Journal of Convex Analysis
Volume 27 (2020), No. 3, 791–810

Sets in the Complex Plane Mapped into Convex Ones by Möbius Transformations

Blagovest Sendov*
*Institute of Information and Communication Technologies,
Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria*
acad@sendov.com

Hristo Sendov†
*Department of Statistical and Actuarial Sciences, Western University,
London, Ont. N6A 5B7, Canada*
hssendov@stats.uwo.ca

Received: July 16, 2018
Revised manuscript received: May 24, 2019
Accepted: May 30, 2019

A set A in the extended complex plane is called convex with respect to a pole u, if for any two points z_1 and z_2 from the set, the arc from z_1 to z_2 on the unique circle through u, z_1, and z_2, opposite of u is contained in A. In that case we say that u is a pole of A. When $u = \infty$, this notion coincides with the usual convexity. Polar convexity, allows one to extend and/or strengthen several classical results about the location of the critical points of polynomials, such as the Gauss-Lucas' and the Laguerre's theorem.
Another way to characterize a pole of a set is through Möbius transformations. A point u is a pole of A if $W(A)$ is a convex set, whenever W is a non-degenerate Möbius transformation, such that $W(u) = \infty$. The goal of this paper is to describe the set of all poles of a given set A with simple, piece-wise smooth, regular boundary.

Keywords: Zeros and critical points of polynomials, Gauss-Lucas' Theorem, Laguerre's Theorem, polar derivative, pole of a set, polar convexity, osculating circle.

2010 Mathematics Subject Classification: 30C10.

1. Introduction

Let $\mathbb{C}$ be the complex plane and let $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$ be the extended complex plane. A *circular domain* (open or closed) is the interior or exterior of a circle, or a half-plane determined by a line in the complex plane. It is well-known, that for any three distinct points z_1, z_2, and u in the extended complex plane $\mathbb{C}^*$, the unique circle C defined by z_1, z_2 and u can be parametrized as follows

$$C = \left\{ z \in \mathbb{C} : z = u + \frac{1}{\frac{1-t}{z_1-u} + \frac{t}{z_2-u}}, \text{ where } t \in \mathbb{R} \cup \{\infty\} \right\}.$$

*The first author was partially supported by Bulgarian National Science Fund #DTK 02/44.
†The second author was partially supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

In particular, when $t \in [0,1]$ the point z goes over the arc of C starting at z_1 and ending at z_2, not containing u. Thus, we let

$$\operatorname{arc}_u[z_1, z_2] := \Big\{ u + \frac{1}{\frac{1-t}{z_1-u} + \frac{t}{z_2-u}} : t \in [0,1] \Big\}, \tag{1}$$

for distinct $z_1, z_2, u \in \mathbb{C}$ and call it *the arc between z_1 and z_2 with respect to u*. It can be seen, see [5], that this formula is continuous in $\mathbb{C}^*$ with respect to z_1, z_2, and u, as long as the three points remain distinct. If any of the points u, z_1, z_2 in $\mathbb{C}^*$ are equal, with common value $v \in \mathbb{C}^*$, then define

$$\operatorname{arc}_u[z_1, z_2] := \{v\}. \tag{2}$$

Definition 1.1. A set $A \subset \mathbb{C}^*$ is called *convex with respect to (pole) $u \in \mathbb{C}^*$* if for any $z_1, z_2, \in A$, we have $\operatorname{arc}_u[z_1, z_2] \subset A$.

Sometimes we say that A is *convex with respect to u* or that it is *u-convex* for short. For example, every circular domain D is convex with respect to any $u \in \operatorname{cl}(\mathbb{C} \setminus D)$. Convex sets (in the ordinary sense) are convex with respect to ∞, because by continuity $\operatorname{arc}_\infty[z_1, z_2] = [z_1, z_2]$.

Denote by $\mathcal{P}(A)$ the set of all poles of A. That is, $u \in \mathcal{P}(A)$, if and only if A is convex with respect to u. A trivial example is $\mathcal{P}(\mathbb{C}^*) = \mathbb{C}^*$ and it is easy to see, that the only set A for which $\mathcal{P}(A) = \mathbb{C}^*$ is $A = \mathbb{C}^*$. The claims in the following theorem are shown in [5].

Theorem 1.2. *For any $A \subsetneqq \mathbb{C}^*$, we have $(\operatorname{int} A) \cap \mathcal{P}(A) = \emptyset$ and $A \subseteq \mathcal{P}(\mathcal{P}(A))$.*

For any $A, U \subset \mathbb{C}^*$, such that $(\operatorname{int} A) \cap U = \emptyset$, let $\operatorname{conv}_U(A)$ denote the smallest set, containing A and convex with respect to every $u \in U$. The minimality in the definition implies, that if $U \subseteq V$, then $\operatorname{conv}_U(A) \subseteq \operatorname{conv}_V(A)$. Analogously, if $A \subseteq B$, then $\operatorname{conv}_U(A) \subseteq \operatorname{conv}_U(B)$. Finally, if $W(z)$ is a non-degenerate Möbius transformation, then

$$W(\operatorname{conv}_U(A)) = \operatorname{conv}_{W(U)}(W(A)). \tag{3}$$

For more details, as well as the proof of the following proposition, see [5].

Proposition 1.3. *Let $\{u_1, \ldots, u_m\}$ and $\{z_1, \ldots, z_n\}$ be non-intersecting sets in $\mathbb{C}^*$. The set $\operatorname{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}$ has empty interior, if and only if the points $\{u_1, \ldots, u_m\}$ and $\{z_1, \ldots, z_n\}$ are on a circle.*

Our interest in the set $\mathcal{P}(A)$ of all poles of a set $A \subseteq \mathbb{C}^*$ is two-fold. First, if $u \in \mathcal{P}(A)$, and $W(z)$ is a non-degenerate Möbius transformation such that $W(u) = \infty$, then by the above, we have that

$$W(A) = W(\operatorname{conv}_u(A)) = \operatorname{conv}_\infty(W(A)), \tag{4}$$

where the latter set, being convex with respect to ∞, is convex in the usual sense. This allows one to characterize all Möbius transformations $W(z)$, such that $W(A)$ is a convex set.

The second motivation arises from a connection between polar convexity with the zeros and critical points of polynomials. We now quickly summarize it, with more details given in [5]. Denote by $\overline{\mathcal{P}}_n$, where $n \geq 0$, the set of all polynomials of degree at most n. (The degree of a polynomial that is a non-zero constant is zero and the degree of 0 is defined to be -1.) Let $\overline{\mathcal{P}}_{-1} := \{0\}$. For any $u \in \mathbb{C}^*$ and $n \geq 0$, the linear operator $\mathcal{D}_u : \overline{\mathcal{P}}_n \to \overline{\mathcal{P}}_{n-1}$, defined by

$$\mathcal{D}_u(p; z) := \begin{cases} np(z) - (z-u)p'(z) & \text{if } u \neq \infty, \\ p'(z) & \text{if } u = \infty, \end{cases} \tag{5}$$

is called the *polar derivative of* $p(z)$ *with pole* u. For $0 \in \overline{\mathcal{P}}_{-1}$, define $\mathcal{D}_u(0; z) := 0 \in \overline{\mathcal{P}}_{-1}$. It is obvious that $\lim_{u\to\infty} \mathcal{D}_u(p; z)/u = p'(z)$, hence the zeros of $\mathcal{D}_u(p; z)$ approach the zeros of $p'(z)$ as u approaches infinity. Note that if $p(z) \in \overline{\mathcal{P}}_n$, for $n \geq 1$, is a non-zero constant polynomial, then $\mathcal{D}_u(p; z) \in \overline{\mathcal{P}}_{n-1}$ is a non-zero constant, whenever $u \neq \infty$. Similarly, for $p(z) \in \overline{\mathcal{P}}_n$, for $n \geq 1$, we have that $\mathcal{D}_u(p; z) \equiv 0$, if and only if $p(z)$ is a constant multiple of $(z-u)^n$, where $u \in \mathbb{C}$. If a polynomial $p(z)$ is considered a member of $\overline{\mathcal{P}}_n$, but has actual degree $m \in \{0, \ldots, n-1\}$, then we consider ∞ to be a zero of $p(z)$ of multiplicity $n-m$. The following proposition was established in [5].

Proposition 1.4. *Let $p(z) \in \overline{\mathcal{P}}_n$, be a polynomial with zeros $z_1, \ldots, z_n \in \mathbb{C}^*$. Let $u \in \mathbb{C}^*$. If $\mathcal{D}_u(p; z) \not\equiv 0$, then all its zeros are in* $\mathrm{conv}_u\{z_1, \ldots, z_n\}$.

Proposition 1.4 extends the classical Gauss-Lucas theorem that can be obtained with $u = \infty$. It also strengthens the classical Laguerre theorem, stating that a circular domain containing the zeros of $p(z)$, but not the point u, contains all zeros of the polar derivative $\mathcal{D}_u(p; z)$ (here $n \geq 2$). Indeed, if D contains the zeros $z_1, \ldots, z_n$ of $p(z)$, but not the pole u, then $\mathrm{conv}_u\{z_1, \ldots, z_n\} \subset D$, since D is convex with respect to u.

In summary, we have the following easy result, concluding the motivation behind our interest in the set of all poles of a given domain.

Theorem 1.5. *Let $A \subset \mathbb{C}^*$ be a set, having a non-empty set of poles $\mathcal{P}(A) \subset \mathbb{C}^*$. Then, for any natural n, any $z_1, \ldots, z_n \in A$, and any $u \in \mathcal{P}(A)$, the zeros of $\mathcal{D}_u(p; z)$ are in A, where $p(z) := (z-z_1)\cdots(z-z_n)$, provided that $\mathcal{D}_u(p; z) \not\equiv 0$.*

2. More notation and conventions

By $C(z_1, z_2, z_3)$ we denote the circle through three distinct points z_1, z_2, and z_3 in $\mathbb{C}$, oriented from z_1 through z_2 to z_3. By $D[z_1, z_2, z_3]$ we denote the closed circular domain, having the circle $C(z_1, z_2, z_3)$ as its boundary, and lying to the left of it. Similarly, $D(z_1, z_2, z_3)$ denotes the open such circular domain. Let $\mathrm{arc}[z_1, u, z_2]$ be the arc on the unique circle, determined by the three distinct points z_1, u, z_2, that starts at z_1, goes through u and ends in z_2. Denote by $\mathrm{line}[z_1, z_2]$ the line through z_1 and z_2. Such lines are assumed oriented from z_1 to z_2, so that we can talk about the left half-plane and right half-plane determined by it. By $B(z; r)$ we denote the open ball with centre z and radius r and by $B[z; r]$ we denote the closed such ball. We say that a point or a set is (strictly) inside a circle C if C is a proper circle (not a line) and the point is contained in the (interior of) bounded circular domain determined by C.

3. The poles of a set with simple piece-wise C^2 boundary

Let $\alpha\colon [0,1] \to \mathbb{C}^*$ be a simple, C^2, regular curve. We allow the possibility that $\alpha(0) = \infty$, or $\alpha(1) = \infty$, or both. Abusing notation, α also denotes the image set $\{\alpha(s) : s \in [0,1]\}$. At finite points $\alpha(s) \in \mathbb{C}$, let $T(s) := \alpha'(s)/|\alpha(s)|$ be the unit tangent vector and let $N(s)$ be the unit normal vector so that $\{T(s), N(s)\}$ is a right-handed basis for $\mathbb{R}^2$ (that is, one turns counterclockwise from $T(s)$ to $N(s)$). Define the curvature $\kappa(s) := \langle T'(s), N(s)\rangle$, and note that it satisfies $T'(s) = \kappa(s)N(s)$. In this way, we have that $k(s) > 0$ when $T(s)$ turns counterclockwise and $k(s) < 0$ when $T(s)$ turns clockwise. The points on the curve α fall into three mutually disjoint groups:

$$\alpha_0 := \{\alpha(s) : \kappa(s) = 0\}, \;\; \alpha_+ := \{\alpha(s) : \kappa(s) > 0\}, \;\text{ and }\; \alpha_- := \{\alpha(s) : \kappa(s) < 0\}.$$

If $\alpha(s) \notin \alpha_0$ consider the circle $C[c(s); r(s)]$ with centre $c(s)$ and radius $r(s)$, given by

$$c(s) := \alpha(s) + \frac{N(s)}{\kappa(s)} \;\text{ and }\; r(s) := \frac{1}{|\kappa(s)|}.$$

In other words, $C[c(s); r(s)]$ is the circle that best approximates the curve α at the point $\alpha(s)$, also known as the *osculating circle*. Let $D[c(s); r(s)]$ be the closed circular domain with boundary $C[c(s); r(s)]$, having $N(s)$ as a normal vector. (That is, $-N(s)$ points towards the interior of $D[c(s); r(s)]$.) We say that $D[c(s); r(s)]$ is the *osculating disk* or the *osculating circular domain* at $\alpha(s)$.

Let $H[\alpha(s)]$ be the closed half-plane, determined by the tangent line to α at $\alpha(s)$, and having $N(s)$ as a normal vector. Note that

$$\begin{aligned} &D[c(s); r(s)] \supseteq H[\alpha(s)], \;\text{ whenever } \alpha(s) \in \alpha_+; \text{ and}\\ &D[c(s); r(s)] \subseteq H[\alpha(s)], \;\text{ whenever } \alpha(s) \in \alpha_-. \end{aligned} \tag{6}$$

If $\alpha(s) \in \alpha_+$, then $D[c(s); r(s)]$ is unbounded circular domain; and if $\alpha(s) \in \alpha_-$, then $D[c(s); r(s)]$ is bounded circular domain. Finally, define

$$D[c(s); r(s)] := H[\alpha(s)], \;\text{ when } \alpha(s) \in \alpha_0.$$

Thus, (6) holds when $\alpha(s) \in \alpha_0$ as well. (We also use the notation $D[c(p); r(p)]$ and $\kappa_\alpha(p)$, where p is a point on the curve α, with the same meaning.) Consider the following set

$$\mathcal{C}(\alpha) := \bigcap_{\alpha(s) \in \alpha \cap \mathbb{C}} D[c(s); r(s)].$$

Now, let A be a connected domain in $\mathbb{C}^*$ with a simple, connected, piecewise C^2 boundary. Assume furthermore that the boundary of A can be partitioned into finitely many simple, C^2, regular curves, $\alpha_i\colon [0,1] \to \mathbb{C}$ for $i = 1, \ldots, n$, where

$\alpha_i(1) = \alpha_{i+1}(0) \in \mathbb{C}$ for $i = 1, \ldots, n-1$, and $\alpha_n(1) = \alpha_1(0) \in \mathbb{C}^*$. Here we make a crucial specification:

The parameterizations of the boundary pieces of a set A in $\mathbb{C}$, are chosen in such a way that vector $N(s)$ always points towards the set A. Consider the set

$$\mathcal{C}(\partial A) := \bigcap_{i=1}^{n} \mathcal{C}(\alpha_i). \tag{7}$$

To simplify the notation, we define $\alpha\colon [0,n] \to \mathbb{C}^*$ by $\alpha(s) := \alpha_i(s-(i-1))$, whenever $s \in [i-1, i]$, for $i = 1, \ldots, n$. The points $\alpha(s)$, $s = 0, 1, \ldots, n$, are possibly non-smooth. At such points, if finite, we talk about the left and the right derivatives, tangents, curvature, osculating circle, osculating circular domain, etc. For example $\alpha'(s+) := \alpha'_{s+1}(0)$ and $\alpha'(s-) := \alpha'_s(1)$ for $s = 1, \ldots, n$, with the convention that $\alpha'(0-) := \alpha'_n(1)$ and $\alpha'(n+) := \alpha'_1(0)$ if both points $\alpha_1(0)$, $\alpha_n(1)$ are finite. The other left and right handed objects are defined analogously.

Assumption 3.1. Assume that at non-smooth points $\alpha(s)$ on the boundary of A, the direction $T(s+)$ can be obtained from the direction $T(s-)$ by a counter-clock wise rotation at an angle less than π.

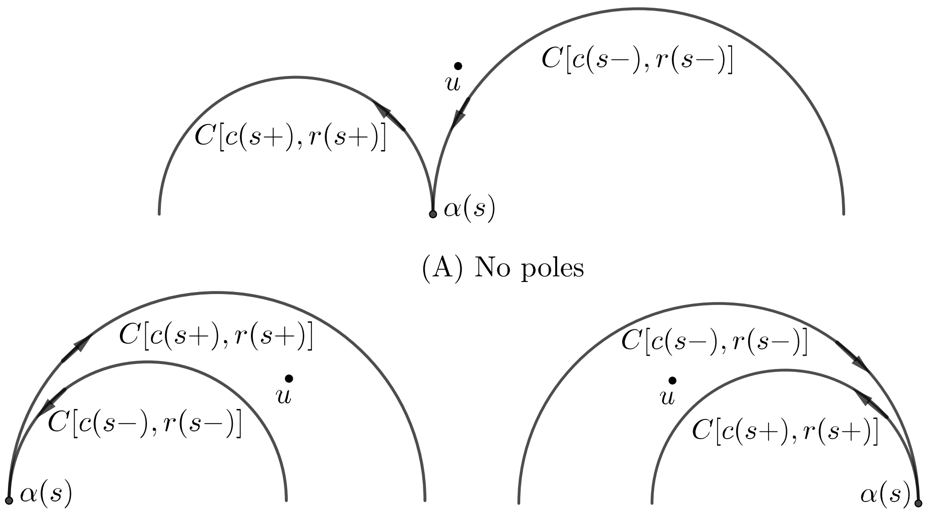
The main result of this work is the following theorem.

Theorem 3.2. *If the boundary of A is a simple, piece-wise C^2 regular curve, then*

$$\mathcal{P}(A) \subseteq \mathcal{C}(\partial A).$$

If, in addition, ∂A is a bounded set and satisfies Assumption 3.1, *then*

$$\mathcal{P}(A) = \mathcal{C}(\partial A). \tag{8}$$



(A) No poles

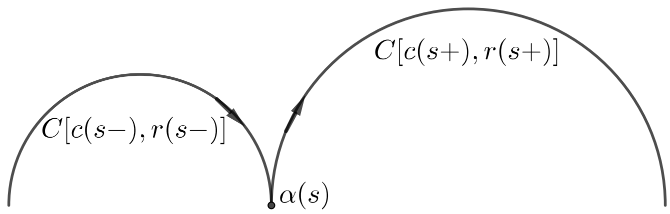
(B) No poles

(C) No poles

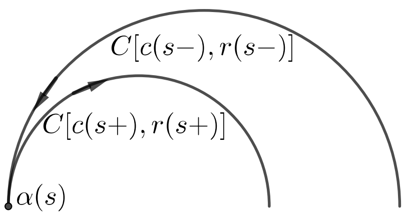
Figure 3.1: The cases without poles in the proof of Proposition 3.3

In the theorem, the requirement that ∂A is a bounded set is not a restriction, at least in theory. As (3) shows, if W is a non-degenerate Möbius transformation, then $W(\mathcal{P}(A)) = \mathcal{P}(W(A))$. Choose W so that $\partial W(A)$ is bounded set, use (8) to find $\mathcal{P}(W(A))$ and then apply W^{-1}. This saves us the trouble of dealing with curvature and osculating circles at infinity.

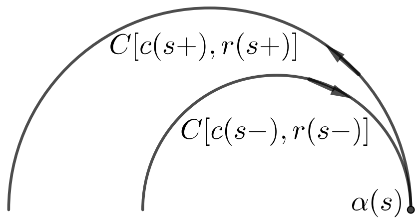
We now investigate the case when, the boundary of A has a non-smooth point $\alpha(s)$, where the angle between the directions $T(s-)$ and $T(s+)$ is π. The osculating circles $C[c(s-);r(s-)]$ and $C[c(s+);r(s+)]$ at $\alpha(s)$ are tangent to each other. There are essentially six possible arrangements between the two osculating circles, taking into account the orientation of the boundary and the sign of the curvatures $\kappa(s-)$ and $\kappa(s+)$. (The orientation of the boundary induces an orientation of the osculating circles.) They are depicted in Figures 3.1 and 3.2. In these six cases, we are mindful of the possibility that $\kappa(s-)$ and $\kappa(s+)$ may be equal and/or zero.



(A) At most one pole



(B) At most one pole



(C) At most one pole

Figure 3.2: The cases with at most one pole in the proof of Proposition 3.3

Proposition 3.3. *Let $\alpha(s)$ be a non-smooth point on the boundary of A, such that the counter-clock wise angle between $T(s-)$ and $T(s+)$ is π.*

(a) *Suppose the point $\alpha(s)$ is of the type depicted on Figures* 3.1A, 3.1B*, or* 3.1C*, then $\mathcal{P}(A) = \emptyset$.*

(b) *Suppose the point $\alpha(s)$ is of the type depicted on Figures* 3.2A, 3.2B*, or* 3.2C *and the rest of the non-smooth points on ∂A satisfy Assumption* 3.1*. Then $\mathcal{P}(A) = \mathcal{C}(\partial A)$, where the latter set is either $\emptyset$ or $\{\alpha(s)\}$.*

We conclude with a simple corollary of Theorem 1.2 and Proposition 1.3.

Corollary 3.4. *For any $A \subset \mathbb{C}^*$ with non-empty interior, the set of its poles $\mathcal{P}(A)$ is either empty, a singleton, or a connected domain.*

Proof. Suppose that $\mathcal{P}(A)$ contains at least two distinct points u_1, u_2. We show that $\mathcal{P}(A)$ is connected and is the closure of its interior. Indeed, since A has non-empty interior, we can find two points $a_1, a_2 \in A$ such that a_1, a_2, u_1, u_2 are not on a circle. By Proposition 1.3 the set $\mathrm{conv}_{\{a_1,a_2\}}\{u_1, u_2\}$ has non-empty interior and is connected. By Theorem 1.2, since a_1, a_2 are poles of $\mathcal{P}(A)$, we conclude that $\mathrm{conv}_{\{a_1,a_2\}}\{u_1, u_2\} \subseteq \mathcal{P}(A)$. □

4. Examples

In many cases, the application of Theorem 3.2 and formula (7) is significantly simplified by the following well-known result, first discovered by Tait [6] and then by Kneser [1].

Theorem 4.1. (Tait-Kneser's Nesting theorem) *Let $\alpha\colon [0,1] \to \mathbb{C}$ be a C^3 curve with monotone non-vanishing curvature. Then, the osculating circles of α are nested.*

With formula (7) in mind, when applying the Tait-Kneser's Nesting theorem, four cases need to be distinguished. If $0 < \kappa(0) \leq \kappa(1)$, then $\mathcal{C}(\alpha) = D[c(0); r(0)]$. If $\kappa(0) \geq \kappa(1) > 0$, then $\mathcal{C}(\alpha) = D[c(1); r(1)]$. If $0 > \kappa(0) \geq \kappa(1)$, then $\mathcal{C}(\alpha) = D[c(1); r(1)]$. If $\kappa(0) \leq \kappa(1) < 0$, then $\mathcal{C}(\alpha) = D[c(0); r(0)]$.

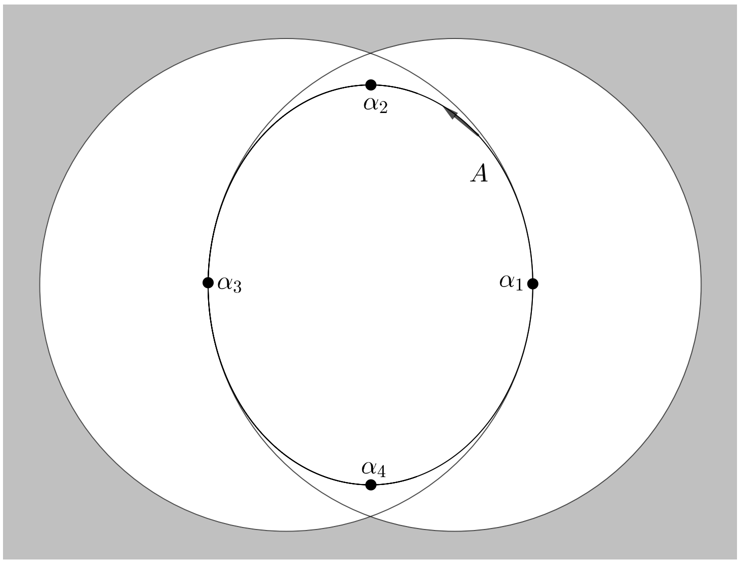


Figure 4.1: The poles of the inside of an ellipse

Example 4.2. (The poles of the inside of an ellipse) Let $A \subset \mathbb{C}$ be a closed bounded domain, having an ellipse for a boundary, see Figure 4.1. A counter clock-wise orientation of the boundary needs to be chosen making the curvature strictly positive. The points α_1, α_2, α_3, and α_4 divide the boundary into four pieces, each with monotone curvature. Assuming that the vertical axis of the ellipse is longer than the horizontal, the curvature increases from α_1 to α_2, then decreases from α_2 to α_3 and so on. Thus,

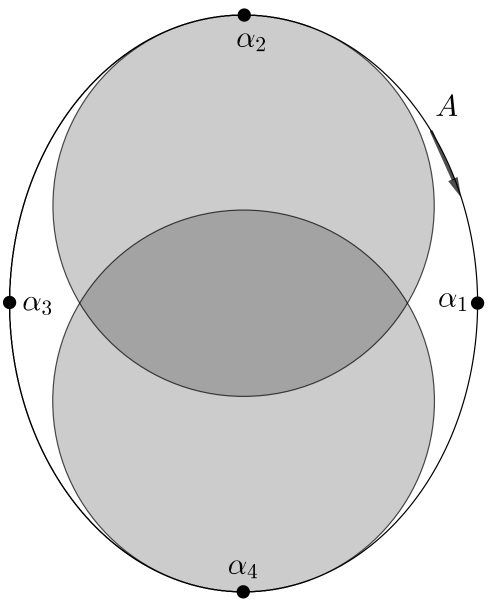
$$\mathcal{P}(A) = \bigcap_{\alpha(s) \in \partial A} D[c(s); r(s)] = D_{\alpha_1} \cap D_{\alpha_3},$$

where D_{α_1} and D_{α_3} are the osculating disks at α_1 and α_3 respectively. Note that these are the unbounded circular domains determined by the corresponding osculating circles. The grey area in Figure 4.1 is that intersection.

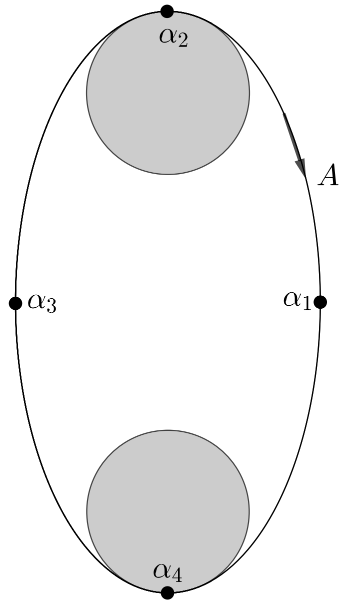
Example 4.3. (The poles of the outside of an ellipse) Let $A \subset \mathbb{C}$ be a closed unbounded domain, having an ellipse for a boundary, see Figure 4.2. A clockwise orientation of the boundary needs to be chosen making the curvature strictly negative. The points α_1, α_2, α_3, and α_4 divide the boundary into four pieces, each with monotone curvature. Assuming that the vertical axis of the ellipse is longer than the horizontal, the curvature decreases from α_1 to α_4, then increases from α_4 to α_3 and so on. Thus,

$$\mathcal{P}(A) = \bigcap_{\alpha(s)\in\partial A} D[c(s); r(s)] = D_{\alpha_2} \cap D_{\alpha_4},$$

where D_{α_2} and D_{α_4} are the osculating disks at α_2 and α_4 respectively. Note that these are the bounded circular domains determined by the corresponding osculating circles. Figure 4.2A shows a situation when the outside of the ellipse has poles: the intersection $D_{\alpha_2} \cap D_{\alpha_4}$ is non-empty; while Figure 4.2B shows a situation when the outside of the ellipse has no poles: the intersection $D_{\alpha_2} \cap D_{\alpha_4}$ is empty.



(A) Some poles



(B) No poles

Figure 4.2: The poles of the outside of an ellipse

Example 4.4. (The poles of the inside of a hyperbola) Let $A \subset \mathbb{C}$ be the closed unbounded domain to the right of the hyperbola on Figure 4.3. A counter clockwise orientation of the boundary needs to be chosen making the curvature strictly positive. The point α_1 divides the boundary into two pieces, each with monotone curvature. The curvature is increasing on the piece before α_1 and is decreasing afterwords. Thus,

$$\mathcal{C}(\partial A) = \bigcap_{\alpha(s)\in\partial A} D[c(s); r(s)] = \lim_{\alpha_2\to\infty} D_{\alpha_2} \cap \lim_{\alpha_3\to\infty} D_{\alpha_3},$$

where D_{α_2} and D_{α_3} are the osculating disks at α_2 and α_3 respectively. In the limits, $\alpha_2, \alpha_3 \to \infty$ means that α_2 goes to infinity along the hyperbola against its orientation, while α_3 goes to infinity along the orientation. Note that D_{α_2} and D_{α_3} are the unbounded circular domains determined by the corresponding osculating circles. The intersection of the limiting osculating discs, shown on Figure 4.3, is the sector determined by the asymptotes of the hyperbola, opposite to the sector containing the hyperbola. Let us denote this sector by S. Since the boundary of A is an unbounded set Theorem 3.2 concludes that $\mathcal{P}(A) \subseteq S$. Straight forward verification can show that every point from S is indeed a pole of A, concluding that $\mathcal{P}(A) = S$.

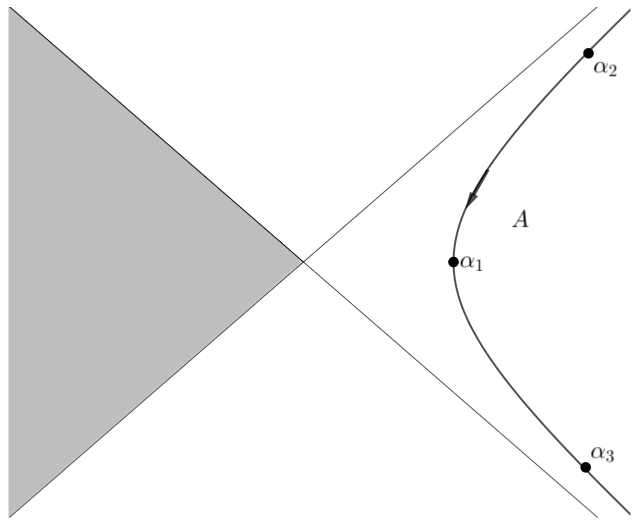


Figure 4.3: The poles of the inside of a hyperbola

Example 4.5. (The poles of the outside of a hyperbola) Let $A \subset \mathbb{C}$ be the closed unbounded domain to the left of the hyperbola on Figure 4.4. A clockwise orientation of the boundary needs to be chosen making the curvature strictly negative. The point α_1 divides the boundary into two pieces, each with monotone curvature. The curvature is increasing on the piece above α_1 and is decreasing on the piece below it. Thus,

$$\mathcal{C}(\partial A) = \bigcap_{\alpha(s)\in\partial A} D[c(s); r(s)] = D_{\alpha_1},$$

where D_{α_1} is the osculating disk at α_1. In this case, D_{α_1} is the bounded circular domain determined by the osculating circle at α_1. Since the boundary of A is an unbounded set Theorem 3.2 concludes that $\mathcal{P}(A) \subseteq D_{\alpha_1}$. It is straight forward to see that no point from D_{α_1} can be a pole of A. (Just take a point $u \in D_{\alpha_1}$ and choose two points $a, b \in A$ that are to the right of u, one above the upper branch of the hyperbola and one below the lower branch. Then, $\mathrm{arc}_u[a, b]$ is not contained in A.) Thus, $\mathcal{P}(A) = \emptyset$.

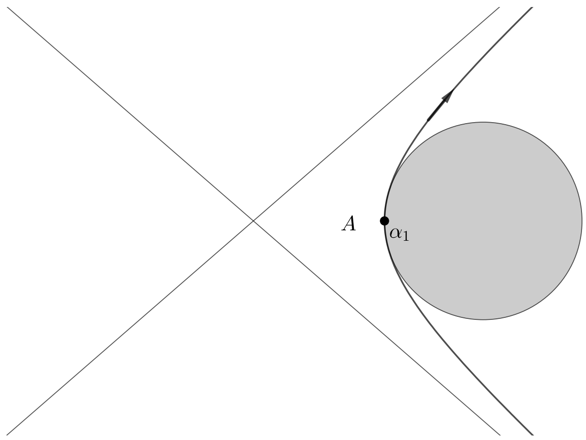
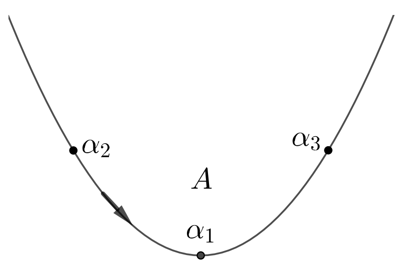


Figure 4.4: The outside of a hyperbola has no poles

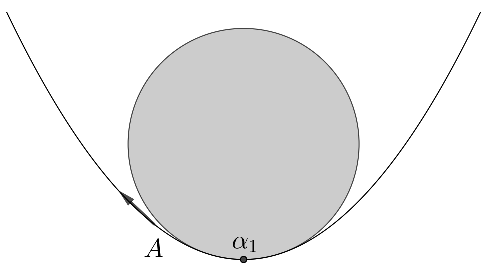
Example 4.6. (The poles of the inside/outside of a parabola) Let $A \subset \mathbb{C}$ be the closed unbounded domain above the parabola on Figure 4.5A. A counter clockwise orientation of the boundary needs to be chosen making the curvature strictly positive. The point α_1 divides the boundary into two pieces, each with monotone curvature. The curvature is increasing on the piece before α_1 and is decreasing afterwords. Thus,

$$\mathcal{C}(\partial A) = \bigcap_{\alpha(s)\in\partial A} D[c(s); r(s)] = \lim_{\alpha_2\to\infty} D_{\alpha_2} \cap \lim_{\alpha_3\to\infty} D_{\alpha_3} = \{\infty\},$$

where D_{α_2} and D_{α_3} are the osculating disks at α_2 and α_3 respectively. In the limits, $\alpha_2, \alpha_3 \to \infty$ means that α_2 goes to infinity along the parabola against its orientation, while α_3 goes along the orientation. Note that D_{α_2} and D_{α_3} are the unbounded circular domains determined by the corresponding osculating circles. A simple calculation can show that the intersection of the limiting osculating discs is $\{\infty\}$. By Theorem 3.2 we conclude that $\mathcal{P}(A) \subseteq \{\infty\}$. Since the set A is convex, we obtain $\mathcal{P}(A) = \{\infty\}$. That is, the inside of a parabola is a convex set but not convex with respect to any finite pole.



(A) The inside of a parabola has only one pole: ∞



(B) The outside of a parabola has no poles

Figure 4.5: The poles of the inside/outside of a parabola

Let $A \subset \mathbb{C}$ be the closed unbounded domain below the parabola on Figure 4.5B. The situation is similar to the one in Example 4.5. One has $\mathcal{C}(\partial A) = D_{\alpha_1}$, which is the shaded disk on Figure 4.5B. It is trivial to see that none of the points in D_{α_1} is a pole. In this case, we have $\mathcal{P}(A) = \emptyset$.

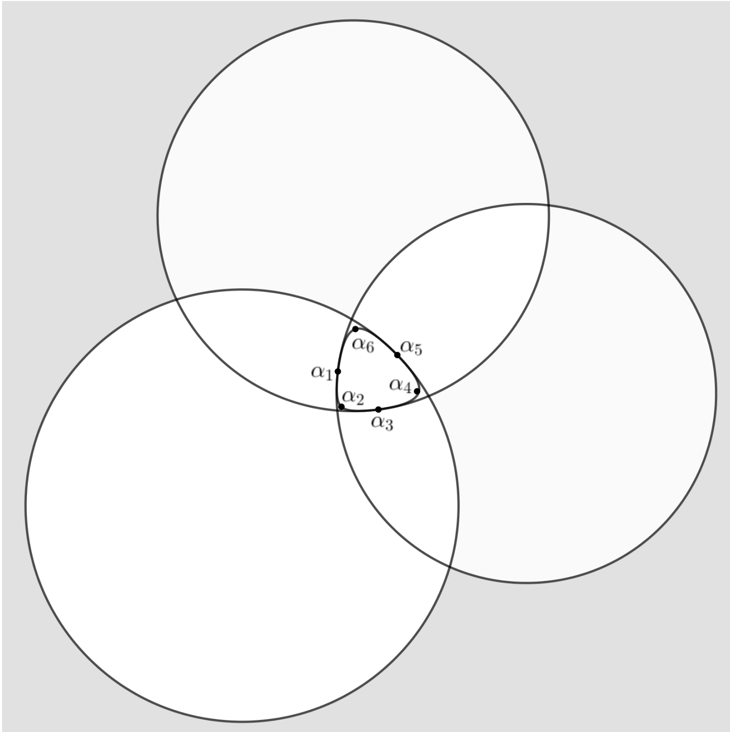


Figure 4.6: The poles of the domain with boundary $(e^{\cos(t)}, e^{\sin(t)})$

Example 4.7. (Miscellaneous example) Let $A \subset \mathbb{C}$ be the bounded domain with boundary specified by the curve $\left(e^{\cos(t)}, e^{\sin(t)}\right)$, for $t \in [0, 2\pi]$, see Figure 4.6. The boundary is given a counter clock-wise orientation. One can show that there are six points on it $\alpha_1, \ldots, \alpha_6$ that divide it into pieces with monotone curvature. Then, by Theorem 3.2, we have $\mathcal{P}(A) = D_{\alpha_1} \cap D_{\alpha_3} \cap D_{\alpha_5}$, where α_1, α_3, α_5 are the points where the curvature achieves a local minimum. Thus, $\mathcal{P}(A)$ is the intersection of three unbounded circular domains determined by the osculating circles at the points α_1, α_3, and α_5.

A variation of this example allows one to see how the set of poles changes when the set is deformed. Consider the bounded domain with boundary given by the curve $\left(e^{2\cos(t)}, e^{2\sin(t)}\right)$, for $t \in [0, 2\pi]$. It is non-convex and has non-empty set of poles, see Figure 4.7. The boundary is given a counter clock-wise orientation. The points $\alpha_1, \ldots, \alpha_6$ on it divide it into pieces with monotone curvature. The points β_1 and β_2 mark the spots where the curvature is zero. We again have $\mathcal{P}(A) = D_{\alpha_1} \cap D_{\alpha_3} \cap D_{\alpha_5}$, where α_1, α_3, α_5 are the points where the curvature achieves a local minimum. This time the set $\mathcal{P}(A)$ is bounded.

Example 4.8. (Non-smooth boundary) Consider the bounded set A defined as the intersection of four closed circular domains (one of them is bounded and the other three are unbounded), as shown on Figure 4.8. Its boundary consists of four circular arcs and it satisfies Assumption 3.1. Theorem 3.2 holds and the set of its poles $\mathcal{P}(A)$ is the circular triangle depicted. For completeness, we mention that the closure of the complement of A has no poles and in fact it does not satisfy Assumption 3.1.

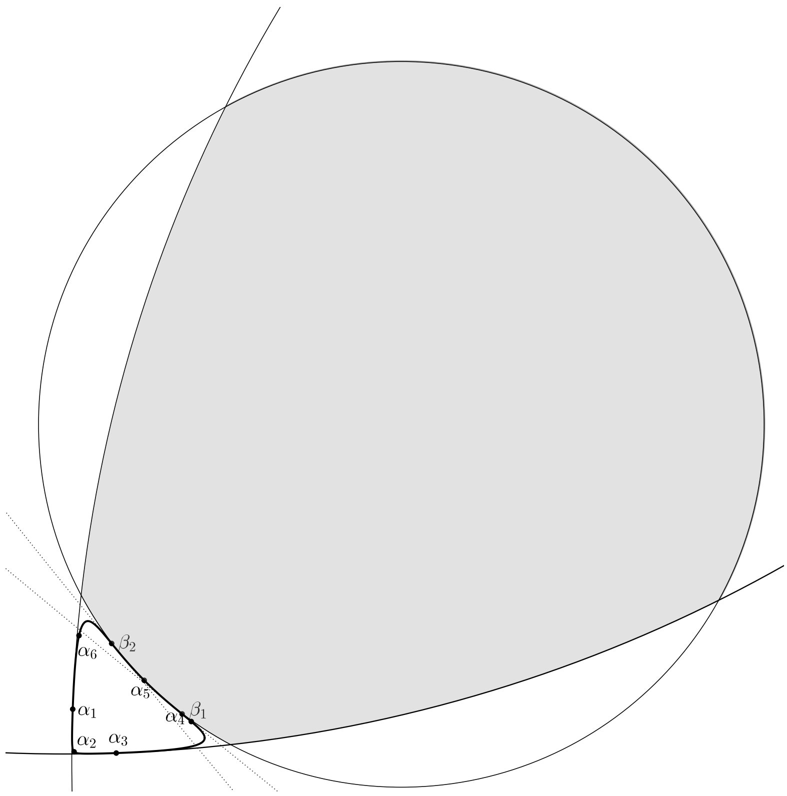


Figure 4.7: The poles of the domain with boundary $(e^{2\cos(t)}, e^{2\sin(t)})$

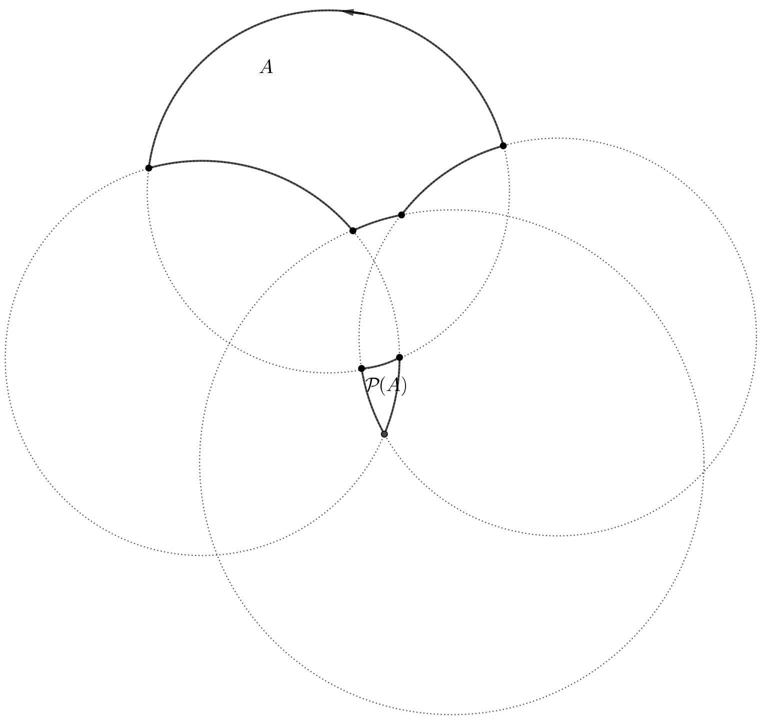
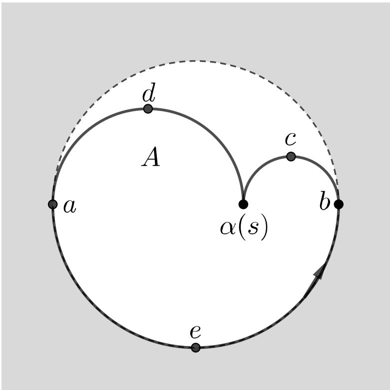
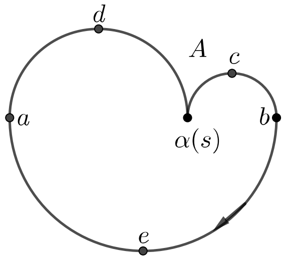


Figure 4.8: Boundary made from circular arcs

Example 4.9. (The relevance of Assumption 3.1) Consider the bounded set A having boundary depicted on Figure 4.9A. The boundary consists of three halfcircles as follows. The points a, $\alpha(s)$, and b are collinear. The circle containing arc$[a, e, b]$ has a centre at the point $(a + b)/2$. The circle containing arc$[b, c, \alpha(s)]$ has a centre at the point $(b+\alpha(s))/2$. Finally, the circle containing arc$[\alpha(s), d, a]$ has a centre at the point $(\alpha(s)+a)/2$. The boundary is given a counter-clock wise orientation, making the curvature positive everywhere. In this case, $\mathcal{C}(\partial A)$ is the unbounded closed circular domain $D[b, e, a]$. This is the grey area on Figure 4.9A. Assumption 3.1 fails at the point $\alpha(s)$, since the positive angle between $T(s-)$ and $T(s+)$ is π. The osculating circles at the point $\alpha(s)$ fall into the case illustrated on Figure 3.1A. Thus, by part a) of Proposition 3.3, we get $\mathcal{P}(A) = \emptyset$.



(A) The bounded component has no poles



(B) The unbounded component has one pole: $\alpha(s)$

Figure 4.9: Illustrating Proposition 3.3

On the other hand, consider the unbounded set A having the same boundary, see Figure 4.9B. A clock-wise orientation of the boundary needs to be given, making the curvature negative everywhere. Then

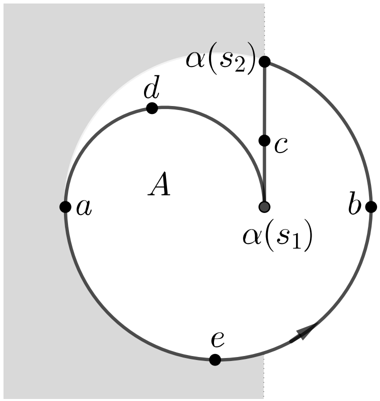
$$\mathcal{C}(\partial A) = D[a, e, b] \cap D[b, c, \alpha(s)] \cap D[\alpha(s), d, a] = \{\alpha(s)\}.$$

The osculating circles at the point $\alpha(s)$ fall into the case illustrated on Figure 3.2A. Thus, by part b) of Proposition 3.3, we get $\mathcal{P}(A) = \{\alpha(s)\}$.

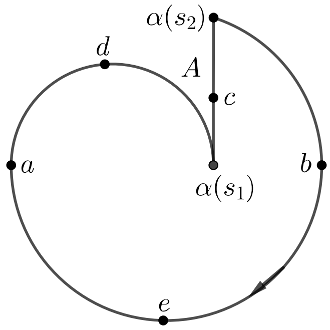
Example 4.10. (Illustrating Proposition 3.3) As a final example, consider a modification of Example 4.9, displayed on Figure 4.10. The points a, $\alpha(s_1)$, and b are collinear. The points a, e, b, and $\alpha(s_2)$ are on a circle with centre $(a + b)/2$. The points $\alpha(s_1)$, d, and a are on a circle with centre $(a+\alpha(s_1))/2$. Finally, the points $\alpha(s_1)$, c, and $\alpha(s_2)$ are collinear.

Let A be the bounded domain determined by that boundary. A counter-clock wise orientation is chosen. The set $\mathcal{C}(\partial A)$ is the grey area on Figure 4.10A: the intersection of $D[b, e, a]$ with the closed half-plane to the right of line$[\alpha(s_2), \alpha(s_1)]$. Assumption 3.1 holds at the point $\alpha(s_2)$ but fails at the point $\alpha(s_1)$. The osculating circles at the point $\alpha(s_1)$ fall into the case illustrated on Figure 3.1A. Thus, by part a) of Proposition 3.3, we get $\mathcal{P}(A) = \emptyset$.

Let A be the unbounded domain determined by the boundary described. A clock wise orientation is chosen, making the curvature negative or zero (on the straight segment). We have $\mathcal{C}(\partial A) = \{\alpha(s_1)\}$, hence $\mathcal{P}(A) \subseteq \{\alpha(s_1)\}$. The counter-clock wise angle between $T(s_1-)$ and $T(s_1+)$ is π and the osculating circles at that point fall into the case illustrated on Figures 3.2A. Since the counter-clock wise angle between $T(s_2-)$ and $T(s_2+)$ is bigger than π, Assumption 3.1 fails at $\alpha(s_2)$. That is, Proposition 3.3 is inapplicable in this case. Direct verifications shows that the image of A under the transformation $W(z) = 1/(z - \alpha(s_1))$ is not convex. Hence, $\mathcal{P}(A) = \emptyset$.



(A) The bounded component has no poles



(B) The unbounded component has no poles

Figure 4.10: Illustrating Proposition 3.3

5. The proofs

Lemma 5.1. *Let $\alpha : [0, 1] \to \mathbb{R}^2$ be a simple, C^2, regular curve. Let $D[c(s); r(s)]$ be the osculating disk at the point $\alpha(s)$. Let $u \in D[c(s); r(s)]$ be a finite point, different from $\alpha(s)$. Consider a Möbius transformation $W(z)$, such that $W(u) = \infty$. Then,*

(1) *$W(D[c(s); r(s)])$ is the osculating disk of $W(\alpha)$ at the point $W(\alpha(s))$.*

(2) *The curvature of $W(\alpha)$ at the point $W(\alpha(s))$ is non-negative.*

Proof. It is well-known that if $\alpha(s)$ is a point on α, for some $s \in (0, 1)$, then the circle $C(\alpha(s - h), \alpha(s), \alpha(s + h))$ converges to the osculating circle $C[c(s); r(s)]$ at $\alpha(s)$ as $h \to 0$. Since W sends $C(\alpha(s - h), \alpha(s), \alpha(s + h))$ into $C(W(\alpha(s - h)), W(\alpha(s)), W(\alpha(s+h)))$ and is continuous, one concludes that W sends the osculating circle $C[c(s); r(s)]$ at $\alpha(s)$ onto the osculating circle of $W(\alpha)$ at $W(\alpha(s))$.

When $s = 0$, we have that the circle $C(\alpha(0), \alpha(h), \alpha(\ell))$ converges to the osculating circle $C[c(0); r(0)]$ at $\alpha(0)$ as $\ell > h > 0$ both approach zero in such a way that $|h/(h - \ell)|$ and $|\ell/(h - \ell)|$ stay bounded. Since we could not find a proof of this claim in the literature, we include the proof in the appendix, see Lemma 6.1. The situation at $s = 1$ is analogous. This shows that W sends the osculating circle $C[c(s); r(s)]$ at $\alpha(s)$ onto the osculating circle of $W(\alpha)$ at $W(\alpha(s))$ for all $s \in [0, 1]$.

Without loss of generality, we may assume $u \in \mathrm{int}D[c(s); r(s)]$, since then one can take the limit as u approaches a finite point on the boundary $C[c(s); r(s)]$. The condition $u \notin C[c(s); r(s)]$ implies that the curvature of $W(\alpha)$ at $W(\alpha(s))$ is not zero (the osculating circle $W(C[c(s); r(s)])$ to $W(\alpha)$ at $W(\alpha(s))$ is a proper circle, not a line).

Extend the curve α by connecting the point $\alpha(1)$ to $\alpha(0)$, so that the resulting curve is closed, simple, C^2, regular curve, surrounding a bounded domain, A. One can do that so that the normal vector $N(s)$ points to the interior of A at all points and so that $u \notin \partial A$. The reason for extending α is so that we can talk about the interior and exterior of the curve α and by that we mean the interior and the exterior of the set A.

The boundary $W(\partial A)$ of $W(A)$ is endowed with orientation from that of ∂A. By considering two cases, depending on whether u is in A or not, one can see that the set $W(A)$ is always to the left of the image of the boundary $W(\partial A)$. Since $u \in D[c(s); r(s)]$, we have that $W(D[c(s); r(s)])$ is an unbounded circular domain.

Case 1. Suppose that $\alpha(s) \in \alpha_+$. In this case, u is in the interior of the unbounded circular domain, determined by the osculating circle $C[c(s); r(s)]$. There are two possibilities for the position of u: a) u and $C[c(s); r(s)]$ are on the same side of the tangent line to α at $\alpha(s)$, or b) u and $C[c(s); r(s)]$ are on different sides of the tangent line to α at $\alpha(s)$. Consider the circle C through the point u, that is tangent to α at $\alpha(s)$. Since we have assumed that $u \notin C[c(s); r(s)]$, then in case a) the curvature of C is strictly smaller than that of α at $\alpha(s)$; and in case b) the circles C and $C[c(s); r(s)]$ are on different sides of the tangent line at $\alpha(s)$. In both cases, locally around the point $\alpha(s)$, the circle C is outside of A (except for the point $\alpha(s)$ only), see [7, Problem 1.7.1].

Now, note that $W(C)$ is the tangent line to $\partial W(A)$ at $W(\alpha(s))$. If the curvature of the boundary of $W(A)$ at $W(\alpha(s))$ is strictly negative, then locally around $W(\alpha(s))$, the tangent line $W(C)$ is in the set $W(A)$. This contradiction shows that the curvature of $\partial W(A)$ at $W(\alpha(s))$ is strictly positive. That is, the osculating disk of $\partial W(A)$ at $W(\alpha(s))$ is the unbounded component determined by $W(C[c(s); r(s)])$. So, it must coincide with $W(D[c(s); r(s)])$.

Case 2. Suppose $\alpha(s) \in \alpha_-$. In this case, u and $C[c(s); r(s)]$ are on the same side of the tangent line to α at $\alpha(s)$. In fact, u is in the interior of the bounded circular domain determined by $C[c(s); r(s)]$. As in Case 1, consider the circle C through the point u, that is tangent to α at $\alpha(s)$. The curvature of C is strictly larger than the absolute value of the curvature of α at $\alpha(s)$, so locally around the point $\alpha(s)$, the circle C is outside of A (except for the point $\alpha(s)$ only). Again, $W(C)$ is the tangent line to $\partial W(A)$ at $W(\alpha(s))$ and one concludes as in Case 1. $\square$

Proof of Theorem 3.2. Let $\alpha\colon [0,1] \to \mathbb{C}$ be a piece-wise C^2 regular parametrization of the boundary of A, so that the normal vector $N(s)$ points towards A. Let

$$\ell(s) := \{\alpha(s) + tT(s) : t \in \mathbb{R}\}$$

be the tangent line to α at the point $\alpha(s)$.

We show first that $\mathcal{P}(A)$ is contained in the set $\mathcal{C}(\partial A)$. Indeed, take a point u not in $\mathcal{C}(\partial A)$, that is $u \notin D[c(s); r(s)]$, for some point $\alpha(s) \in \partial A$.

Suppose first that the point $\alpha(s)$ is a smooth point and consider several cases.

Case 1. Suppose $\alpha(s) \in \alpha_+$. That is, u is in the interior of the bounded circular domain determined by the osculating circle at $\alpha(s)$. For an $\epsilon > 0$ small enough, the set $A \cap B(\alpha(s); \epsilon)$ is on the same side as u with respect to the tangent line $\ell(s)$ at $\alpha(s)$. Consider the circle C through u and $\alpha(s)$ that is tangent to $\ell(s)$. Since u is strictly inside the osculating circle at $\alpha(s)$, so is the circle C (except the point $\alpha(s)$ only). That is, the curvature of C is strictly larger than $\kappa(s)$. This implies that for $h > 0$ small enough, the points $\alpha(s-h)$, $\alpha(s+h)$ are outside of C, see [7, Problem 1.7.1]. Therefore, for $h > 0$ small enough, the arc $\mathrm{arc}_u[\alpha(s-h), \alpha(s+h)]$ contains points in $B(\alpha(s); \epsilon)$, that are on the other side of $\ell(s)$. That is, $\mathrm{arc}_u[\alpha(s-h), \alpha(s+h)]$ is not contained in A, implying that $u \notin \mathcal{P}(A)$.

Case 2. Suppose $\alpha(s) \in \alpha_-$. That is, $D[c(s); r(s)]$ is a bounded circular domain and u is in the interior of its complement. Let C_1 be a circle through $\alpha(s)$, tangent to $\ell(s)$, and strictly contained in $D[c(s); r(s)]$ (except for the point $\alpha(s)$). For an $\epsilon > 0$ small enough, the intersection between $B(\alpha(s); \epsilon)$ and the interior of C_1 is in the complement of A. We consider two subcases.

Case 2.a. Suppose u is on the same side as $D[c(s); r(s)]$ with respect to the tangent line $\ell(s)$ at $\alpha(s)$, but suppose that $u \notin \ell(s)$. Consider the circle C through the points u and $\alpha(s)$, tangent to $\ell(s)$. Since $u \notin D[c(s); r(s)]$, the latter set is strictly contained in C, except for the point $\alpha(s)$. Thus, the curvature of C is strictly smaller than $|\kappa(s)|$, while that of C_1 is strictly larger than $|\kappa(s)|$. This implies that for $h > 0$ small enough, the points $\alpha(s-h)$, $\alpha(s+h)$ are inside of C but outside of C_1. Hence, the whole arc $\mathrm{arc}_u[\alpha(s-h), \alpha(s+h)]$ is in C. These arcs cross the interior of C_1 and converge to $\alpha(s)$ as h approaches 0. Hence, for small h, $\mathrm{arc}_u[\alpha(s-h), \alpha(s+h)]$ is not entirely contained in A. That is, $u \notin \mathcal{P}(A)$.

Case 2.b. Suppose u is on the opposite side from $D[c(s); r(s)]$ with respect to the tangent line $\ell(s)$ at $\alpha(s)$, or $u \in \ell(s)$, then the arc $\mathrm{arc}_u[\alpha(s-h), \alpha(s+h)]$ contains points from the interior of C_1 for all $h > 0$ small enough. Again, we have $u \notin \mathcal{P}(A)$.

Case 3. Finally, suppose $\alpha(s) \in \alpha_0$. That is, $D[c(s); r(s)] = H[\alpha(s)]$ is a closed half-plane. Let C_1 be a circle through $\alpha(s)$, tangent to $\ell(s)$, and strictly contained in $D[c(s); r(s)]$ (except for the point $\alpha(s)$). The centre of C_1 is on the ray $\{\alpha(s) - tN(s) : t \geq 0\}$ and if it is close enough to $\alpha(s)$, the entire C_1 is contained in the complement of A (except for the point $\alpha(s)$). Consider the circle C through the points u and $\alpha(s)$, tangent to $\ell(s)$. Circle C is contained entirely in the complement of $D[c(s); r(s)]$, except for the point $\alpha(s)$. The curvature of C is strictly positive, hence for $h > 0$ small enough, the points $\alpha(s-h)$, $\alpha(s+h)$ are outside of C. Thus, $\mathrm{arc}_u[\alpha(s-h), \alpha(s+h)]$ is outside C and crosses C_1.

Suppose now that $\alpha(s)$ is not a smooth point on the boundary of A and $u \notin D[c(s+); r(s+)]$. (The case when $u \notin D[c(s-); r(s-)]$ is analogous.) Since the osculating disk changes in a continuous fashion, we have $u \notin D[c(s+h); r(s+h)]$ for all small enough h. For such small h, $\alpha(s+h)$ is a smooth point on the boundary of A and the above arguments apply. All that shows $\mathcal{P}(A) \subseteq \mathcal{C}(\partial A)$.

For the opposite inclusion, suppose in addition, that ∂A is a bounded set and satisfies Assumption 3.1. Take a point u in $\mathcal{C}(\partial A)$. Consider the Möbius transformation $W(z) = 1/(z-u)$. By Lemma 5.1, every (finite) point on the boundary of $W(A)$ has a non-negative curvature. By Assumption 3.1, at every non-smooth point $W(\alpha(s))$ on the boundary of $W(A)$, the one-sided tangent $T(W(\alpha(s+)))$ is obtained from $T(W(\alpha(s-)))$ by a counter-clock wise rotation at an angle of size less than π. (Here, we use the fact that ∂A is a bounded set. Otherwise, $W(\infty)$ may be a new non-smooth point on the boundary of $W(A)$.) Since the boundary of $W(A)$ is a simple curve, this implies that $W(A)$ is a convex set, that is, $\infty \in \mathcal{P}(W(A))$. Using (4), one concludes that $u \in \mathcal{P}(A)$. □

Proof of Proposition 3.3. (a) We proof this part, only in the case depicted on Figures 3.1A, since the other two cases are analogous. Let C_ϵ^-, for $\epsilon > 0$, be a circle tangent to ∂A at $\alpha(s-)$, to the left of the (oriented) tangent line $\{\alpha(s) + tT(s-) : t \in \mathbb{R}\}$ and radius $r(s-) - \epsilon$. (In the case when $r(s-) = \infty$, we take the radius of C_ϵ^- to be $1/\epsilon$.) Analogously, we let C_ϵ^+ be the circle to the right of the same tangent line with radius $r(s+) - \epsilon$, if finite number, or radius $1/\epsilon$, if $r(s+) = \infty$. The situation is illustrated on Figure 5.1.

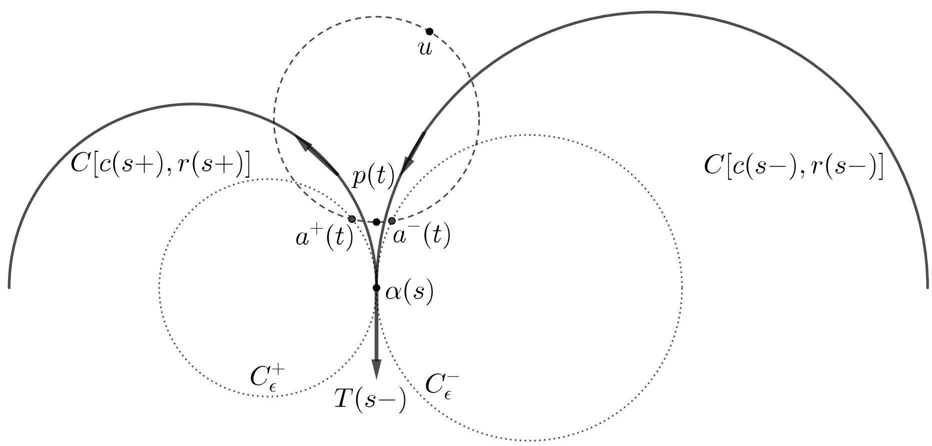


Figure 5.1: Illustrating the proof of Proposition 3.3, part (a).

Since the set A is to the left of its boundary, locally around $\alpha(s)$, the circles C_ϵ^- and C_ϵ^+ are in A. Moreover, if D_ϵ^- and D_ϵ^+ are the closed bounded circular disks determined by these circles, the set $B(\alpha(s), \delta) \cap (D_\epsilon^- \cup D_\epsilon^+)$ is in A for all small $\delta > 0$.

By Theorem 3.2, we have

$$\mathcal{P}(A) \subseteq \mathcal{C}(\partial A) \subseteq D[c(s-); r(s-)] \cap D[c(s+); r(s+)]. \tag{9}$$

The last inclusion holds since the latter two osculating disks are part of the intersection, defining $\mathcal{C}(\partial A)$. Let u be any point in $D[c(s-); r(s-)] \cup D[c(s+); r(s+)]$ and let $p(t) := \alpha(s) + tT(s-)$ be a point on the tangent line at $\alpha(s)$. Consider the circle $C(t)$, through u, $p(t)$, that is perpendicular to the tangent line at $\alpha(s)$.

For $t \in (-\eta, \eta)$, where $\eta > 0$ is small enough, $C(t)$ intersects, C_ϵ^- and C_ϵ^+ at the points $a^-(t), a^+(t)$, respectively. (Since there are multiple intersection points, choose $a^-(t), a^+(t)$ to be those closest to $\alpha(s)$, one on each side of the normal line $\{\alpha(s) + tN(s-) : t \in \mathbb{R}\}$.) If the arc $\mathrm{arc}_u[a^-(t), a^+(t)]$ is entirely in A, for all $t \in (-\eta, \eta)$, then one can conclude that $\alpha(s)$ is an interior point of A, a contradiction. Otherwise, we find an arc $\mathrm{arc}_u[a^-(t), a^+(t)]$, for some t, that is not in A, but has the property that $a^-(t), a^+(t) \in A$. Hence, $u \notin \mathcal{P}(A)$, concluding that $\mathcal{P}(A) = \emptyset$.

(b) We proof this part, only in the case depicted on Figures 3.2A, since the other two cases are analogous. Note that inclusion (9) still holds. But this time

$$D[c(s-); r(s-)] \cap D[c(s+); r(s+)] = \begin{cases} \{\alpha(s) + tT(s-) : t \in \mathbb{R}\} & \text{if } \kappa(s+) = \kappa(s-) = 0, \\ \{\alpha(s)\} & \text{otherwise.} \end{cases}$$

Thus, if $|\kappa(s+)| + |\kappa(s-)| > 0$, then $\mathcal{C}(\partial A) \subseteq \{\alpha(s)\}$.

In general, we have $\mathcal{P}(A) \subseteq \{\alpha(s)\}$. Indeed, let $u \in \mathcal{P}(A)$ be different from $\alpha(s)$. Then, u cannot be a pole for A, or else with $W(z) := 1/(z-u)$ we have that $W(A)$ is a convex set with a non-empty interior, having a non-smooth boundary point $W(\alpha(s))$, at which the tangent vector changes its direction at an angle π. There are no such convex sets.

Thus, if $\alpha(s) \in \mathcal{C}(\partial A)$, the Möbius transformation $W(z) = 1/(z - \alpha(s))$ maps A into an unbounded set $W(A)$. By Lemma 5.1 every point on the boundary of $W(A)$ has a non-negative curvature and the non-smooth points satisfy Assumption 3.1. Hence, $W(A)$ is a convex set and $\mathcal{P}(A) = \mathcal{C}(\partial A) = \{\alpha(s)\}$. If $\alpha(s) \notin \mathcal{C}(\partial A)$, then $\mathcal{P}(A) = \mathcal{C}(\partial A) = \emptyset$. □

6. Appendix

The purpose of this section is to present the proof of the following lemma.

Lemma 6.1. *Suppose $\alpha\colon [0,1] \to \mathbb{R}^2$ is a simple, C^2, regular curve. The circle $C(\alpha(0), \alpha(h), \alpha(\ell))$ converges to the osculating circle $C[c(0); r(0)]$ at $\alpha(0)$ as $\ell > h > 0$ both approach zero in such a way that $|h/(h-\ell)|$ and $|\ell/(h-\ell)|$ stay bounded.*

Proof. Let $(x(s), y(s))$ be a parametrization of the curve α, then

$$|\kappa_\alpha(s)| = \frac{|x'(s)y''(s) - x''(s)y'(s)|}{(x'(s)^2 + y'(s)^2)^{3/2}}.$$

Let (x_1, y_1) be the Cartesian coordinates of the point $\alpha(0)$; let (x_2, y_2) be those of $\alpha(h)$; finally let (x_3, y_3) be those of $\alpha(\ell)$. The equation of the circle passing through $\alpha(0), \alpha(h)$, and $\alpha(\ell)$ is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Expanding along the first row and grouping appropriate terms, one can see that the radius of this circle is

$$r = \sqrt{\Big(\frac{X}{2D}\Big)^2 + \Big(\frac{Y}{2D}\Big)^2 + \frac{B}{D}},$$

where
$$X := \begin{vmatrix} x_1^2+y_1^2 & y_1 & 1 \\ x_2^2+y_2^2 & y_2 & 1 \\ x_3^2+y_3^2 & y_3 & 1 \end{vmatrix}, \quad Y := \begin{vmatrix} x_1^2+y_1^2 & x_1 & 1 \\ x_2^2+y_2^2 & x_2 & 1 \\ x_3^2+y_3^2 & x_3 & 1 \end{vmatrix},$$

$$B := \begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 \\ x_2^2+y_2^2 & x_2 & y_2 \\ x_3^2+y_3^2 & x_3 & y_3 \end{vmatrix}, \quad D := \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Thus,
$$\frac{1}{r} = \frac{2|D|}{\sqrt{X^2+Y^2+BD}}$$
$$= \frac{2|x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2|}{\sqrt{((x_1-x_2)^2+(y_1-y_2)^2)((x_1-x_3)^2+(y_1-y_3)^2)((x_2-x_3)^2+(y_2-y_3)^2)}}.$$

In the last formula, substitute (x_1, y_1) by $(x(0), y(0))$; substitute (x_2, y_2) by the second order Taylor expansion of $(x(s), y(s))$ at $s = 0$

$$(x(0) + x'(0)h + x''(0)\frac{h^2}{2} + o(h^2), y(0) + y'(0)h + y''(0)\frac{h^2}{2} + o(h^2));$$

and substitute (x_3, y_3) by

$$(x(0) + x'(0)\ell + x''(0)\frac{\ell^2}{2} + o(\ell^2), y(0) + y'(0)\ell + y''(0)\frac{\ell^2}{2} + o(\ell^2)).$$

The result is

$$\begin{aligned} D = {} & \frac{1}{2}(x'(0)y''(0) - x''(0)y'(0))(h\ell^2 - h^2\ell) + (x'(0) - y'(0))(ho(\ell^2) - o(h^2)\ell) + \\ & + \frac{1}{2}(x''(0) - y''(0))(h^2o(\ell^2) - o(h^2)\ell^2), \\ & (x_1-x_2)^2 + (y_1-y_2)^2 = (x'(0)^2 + y'(0)^2)h^2 + o(h^2), \\ & (x_1-x_3)^2 + (y_1-y_3)^2 = (x'(0)^2 + y'(0)^2)\ell^2 + o(\ell^2), \\ & (x_2-x_3)^2 + (y_2-y_3)^2 = (x'(0)^2 + y'(0)^2)(h-\ell)^2 + \end{aligned}$$

$$+ (x'(0)x''(0)+y'(0)y''(0))(h-\ell)(h^2-\ell^2)+2(x'(0)+y'(0))(h-\ell)(o(h^2)-o(\ell^2))$$
$$+ \frac{1}{4}(x''(0)^2 + y''(0)^2)(h^2 - \ell^2)^2 + (x''(0) + y''(0))(h^2 - \ell^2)(o(h^2) - o(\ell^2))$$
$$+ 2(o(h^2) - o(\ell^2))^2.$$

Substitute everything in the formula for $1/r$. Divide the numerator and the denominator by $|h\ell(h-\ell)|$. Finally, take the limit as h and ℓ approach zero in such a way that $|h/(h-\ell)|$ and $|\ell/(h-\ell)|$ stay bounded. Elementary manipulations of the little-o functions show that the quantity $1/r$ converges to $|\kappa_\alpha(0)|$. This completes the proof. □

References

[1] A. Kneser: *Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nichteuklidischen Geometrie*, in: Festschrift zum 70. Geburtstag von H. Weber, Leipzig-Berlin (1912) 170–180.

[2] G. Pólya, G. Szegö: *Problems and Theorems in Analysis II*, Springer, Berlin (1976).

[3] Q. I. Rahman, G. Schmeisser: *Analytic Theory of Polynomials*, Oxford University Press, New York (2002).

[4] R. T. Rockefellar: *Convex Analysis*, Princeton University Press, Princeton (1970).

[5] B. Sendov, H. Sendov, C. Wang: *Polar convexity and critical points of polynomials*, J. Convex Analysis 26(2) (2019) 635–660.

[6] P. G. Tait: *Note on the circles of curvature of a plane curve*, Proc. Edinburgh Math. Soc. 14 (1896) 403.

[7] V. A. Toponogov: *Differential Geometry of Curves and Surfaces*, Birkhäuser, Boston (2006).