Journal of Convex Analysis
Volume 27 (2020), No. 2, 777–790

A Partial Condition Number Theorem in Mathematical Programming

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

Received: February 2, 2019
Revised manuscript received: April 25, 2019
Accepted: April 27, 2019

A condition number of mathematical programming problems is defined as a measure of the sensitivity of their global optimal solutions under general perturbations described by parameters acting on their data. A (pseudo-) distance among problems fulfilling prescribed bounds is defined via the corresponding augmented Kojima functions. A characterisation of well-conditioning is obtained. It is shown that the distance to ill-conditioning is bounded from above by a multiple of the reciprocal of the condition number. This upper bound extends to general perturbed problems known results dealing with canonical perturbations.

Keywords: Condition number, condition number theorem, mathematical programming with data perturbations.

2010 Mathematics Subject Classification: 90C30, 90C31.

1. Introduction

We consider finite-dimensional smooth mathematical programmig problems with inequality constraints, perturbed by parameters acting on their data in general form. A fixed value of the parameter, say equal to zero, corresponds to the unperturbed problem, whose condition number, with respect to such a class of perturbations, we want to consider. For any given parameter we consider the set of the global optimal solutions (minimizers and multipliers) of the corresponding problem. The (absolute) condition number is then defined as the Lipschitz modulus of the set-valued global optimal solutions mapping, at the parameter defining the unperturbed problem, with respect to the Hausdorff distance.

The condition number of some optimization problems, with respect to suitable data perturbations, is known to be related to a properly defined distance of the problems to ill-conditioning. The simplest result of this type is the optimization version (see [15]) of the so called Eckart-Young theorem (see [4, Theorem 1.7]), dealing with the unconstrained optimization of quadratic forms on Banach spaces. There we find that the distance to ill-conditioning equals the reciprocal

ISSN 0944-6532 / $ 2.50

of the condition number. Such a basic property has been extended to the constrained setting for more general optimization problems in [16, 17, 18, 19 and 20]. These extensions are obtained in the form of a lower and an upper bound of the distance to ill-conditioning in terms of the reciprocal of the condition number, up to constants depending on problem's data. Such a pair of inequalities will be called a condition number theorem. In the mathematical programming setting, we define the distance among problems by the Lipschitz constant of their augmented Kojima functions (see [8, Chapter 7]), within the class of all problems which satisfy given bounds on their optimal solutions.

Such results are of interest for the analysis of the computational complexity (see [3 and 4]), for the sensitivity and stability properties of the considered problems, and for evaluating the performance of numerical methods. In [17] it is shown, however, that there exist minimum norm least squares problems for which a condition number theorem in the above mentioned form cannot hold. This adds interest in finding classes of optimization problems for which a condition number theorem is available. We remark that the results of Renegar (see [12 and 13]) deal with the distance to instability (a different notion than ill-conditioning) within linear programming problems, following a different approach (in a sense, opposite to the one pursued here). In [1] a condition number theorem is obtained in the infinite-dimensional setting for unconstrained problems. Extensions to multiobjective optimization are dealt with in [2].

The objective of this paper is to obtain a partial extension of the condition number theorem, already known for canonical perturbations of nonconvex mathematical programming problems (see [20]), to general data perturbations. We find conditions such that the distance to ill-conditioning is bounded above by a multiple of the reciprocal of the condition number. As far as we know, no such results are known beyond the case of canonical perturbations. In section 2 we describe the problem, the basic definitions and assumptions. In section 3 we characterize well-conditioning via the augmented Kojima function. In section 4 we obtain an upper bound of the distance to ill-conditioning in terms of the reciprocal of the condition number. Section 5 is devoted to an example, and Section 6 contains the proofs.

2. Notations, problem statement and assumptions

Given positive integers s, n, q we consider a fixed compact set $\Omega \subset R^s$ containing 0 as a cluster point, a closed ball $B \subset R^s$ containing Ω, and functions

$$f, g_1, ..., g_q \colon B \times R^n \to R.$$

For each parameter $p \in \Omega$ we consider the mathematical programming problem $Q(p)$, of minimizing $f(p, x)$ with respect to x, subject to the constraints

$$g_1(p, x) \leq 0, ..., g_q(p, x) \leq 0. \tag{1}$$

We denote by g the vector of components $g_1, ..., g_q$ and by $Q = (f, g)$ the family $Q(p), p \in \Omega$, of the mathematical programming problems described above. The inequalities (1) defining the feasible region of $Q(p)$ will be written also as $g(p, x) \leq 0$.

Here $Q(0)$ is to be considered as the unperturbed problem, while $Q(p), p \in \Omega$, model the perturbations acting on $Q(0)$. For sake of simplicity we assume that all parametric mathematical programming problems we consider have the same parameter set Ω. For each Q and $p \in \Omega$ let $S(p)$ denote the set of all *global optimal solutions* (if any) of $Q(p)$, namely all pairs $(x, u) \in R^{n+q}$ where x is a global minimizer of $Q(p)$, and u is a (Kuhn-Tucker) multiplier of $Q(p)$ corresponding to x. We denote by $M(p)$ the set of all global minimizers x of $Q(p)$, and by $U(x, p)$ the set of all such multipliers corresponding to x. We consider also the *optimal value function* of $Q(p)$, namely $v\colon \Omega \to [-\infty, +\infty]$ given by

$$v(p) = \inf \{f(p, x) : g(p, x) \leq 0\}, p \in \Omega. \tag{2}$$

For $y = (y_1, ..., y_m)^T \in R^m$ we write $|y| = \sum_{j=1}^{m} |y_j|$. If $T \subset R^m$ and $y \in R^m$ we write

$$\operatorname{dist}(y, T) = \inf \{|y - x| : x \in T\}$$

and if also $B \subset R^m$ we consider

$$e(T, B) = \sup \{ \operatorname{dist}(x, B) : x \in T\}, \text{ hausd } (T, B) = \max \{e(T, B), e(B, T)\}.$$

Given Q, the *condition number* of Q is defined by

$$\operatorname{cond} Q = \limsup_{p', p'' \to 0} \frac{\text{hausd } [S(p'), S(p'')]}{|p' - p''|} \tag{3}$$

where $p', p'' \in \Omega$. Problem Q will be called *well-conditioned* if $S(p) \neq \emptyset$ for all sufficiently small $p \in \Omega$ and cond $Q < +\infty$, *ill-conditioned* otherwise. We denote by IC the set of ill-conditioned problems. Thus $Q \in IC$ iff either each $Q(p_j)$ has no global optimal solutions for some sequence $p_j \to 0$ in Ω, or $Q(p)$ has global optimal solutions for every sufficiently small $p \in \Omega$ and cond $Q = +\infty$.

Denote by $\bigtriangledown$ the gradient, or Jacobian, acting on the x-variables. The following conditions will be used.

(A1) g is continuous, and on every set $\Omega \times T$ with T bounded in $R^n, f, \bigtriangledown f, \bigtriangledown g$ are Lipschitz continuous.

(A2) $-\infty < v(p) < +\infty$ for each sufficiently small $p \in \Omega$.

(A3) For each $p \in \Omega$ and $x \in M(p)$ there exist multipliers of $Q(p)$ corresponding to x.

(A4) There exists $L > 0$ such that

$$|g(p'', x'') - g(p', x')| \leq L(|p'' - p'| + |x'' - x'|)$$

for each $p', p'' \in \Omega$ and $x' \in M(p'), x'' \in M(p'')$.

We write "assumptions (A)", instead of "assumptions (A1), (A2), (A3) and (A4)". We start by characterizing global optimality and well-conditioning making use of the Kojima function.

Given $Q = (f, g)$, its *Kojima function* (see [8]) $K : \Omega \times R^{n+q} \to R^{n+q}$ is defined by

$$K(p, x, y) = (\bigtriangledown f(p, x) + \sum_{j=1}^{q} y_j^+ \bigtriangledown g_j(p, x), g(p, x) - y^-) \tag{4}$$

where $z^+ = \max\{z, 0\}$, $z^- = \min\{z, 0\}$ for any $z \in R$, and $y^- = (y_1^-, ..., y_q^-)^T$, where T denotes transpose. We also consider the *augmented Kojima function* $H : \Omega \times R^{n+q} \to R^{n+q+1}$ given by

$$H(p, x, y) = [K(p, x, y), f(p, x)]. \tag{5}$$

3. Characterization of well-conditioned problems

The following theorem characterizes global optimality making use of the augmented Kojima function H of (5).

Theorem 3.1. *Suppose that* $Q = (f, g)$ *fulfills* (A1), (A2) *and* (A3). *Given* $(p, x, y) \in \Omega \times R^{n+q}$, *the following are equivalent:*

$$H(p, x, y) = [0, v(p)]; \tag{6}$$

$$x \in M(p),\ y = u + g(p, x) \text{ for some } u \in U(p, x). \tag{7}$$

In order to characterize well-conditioning, given $Q = (f, g)$ we consider the set-valued mapping $\Sigma : \Omega \to R^{n+q}$ defined by

$$\Sigma(p) = \{(x, y) \in R^{n+q} : [x, y - g(p, x)] \in S(p)\}. \tag{8}$$

Theorem 3.2. *Let* Q *fulfill assumptions* (A). *Then the following properties are equivalent:*

$$Q \text{ is well-conditioned };\tag{9}$$

$$\Sigma \text{ is Lipschitz near } p = 0. \tag{10}$$

Equivalent conditions, or related, to well-conditioning of Q, namely to the Lipschitz continuity of the set-valued map S near $p = 0$ (see (3)), under various types of data perturbations, are known, see e.g. [5, 6, 8, 9, 10, 14]. Well-conditioning is also related to local Lipschitz continuity of the optimal value function.

Proposition 3.3. *Suppose that* Q *fulfills* (A1), (A2) *and* (A3). *Let* f *be Lipschitz on* $\Omega \times \cup\{M(p) : p \in \Omega$ *and sufficiently small*$\}$. *If* Q *is well-conditioned, then its optimal value function is Lipschitz near* 0.

4. Upper bound

We need a suitable notion of distance between mathematical programming problems fulfilling suitable bounds, as follows. Fix any $r > 0$ and consider the class N of all problems Q such that

$$|x| + |y| \le r \text{ for every } (x, y) \in \Sigma(p) \text{ and every } p \in \Omega \tag{11}$$

where $\Sigma(p)$ is defined by (8). Consider the closed ball D of center 0 and radius r in R^{n+q}. Given $Q_1 = (f_1, g_1)$, $Q_2 = (f_2, g_2)$ both in N, with the corresponding Kojima functions K_1, K_2, let $K = K_1 - K_2$, $f = f_1 - f_2$ and write $z' = (x', y')$, $z'' = (x'', y'')$. Then the distance between Q_1 and Q_2 within N is defined by

$$\mathrm{dist}(Q_1, Q_2) = \tag{12}$$

$$= \sup\left\{ \frac{|K(p'', z'') - K(p', z')| + |f(p'', x'') - f(p', x')|}{|p'' - p'| + |z'' - z'|} : p', p'' \in \Omega, z', z'' \in D \right\}.$$

It is easily seen that dist $(\cdot, \cdot)$ is a pseudo-distance on the space N.

(If dist $(Q_1, Q_2) = 0$, then the objective functions f_1, f_2 differ by a constant, and the constraint functions g_1, g_2 differ by an affine function of the x- variables.)

Remark 4.1. For canonical perturbations of convex problems (see [19]) or non-convex mathematical programming (see [20]), we used a different (however similar) definition of distance, making use of the Kojima functions (4). The modification of the definition, used here, is due to technical reasons, since the class of perturbations allowed is more general.

Given $r > 0$ and $Q \in N$ we define

$$\text{dist } (Q, IC) = \inf \{ \text{ dist } (Q, \bar{Q}) : \bar{Q} \in IC \cap N\}. \tag{13}$$

Given Q fulfilling (11), there are infinitely many classes N of problems such that $Q \in N$, so that, given Q, the choice of N is up to some extent arbitrary. Of course the definition of distance (12) as the Lipschitz constant of $H_1 - H_2$ on $\Omega \times D$ (H_1, H_2 being the extended Kojima functions of Q_1, Q_2) depends on the choice of N through D. However, as stated above, we develop our results having fixed r and the corresponding class N, to which all problems we consider belong (if no other conditions are specified). We write

$$\delta = \sup \{|p| : p \in \Omega\} \tag{14}$$

and we assume without restriction that

$$\delta < \frac{r}{2(q+1)}. \tag{15}$$

The following example shows that definition (13) makes sense for every choice of r, if the following condition on Ω is fulfilled:

$$\left\{ \begin{matrix} s = n + q \text{ and there exists a sequence } (a_m, b_m) \to 0 \text{ in } \Omega, \\ a_m \in R^n, b_m \in R^q, \text{ with } a_{mj} \neq 0 \text{ for some } j \text{ and every } m. \end{matrix} \right\} \tag{16}$$

Example 4.2. Let (15) and (16) hold. Given r, consider the canonical perturbations of minimizing $(\alpha/4) \sum_{j=1}^{n} x_j^4$ subject to the trivial constraint $-\beta \leq 0$, where

α is a positive parameter, $\beta \in R^q$ is the vector with all components equal to a positive parameter we still denote by $\beta : \alpha, \beta$ to be fixed later. Then if $p = (a, b) \in \Omega$ we get the problem $\bar{Q} = (\bar{f}, \bar{g})$ with

$$\bar{f}(p, x) = \frac{\alpha}{4} \sum_{j=1}^{n} x_j^4 - a^T x, \bar{g}_j(p, x) = -\beta - b_j,$$

hence $\bar{Q}(p)$ is an unconstrained problem provided $\beta + b_j \geq 0, j = 1, ..., q$. Its unique global optimal solution is given by

$$\bar{x} = \sqrt[3]{\frac{a}{\alpha}}, \bar{u} = 0, \text{ where } a = (a_1, ..., a_n)^T$$

and $\sqrt[3]{a}$ denotes the vector with components $\sqrt[3]{a_j}$. Thus $\bar{Q}$ is ill-conditioned (remembering (16)). Then by (8)

$$(x, y) \in \bar{\Sigma}(p) \text{ iff } x = \sqrt[3]{\frac{a}{\alpha}}, y = -\beta - b.$$

It follows that, remembering (14),

$$|x| + |y| \leq \frac{n}{\sqrt[3]{\alpha}} \sqrt[3]{\delta} + q\beta + \delta$$

and $|x| + |y| \leq r$ provided $\alpha \geq \dfrac{8n^3\delta}{r^3}, \beta \leq \dfrac{r}{2q} - \dfrac{\delta}{q}$; by (15), any β such that

$$\delta \leq \beta \leq (\frac{r}{2} - \delta)\frac{1}{q}$$

will do, so that $\bar{Q} \in N \cap IC$, hence $N \cap IC \neq \emptyset$.

An upper estimate of the distance to ill-conditioning within a fixed class N will be obtained now making use of some constants depending on problem's data, as follows. Given $Q = (f, g)$ fulfilling (A1), let $p = (a, b) \in \Omega$ be with $a \in R^n, b \in R^q$. Then consider

L_1 the Lipschitz constant of

$$(\bigtriangledown f(p, x) - \frac{8n^3}{r^3}\delta x^3 + a + \sum_{j=1}^{q} y_j^+ \bigtriangledown g_j(p, x), g(p, x) + \delta + b - y^-), \tag{17}$$

where δ denotes the vector whose components are equal to the positive constant δ;

L_2 the Lipschitz constant of

$$f(p, x) - \frac{2n^3}{r^3} \sum_{j=1}^{n} {x_j}^4 + a^T x, \tag{18}$$

both as $(p, x, y) \in \Omega \times D$.

Proposition 4.3. *Let Q fulfill* (A1), (15) *and* (16). *Then*

$$dist\ (Q, IC) \leq L_1 + L_2 \tag{19}$$

where L_1, L_2 are given by (17) *and* (18).

The final step is to obtain an upper bound of the condition number in terms of problem's data.

Lemma 4.4. *Let Q fulfill assumptions* (A)*. Suppose there exists a constant $c > 0$ such that*

$$hausd\ [\Sigma(p''), \Sigma(p')] \leq c|p'' - p'| \tag{20}$$

for every sufficiently small p', p'' in Ω. Then

$$cond\ Q \leq L(c + 1) + c. \tag{21}$$

In order to obtain an upper bound of the condition number, we need the following. For any set-valued map $W\colon R^N \to R^M$ and $z_0 \in R^N$ we write

$$\text{Lip } (W, z_0) = \text{ lim sup}_{p',p''\to 0} \frac{\text{hausd } [W(p''), W(p')]}{|p'' - p'|}. \tag{22}$$

By Lemma 4.4 it suffices to get an upper bound of $\text{Lip}(\Sigma, 0)$. We denote by $(\text{lip }\Sigma)(0, z)$ the infimum of the constants in the Aubin property for Σ at $(0, z)$, $z \in \Sigma(0)$, see [7, 3 E] (called there the Lipschitz modulus of Σ at 0 for z). Moreover we consider $G\colon \Omega \times R^{n+q} \to R^{n+q+1}$ given by

$$G(p, x, y) = H(p, x, y) - [0, v(p)] \tag{23}$$

with v given by (2), the $q \times n$ Jacobian matrix $\bigtriangledown g(p, x)$ with respect to the x-variables, and the $(n + q) \times (n + 2q)$ matrix

$$A(p, x, y) = \begin{pmatrix} \bigtriangledown^2 f(p, x) + \sum_{j=1}^{q} y_j^+ \bigtriangledown^2 g_j(p, x) & \bigtriangledown g(p, x)^T & 0 \\ \bigtriangledown g(p, x) & 0 & I \end{pmatrix}, \tag{24}$$

where the Hessian $\bigtriangledown^2$ acts on the x-variables, the north-east 0 is $n \times q$, the other is $q \times q$ as well as the identity matrix I. Given y and w_2 in R^q, let $\alpha \in R^q$ be with the i-th component given by

$$0 \text{ if } y_i < 0, w_{2i}^+ \text{ if } y_i = 0, w_{2i} \text{ if } y_i > 0, \tag{25}$$

and denote by $\beta \in R^q$ the vector whose i-th component is given by

$$-w_{2i} \text{ if } y_i < 0, -w_{2i}^- \text{ if } y_i = 0, 0 \text{ if } y_i > 0. \tag{26}$$

We shall need a positive constant λ such that

$$\sup_{|v|\leq 1} \inf \{|w| : v_1 = A(p, x', y')(w_1, \alpha, \beta),\ v_2 = \bigtriangledown f(p, x')^T w_1\} \leq \lambda, \tag{27}$$

where

$$w = (w_1, w_2) \text{ with } w_1 \in R^n,\ w_2 \in R^q,\ v = (v_1, v_2) \text{ with } v_1 \in R^{n+q} \text{ and } v_2 \in R. \tag{28}$$

Finally we consider a positive constant Λ such that

$$|G(p', x', y') - G(p'', x', y')| \leq \Lambda|p' - p''|; \tag{29}$$

here (27) and (29) are required to hold for every $(x, y) \in \Sigma(0)$ and (x', y') sufficiently close to $(x, y), p, p'$ and p'' sufficiently small in Ω.

Theorem 4.5. *Suppose that $Q = (f, g)$ is well-conditioned and fulfills assumptions* (A) *with $f(p, \cdot)$, $g(p, \cdot) \in C^2(R^n)$ for every p. Let $\cup\{S(p) : p \in \Omega$ and sufficiently small$\}$ be bounded, and let Λ be given by* (29)*. Suppose there exists a constant λ verifying* (27)*. Then*

$$cond\, Q \leq (L + 1)(\lambda\Lambda + 1). \tag{30}$$

As an immediate corollary of Proposition 4.3 and Theorem 4.5, we obtain the following partial condition number theorem.

Theorem 4.6. *Let Q fulfill the assumptions of Proposition* 4.3 *and Theorem* 4.5*. Then if cond $Q > 0$ we have*

$$dist\ (Q, IC) \leq \frac{(L_1 + L_2)(L + 1)(\lambda\Lambda + 1)}{cond\ Q},$$

where L_1, L_2 are as in Proposition 4.3*, λ and Λ as in Theorem* 4.5*.*

5. An example

Example 5.1. In this illustrative example we evaluate a constant λ as in (27). Let $Q(p)$ consist in minimizing x_1 under the constraint $x_1^2 + x_2^2 \leq 1 + p^2, |p| \leq 1$. Here $s = q = 1, n = 2$. We have

$$\Sigma(p) = \left\{(-\sqrt{1 + p^2}, 0, \frac{1}{2\sqrt{1 + p^2}})\right\},$$

hence Q is well-conditioned. Following (27) we consider

$$x_1 = -1,\ x_2 = 0,\ y = \frac{1}{2},\ |x_1' - x_1| \leq \frac{1}{4},\ |x_2'| \leq \frac{1}{4},\ |y' - y| \leq \frac{1}{4},\ |p'| \leq \frac{1}{2},\ |p| \leq \frac{1}{2}.$$

Let $w = (w_{11}, w_{12}, w_2) \in R^3$, $v_1 = (v_{11}, v_{12}, v_{13}) \in R^3$, $v_2 \in R$.

Then $y' > 0$ and by (24)

$$A(p, x', y') = \begin{pmatrix} 2y' & 0 & 2x_1' & 0 \\ 0 & 2y' & 2x_2' & 0 \\ 2x_1' & 2x_2' & 0 & 1 \end{pmatrix}$$

so that $v_1 = A(p, x', y')(w_1, \alpha, \beta)$, $v_2 = \bigtriangledown f(p', x')^T w_1$, $w_1 = (w_{11}, w_{12})$, iff

$$v_{11} = 2y' w_{11} + 2x_1' w_2, v_{12} = 2y' w_{12} + 2x_2' w_2,$$

$$v_{13} = 2x_1' w_{11} + 2x_2' w_{12}, v_2 = w_{11}. \tag{31}$$

Then for every v such that $|v| \leq 1$ we have $|w_2| \leq 5/3$ by the first equality in (31), and the infimum in (27) is bounded above by $8/3 + (1 + 2|x_2'||w_2|)/2y' \leq 6.4$, hence $\lambda = 6.4$ works.

6. Proofs and auxiliary results

Proof of Theorem 3.1. By (A2) and (A3), if $x \in M(p)$ then, by the multipliers theorem, there exists $u \in U(p,x)$ such that

$$\nabla f(p,x) + \sum_{j=1}^{q} u_j \nabla g_j(p,x) = 0, \tag{32}$$

$$u_j \geq 0,\ g_j(p,x) \leq 0,\ u_j g_j(p,x) = 0 \text{ for all } j = 1,...,q; \tag{33}$$

$$f(p,x) = v(p). \tag{34}$$

Assume (6). Let $u_j = y_j - g_j(p,x)$, $j = 1,...,q$. Then by (6)

$$\nabla f(p,x) + \sum_{j=1}^{q} [u_j + g_j(p,x)]^+ \nabla g_j(p,x) = 0. \tag{35}$$

Let l be such that (without loss of generality)

$$u_j + g_j(p,x) \geq 0 \text{ i } j = 1,...,l; \quad u_j + g_j(p,x) < 0 \text{ if } j = l+1,...,q.$$

Then
$$\nabla f(p,x) + \sum_{j=1}^{q} u_j \nabla g_j(p,x) =$$

$$= \nabla f(p,x) + \sum_{j=1}^{q}[u_j + g_j(p,x)] \nabla g_j(p,x) - \sum_{j=1}^{q} g_j(p,x) \nabla g_j(p,x) =$$

$$= \nabla f(p,x) + (\sum_{j=1}^{l} + \sum_{j=l+1}^{q})[u_j + g_j(p,x)] \nabla g_j(p,x) - \sum_{j=1}^{q} g_j(p,x) \nabla g_j(p,x) =$$

$$= \nabla f(p,x) + \sum_{j=1}^{l} y_j^+ \nabla g_j(p,x) + \sum_{j=l+1}^{q} u_j \nabla g_j(p,x) - \sum_{j=1}^{l} g_j(p,x) \nabla g_j(p,x) =$$

$$= \text{ (by (35)) } \sum_{j=l+1}^{q} u_j \nabla g_j(p,x) - \sum_{j=1}^{l} g_j(p,x) \nabla g_j(p,x). \tag{36}$$

By (6) we get $g_j(p,x) = 0$ if $j = 1,...,l$, hence $u_j = 0$ if $j = l+1,...,q$, thus (36) yields (32). Moreover $u_j = 0$ if $j = l+1,...,q$, and

$$u_j \geq -g_j(p,x) = 0 \text{ if } j = 1,...,l; g_j(p,x) < -u_j = 0 \text{ if } j = l+1,...,q,$$

thus (33) is proved. It follows that x is a feasible point of $Q(p)$, and $x \in M(p)$ by (34), $u \in U(p,x)$ by (32) and (33). Conversely assume (7). Then the multipliers theorem holds for $[x, y - g(p,x)]$ corresponding to $Q(p)$. Hence (32), (33) and (34) hold with

$$u = y - g(p,x). \tag{37}$$

We need only prove that $K(p, x, y) = 0$. With l as before we obtain

$$y_j \geq 0 \text{ if } j = 1, ..., l; \quad y_j < 0 \text{ if } j = l+1, ..., q.$$

Then
$$\nabla f(p,x) + \sum_{j=1}^{q} y_j^+ \nabla g_j(p,x) =$$

$$= \nabla f(p,x) + \sum_{j=1}^{l} [y_j - g_j(p,x)] \nabla g_j(p,x) + \sum_{j=1}^{l} g_j(p,x) \nabla g_j(p,x) =$$

$$= \nabla f(p,x) + \left(\sum_{j=1}^{q} - \sum_{j=l+1}^{q}\right)[y_j - g_j(p,x)] \nabla g_j(p,x) + \sum_{j=1}^{l} g_j(p,x) \nabla g_j(p,x). \quad (38)$$

If $1 \leq j \leq l$, by (33) and (37)

$$0 = -g_j(p,x)u_j \geq g_j(p,x)^2 \quad (39)$$

hence the last sum of (38) is 0. Then by (38) and (32)

$$\nabla f(p,x) + \sum_{j=1}^{q} y_j^+ \nabla g_j(p,x) =$$

$$= \nabla f(p,x) + \sum_{j=1}^{q} [y_j - g_j(p,x)] \nabla g_j(p,x) - \sum_{j=l+1}^{q} [y_j - g_j(p,x)] \nabla g_j(p,x) =$$

$$= - \sum_{j=l+1}^{q} [y_j - g_j(p,x)] \nabla g_j(p,x). \quad (40)$$

If $l+1 \leq j \leq q$, by (33) $\quad 0 \leq u_j^2 \leq -u_j g_j(p,x) = 0, \quad (41)$

then $u_j = y_j - g_j(p,x) = 0$, hence by (40)

$$\nabla f(p,x) + \sum_{j=1}^{q} y_j^+ \nabla g_j(p,x) = 0.$$

If $l+1 \leq j \leq q$, then $y_j < 0$ hence by (41)

$$g_j(p,x) - y_j^- = g_j(p,x) - y_j = 0$$

and if $1 \leq j \leq l$ then $y_j \geq 0$, and by (39)

$$g_j(p,x) - y_j^- = g_j(p,x) = 0$$

proving that $g(p,x) = y^-$. This ends the proof. □

In the proof of Theorem 3.2 we make use of the following Lemma, which is obvious by (8).

Lemma 6.1. *Let $F\colon R^{n+q} \to R$ be bounded from below. Then for every $p \in \Omega$*

$$\mathit{inf}\,\{F(x,y) : (x,y) \in \Sigma(p)\} = \mathit{inf}\,\{F[x, u + g(p,x)] : (x,u) \in S(p)\};$$

and

$$\mathit{inf}\,\{F(x,u) : (x,u) \in S(p)\} = \mathit{inf}\,\{F[x, y - g(p,x)] : (x,y) \in \Sigma(p)\}.$$

Proof of Theorem 3.2. Let Q be well-conditioned, then there exists $N > 0$ such that

$$e[S(p''), S(p')] + e[S(p'), S(p'')] \leq N|p'' - p'| \tag{42}$$

for each sufficiently small p', p'' in Ω. Then $\Sigma(p) \neq \emptyset$ by (9) if p is sufficiently small, and for such p', p'' in Ω we have

$$e[\Sigma(p''), \Sigma(p')] = \sup\,\{\ \mathrm{dist}\ [z'', \Sigma(p')] : z'' \in \Sigma(p'')\}.$$

For each $z' \in \Sigma(p')$ we have, by (8), $z' = (x', y')$ with

$$x' \in M(p'), u' = y' - g(p', x') \in U(p', x');$$

similarly for $z'' \in \Sigma(p'')$. We get

$$\mathrm{dist}\ [z'', \Sigma(p')] = \inf\{|z'' - z'| : z' \in \Sigma(p')\} =$$

$$= \inf\,\{|x'' - x'| + |y'' - y'| : (x', y') \in \Sigma(p')\} \leq$$

$$\leq \text{ (by Lemma 6.1) } \inf\{|x''-x'|+|u''+g(p'',x'')-u'-g(p',x')| : (x',u') \in S(p')\} \leq$$

$$\leq \text{(by (A4)) } L|p'' - p'| + \inf\{|x'' - x'| + |u'' - u'| + L|x'' - x'| : (x', u') \in S(p')\} \leq$$

$$\leq L|p'' - p'| + (L+1)\inf\{|x'' - x'| + |u'' - u'| : (x', u') \in S(p')\}.$$

Then, remembering (42),

$$\mathrm{dist}\,[z'', \Sigma(p')] \leq L|p'' - p'| + (L+1)\ \mathrm{dist}\ [(x'', u''), S(p')] \leq$$

$$\leq (L+1)e[S(p''), S(p')] + L|p'' - p'| \leq [(L+1)N + L]|p'' - p'|$$

whence

$$e[\Sigma(p''), \Sigma(p')] \leq [(L+1)N + L]|p'' - p'|;$$

similarly for $e[\Sigma(p'), \Sigma(p'')]$. It follows that Σ is Lipschitz near $p = 0$ and (10) is proved. Conversely assume (10). The definition of Lipschitz continuity implies that $\Sigma(p) \neq \emptyset$ for sufficiently small $p \in \Omega$, hence $S(p) \neq \emptyset$ by (8). By (10) there exists $N > 0$ such that

$$e[\Sigma(p''), \Sigma(p')] + e[\Sigma(p'), \Sigma(p'')] \leq N|p'' - p'| \tag{43}$$

for all sufficiently small p', p'' in Ω. We have

$$e[S(p''), S(p')] = \sup\,\{\ \mathrm{dist}\ [w'', S(p')] : w'' \in S(p'')\} =$$

$$= \sup{}_{w'' \in S(p'')} \inf\,\{|w'' - w'| : w' \in S(p')\}.$$

Every $w' \in S(p')$ can be written as $w' = (x', u')$ where $x' \in M(p'), u' \in U(p', x')$, and similarly for $w'' = (x'', u'') \in S(p'')$. Consider

$$y'' = u'' + g(p'', x''), y' = u' + g(p', x'),$$

then we have $(x', y') \in \Sigma(p'), (x'', y'') \in \Sigma(p'')$. Moreover by (A4)

$$|w'' - w'| = |x'' - x'| + |u'' - u'| \leq |x'' - x'| + |y'' - g(p'', x'') - y' + g(p', x')| \leq$$

$$\leq |x'' - x'| + |y'' - y'| + L(|p'' - p'| + |x'' - x'|).$$

Then by Lemma 6.1

$$\text{dist}\ [w'', S(p')] = \inf\ \{|w'' - w'| : w' \in S(p')\} \leq$$

$$\leq \inf\ \{(L+1)|x'' - x'| + |y'' - y'| : (x', y') \in \Sigma(p')\} + L|p'' - p'| \leq$$

$$\leq (L+1) \inf\ \{|x'' - x'| + |y'' - y'| : (x', y') \in \Sigma(p')\} + L|p'' - p'| =$$

$$= (L+1)\ \text{dist}\ [(x'', y''), \Sigma(p')] + L|p'' - p'| \leq (L+1)e[\Sigma(p''), \Sigma(p')] + L|p'' - p'| \quad (44)$$

hence by (43) and (44)

$$\text{dist}\,[w'', S(p')] \leq (L+1)N|p'' - p'| + L|p'' - p'|$$

whence $$e[S(p''), S(p')] \leq \text{ (const.) } |p'' - p'|;$$

similarly for $e[S(p'), S(p'')]$. Then we get (9), and the proof is completed. □

Proof of Proposition 3.3. Given sufficiently small points $p', p'' \in \Omega$ let $x'' \in M(p'')$. Let $x' \in M(p')$ be one of the closest points to x'' in $M(p')$. Let F be a Lipschitz constant of f. Then

$$|v(p'') - v(p')| = |f(p'', x'') - f(p', x')| \leq F(|p'' - p'| + |x'' - x'|) =$$

$$= F|p'' - p'| + F\ \text{dist}\ [x'', M(p')] \leq F|p'' - p'| + Fe[M(p''), M(p')]. \quad (45)$$

Let $u'' \in U(p'', x'')$, then, since every $S(p)$ is a closed set, there exists $(\bar{x}', \bar{u}') \in S(p')$ such that

$$\text{dist}\,[(x'', u''), S(p')] = |x'' - \bar{x}'| + |u'' - \bar{u}'|$$

hence $$|x'' - \bar{x}'| \leq e[S(p''), S(p')] \leq \text{ (const.) } |p'' - p'|$$

by well-conditioning. It follows that

$$\text{dist}\,[x'', M(p')] = \inf\{|x'' - x| : x \in M(p')\} \leq |x'' - \bar{x}'| \leq \text{ (const.) } |p'' - p'|$$

hence $$e[M(p''), M(p')] \leq \text{ (const.) } |p'' - p'|$$

so that by (45) $$|v(p'') - v(p')| \leq \text{ (const.) } |p'' - p'|$$

ending the proof. □

Proof of Proposition 4.3. We consider again problem $\bar{Q}$ of Example 4.2 with $\alpha = 8n^3\delta/r^3, \beta = \delta$, then $\bar{Q} \in N \cap IC$. Then (17) and (18) imply by (12) that dist $(Q, \bar{Q}) \leq L_1 + L_2$, hence (19) holds, ending the proof. □

Proof of Lemma 4.4. From (44) we have

$$e[S(p''), S(p')] \leq (L+1)e[\Sigma(p''), \Sigma(p')] + L|p'' - p'|$$

for all sufficiently small p', p'' in Ω, and similarly for $e[S(p'), S(p'')]$. Then by (20)

$$\text{hausd } [S(p''), S(p')] \leq (L(c+1) + c)|p'' - p'|$$

which yields (21). □

Proof of Theorem 4.5. By Lemma 4.4, we need to find an upper bound of Lip $(\Sigma, 0)$ given by (22). We apply [11, Theorems 5.11 and 5.7] since Σ is Lipschitz near $p = 0$ by Theorem 3.2, and S too by (44), thus Σ has closed graph at $p = 0$, as requested. Then we have

$$\text{Lip } (\Sigma, 0) = \sup \{(\text{ lip } \Sigma)(0, z) : z \in \Sigma(0)\}, \tag{46}$$

thus we need an uniform upper bound of each (lip $\Sigma)(0, z), z \in \Sigma(0)$. We apply [7, Theorem 4B. 6] to the system of implicit equations

$$K(p, x, y) = 0, f(p, x) = v(p), \text{ i.e. by (23) } G(p, x, y) = 0$$

remarking that, by Theorem 3.1, $(x, y) \in \Sigma(p)$ iff $G(p, x, y) = 0$. We check condition (25) of [7, Theorem 4B. 6]. By [19, Lemma 5.4] we have

$$v \in DG(p, x, y)(w) = DH(p, x, y)(w)$$

if and only if

$$v_1 = A(p, x, y)(w_1, \alpha, \beta), v_2 = \bigtriangledown f(p, x)^T w_1,$$

where the graphical derivative D (see [7, p.199]) is taken with respect to (x, y), and v, w_1, α, β are as in (25), (26) and (28). Since H is single-valued, we write $DH(p, x, y)$ instead of $DH(p, x, y|t)$ with $t = H(p, x, y)$. Then condition (25) of [6, Theorem 4B.6] is fulfilled by (27) here, remembering [7, (8) p. 202]: indeed

$$|DG(p, z')^{-1}|^{-} = \sup_{|v| \leq 1} \inf \{|w| : w \in DG(p, z')^{-1}(v)\} =$$

$$= \sup_{|v| \leq 1} \inf \{|w| : v \in DG(p, z')(w)\}.$$

It follows that for every $z \in \Sigma(0)$ we have by (29)

$$(\text{ lip } \Sigma)(0, z) \leq \lambda\Lambda \tag{47}$$

since, by (23), Λ is a Lipschitz constant of $K(\cdot, x', y'), f(\cdot, x')$ and of the optimal value function near $p = 0$, as (x', y') is sufficiently close to $(x, y) \in \Sigma(0)$. The optimal value function is locally Lipschitz due to Proposition 3.3. The existence of Λ follows by (A1) and boundedness of $\Sigma(0)$. Then, by (46) and (47), we get Lip $(\Sigma, 0) \leq \lambda\Lambda$, so that (30) holds by Lemma 4.4, ending the proof. □

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