v = u + (d_S(u) - d_S(v))c^* + (v - u - \langle v-u, c^*\rangle c^*) - f_{c,\gamma}(v - u - \langle v-u, c^*\rangle c^*)c^*.$$

It follows from Theorem 3.1 that for $\alpha := [2(4r - (d_S(v) - d_S(u)))]^{-1}$ and for all $d^* \in \partial f_{c,\gamma}(v - u - \langle v-u, c^*\rangle c^*)$ we have for all $h \in \mathbb{B}_Y[0,r]$,

$$\begin{aligned} \langle d^*, h\rangle &\le \alpha((f_{c,\gamma}(v - u - \langle v-u, c^*\rangle c^* + h) - f_{c,\gamma}(0))^2 \\ &\quad + \|v - u - \langle v-u, c^*\rangle c^* + h\|^2) \le 2\alpha \|v - u - \langle v-u, c^*\rangle c^* + h\|^2, \end{aligned} \tag{25}$$

hence $$\|d^*\| \le 8\alpha \|v - u - \langle v-u, c^*\rangle c^*\|. \tag{26}$$

Notice that $v^* - \langle v^*, c^*\rangle c^* \in Y$. Since

$$\forall y \in \mathbb{B}_Y(0,r),\ (u + (d_S(u) - d_S(v))c^* + y - f_{c,\gamma}(y)c^*) \in \mathrm{fr}\, S_v,$$

by Proposition 2.10 and Lemma 2.9 we have for all $y \in \mathbb{B}_Y(0,r)$,

$$\begin{aligned}0 &\geq \langle v^*, (u + (d_S(u) - d_S(v))c^* + y - f_{c,\gamma}(y)c^*) - v\rangle \\ &= \langle v^*, (u + (d_S(u) - d_S(v))c^* + y - f_{c,\gamma}(y)c^*) - (u + (d_S(u) - d_S(v))c^* \\ &\qquad + v - u - \langle v - u, c^*\rangle c^* - f_{c,\gamma}(v - u - \langle v - u, c^*\rangle c^*)\rangle \\ &= \langle v^*, (y - f_{c,\gamma}(y)c^*) - (v - u - \langle v - u, c^*\rangle c^* - f_{c,\gamma}(v - u - \langle v - u, c^*\rangle c^*)\rangle \\ &= \langle v^*, y - (v - u - \langle v - u, c^*\rangle c^*)\rangle + \langle v^*, c^*\rangle(f_{c,\gamma}(v - u - \langle v - u, c^*\rangle c^*) - f_{c,\gamma}(y)) \\ &= \langle v^* - \langle v^*, c^*\rangle c^*, y - (v - u - \langle v - u, c^*\rangle c^*)\rangle \\ &\qquad - \langle v^*, c^*\rangle(f_{c,\gamma}(y) - f_{c,\gamma}(v - u - \langle v - u, c^*\rangle c^*)). \end{aligned} \tag{27}$$

Suppose that $\langle v^*, c^*\rangle < 0$. We recall that $\mathbb{H}\setminus S_v$ is convex nonempty and open with $B_{\mathbb{H}}[z, 2r] \subset (\mathbb{H} \setminus S_v)$. Choose any $t_v > 0$ such that $v - t_v c^* \notin S_v$. By Proposition 2.10 we have

$$0 \geq \langle v^*, v - t_v c^* - v\rangle = -t_v\langle v^*, c^*\rangle > 0,$$

a contradiction. If $\langle v^*, c^*\rangle = 0$, then by (27) we get

$$\partial(-d_S)(v) \ni v^* = (v^* - \langle v^*, c^*\rangle c^*) = 0,$$

thus $\gamma = d_S(v) \geq d_S(u)$ and $-d_S(v) = -d_S(u + (d_S(u) - d_S(v))c^*)$.

Since $F_S(u+(d_S(u)-d_S(v))c^*) > 0$, so by Lemma 2.1 $D_F d_S(u+(d_S(u)-d_S(v))c^*)$ exists. This is impossible since

$$v^* = 0,\ v^* \in \partial(-d_S)(v) \quad \text{and} \quad -d_S(v) = -d_S(u + (d_S(u) - d_S(v))c^*)$$

imply $0 \in \partial(-d_S)(u + (d_S(u) - d_S(v))c^*)$. If $\langle v^*, c^*\rangle > 0$, then by (27) we have $\langle v^*, c^*\rangle^{-1}(v^* - \langle v^*, c^*\rangle c^*) \in \partial f_{c,\gamma}(v - u - \langle v - u, c^*\rangle c^*)$, and hence by (26) we get

$$\|\frac{1}{\langle v^*, c^*\rangle} v^* + D_F d_S(u)\| \leq 8\alpha\|v - u - \langle v - u, c^*\rangle c^*\| \leq \frac{5}{F_S(u)}\|v - u\|.$$

Notice that, see (24), if $\left(\frac{1}{\langle v^*, c^*\rangle} + 1\right)\|v^*\|^2 - 2\langle v^*, c^*\rangle \geq 0,$ then

$$\begin{aligned}\|\frac{1}{\langle v^*, c^*\rangle} v^* - c^*\|^2 - \|v^* - c^*\|^2 &= \langle\left(\frac{1}{\langle v^*, c^*\rangle} - 1\right) v^*, \left(\frac{1}{\langle v^*, c^*\rangle} + 1\right) v^* - 2c^*\rangle \\ &= \left(\frac{1}{\langle v^*, c^*\rangle^2} - 1\right)\|v^*\|^2 - 2\left(\frac{1}{\langle v^*, c^*\rangle} - 1\right)\langle v^*, c^*\rangle \geq 0.\end{aligned}$$

Hence
$$\|\frac{1}{\langle v^*, c^*\rangle} v^* - c^*\| \geq \|v^* - c^*\|$$

and we are done. □

Let us observe that (24) holds true for all $v^* \in \mathbb{H}$ such that $\|v^*\|^2 \geq -\langle v^*, D_F d_S(u)\rangle$, in particular, whenever $v^* \in S_{\mathbb{H}}[0, 1]$. In fact we have

$$\begin{aligned}&\left(-\frac{1}{\langle v^*, D_F d_S(u)\rangle} + 1\right)\|v^*\|^2 + 2\langle v^*, D_F d_S(u)\rangle \\ &\geq -\left(-\frac{1}{\langle v^*, D_F d_S(u)\rangle} + 1\right)\langle v^*, D_F d_S(u)\rangle + 2\langle v^*, D_F d_S(u)\rangle\end{aligned}$$

$$= -\langle v^*, D_F d_S(u)\rangle \left(-\frac{1}{\langle v^*, D_F d_S(u)\rangle} - 1\right) \geq 0.$$

The measurability, with respect to the Borel σ-field, of the set

$$\{u \in U \mid P_S(u) \neq \emptyset \text{ and } F_S(u) = 0\}$$

becomes a sound question, whenever measure theory is to be involved in the investigation of properties of distance functions. Of course, there are no issues in finite dimensions: the local compactness implies that $U = \{u \in U \mid P_S(u) \neq \emptyset\}$ is open, and hence is Borel. However, in the infinite dimensional setting, in view of Example 2.6, the question needs an answer. To the best of the author's knowledge, little is known on this subject. Some facts are stated below, however we are far from understanding the problem.

Any subset $X \subset \mathbb{H}$ becomes a metric space with the induced metric. The smallest σ-algebra of subsets of X which contains all open sets of X is denoted by $B(X)$ and is called the Borel σ-field and elements of $B(X)$ are called Borel sets see [18, Chapter I, Definition 1.15] for the definition of Borel σ-algebra, we refer also to [17, Chapter I, General Properties of Borel Sets] for several properties of Borel sets. If X is a metric space and $Y \subset X$ is a Borel set in X, then $B(Y) = \{E \cap Y \mid E \in B(X)\}$ and $B(Y)$ is precisly the class of all subsets of Y which are Borel sets in X; see [17, Chapter I, Theorem 1,9].

Following [17, page 9] we write N for the space of all sequences $\{n_i\}_{i\in\mathbb{N}}$, where $n_1, n_2, \ldots$ are integers (≥ 1). N is the product of countably many copies of the set of positive integers, and a complete separable metric space in the product topology, the integers being given the discrete topology. The following definition of analytic set is taken from [17, Chapter I, Definition 3.1].

Definition 3.5. Let X be a complete separable metric space. A subset $E \subset X$ is called *analytic* if there is a continuous map ϕ of N onto E.

Every Borel set in X is analytic, whenever X is a complete separable metric space; see [17, Chapter I, Theorem 3.1]. Moreover, if X, Y are complete separable metric spaces, $E \subset X$ is a Borel set and $\phi\colon E \longrightarrow Y$ is a continuous one-one map, then $\phi(E)$ is a Borel set (it belongs to $B(Y)$; see [17, Chapter I, Theorem 3.9 and Theorem 1.5].

Corollary 3.6. *Let $\mathbb{H}$ be a separable Hilbert space, $U \subset \mathbb{H}$ a convex nonempty and open subset, $\bar{u} \in U$, and $\eta > 0$ be given such that $\mathbb{B}_{\mathbb{H}}[\bar{u}, 3\eta] \subset U$. Then $\{u \in \mathbb{B}_{\mathbb{H}}[\bar{u}, \eta] \mid P_S(u) \neq \emptyset$ and $F_S(u) = 0\}$ is an analytic set, where $S = \mathbb{H} \setminus U$.*

Moreover, if for all $u \in \mathbb{B}_{\mathbb{H}}[\bar{u}, \eta]$ the set $P_S(u)$ contains at most one-element, that is, the set is either empty or a singleton, then both sets

$$\{u \in \mathbb{B}_{\mathbb{H}}[\bar{u}, \eta] \mid P_S(u) = \emptyset,\ F_S(u) = 0\} \quad \textit{and} \quad \{u \in \mathbb{B}_{\mathbb{H}}[\bar{u}, \eta] \mid P_S(u) \neq \emptyset,\ F_S(u) = 0\}$$

are Borel sets.

Proof. Fix $\kappa \in]0, \min\{\frac{\eta}{8}, \frac{\eta}{\sup_{u\in\mathbb{B}_{\mathbb{H}}[\bar{u},\eta]} d_S(u)}, \frac{1}{4}, \}[$ and define (see Lemma 2.1)

$$\begin{aligned} A &:= \{u \in \mathbb{B}_{\mathbb{H}}[\bar{u}, 2\eta] \mid F_S(u) \geq \kappa\eta\} \quad \text{and for all } \ a \in A, \\ G(a) &:= a + \frac{\kappa}{1-\kappa} d_S(a) D_F d_S(a), \quad \text{and} \\ E &:= \{a \in A \mid F_S(a) = \frac{\kappa}{1-\kappa} d_S(a)\}. \end{aligned}$$

The set A is closed since F_S is upper semi-continuous, see Proposition 2.4, and E is a Borel set. It follows from Proposition 3.4 that G is a continuous mapping. Take any $v \in \{u \in \mathbb{B}_{\mathbb{H}}[\bar{u}, \eta] \mid P_S(u) \neq \emptyset \text{ and } F_S(u) = 0\}$, $s \in P_S(v)$ and define $w := v - \kappa(v - s)$. We have $F_S(w) \geq \kappa\eta$ and $w \in A$. Also $F_S(w) = \frac{\kappa}{1-\kappa} d_S(w)$, so $w \in E$ and $G(E) = \{u \in \mathbb{B}_{\mathbb{H}}[\bar{u}, \eta] \mid P_S(u) \neq \emptyset \text{ and } F_S(u) = 0\}$, and thus $\{u \in \mathbb{B}_{\mathbb{H}}[\bar{u}, \eta] \mid P_S(u) \neq \emptyset \text{ and } F_S(u) = 0\}$ is an analytic set.

If, additionally, for all $u \in \mathbb{B}_{\mathbb{H}}[\bar{u}, \eta]$ the set $P_S(u)$ contains at most one-element, then G is surjective on E, so $\{u \in \mathbb{B}_{\mathbb{H}}[\bar{u}, \eta] \mid P_S(u) \neq \emptyset \text{ and } F_S(u) = 0\}$ is a Borel set as well as $\{u \in \mathbb{B}_{\mathbb{H}}[\bar{u}, \eta] \mid P_S(u) = \emptyset \text{ and } F_S(u) = 0\}$, since the set $\{u \in \mathbb{B}_{\mathbb{H}}[\bar{u}, \eta] \mid F_S(u) = 0\}$ is Borel. □

The next two results can be inferred from [15, Theorem 4], see also [22, Proposition 4.3 and Proposition 4.5], where they were stated explicitly. In the Proposition below it is recalled that for a given point $u \notin S$ there is a point u' close to the given point such that $P_S(u')$ is a singleton and $u' \in D_{P_S(u')}(S)$.

Proposition 3.7. *Let $\mathbb{H}$ be a real Hilbert space, $U \subset \mathbb{H}$ be an open nonempty subset, and $w \in U$ be fixed. For every $\mu > 0$ there are $x \in U \cap \mathbb{B}_{\mathbb{H}}[w, \mu]$, $s \in \operatorname{fr} U$ such that $d_{\mathbb{H}\setminus U}(x) = \|x - s\|$ and any sequence $\{w_i\}_{i\in\mathbb{N}}$ in $\mathbb{H} \setminus U$ such that $\|w_i - x\| \longrightarrow d_{\mathbb{H}\setminus U}(x)$ has a convergent subsequence (with respect to the norm topology), say $\{w_{i_n}\}_{n\in\mathbb{N}}$, for which $s = \lim_{n\longrightarrow\infty} w_{i_n}$.*

Let us also recall that the set of points from the boundary of some set, which are from spheres contained in the set, is dense in the boundary of the set.

Proposition 3.8. *Let $\mathbb{H}$ be a real Hilbert space, $U \subset \mathbb{H}$ be an open nonempty subset, and $w \in \operatorname{cl} U \setminus U$ be fixed. For every $\mu > 0$ there are $u \in \big(\operatorname{cl} U \setminus U\big) \cap \mathbb{B}_{\mathbb{H}}[w, \mu]$, $x \in U$ such that $d_{\mathbb{H}\setminus U}(x) = \|u - x\|$ and any sequence $\{w_i\}_{i\in\mathbb{N}}$ in $\mathbb{H} \setminus U$ such that $\|w_i - x\| \longrightarrow d_{\mathbb{H}\setminus U}(x)$ has a convergent subsequence (with respect to the norm topology), say $\{w_{i_n}\}_{n\in\mathbb{N}} \subset \{w_i\}_{i\in\mathbb{N}}$, for which $d_{\mathbb{H}\setminus U}(x) = \|x - \lim_{n\longrightarrow\infty} w_{i_n}\|$.*

A direct consequence of Proposition 3.8 and Lemma 3.2 is that the set of points at which the subdifferential mapping of a convex continuous function admits Aubin Lipschitz-like property is dense in its domain.

Corollary 3.9. *Let $\mathbb{H}$ be a Hilbert space and $f \colon \mathbb{H} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function, which is continuous on a nonempty open convex subset $V \subset \mathbb{H}$. There is a dense subset $D \subset V$ such that for all $z \in D$ statements* (i)–(iv) *of Theorem* 3.1 *hold true.*

Proof. Let us fix $v \in V$ and put $b := (v, f(v))$, $U := \{(a, r) \in V \times \mathbb{R} \mid f(a) < r\}$. The set U is an open nonempty and convex subset of the Hilbert space $\mathbb{H} \times \mathbb{R}$ (with the norm $\sqrt{\|\cdot\|^2 + |\cdot|^2}$). Take any $\mu > 0$ such that $\mathbb{B}_{\mathbb{H}}[v, 2\mu] \subset V$. It follows from Proposition 3.8 that for some $u = (x, f(x)) \in \Big(\operatorname{cl} U \setminus U\Big) \cap \mathbb{B}_{\mathbb{H}}[b, \mu]$ and $w = (y, \beta) \in U$ we have $d_{\mathbb{H}\setminus U}(w) = \|u - w\|$. In order to finish the proof it is enough to observe that conditions (17) and (18) are satisfied. It follows from Lemma 3.2 that $\beta > \max\{f(x), f(y)\}$ and that (i)–(iv) of Theorem 3.1 hold true. $\square$

4. How large is the set of points of Lipschitz continuity of a subdifferential mapping of a distance function?

Let us recall that the set $\{u \in U \mid F_S(u) = 0\}$ is G_δ, whenever U is an open nonempty and convex subset of a Hilbert space. It is a question, when this set is large or small (preferable). Below a discussion on this problem is presented.

Corollary 4.1. *Let $U \subset \mathbb{H}$ be a nonempty open and convex subset of a Hilbert space $\mathbb{H}$ with $S := \mathbb{H} \setminus U$ nonempty. Then the function F_S is continuous at all points of the set $Z_0 := \{u \in U \mid F_S(u) = 0\}$ and Z_0 is a G_δ set (it is a countable intersection of open sets); consequently the set of points at which* (23) *holds true, is a dense F_σ subset.*

Proof. Fix $z \in Z_0$. Since F_S has nonnegative values, so the upper semicontinuity of F_S, see Proposition 2.4, yields

$$0 \le \liminf_{u \to z, u \in \operatorname{cl} U} F_S(u) \le \limsup_{u \to z, u \in \operatorname{cl} U} F_S(u) \le F_S(z) = 0,$$

which translates the continuity of F_S at z. Observe that the sets $U_n := \{u \in U \mid F_S(u) < n^{-1}\}$ are open and $Z_0 = \bigcap_{n \in \mathbb{N}} U_n$. The density of $U \setminus Z_0$ is then a direct consequence of Proposition 3.7, see also Proposition 3.4. $\square$

A direct consequence of Corollary 4.1 and Proposition 3.4 is the density of the set of all $u \in U$ for which $D_F d_S(u)$ exists and $P_S(u)$ is a singleton. This result is known in much more general setting, see [2, Theorem 1].

Corollary 4.2. *Let $U \subset \mathbb{H}$ be a nonempty open and convex set with $S := \mathbb{H} \setminus U$ nonempty, and let $u \in \operatorname{cl} U$ be given. Then the set $\{u \in U \mid F_S(u) > 0\}$ is a dense F_σ subset of U. Moreover, the Fréchet derivative of the distance function exists for all u from this set, that is, $D_F d_S(u)$ exists, whenever $F_S(u) > 0$.*

In the next proposition, it is established that the set of points, at which Lipschitz behavior of subdifferential of $-d_S$ is preserved (in other words Lipschitz behavior of the Clarke subdifferential of d_S), has the full Lebesgue measure, whenever the finite dimensional setting is under consideration. In fact, this property can be extended to a more general setting. Let us fix an open convex nonempty subset $U \subset \mathbb{H}$ such that $S := (\mathbb{H} \setminus S) \neq \emptyset$. Define the continuous mapping $G : (U \setminus \{u \in U \mid F_S(u) = 0\}) \longrightarrow \mathbb{H}$, see Proposition 3.4, as follows

$$\forall a \in \{u \in U \mid F_S(u) > 0\},\ G(a) := d_S(a) D_F d_S(a). \tag{28}$$

For all $t \in [0, \frac{1}{4}]$ put $Q_t := \{z \in U \mid F_S(z) = \frac{t}{1-t} d_S(z)\}$. Here, it is obtained that $\{u \in U \,|\, F_S(u) = 0 \text{ and } P_S(u) \neq \emptyset\}$ is a null set for every measure on the Borel σ-algebra, say for $m\colon B(U) \longrightarrow [0, \infty[\cup \{\infty\}$ on $B(U)$, where $U := \mathbb{H} \setminus S$, having the following two properties (with G as in (28)):

(m1) $\forall u \in U, \exists \delta > 0 : \forall \mu \in]0, \delta],\ \ m(\mathbb{B}_{\mathbb{H}}[u, \mu] \cap U) < \infty;$

(m2) $\forall t \in [0, \frac{1}{4}], \forall D \in B(Q_t), \Big(m(D) = 0, \text{ diam}\, D < \frac{\inf_{d\in D} F_S(d)}{16}$, see (23),
$\Longrightarrow \exists D_t \in B(U), m(D_t) = 0 \text{ and } (\text{id} + \frac{t}{1-t} G)(D) \subset D_t\Big),$
where $id(x) = x$ for all $x \in \mathbb{H}$.

Let us fix $t_0 \in [0, \frac{1}{4}]$ and $D \subset Q_{t_0}$. Suppose that $m(D) = 0$, $\kappa := \inf_{d \in D} F_S(d) > 0$ and that $\mathbb{H}$ is separable. By the separability of Q_{t_0} and Proposition 3.4, see also the comment following after it, we can find countable numbers of subsets of D, say $C_1 \subset D, C_2 \subset D \ldots$, such that $D \subset \bigcup_{j \in \mathbb{N}} C_j$, $\text{diam}\, C_j < \frac{\inf_{d\in D} F_S(d)}{16}$ and

$$\forall j \in \mathbb{N}, \forall a, b \in C_j, \|D_F d_S(a) - D_F d_S(b)\| \leq \frac{5}{\kappa}\|a - b\|,$$

which implies the existence of $M > 0$ such that for all $j \in \mathbb{N}$ and for all $a, b \in C_j$,

$$\|(a + \frac{t_0}{1-t_0} d_S(a) D_F d_S(a)) - (b + \frac{t_0}{1-t_0} d_S(b) D_F d_S(b))\| \leq M\|a - b\|.$$

Thus, mappings $(\text{id} + \frac{t_0}{1-t_0} G)$ are M-Lipschitz on sets C_j, for all $j \in \mathbb{N}$. Now, it becomes obvious that Lebesgue measure has properties $m(1)$ and $m(2)$, whenever $\mathbb{H} = \mathbb{R}^n$; see [19, Theorem 3.33] – in the proof of this theorem it is shown that a Lipschitz transformation sends sets of measure zero into sets of measure zero. Assumption (m1) is not a restrictive one, see for example [9, Definition 2.1]. It seems that (m2) can be easily verified only in finite-dimensional spaces. In infinite dimensions we should impose additional assumptions on sets and mappings to get this property. Let us recall that $\{u \in U \,|\, F_S(u) = 0\}$ is a Borel set, thus $\{u \in U \,|\, F_S(u) = 0 \text{ and } P_S(u) \neq \emptyset\}$ is also a Borel set, whenever $P_S(u) \neq \emptyset$ for all $u \in U$. In Corollary 3.6 it is shown that $\{u \in U \,|\, F_S(u) = 0 \text{ and } P_S(u) \neq \emptyset\}$ is also a Borel set, whenever the map $(\text{id} + \frac{t}{1-t} G)$ is one-to-one.

Proposition 4.3. *Let $\mathbb{H}$ be a separable Hilbert space, $U \subset \mathbb{H}$ be a convex nonempty and open subset such that $S := (\mathbb{H} \setminus U) \neq \emptyset$. Assume that $m\colon \mathbb{B}(U) \longrightarrow [0, \infty[\cup \{\infty\}$ is a measure on the Borel σ-algebra $\mathbb{B}(U)$ such that conditions* (m1) *and* (m2) *hold true.*

Then there is $D \in B(U)$ such that $\{u \in U \,|\, F_S(u) = 0$ and $P_S(u) \neq \emptyset\} \subset D$ and $m(D) = 0$. Consequently the complement of the set of points at which (23) *holds true, has the null measure, whenever*

$$\forall z \in U,\ P_S(z) \neq \emptyset.$$

Proof. Put $Z := \{u \in U \,|\, F_S(u) = 0 \text{ and } P_S(u) \neq \emptyset\}$. For all $z \in Z$ take $r(z) \in]0, \infty[$ such that $\mathbb{B}_{\mathbb{H}}[z, 3r(z)] \subset U$, $m(\mathbb{B}_{\mathbb{H}}[z, 3r(z)]) < \infty$, see (m1), and

$F_S(u) \le \frac{1}{4}\min\{1, \inf_{v\in\mathbb{B}_{\mathbb{H}}[z,r(z)]} d_S(v)\}$ for all $u \in \mathbb{B}_{\mathbb{H}}[z, r(z)]$, see Proposition 2.4.

We have
$$Z \subset \bigcup_{z\in Z} \mathbb{B}_{\mathbb{H}}(z, r(z)),$$
thus by the separability we can choose $\{z_1, z_2, \ldots\} \subset Z$ such that
$$Z \subset \bigcup_{k\in\mathbb{N}} \mathbb{B}_{\mathbb{H}}(z_k, r(z_k)),$$
and for all $k \in \mathbb{N}$ and for all $u \in \mathbb{B}_{\mathbb{H}}[z_k, r(z_k)]$,
$$\mathbb{B}_{\mathbb{H}}[z_k, 3r(z_k)] \subset U, \quad m(\mathbb{B}_{\mathbb{H}}[z_k, 3r(z_k)]) < \infty, \quad F_S(u) \le \frac{1}{4}\min\{1, \inf_{v\in\mathbb{B}_{\mathbb{H}}[z_k,r(z_k)]} d_S(v)\}.$$

Fix k_0 and put $W_{k_0} := \mathbb{B}_{\mathbb{H}}[z_{k_0}, r_{k_0}] \cap Z$ and, for $t \in [0, \frac{1}{4}]$,
$$Q_t^{k_0} := \{z \in \mathbb{B}_{\mathbb{H}}[z_{k_0}, 2r_{k_0}] \mid F_S(z) = \frac{t}{1-t} d_S(z)\}.$$

Notice that $Q_t^{k_0}$ are Borel sets for $t \in [0, \frac{1}{4}]$, since F_S is upper semicontinuous. Choose $\epsilon > 0$ such that for all $b \in W_{k_0}$,
$$(1-\epsilon)b + \epsilon P_S(b) \subset \mathbb{B}_{\mathbb{H}}[z_{k_0}, 2r_{k_0}]. \tag{29}$$

Suppose that for all $t \in]0, \epsilon]$, $m(Q_t^{k_0}) > 0$. Then, we choose $\epsilon > t_1 > t_2 > \ldots$, $t_i > 0$ for all $i \in \mathbb{N}$, such that
$$\exists \mu > 0, \forall i \in \mathbb{N}, m(Q_{t_i}^{k_0}) \ge \mu.$$

Since $\bigcup_{i\in\mathbb{N}} Q_{t_i}^{k_0} \subset \mathbb{B}_{\mathbb{H}}[z_{k_0}, 2r_{k_0}]$ and $Q_{t_i}^{k_0} \cap Q_{t_j}^{k_0} = \emptyset$ for all $i \ne j$, we get the following contradiction:
$$\infty = \sum_{i\in\mathbb{N}} m(Q_{t_i}^{k_0}) = m(\bigcup_{i\in\mathbb{N}} Q_{t_i}^{k_0}) \le m(\mathbb{B}_{\mathbb{H}}[z_{k_0}, 2r_{k_0}]) < \infty.$$
Thus, $m(Q_{t_0}^{k_0}) = 0$ for some $t_0 \in]0, \epsilon]$. Observe that
$$W_{k_0} \subset \bigcup_{a\in Q_{t_0}^{k_0}} \{a + \frac{t_0}{1-t_0} d_s(a) D_F d_S(a)\} = \bigcup_{a\in Q_{t_0}^{k_0}} \{(\mathrm{id} + \frac{t_0}{1-t_0} G)(a)\}, \tag{30}$$
where G is as in $m(2)$. In fact, for any $q_0 \in W_{k_0}$ and $s_0 \in P_S(q_0)$ by (29) we get
$$a_0 := (1-t_0)q_0 + t_0 s_0 \in \mathbb{B}_{\mathbb{H}}[z_{k_0}, 2r_{k_0}].$$

Moreover, $F_S(a_0) = \|a_0 - q_0\| = t_0 d_S(q_0)$, $d_S(a_0) = \|a_0 - s_0\| = (1-t_0) d_S(q_0)$.

Thus $F_S(a_0) = \frac{t_0}{1-t_0} d_S(a_0)$, $a_0 \in Q_{t_0}^{k_0}$, and $q_0 = a_0 + \frac{t_0}{1-t_0} d_s(a_0) D_F d_S(a_0)$, which guarantees the inclusion in (30).

There is $\kappa \ge \frac{t_0}{1-t_0} r$ (keep in mind the inclusion $\mathbb{B}_{\mathbb{H}}[z_{k_0}, 3r_{k_0}] \subset U$) such that
$$\forall b \in Q_{t_0}^{k_0}, F_S(b) \ge \kappa.$$

By the separability of $Q_{t_0}^{k_0}$ and Proposition 3.4 we can find countable numbers of subsets of $Q_{t_0}^{k_0}$, say $C_1^{k_0} \subset Q_{t_0}^{k_0}, C_2^{k_0} \subset Q_{t_0}^{k_0} \ldots$, such that $Q_{t_0}^{k_0} \subset \bigcup_{j\in\mathbb{N}} C_j^{k_0}$ and

$$\forall j \in \mathbb{N}, \operatorname{diam} C_j^{k_0} < \frac{\kappa}{16},$$

which, by $m(2)$, implies the existence $D_1^{k_0} \subset B(U), D_2^{k_0} \subset B(U), \ldots$ such that $m(D_j^{k_0}) = 0$ and $(\mathrm{id} + \frac{t_0}{1-t_0}G)(C_j^{k_0}) \subset D_j^{k_0}$ for all $j \in \mathbb{N}$. Thus, by (30), $W_{k_0} \subset \bigcup_{j\in\mathbb{N}} D_j^{k_0}$ and $m(\bigcup_{j\in\mathbb{N}} D_j^{k_0}) = 0$. Since this holds true for all $k_0 \in \mathbb{N}$, we get

$$Z \subset \bigcup_{k\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} D_j^k \text{ and } m(\bigcup_{k\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} D_j^k) = 0$$

and we are done. □

The continuity from below of F_S can be related to the local Lipschitz continuity of the Fréchet derivative of the convex continuous function, for S being its hypograph. In fact, the continuity assumption (of the function under investigation) can be relaxed. However for the sake of simplicity, only convex continuous functions are explored.

Proposition 4.4. *Let $\mathbb{H}$ be a Hilbert space and $f\colon \mathbb{H} \longrightarrow \mathbb{R}$ be a convex continuous function, and $S := \{(u,\xi) \in \mathbb{H}\times\mathbb{R} \,|\, \xi \leq f(u)\}$, $(y,\beta) \notin S$ be given such that $F_S((y,\beta)) > 0$. If F_S is continuous at (y,β), then there are $M \geq 0$, $r > 0$ and $x \in \mathbb{H}$ such that $D_F f(u)$ exists for all $u \in \mathbb{B}_{\mathbb{H}}(x,r)$ and $\|D_F f(u) - D_F f(v)\| \leq M\|u-v\|$ for all $u,v \in \mathbb{B}_{\mathbb{H}}(x,r)$.*

Proof. Take any $(d,\gamma) \in D_{(y,\beta)}(S_{(y,\beta)}) \setminus \{(y,\beta)\}$ and choose $t \geq 0$ such that $(x,f(x)) := (y,\beta) - t((d,\gamma)-(y,\beta)) \in \operatorname{fr} S$. We have $\|(y,\beta)-(x,f(x))\| = d_S((y,\beta))$. Choose $\delta > 0$ such that if $(a,\sigma) \in (\operatorname{fr} S_{(y,\beta)}) \cap \mathbb{B}_{\mathbb{H}}((y,\beta),\delta)$ then $F_S(a,\sigma)) > 2^{-1}F_S((y,\beta))$. The set $\mathrm{cl}(\mathbb{H}\setminus S_{(y,\beta)})$ is closed and convex, thus the metric projection mapping $P_{\mathrm{cl}(\mathbb{H}\setminus S_{(y,\beta)})}(\cdot)$ is a Lipschitz continuous mapping and if

$$(u,f(u)) \in \left(P^{-1}_{\mathrm{cl}(\mathbb{H}\setminus S_{(y,\beta)})}\left((\operatorname{fr} S_{(y,\beta)}) \cap \mathbb{B}_{\mathbb{H}}((y,\beta),\delta)\right)\right) \cap \operatorname{fr} S,$$

then $F_S((u,f(u)) > 2^{-1}F_S((y,\beta))$. The projection mapping $P\colon \mathbb{H}\times\mathbb{R} \longrightarrow \mathbb{H}$ is an open mapping, see [20, 5. The Open Maping Theorem], where $P((c,\theta)) = c$. Thus for some $r > 0$ we have if $u \in B_{\mathbb{H}}(x,r)$, then $F_S((u,f(u)) > 2^{-1}F_S((y,\beta))$. We may assume that r is such that f is Lipschitz continuous on $B_{\mathbb{H}}(x,r)$, say L-Lipschitz. It follows from Theorem 3.1 that $D_F f(u)$ exists for all $u \in B_{\mathbb{H}}(x,r)$. Moreover, taking $r > 0$ small enough, we may additionally assume, using the continuity of the metric projection on $\mathrm{cl}(\mathbb{H}\setminus S_{(y,\beta)})$ and the continuity of the projection mapping (defined above), that if $(a,\sigma) \in (\operatorname{fr} S_{(y,\beta)}) \cap D_{(u,f(u)}(S)$ and $u \in B_{\mathbb{H}}(x,r)$, then $\|a-u\| < 1 + \|x-y\|$ and $\sigma - f(u) > \frac{\beta - f(x)}{2} > 0$, see also Lemma 3.2. Applying Theorem 3.1(ii) to every $u \in B_{\mathbb{H}}(x,r)$ we get that for all $u,v \in B_{\mathbb{H}}(x,r)$ and all $u^* \in \partial f(u)$, $v^* \in \partial f(v)$ we have

$$\|u^* - v^*\| \leq \frac{8(L^2+1)}{\beta - f(x)}\|u-v\|;$$

and we are done. □

References

[1] J.-P. Aubin: *Lipschitz behavior of solutions to convex minimization problems*, Math. Oper. Res. 9 (1984) 87–111.

[2] V. S. Balaganskii: *On approximation properties of sets with convex complement*, Math. Notes 57 (1995) 15–20.

[3] V. Barbu, T. Precupanu: *Convexity and Optimization in Banach Spaces*, 4th ed., Springer, Berlin (2012).

[4] H. H. Bauschke, P. L. Combettes: *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, Berlin (2011).

[5] J. M. Borwein: *Future challenges for variational analysis, analysis and generalized differentiation in optimization and control*, in: Springer Optimization and its Applications 47, Springer, Cham (2010) 95–105.

[6] J. M. Borwein, J. D. Vanderwerff: *Convex Functions: Constructions, Characterizations and Counterexamples*, Cambridge University Press, Cambridge (2010).

[7] A. Cabot, A. Jourani, L. Thibault, D. Zagrodny: *The attainment set of ϕ-envelope and genericity properties*, Mathematica Scandinavica 124 (2019) 203–246.

[8] S. Fitzpatrick: *Metric projections and the differentiability of distance functions*, Bull. Austral. Math. Soc. 22 (1980) 291–312.

[9] R. J. Gardner, W. F. Pfeffer: *Borel Measures*, in: Handbook of Set-Theoretic Topology, Chapter 22, K. Kunen, J. E. Vaughan (eds.), North-Holland, Amsterdam (1984).

[10] J.-B. Hiriart-Urruty: *New concepts in nondifferentiable programming*, Mém. Soc. Math. France 60 (1979) 57–85.

[11] A. D. Ioffe, V. M. Tikhomirov: *Theory of Extremal Problems*, North-Holland, Amsterdam (1979).

[12] A. Jourani, L. Thibault, D. Zagrodny: *$C^{1,\omega(\cdot)}$-regularity and Lipschitz-like properties of subdifferential*, Proc. London Math. Soc. (3) 105 (2012) 189–223.

[13] A. Jourani, L. Thibault, D. Zagrodny: *Differential properties of the Moreau envelope*, J. Func. Analysis 266 (2014) 1185–1237.

[14] A. Jourani, L. Thibault, D. Zagrodny: *The NSLUC property and Klee envelope*, Math. Ann. V. 365 I. 3-4 (2016) 923–967.

[15] K.-S. Lau: *On almost Chebyshev subsets in reflexive Banach spaces*, Indiana Univ. Math. J. 27 (1978) 791–795.

[16] J. C. Oxtoby: *Measure and Category. A Survey of the Analogies between Topological and Measure Spaces*, 2nd ed., Springer, New York(1980).

[17] K. R. Parthasarathy: *Probability Measures on Metric Spaces*, Academic Press, New York (1967).

[18] D. A. Salamon: *Measure and Integration*, European Mathematical Society, Zurich (2016).

[19] R. L. Wheeden, A. Zygmund: *Measure and Integral. An Introduction to Real Analysis*, Marcel Dekker, New York et al. (1977).

[20] K. Yosida: *Functional Analysis*, 6th ed., Springer, Berlin (1980).

[21] D. Zagrodny: *Convergences of subgradients of sequences of convex functions*, Nonlinear Analysis 84 (2013) 84–90.

[22] D. Zagrodny: *On closures of preimages of metric projection mappings in Hilbert spaces*, Set-Valued Var. Analysis 23 (2015) 581–612.

[23] D. Zagrodny: *On the strong convergence of subgradients of convex functions*, J. Optim. Theory Appl. 178(2) (2018) 411–423.

[24] D. Zagrodny: *The strong convergence of subgradients of convex functions along directions: Perspectives and open problems*, J. Optim. Theory Appl. 178(2) (2018) 660–671.

[25] N. V. Zhivkov: *Metric projections and antiprojections in strictly convex normed spaces*, C. R. Acad. Bulgar. Sci. 31(4) (1978) 369–372.