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Existence of Solutions for Sweeping Processes with Local Conditions

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Sweeping processes, introduced by J. J. Moreau, have been studied significantly in recent years due to their interesting applications in practice. However, most of the researches use the global Hausdorff distance to describe the movement of the moving set, which is probably not satisfied for unbounded cases. In this paper, we study the existence of solutions for sweeping processes by using only local excess condition. The result obtained can be applied to analyze a class of time-varying evolution variational inequalities.

Keywords: Existence of solutions, sweeping processes, local excess condition, time-varying evolution variational inequalities.

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1. Introduction

Sweeping processes were proposed and thoroughly studied by J. J. Moreau in the seventies [20, 21, 22, 23, 24, 25], which can be written as follows

$$\begin{cases} \dot{u}(t) \in -N_{C(t)}(u(t)) \text{ a.e. } t \in [0,T], \\ u(0) = u_0 \in C(0), \end{cases} \tag{1}$$

where $N_{C(t)}(\cdot)$ denotes the normal cone operator of the closed, convex set $C(t)$ in the sense of convex analysis. It is known that if the moving set $C(\cdot)$ varies in a "continuous" way in the sense that

$$d_H(C(t), C(s)) \leq |a(t) - a(s)|, \quad \forall\, 0 \leq s, t \leq T, \tag{2}$$

or

$$\operatorname{exc}(C(s), C(t)) \leq |a(t) - a(s)|, \quad \forall\, 0 \leq s \leq t \leq T, \tag{3}$$

where $\operatorname{exc}(A,B) := \sup_{x\in A} d(x,B)$ denotes the excess of A over B, furthermore $d_H(A,B) = \max\{\operatorname{exc}(A,B), \operatorname{exc}(B,A)\}$ denotes the Hausdorff distance and $a\colon [0,T] \to [0,+\infty[$ is an absolutely continuous function, then (1) has a unique absolutely continuous solution. If a is Lipschitz continuous (or right-continuous bounded variation) then the unique solution is also Lipschitz continuous (bounded variation, respectively). The original motivation is to model quasi-static evolution in elastoplasticity, contact dynamics, friction dynamics, granular material

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(see, e.g., [24, 25]). Nowadays, abundant applications of sweeping processes can be also found in nonsmooth mechanics, convex optimization, mathematical economics, dynamic networks, switched electrical circuits, modeling of crowd motion (see, e.g., [1, 18, 19, 28, 29, 30] and references therein).

Existence and uniqueness of solutions for such systems and their classical extensions (subjected to perturbation forces, non-convex state dependent moving set $C(t,x)$, second order sweeping processes, considered in infinite dimensional Banach spaces, stochastic sweeping processes ...) have been considered by many authors in the literature (see, e.g., [4, 5, 7, 8, 16, 18, 29]). However, most of the works used global Hausdorff distance as in (2) which is very limited for unbounded sets since the Hausdorff distance of two unbounded set can be infinity. The main motivation of our work is to study the way C moves by using only the local excess condition.

In [17], Kenmochi gave a sufficient condition for the existence and uniqueness of solutions for the nonlinear evolution equations of the form

$$\dot{u}(t) \in -\partial\phi^t(u(t)) + f(t), \quad \text{a.e. } t \in [0,T], \tag{4}$$

where $\partial\phi^t$ denotes the subdifferential operator of convex function ϕ^t. Recently, Tolstonogov [26] applied Kenmochi's work for $\phi^t = I_{C(t)}$, the indicator function of $C(t)$, to provide a kind of sufficient condition for problem (1): for each $r > 0$, there exists a function $a_r \in W^{1,2}([0,T];\mathbb{R})$ such that for all $0 \le s \le t \le T$, for all $x \in C(s) \cap r\mathbb{B}$ (where $\mathbb{B}$ denotes the unit ball), there exists $y \in C(t)$ such that

$$\|x - y\| \le |a_r(t) - a_r(s)|,$$

which is equivalent to

$$\operatorname{exc}_r(C(s), C(t)) := \operatorname{exc}(C(s) \cap r\mathbb{B};\, C(t)) \le |a_r(t) - a_r(s)| \quad \forall\, 0 \le s \le t \le T. \tag{5}$$

Tolstonogov realized that set-valued mappings whose values are hyperplanes, half-spaces may satisfy (5) rather than (2). It is natural to ask whether the result is still true when a_r only belongs to $W^{1,1}([0,T];\mathbb{R})$, the space of absolutely continuous functions. In [13], Colombo et al. assumed that the function $v(t) := \operatorname{proj}_{C(t)}(0)$, the unique projection of the origin onto $C(t)$, is absolutely continuous and for each $r > 0$ and $\epsilon > 0$ there exists $\delta = \delta(r,\epsilon) > 0$ such that

$$\sum_{i=1}^{l} \max_{z \in K(\alpha_i) \cap r\mathbb{B}} d(z, K(\beta_i)) \le \epsilon, \tag{6}$$

for every collection of mutually disjoint subintervals

$$\Big\{[\alpha_i, \beta_i] | i = 1, \dots, l\Big\} \text{ of } [0,T] \text{ with } \sum_{i=1}^{l} |\beta_i - \alpha_i| \le \delta$$

where $K(t) := C(t) - v(t)$ for all $0 \le s \le t \le T$. We observe that the absolute continuity of the function $v(\cdot)$ can be improved and one can consider directly the original moving set C instead of the translated set K. Our first result Theorem 3.1

only requires an absolutely continuous selection of C (which is also a necessary condition) and for each $r > 0$, there exists an absolutely continuous function $a_r\colon [0,T] \to \mathbb{R}$ such that

$$\operatorname{exc}_r(C(s), C(t)) \leq |a_r(t) - a_r(s)| \ \ \forall\, 0 \leq s \leq t \leq T. \tag{7}$$

However, the existence of an absolutely continuous selection of C is not easy to verify in general. It motivates us to propose another practical version. Indeed Theorem 3.4 states that if the minimal norm element $C^0(t)$ is uniformly bounded by some $M > 0$ and for each $r > M$, there exists an absolutely continuous function $a_r\colon [0,T] \to \mathbb{R}$ such that for all $0 \leq s \leq t \leq T$, we have

$$\operatorname{exc}^s_r(C(s), C(t)) := \operatorname{exc}(C(s) \cap r\mathbb{B}, C(t) \cap r\mathbb{B}) \leq |a_r(t) - a_r(s)|, \tag{8}$$

then for given initial condition, problem (1) has a unique absolutely continuous solution. Obviously the boundedness of $C^0(\cdot)$ is a necessary condition and can not be implied by the excess condition (8). It is interesting to say that we can convert the unbounded problem to be the bounded one. Note that the convexity of C can be relaxed and one can add perturbation terms with state-dependent moving set to form various variants similarly as in the literature [4, 5, 8, 14, 16, 18, 29, 30]. Finally, we provide an application in Section 4 to study for the first time the existence of solutions for a class of time-varying evolution variational inequalities by using only local conditions.

2. Notation and preliminaries

We begin with some notation used in the paper. Let H be a real Hilbert space. Denote by $\langle \cdot, \cdot \rangle$, $\|\cdot\|$ the *scalar product* and the corresponding *norm* in H. Denote by $\mathbb{B}$ the *closed unit ball* in H. The *distance* from a point x to a set C is denoted by $d(x, C)$. If C is nonempty closed and convex, then for each $x \in H$, there exists uniquely a point $y \in C$ which is nearest to x, denoted by $\operatorname{proj}(C; x)$ or $\operatorname{proj}_C(x)$.

The *minimal norm element* of C is denoted by $C^0 := \operatorname{proj}(C; 0)$. The *normal cone* of C is given by

$$N_C(x) := \{p \in H : \langle p, y - x\rangle \leq 0, \text{ for all } y \in C\}.$$

The *r-excess* and the *symmetric r-excess* of A over B are defined respectively by

$$\operatorname{exc}_r(A, B) := \operatorname{exc}(A \cap r\mathbb{B}, B) \ \text{ and } \ \operatorname{exc}^s_r(A, B) := \operatorname{exc}(A \cap r\mathbb{B}, B \cap r\mathbb{B}),$$

where $\operatorname{exc}(A, B) := \sup_{x \in A} d(x, B)$ is the *excess of A over B*. Conventionally, $\operatorname{exc}(A, B)=0$ if A is empty. A linear bounded operator $P\colon H \to H$ is called *coercive* if there exists $\alpha > 0$ such that

$$\langle Px, x\rangle \geq \alpha \|x\|^2, \ \forall\, x \in H.$$

Let $T > 0$ be given. A set-valued map $C\colon [0,T] \rightrightarrows H$ is said to have an *absolutely continuous selection* if there exists an absolutely continuous function

$x\colon [0,T] \to H$ such that $x(t) \in C(t)$ for all $t \in [0,T]$. It is easy to see that if (1) has an absolutely continuous solution then C has an absolutely continuous selection. Similarly, C has a bounded selection if there exist $M > 0$ and a mapping $x\colon [0,T] \to H$ such that $x(t) \in C(t)$ and $\|x(t)\| \leq M$ for all $t \in [0,T]$. Clearly C has a bounded selection if and only if minimal norm map C^0 is bounded.

3. Main results

In this section, we provide an existence result for sweeping processes by using local conditions with a comparison with previous works in the literature. The first result Theorem 3.1 requires an absolute continuous selection of the moving set C which locally varies in an absolute continuous way. The main idea of the proof is to prove an existence of a local solution and then extend it globally. On the other hand, it is not easy to verify the existence of an absolute continuous selection of C in general. It motivates us to give a similar result Theorem 3.4 with only checking an bounded selection. Let us first make the following assumption.

Assumption $(\mathcal{H}_1)$:

(a) The given set-valued map $C\colon [0,T] \rightrightarrows H$ has an absolutely continuous selection.

(b) For all $t \in [0,T]$, $C(t)$ is non-empty, closed, convex and for each $r > 0$ there exists an absolutely continuous function $a_r\colon [0,T] \to [0,+\infty[$ such that

$$\mathrm{exc}_r(C(s), C(t)) \leq |a_r(t) - a_r(s)|, \quad \text{for all } 0 \leq s \leq t \leq T. \tag{9}$$

Now we are ready for the main results.

Theorem 3.1. *Let the assumption $(\mathcal{H}_1)$ hold. Then, for given initial condition $u_0 \in C(0)$, there exists uniquely a solution u in the following sense:*

(1) *Process (1) is satisfied for a.e. $t \in [0,T]$,*

(2) $u(0) = u_0$ *,*

(3) *u is absolutely continuous.*

Proof. The uniqueness of solutions can be seen easily due to the monotonicity of the normal cone of a convex set. For the existence, we divide the proof into 2 steps: step 1 for the existence of a local solution and step 2 for extending to a global solution. Without loss of generality, for each $r > 0$, we assume that the absolutely continuous function a_r is nondecreasing since we can replace a_r by $b_r\colon [0,T] \to [0,+\infty[$ defined by $b_r(t) = \int_0^t |\dot{a}(s)| ds$ to obtain such property.

Step 1: Let $\rho := \|u_0\| + 1$ and $v := a_\rho$. We can choose $0 < T_1 \leq T$ such that $\rho_0 := \|u_0\| + v(T_1) - v(0) < \rho$. Then

$$\mathrm{exc}_\rho(C(s), C(t)) \leq v(t) - v(s), \quad \text{for all } 0 \leq s \leq t \leq T_1. \tag{10}$$

We will apply [28, Corollary 4.5] for the interval $[0,T_1]$. It remains to check that for all $0 < t_1 < t_2 < \ldots < t_k < T_1$, we have

$$\|\big(\mathrm{proj}_{C(t_k)} \circ \ldots \mathrm{proj}_{C(t_1)}\big)(u_0)\| \leq \rho_0. \tag{11}$$

Let $$u_i := \big(\mathrm{proj}_{C(t_i)} \circ \ldots \mathrm{proj}_{C(t_1)}\big)(u_0),\ i \in \{1, 2, \ldots, k\}.$$

We prove by induction that

$$\|u_i\| \leq \rho_0 \text{ for all } i \in \{0, 1, 2, \ldots, k\}. \tag{12}$$

Indeed, we have $\|u_0\| \leq \rho_0 < \rho$. Thus,

$$\|u_1 - u_0\| = d(u_0, C(t_1)) \leq \mathrm{exc}_\rho(C(0), C(t_1)) \leq v(t_1) - v(0),$$

which implies that

$$\|u_1\| \leq \|u_0\| + v(t_1) - v(0) \leq \|u_0\| + v(T_1) - v(0) = \rho_0.$$

Suppose that $\|u_j\| \leq \rho_0$, for all $j = 0, 1, \ldots, i$ $(1 \leq i \leq k-1)$. We have

$$\begin{aligned}\|u_{i+1} - u_0\| &\leq \sum_{j=0}^{i} \|u_{j+1} - u_j\| = \sum_{j=0}^{i} d(u_j, C(t_{j+1})) \leq \sum_{j=0}^{i} \mathrm{exc}_\rho(C(t_j), C(t_{j+1})) \\ &\leq \sum_{j=0}^{i} (v(t_{j+1}) - v(t_j)) = v(t_{i+1}) - v(0) \leq v(T_1) - v(0).\end{aligned}$$

Thus

$$\|u_{i+1}\| \leq \|u_0\| + v(T_1) - v(0) = \rho_0.$$

By induction we obtain (12) and in particular we have (11). Using [28, Corollary 4.5], one deduce that there exists a unique solution of (1) on $[0, T_1]$.

Step 2: Using similar arguments in Step 1 with the new initial time at T_1, we obtain a unique solution of (1) on $[T_1, T_2]$ for some $T_2 > T_1$ and so on. Consequently, one can define the maximal solution u on some interval I where $I \subset [0, T]$. If $I \neq [0, T]$ then I must be $[0, T_{max}[$ for some $0 < T_{max} \leq T$. Suppose that $\limsup_{t \to T_{max}} \|u(t)\| = +\infty$. For almost all $t \in [0, T_{max}[$, one has $\dot{u}(t) \in -N_{C(t)} u(t)$ which implies

$$\langle \dot{u}(t), u(t) - a(t) \rangle \leq 0, \tag{13}$$

where $a\colon [0, T] \to H$ is an absolutely continuous selection of C by using ($\mathcal{H}_1$-a). Integrating the inequality above from 0 to t $(0 < t < T_{max})$ and using integration by parts, we get

$$\frac{1}{2}(\|u(t)\|^2 - \|u_0\|^2) \leq \langle u(t), a(t) \rangle - \langle u_0, a(0) \rangle - \int_0^t \langle u(s), \dot{a}(s) \rangle ds. \tag{14}$$

Since $\limsup_{t \to T_{max}} \|u(t)\| = +\infty$, one can find a strictly increasing sequence of positive numbers (T_n) such that $T_n \to T_{max}$, $\|u(T_n)\| = \max_{s \in [0, T_n]} \|u(s)\|$ and $\lim_{n \to +\infty} \|u(T_n)\| = +\infty$. Let $t = T_n$ in (14) and divide both sides by $\|u(T_n)\|$, one deduces that

$$\begin{aligned}\|u(T_n)\| - \frac{\|u_0\|^2}{\|u(T_n)\|} &\leq 2(\|a(T_n)\| + \|a(0)\| + \int_0^{T_n} \|\dot{a}(s)\| ds) \\ &\leq 4 \sup_{t \in [0, T]} \|a(t)\| + 2 \int_0^T \|\dot{a}(s)\| ds < +\infty,\end{aligned} \tag{15}$$

a contradiction by letting $n \to +\infty$. Thus $\limsup_{t \to T_{max}} \|u(t)\| < +\infty$ and we can find some real number $R > 0$ such that

$$\sup_{t \in [0, T_{max}[} \|u(t)\| < R. \tag{16}$$

Let $\rho := R + 1$. By using ($\mathcal{H}_1$-b), there exists a nondecreasing absolutely continuous function $v := a_\rho \colon [0, T] \to [0, +\infty[$ such that

$$\operatorname{exc}_\rho(C(s), C(t)) \le v(t) - v(s), \quad \text{for all } 0 \le s \le t \le T. \tag{17}$$

Since v is absolutely continuous we can find some positive real constant $\delta < T$ such that $v(t + \delta) - v(t) \le 1$ for all $t \in [0, T - \delta]$. Let $T_k = k\delta$ and let m be the smallest positive integer such that $T_m \ge T_{max}$.

We consider the following differential inclusion

$$\begin{cases} \dot{y}(t) \in -N_{C(t)}(y(t)) \text{ a.e. } t \in [T_{m-1}, T_m], \\ y(T_{m-1}) = u(T_{m-1}) \in C(T_{m-1}) \cap R\mathbb{B}. \end{cases} \tag{18}$$

Let $\rho_0 := \|u(T_{m-1})\| + v(T_m) - v(T_{m-1}) < R + 1 = \rho$. Then we can prove similarly as in Step 1 using [28, Corollary 4.5] to obtain a unique solution $y(\cdot)$ of (18). Define the function $\tilde{u} \colon [0, T_m] \to H$ as follows

$$\tilde{u}(t) = \begin{cases} u(t), & t \in [0, T_{m-1}], \\ y(t), & t \in [T_{m-1}, T_m]. \end{cases} \tag{19}$$

Then $\tilde{u}(\cdot)$ is the unique solution of problem (1) on $[0, T_{max}]$, which is a contradiction. Consequently, we must have $I = [0, T]$ and the conclusion follows. □

Remark 3.2. (i) Note that ($\mathcal{H}_1$-a) is a necessary condition for the existence of solutions of (1) since if (1) has a solution $u(\cdot)$ then $u(\cdot)$ is an absolutely continuous selection of C. Therefore the assumption ($\mathcal{H}_1$) is weaker than the one used in [26] since $W^{1,2}([0, T]; \mathbb{R}) \subset W^{1,1}([0, T]; \mathbb{R})$, the space of absolutely continuous functions. In addition, it is also more flexible than the one used in [13, Theorem 2.1] where the authors supposed that the function $v(t) := \operatorname{proj}_{C(t)}(0)$ is absolutely continuous and ($\mathcal{H}_1$-b) is satisfied for the translated set $K(t) = C(t) - v(t)$.

(ii) There are some other existence results by using local conditions, for examples [4] (when C moves absolutely continuously) and [28] (when C moves in a ways of bounded variation). Particularly in [28, Corollary 4.6], L. Thibault observed that for given initial point $u_0 \in C(0)$, it is sufficient to find a positive Radon measure μ on $[0, T]$, a real $\rho_0 \ge \|u_0\| + \mu(]0, T])$, and an extended real $\rho > \rho_0$ such that, for all $s, t \in [0, T]$ with $s \le t$

$$\operatorname{exc}_\rho(C(s), C(t)) \le \mu(]s, t]). \tag{20}$$

However, we can see that ρ and μ are interdependence. Thus it is not easy to find ρ and μ in general. On the contrary, in Theorem 3.1, the function a_r depends on r, but r does not depend on a_r.

(iii One can use similar arguments for the case of Lipschitz continuous perturbation.

(iv) In general, it is not easy to verify the existence of an absolutely continuous selection of the moving set C. It motivates us to provide another practical version without checking absolutely continuous selections but only bounded selections.

Assumption ($\mathcal{H}_2$) :

(a) $C^0(t) := \mathrm{proj}_{C(t)}(0)$ is uniformly bounded by some constant $M > 0$ for all $t \in [0, T]$.

(b) For all $t \in [0, T]$, $C(t)$ is non-empty, closed, convex and for each $r > M$ there exists an absolutely continuous function $b_r\colon [0, T] \to [0, +\infty[$ such that for all $0 \le s \le t \le T$, one has

$$\mathrm{exc}^s_r(C(s), C(t)) = \mathrm{exc}(C(s) \cap r\mathbb{B}, C(t) \cap r\mathbb{B}) \le |b_r(t) - b_r(s)|. \tag{21}$$

Lemma 3.3. *Let be given $t, s \in [0, T]$ and $s \le t$. Suppose that $\|C^0(t)\| \le M$. If $x \in C(s) \cap r\mathbb{B}$ for some $r > 0$ then*

$$d(x, C(t)) = d(x, C(t) \cap (2r + M)\mathbb{B}). \tag{22}$$

In particular, one has

$$\mathrm{exc}_r(C(s), C(t)) = \mathrm{exc}_r(C(s), C(t) \cap (2r + M)\mathbb{B}). \tag{23}$$

Proof. Let $y := \mathrm{proj}(x, C(t))$. Let $z := C^0(t)$. Then

$$\|y\| \le \|x\| + \|x - y\| \le r + \|x - z\| \le r + \|x\| + \|z\| \le 2r + M,$$

and the conclusion follows. □

Theorem 3.4. *Let the assumption ($\mathcal{H}_2$) hold. Then, for given initial condition $u_0 \in C(0)$, problem* (1) *has a unique solution on* $[0, T]$.

Proof. Let us check that the assumption ($\mathcal{H}_1$) is satisfied. Given $u_0 \in C(0)$ and let $r = M + \|u_0\|, a = b_r$ and $D(t) = C(t) \cap r\mathbb{B}$ for all $t \in [0, T]$. Then $D(t)$ is non-empty, closed and convex set for all $t \in [0, T]$. In addition, $u_0 \in D(0)$ and for all $0 \le s \le t \le T$ one has

$$\mathrm{exc}(D(t), D(s)) \le |a(t) - a(s)|. \tag{24}$$

Then classically (see, e.g., [22]) we obtain an absolutely continuous solution $y(\cdot)$ of the differential inclusion

$$\begin{cases} \dot{y}(t) \in -N_{D(t)}(y(t)) \text{ a.e. } t \in [0, T], \\ y(0) = u_0 \in D(0). \end{cases} \tag{25}$$

In particular, $y(t) \in D(t) \subset C(t)$ for all $t \in [0, T]$ which shows that C has an absolutely continuous selection. Next we verify that ($\mathcal{H}_1$-b) also holds.

Indeed, for all $r>0$, by using Lemma 3.3, we have

$$\begin{aligned}\operatorname{exc}_r(C(s),C(t)) &= \operatorname{exc}(C(s)\cap r\mathbb{B}, C(t)\cap(2r+M)\mathbb{B})\\ &\le \operatorname{exc}(C(s)\cap(2r+M)\mathbb{B}, C(t)\cap(2r+M)\mathbb{B})\\ &\le \operatorname{exc}^s_{2r+M}(C(s),C(t)) \le |b_{2r+M}(t)-b_{2r+M}(s)|.\end{aligned}$$

Then the conclusion follows by putting $a_r := b_{2r+M}$. □

Remark 3.5. (i) The assumption ($\mathcal{H}_2$-a) is a natural necessary condition which is easy to check. In particular, it is more flexible than the uniform boundedness of

$$\Big(\operatorname{proj}_{C(t_k)}\circ\ldots\circ\operatorname{proj}_{C(t_1)}\Big)(u_0)$$

for all $0\le t_1<\ldots<t_k\le T$, which was used in [28, Theorem 4.3].

(ii) By using the symmetrically local excess in Theorem 3.4, it is interesting to say that we can convert unbounded problems to be bounded ones.

4. Application

In this section, we apply the result obtained above to study the existence of solutions for a class of evolution variational inequalities (also called Lur'e dynamical systems), which consists of ordinary differential equations describing the motion of the state variable and variational inequalities representing some constraints or some relations fulfilled by the state variable. It is known that evolution variational inequalities appear in many optimization problems, control theory, engineering and applied mathematics (see, e.g., [2, 3, 9, 10, 11, 12, 27] and references therein). Here we are interested in the case when the variational inequalities is time-varying, in particular involves with the normal cone of a moving set.

In detail, let be given some linear bounded operators $A\colon H_1\to H_1$, $B\colon H_2\to H_1$, $G\colon H_1\to H_2$ and a set-valued map $C(t)\colon[0,T]\rightrightarrows H_2$ with non-empty closed convex values satisfying $\mathcal{R}(G)\cap C(t)\neq\emptyset$ for all $t\in[0,T]$, where H_1,H_2 are two Hilbert spaces and $\mathcal{R}(G)$ denotes the range of G. Then, for some initial point $x_0\in H_1$ such that $Gx_0\in C(0)$, we want to find an absolutely continuous function $x(\cdot)$ defined on $[0,T]$ such that

$$\begin{cases}\dot{x}(t)=Ax(t)+B\lambda(t) \text{ a.e. } t\in[0,T],\\ \lambda(t)\in -N_{C(t)}(Gx(t)),\ t\ge 0;\\ x(0)=x_0.\end{cases}\tag{26}$$

We can rewrite the system (26) in the form of first order differential inclusion as follows

$$\dot{x}(t)\in Ax(t)-BN_{C(t)}(Gx(t))\ \ a.e.\ \ t\ge 0.\tag{27}$$

Suppose that there exists a coercive symmetric linear bounded operator P such that $PB=G^T$, where G^T denotes the adjoint operator of G. This kind of assumption is largely used in the literature (see, e.g., [9, 10, 11, 12, 27]). By letting

$z(t) = Rx(t)$ where R is the unique positive symmetric linear bounded operator such that $P = R^2$, classically we can transform (27) into new coordinates

$$\dot{z}(t) \in RAR^{-1}z(t) - R^{-1}G^T N_{C(t)}(GR^{-1}z(t)) = f(z(t)) - N_{D(t)}(z(t)) \;\; a.e.\;\; t \geq 0,$$

where $f := RAR^{-1}$ is Lipschitz continuous and $D(t) := \{z \in H_1 : GR^{-1}z \in C(t)\}$ is a non-empty, closed and convex set. In general, $D(t)$ or $C(t)$ can be unbounded. Thus it is quite limited if we use the global Hausdorff distance (see, e.g., [12, 27]) since the Hausdorff distance of unbounded sets can equal infinity. Our work can be considered as the first one to study this kind of systems by only using local conditions. Without loss of generality, one can assume that P is the identity operator, i.e., $B = G^T$.

Assumption ($\mathcal{H}_3$) :

(a) There exists an absolutely continuous selection of $C(\cdot) \cap \mathcal{R}(G)$.

(b) For all $t \in [0,T]$, $C(t)$ is non-empty, closed, convex and for each $r > 0$ there exists an absolutely continuous function $a_r\colon [0,T] \to [0,+\infty[$ such that

$$\mathrm{exc}_r(C(t)\cap\mathcal{R}(G), C(s)\cap\mathcal{R}(G)) \leq |a_r(t) - a_r(s)|, \;\; \text{for all } 0 \leq t \leq s \leq T. \tag{28}$$

(c) The range of G^TG is closed.

Theorem 4.1. *Let the assumption ($\mathcal{H}_3$) hold. Then for given $x_0 \in H_1$ such that $Gx_0 \in C(0)$, the problem (26) has a unique solution.*

Proof. We plan to apply Theorem 3.1 and Remark 1 to obtain the existence and uniqueness result.

Let us first check that $D(\cdot) = G^{-1}C(\cdot), t \in [0,T]$ has an absolutely continuous selection. Indeed, from ($\mathcal{H}_3$-a), there exists an absolutely continuous function $z\colon [0,T] \to H_2$ such that $z(t) \in C(t) \cap \mathcal{R}(G)$. Thus for each $t \in [0,T]$, one can find some $y(t) \in H_1$ such that $z(t) = Gy(t)$ and hence $y(t) \in D(t)$. We can suppose that $y_t \in \mathcal{R}(G^TG)$ since if not, we replace y_t by the projection of y_t onto $\mathcal{R}(G^TG)$ due to the fact that $\ker(G^TG) = \ker(G)$, where $\ker(G)$ denotes the kernel of G. Using ($\mathcal{H}_3$-c) and the fact that

$$\langle G^TGx, x\rangle = \langle Gx, Gx\rangle \geq 0, \;\; \forall x \in H_1,$$

there exists $c > 0$ (see, e.g., [15, Proposition 1.2]) such that

$$\langle G^TGy, y\rangle \geq c\|y\|^2 \;\; \forall\, y \in \mathcal{R}(G^TG).$$

Consequently, for all $t, s \in [0,T]$, one has

$$\begin{aligned} c\|y(t) - y(s)\|^2 &\leq \langle G^TG(y(t) - y(s)), y(t) - y(s)\rangle \\ &\leq \|G^T\|\|z(t) - z(s)\|\|y(t) - y(s)\|, \end{aligned}$$

which implies that
$$\|y(t) - y(s)\| \leq \frac{\|G^T\|}{c}\|z(t) - z(s)\|. \tag{29}$$

Consequently, $y(\cdot)$ is an absolutely continuous selection of $D(\cdot)$.

Next, for each $r > 0$, one wants to estimate the term $\mathrm{exc}_r(D(t), D(s))$ for all $0 \le t \le s \le T$. Let $x_t \in D(t) \cap r\mathbb{B}$ and $x_s := \mathrm{proj}(D(s), x_t)$. Let us decompose $x_t = x_t^1 + x_t^2$, $x_s = x_s^1 + x_s^2$ where $x_t^1, x_s^1 \in \mathcal{R}(G^T G)$ and $x_t^2, x_s^2 \in \ker(G^T G) = \ker(G)$. Then $x_t^1 \in D(t) \cap r\mathbb{B}, x_s^1 \in D(s)$. One has

$$d^2(x_t, D(s)) = \|x_t - x_s\|^2 = \|x_t^1 - x_s^1 + x_t^2 - x_s^2\|^2 = \|x_t^1 - x_s^1\|^2 + \|x_t^2 - x_s^2\|^2. \quad (30)$$

Suppose that $x_t^2 - x_s^2 = d \in \ker(\mathrm{G}) \setminus \{0\}$. Let $x_s^* := x_s^1 + x_s^2 + d \in D(s)$. Then

$$\|x_t - x_s^*\|^2 = \|x_t^1 - x_s^1\|^2 + \|x_t^2 - x_s^2 - d\|^2 = \|x_t^1 - x_s^1\|^2 < d^2(x_t, D(s)),$$

which is a contradiction. Thus we must have $x_t^2 = x_s^2$ and hence

$$d(x_t, D(s)) = \|x_t - x_s\| = \|x_t^1 - x_s^1\| = d(x_t^1, D(s)). \quad (31)$$

Let $w_t := Gx_t^1 = Gx_t$ and $w_s := \mathrm{proj}(C(s) \cap \mathcal{R}(G), w_t)$ then $w_t \in C(t) \cap \mathcal{R}(G) \cap r_1\mathbb{B}$ where $r_1 := \|G\| r$ and

$$\begin{aligned} \|w_t - w_s\| &= d(w_t, C(s) \cap \mathcal{R}(G)) \\ &\le \mathrm{exc}_{r_1}(C(s) \cap \mathcal{R}(G), C(t) \cap \mathcal{R}(G)) \le |a_{r_1}(t) - a_{r_1}(s)|, \quad (32) \end{aligned}$$

where the last inequality is due to ($\mathcal{H}_3$-b). Since $w_s \in C(s) \cap \mathcal{R}(G)$, we can find some $\bar{x}_s \in D(s) \cap \mathcal{R}(G^T G)$ such that $w_s = G\bar{x}_s$. Similarly as in (29) and using (31), we have

$$\begin{aligned} \|w_t - w_s\| &= \|Gx_t^1 - G\bar{x}_s\| \ge \frac{c}{\|G^T\|}\|x_t^1 - \bar{x}_s\| \\ &\ge \frac{c}{\|G^T\|}\|x_t^1 - x_s^1\| = \frac{c}{\|G^T\|} d(x_t, D(s)). \quad (33) \end{aligned}$$

Since x_t is arbitrary in $D(t) \cap r\mathbb{B}$, from (32) and (33), we deduce that

$$\mathrm{exc}_r(D(t), D(s)) \le \frac{\|G^T\|}{c}|a_{r_1}(t) - a_{r_1}(s)| = |b_r(t) - b_r(s)|$$

where $b_r(\cdot) := \frac{\|G^T\|}{c} a_{r_1}(\cdot)$. The proof is therefore complete. $\square$

Remark 4.2. (i) Note that the assumption ($\mathcal{H}_3$-a) is also a necessary condition since if the problem (26) has an absolutely continuous $x(\cdot)$ then $Gx(\cdot)$ is an absolutely continuous selection of $C(\cdot) \cap \mathcal{R}(G)$.

(ii) Assumption ($\mathcal{H}_3$-c) is satisfied automatically in finite dimensional spaces.

(iii) We don't need to assume that $C(t) \subset \mathcal{R}(G)$ for all $t \in [0, T]$ as in [27]. Furthermore, in ($\mathcal{H}_3$-b), we only use the local Hausdorff distance for smaller set $C(t) \cap \mathcal{R}(G)$ rather than the whole original set $C(t)$. Clearly, if $C(t) \subset \mathcal{R}(G)$ then $C(t) \cap \mathcal{R}(G) = C(t)$.

5. Conclusions

In this paper, we provide an existence result for sweeping processes by using local excess condition which allows unbounded moving sets. Roughly speaking, now we can convert the unbounded problem into the bounded one. The result obtained is used to study a class of time-varying evolution variational inequalities. It would be interesting to consider a local version for differential inclusions associated with time-varying maximal monotone operators. It is out of scope of the current paper and will be the subject of a future project.

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