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Characterization of Approximate Birkhoff-James Orthogonality Sets in a Banach Space*

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There are two notions of approximate Birkhoff-James orthogonality sets in a Banach space. A geometric study of them has recently been done by D. Sain, K. Paul and A. Mal [*On approximate Birkhoff-James orthogonality and normal cones in a normed space*, J. Convex Analysis 26 (2019) 341–351]. We here characterize completely one of these sets in terms of normal cones. We obtain a uniqueness criterion between approximate orthogonality sets and normal cones in a Banach space. We also investigate the action of a bounded linear operator on approximate Birkhoff-James orthogonality sets.

Keywords: Approximate Birkhoff-James orthogonality, normal cone, smoothness, strictly convex space.

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1. Introduction

Birkhoff-James orthogonality plays a very important role in the study of geometry of Banach spaces. This notion of orthogonality enlightens the intricacy of Banach space, the norm of which is not necessarily induced by an inner product. Due to its importance, Birkhoff-James orthogonality has been generalized in [2, 4] to introduce the notion of approximate Birkhoff-James orthogonality. The purpose of this paper is to explore approximate Birkhoff-James orthogonality from the geometric point of view, the study of which was initiated by Sain et al. in [9].

Let X denote a real Banach space. Let B_X and S_X denote the *unit ball* and the *unit sphere* of X, respectively: $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$.

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$B(X)$ denotes the *space of all bounded linear operators* on X. The space X is said to be *strictly convex* if $x, y \in S_X$ and $y \neq \pm x \Rightarrow \|tx + (1-t)y\| < 1$ for all $t \in (0,1)$. Let X^* denote the *dual space* of X. For $x \in X$, let $J(x)$ denote the set of all *supporting functionals* of x, i.e., $J(x) = \{\phi \in S_{X^*} : \phi(x) = \|x\|\}$. An element $x \in S_X$ is said to be a *smooth point* of X if $J(x)$ is singleton. The space X is said to be a *smooth space* if every element of S_X is smooth. For x, $y \in X$, x is said to be *orthogonal* to y in the sense of Birkhoff-James [1, 5], written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all real scalars λ. In [5, Th. 2.1], James proved that for $x, y \in X$ and $x \neq 0$, $x \perp_B y$ if and only if there exists $\phi \in J(x)$ such that $\phi(y) = 0$.

In [7] Sain introduces the notion of x^+ and x^-. For $x, y \in X$ we say that $y \in x^+$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \geq 0$ and $y \in x^-$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \leq 0$. In an inner product space the notion of Birkhoff-James orthogonality coincides with the usual notion of orthogonality. In an inner product space approximate orthogonality [2] is defined as: for $x, y \in X$ and $\epsilon \in [0,1)$, $x \perp^\epsilon y$ if and only if $|\langle x, y\rangle| \leq \epsilon\|x\|\|y\|$. Dragomir [4] and Chmieliński [2] have introduced and studied approximate Birkhoff-James orthogonality in the setting of general Banach space.

For x, $y \in X$ and $\epsilon \in [0,1)$, x is said to be *approximate Birkhoff-James orthogonal* to y, written as "$x \perp_B^\epsilon y$", if and only if $\|x + \lambda y\|^2 \geq \|x\|^2 - 2\epsilon\|x\|\|\lambda y\|$ for all $\lambda \in \mathbb{R}$ and x is said to be approximate Birkhoff-James orthogonal to y in the sense of Dragomir, written as "$x \perp_D^\epsilon y$", if and only if $\|x + \lambda y\| \geq \sqrt{1-\epsilon^2}\|x\|$ for all $\lambda \in \mathbb{R}$. Note that these two types of approximate Birkhoff-James orthogonality coincide with approximate orthogonality in an inner product space, i.e., in an inner product space, $x \perp_B^\epsilon y \Leftrightarrow x \perp^\epsilon y$ and $x \perp_D^\epsilon y \Leftrightarrow x \perp^\epsilon y$. Thus, in an inner product space, these two notions are equivalent. In [2], Chmieliński exhibited an example of a non-strictly convex, non-smooth Banach space (ℓ_∞^2), where $\perp_B^\epsilon$ and $\perp_D^\epsilon$ are not same.

Later on we will give an example to establish that these two notions are different even if the space is smooth and strictly convex. This motivates us to study approximate Birkhoff-James orthogonality sets in the setting of general Banach space. We note that both types of approximate Birkhoff-James orthogonality are homogeneous, i.e., $x \perp_B^\epsilon y \Rightarrow ax \perp_B^\epsilon by$ and $x \perp_D^\epsilon y \Rightarrow ax \perp_D^\epsilon by$ for all $a, b \in \mathbb{R}$. Some characterizations of approximate Birkhoff-James orthogonality are obtained in [3, 6]. Among these we need the following two characterizations.

Theorem 1.1. [3, Th. 2.3] *Let X be a Banach space. For $x, y \in X$ and $\epsilon \in [0,1)$,*

$$x \perp_B^\epsilon y \Leftrightarrow \exists\ \phi \in J(x) : |\phi(y)| \leq \epsilon\|y\|.$$

Theorem 1.2. [6, Lemma 3.2] *Let X be a Banach space. For $x, y \in X$ and $\epsilon \in [0,1)$,*

$$x \perp_D^\epsilon y \Leftrightarrow \exists\ \phi \in S_{X^*} : \phi(x) \geq \sqrt{1-\epsilon^2}\|x\|,\ \phi(y) = 0.$$

In order to study approximate Birkhoff-James orthogonality, Sain et al. [9] introduced the following two notations. For $x \in S_X$ and $\epsilon \in [0,1)$,

$$G(x,\epsilon) = \{y \in X : x \perp_B^\epsilon y\}, \quad F(x,\epsilon) = \{y \in X : x \perp_D^\epsilon y\}.$$

In [9], the authors studied both types of approximate Birkhoff-James orthogonality from a geometric point of view. For this study the notion of normal cones in a Banach space was necessary. A subset K of X is said to be a *normal cone* in X if

$$\text{(i) } K+K\subseteq K,\ \text{ (ii) } \alpha K\subseteq K \text{ for all } \alpha\geq 0 \text{ and (iii) } K\cap(-K)=\{\theta\}.$$

Observe that a normal cone K in a two-dimensional Banach space X is determined by its intersection with S_X. Following [9], we say that a normal cone K in a two-dimensional Banach space X is determined by $v_1, v_2 \in S_X$, if

$$K\cap S_X=\Big\{\frac{(1-t)v_1+tv_2}{\|(1-t)v_1+tv_2\|}:t\in[0,1]\Big\}.$$

Clearly, in this case, $K=\{av_1+bv_2 : a,b\geq 0\}$.

We first characterize $G(x,\epsilon)$ in terms of normal cones in a two-dimensional Banach space. Then we show that in arbitrary Banach space $G(x,\epsilon)$ is union of two dimensional normal cones. Then we obtain some uniqueness theorem for $G(x,\epsilon)$. Using these results, we obtain some relation between $T(G(x,\epsilon))$ and $G(Tx,\delta)$, where $T\in B(X)$, $\|T\|=1$ and $\|Tx\|=1$. We also study some interconnection among normal cones, $F(x,\epsilon)$ and norm one linear functionals in a two-dimensional Banach space.

2. Main results

We begin with an example to show that these two types of approximate Birkhoff-James orthogonality ($\perp_B^\epsilon$ & $\perp_D^\epsilon$) may not be same, even if the space is strictly convex and smooth.

Example 2.1. Consider the space $X=l_4^2$. Let $x=(1,0), v=(\frac{1}{2},\frac{(15)^{\frac14}}{2})$ and $\epsilon=\frac12$. Let $\psi\in X^*$ be defined by $\psi(a,b)=a$. Then $\psi\in J(x)$. Clearly, $\psi(v)=\frac12=\epsilon\|v\|$. Therefore, by Theorem 1.1, $x\perp_B^\epsilon v$. If possible, suppose that $x\perp_D^\epsilon v$. Then by Theorem 1.2, there exists $\xi\in S_{X^*}$ such that $\xi(x)\geq\sqrt{1-\epsilon^2}\|x\|$ and $\xi(v)=0$. Again we see that

$$((15)^{\frac{1}{12}},-1)\perp_B(\frac12,\frac{(15)^{\frac14}}{2})=v.$$

Therefore, by [5, Th. 2.1], there exists $\sigma\in S_{X^*}$ such that

$$\sigma((15)^{\frac{1}{12}},-1)=\|((15)^{\frac{1}{12}},-1)\|=(1+(15)^{\frac13})^{\frac14}$$

and $\sigma(v)=0$. Since X is two-dimensional, $\xi=\sigma$. Thus,

$$\xi(x)=\sigma(x)=\sigma(1,0)=\frac{(15)^{\frac14}}{(1+(15)^{\frac13})^{\frac34}}<\frac{\sqrt3}{2}=\sqrt{1-\epsilon^2}\|x\|,$$

a contradiction. Therefore, $x\not\perp_D^\epsilon v$.

Now, let $y=(0,1), u=(2^{-\frac14},2^{-\frac14})$ and $\epsilon=2^{-\frac{9}{32}}$. Let $\phi\in X^*$ be defined by $\phi(a,b)=2^{-\frac34}(-a+b)$. Then $\|\phi\|=1$, $\phi(y)=\phi(0,1)=2^{-\frac34}>\sqrt{1-\epsilon^2}$ and $\phi(u)=0$. Thus, by Theorem 1.2, $y\perp_D^\epsilon u$. Now, $J(y)=\{\psi\}$, where ψ is defined by $\psi(a,b)=b$. Clearly, $\psi(u)=2^{-\frac14}>\epsilon$. Hence, by Theorem 1.1, $y\not\perp_B^\epsilon u$.

In [9, Th. 2.14], Sain et al. proved that if X is a two-dimensional Banach space, then for a smooth point $x \in S_X$ and $\epsilon \in [0,1)$, $G(x,\epsilon) = K \cup (-K)$, where K is a normal cone in X. In this paper we prove the same result without assuming the smoothness of x. Before doing so, we prove two easy propositions which explore nice relation among $G(x,\epsilon)$, normal cone and linear functionals.

Proposition 2.2. *Let X be a two-dimensional Banach space. Let $x \in S_X$ be a smooth point and $\epsilon \in [0,1)$. Let $G(x,\epsilon) = K \cup (-K)$, where K is the normal cone determined by $v_1, v_2 \in S_X$. Then for $\phi \in J(x)$, $|\phi(v_1)| = |\phi(v_2)| = \epsilon$. Moreover, if $\epsilon > 0$, then $\phi(v_1)\phi(v_2) < 0$.*

Proof. Since $v_1, v_2 \in K \subseteq G(x,\epsilon)$, $x \perp_B^\epsilon v_1$ and $x \perp_B^\epsilon v_2$. If $\epsilon = 0$, then by [5, Th. 2.1], $\phi(v_1) = \phi(v_2) = 0$. Let $\epsilon > 0$. Then by Theorem 1.1, we have, for $\phi \in J(x)$, $|\phi(v_1)| \leq \epsilon$. We claim that $|\phi(v_1)| = \epsilon$. If possible, let $|\phi(v_1)| < \epsilon$. Since ϕ is continuous linear, we can say that there exists a neighbourhood $B(v_1,\delta)$ of v_1 such that $|\phi(y)| < \epsilon$ for all $y \in B(v_1,\delta) \cap S_X$ for some $\delta > 0$. Hence, $y \in G(x,\epsilon)$ for all $y \in B(v_1,\delta) \cap S_X$. Since the cone K is determined by v_1, v_2, $B(v_1,\delta) \cap S_X$ contains a vector y which is not in $K \cup (-K)$. This shows that $G(x,\epsilon) \nsubseteq K \cup (-K)$, a contradiction. So $|\phi(v_1)| = \epsilon$. Similarly we can show that $|\phi(v_2)| = \epsilon$.

Now, we show that $\phi(v_1)\phi(v_2) < 0$. Choose $z \in S_X$ such that $\phi(z) = 0$. Clearly, $x \perp_B^\epsilon z$. Hence, $z \in G(x,\epsilon) = K \cup (-K)$. Without loss of generality, we may assume that $z \in K$. Then $z = a\{(1-t)v_1 + tv_2\}$ for some $t \in [0,1]$ and $a > 0$. If $t = 1$, then $z = av_2 \Rightarrow \phi(z) = a\phi(v_2)$, a contradiction, since $\phi(z) = 0$ and $|\phi(v_2)| = \epsilon > 0$. Hence, $t < 1$. Similarly, $t > 0$. Now, $\phi(z) = 0 \Rightarrow a(1-t)\phi(v_1) + at\phi(v_2) = 0 \Rightarrow \phi(v_1)\phi(v_2) < 0$, since $a > 0$ and $t \in (0,1)$. This completes the proof of the proposition. □

The following proposition shows that the converse of Proposition 2.2 also holds.

Proposition 2.3. *Let X be a two-dimensional Banach space. Let $x \in S_X$ be a smooth point and $\epsilon \in (0,1)$ be such that for $\phi \in J(x)$, $|\phi(v_1)| = |\phi(v_2)| = \epsilon$ for some v_1, $v_2 \in S_X$ with $v_1 \neq \pm v_2$ and $\phi(v_1)\phi(v_2) < 0$. Then $G(x,\epsilon) = K \cup (-K)$, where K is the normal cone determined by v_1 and v_2.*

Proof. Without loss of generality, let $\phi(v_1) = \epsilon$ and $\phi(v_2) = -\epsilon$. We now show that $G(x,\epsilon) = K \cup (-K)$. Let $w \in K$. Then $w = a((1-t)v_1 + tv_2)$ for some $a \geq 0$ and $t \in [0,1]$. Now, $\phi(w) = a((1-t)\phi(v_1) + t\phi(v_2)) = a(1-t)\epsilon + at(-\epsilon) = a(1-2t)\epsilon$. Therefore, $|\phi(w)| = a|(1-2t)|\epsilon$. Again, $\|w\| = a\|(1-t)v_1 + tv_2\| \geq a|\|(1-t)v_1\| - t\|v_2\|| = a|1-2t|$. So $|\phi(w)| = a|(1-2t)|\epsilon \leq \epsilon\|w\|$. So by Theorem 1.1, we have, $x \perp_B^\epsilon w$. Therefore, $w \in G(x,\epsilon)$. Similarly, $w \in (-K) \Rightarrow w \in G(x,\epsilon)$. Thus, $K \cup (-K) \subseteq G(x,\epsilon)$.

Now, we show that $G(x,\epsilon) \subseteq K \cup (-K)$. Clearly, $x \notin K \cup (-K)$, since $x \notin G(x,\epsilon)$. Without loss of generality we may assume $x = ((1-s)v_1 - sv_2)/(\|(1-s)v_1 - sv_2\|$ for some $s \in (0,1)$. Let $w \notin K \cup (-K)$. Then either $w = a((1-t)x - tv_2)$ where $a \neq 0$ and $t \in (0,1)$ or $w = a((1-t)v_1 + tx)$ where $a \neq 0$ and $t \in (0,1)$. For the first case, $\phi(w) = a(1-t) + at\epsilon$. So $|\phi(w)| = |a||1-t+t\epsilon| > |a|\epsilon$ as $0 < \epsilon < 1$. Again, $\|w\| = |a|\|(1-t)x - tv_2\| \leq |a|(1-t+t) = |a|$. So $|\phi(w)| > |a|\epsilon \geq \epsilon\,\|w\|$, which by Theorem 1.1, shows that $w \notin G(x,\epsilon)$. Again, for the second case, $\phi(w) = a(1-t)\epsilon + at$. So $|\phi(w)| = |a||(1-t)\epsilon + t| > |a|\epsilon$ and $\|w\| = |a|\ \|(1-t)v_1 + tx\| \leq |a|(1-t+t) = |a|$. Therefore, $|\phi(w)| > |a|\epsilon \geq \epsilon\ \|w\|$, which again by Theorem 1.1, shows that $w \notin G(x,\epsilon)$. So $w \in G(x,\epsilon)$ implies $w \in K \cup (-K)$. Therefore, $G(x,\epsilon) \subseteq K \cup (-K)$. Thus, $G(x,\epsilon) = K \cup (-K)$. This completes the proof of the proposition. □

In the following theorem, we characterize the set $G(x,\epsilon)$ through normal cones in any two dimensional Banach space. This result generalizes [9, Th. 2.14] by removing the additional condition of smoothness of the point $x \in S_X$.

Theorem 2.4. *Let X be a two-dimensional Banach space. Then for any $x \in S_X$ and $\epsilon \in [0,1)$, there exists a normal cone K in X such that $G(x,\epsilon) = K \cup (-K)$.*

Proof. By [9, Th. 2.1], for each $\epsilon \in [0,1)$, there exists a normal cone K in X such that $F(x,\epsilon) = K \cup (-K)$. Therefore, $F(x,0) = K_0 \cup (-K_0)$, where K_0 is a normal cone in X. Let the cone K_0 be determined by $u_1, u_2 \in S_X$. Then $x \perp_B u_i$, for $i = 1,2$. Thus, by [5, Th. 2.1], for $i = 1,2$, there exist $\phi_i \in J(x)$ such that $\phi_i(u_i) = 0$. Without loss of generality, we may assume that $\phi_1(u_2) \leq 0$. Clearly, there exist $t_1, t_2 \in (0,1)$ such that

$$v_1 = \tfrac{(1-t_1)x + t_1 u_1}{\|(1-t_1)x + t_1 u_1\|},\ \ v_2 = \tfrac{-(1-t_2)x + t_2 u_2}{\|-(1-t_2)x + t_2 u_2\|}$$

and $\phi_1(v_1) = \epsilon, \phi_2(v_2) = -\epsilon$. Let K be the normal cone determined by $v_1, v_2 \in S_X$. Clearly, $K_0 \subseteq K$ and $x \notin K \cup (-K)$. We show that $G(x,\epsilon) = K \cup (-K)$. Let $v \in K \cap S_X$. If $v \in K_0$, then clearly $v \in G(x,\epsilon)$. Let $v = \frac{(1-t)v_1 + tu_1}{\|(1-t)v_1 + tu_1\|}$ for some $t \in [0,1]$. Then $\phi_1(v) = \frac{(1-t)\epsilon}{\|(1-t)v_1 + tu_1\|}$. If possible, let $\|(1-t)v_1 + tu_1\| < (1-t)$. Then $u_1 \notin v_1^+ \Rightarrow u_1 \in v_1^-$. Now, there exists $s \in (0,1)$ such that $x = \frac{(1-s)v_1 - su_1}{\|(1-s)v_1 - su_1\|}$. Therefore, $1 = \phi_1(x) = \frac{(1-s)\epsilon}{\|(1-s)v_1 - su_1\|} \leq \frac{(1-s)\epsilon}{1-s} = \epsilon$, a contradiction. Thus, we have $\|(1-t)v_1 + tu_1\| \geq (1-t)$. So, $\phi_1(v) = \frac{(1-t)\epsilon}{\|(1-t)v_1 + tu_1\|} \leq \frac{(1-t)\epsilon}{(1-t)} = \epsilon$. Thus, $v \in G(x,\epsilon)$. Similarly, $v = \frac{(1-t)v_2 + tu_2}{\|(1-t)v_2 + tu_2\|}$ for some $t \in [0,1]$, implies that $|\phi_2(v)| \leq \epsilon$. Thus, $v \in G(x,\epsilon)$.

Hence, by homogeneity of approximate Birkhoff-James orthogonality, we have, $K \cup (-K) \subseteq G(x,\epsilon)$. Now, let $v \in S_X \setminus (K \cup (-K))$. Then either $v = \pm\frac{(1-t)x + tv_1}{\|(1-t)x + tv_1\|}$ or $v = \pm\frac{(1-t)x - tv_2}{\|(1-t)x - tv_2\|}$ for some $t \in [0,1)$. First let $v = \pm\frac{(1-t)x + tv_1}{\|(1-t)x + tv_1\|}$ for some $t \in [0,1)$. If possible, let $v \in G(x,\epsilon)$. Then there exists $\phi \in J(x)$ such that $|\phi(v)| \leq \epsilon$. Thus, $|1 - t + t\phi(v_1)| \leq |\frac{(1-t) + t\phi(v_1)}{\|(1-t)x + tv_1\|}| \leq \epsilon$. This gives that $\phi(v_1) < \epsilon \Rightarrow \frac{(1-t_1) + t_1\phi(u_1)}{\|(1-t_1)x + t_1 u_1\|} < \epsilon = \phi_1(v_1) = \frac{(1-t_1)}{\|(1-t_1)x + t_1 u_1\|}$. Thus, $\phi(u_1) < 0$. Therefore, there exists $r \in (0,1)$ such that $\phi(\frac{(1-r)x + ru_1}{\|(1-r)x + ru_1\|}) = 0$. Hence, $x \perp_B (1-r)x + ru_1$, i.e., $(1-r)x + ru_1 \in F(x,0)$ but $(1-r)x + ru_1 \notin K_0 \cup (-K_0)$, a contradiction. Therefore, $v \notin G(x,\epsilon)$. Similarly, $v = \pm\frac{(1-t)x - tv_2}{\|(1-t)x - tv_2\|}$ for some $t \in [0,1)$ implies that $v \notin G(x,\epsilon)$. Thus, $G(x,\epsilon) = K \cup (-K)$. □

Remark 2.5. Looking at the proof of Theorem 2.4 it would not be an exaggeration to remark that even for two dimensional Banach spaces, analytical conversion of geometrical concepts are very much intricate.

For $x, y \in X$ and $\epsilon \in [0, 1)$, let $Q_{x,y}(\epsilon)$ denote the restriction of $G(x, \epsilon)$ to the subspace spanned by x and y. Using this notation and the description of the set $G(x, \epsilon)$ in a two-dimensional Banach space, we now obtain the following immediate description of the set $G(x, \epsilon)$ in arbitrary Banach space.

Theorem 2.6. *Let X be arbitrary Banach space. Let $x \in X$ and $\epsilon \in [0, 1)$. Then $G(x, \epsilon) = \bigcup_{y \in X} Q_{x,y}(\epsilon)$, i.e., $G(x, \epsilon)$ is union of two-dimensional normal cones.*

Now, we turn our attention to the converse of Theorem 2.4. We study whether for a given cone K in a two-dimensional Banach space X, there is any $x \in S_X$ and $\epsilon \in [0, 1)$ such that $G(x, \epsilon) = K \cup (-K)$. The following example shows that it is not true in general.

Example 2.7. Let $X = l_\infty^2$. Let K be the normal cone determined by $(-\frac{1}{2}, 1)$ and $(-1, 1)$. If possible , suppose that there exist $x \in S_X$ and $\epsilon \in [0, 1)$ such that $G(x, \epsilon) = K \cup (-K)$. Let $x \perp_B z$ and $z \in S_X$. Then $z \in K \cup (-K)$. Without loss of generality assume that $z \in K$. Then $z = (1-t)(-\frac{1}{2}, 1) + t(-1, 1)$ for some $t \in [0, 1]$. Then clearly $x = \pm(1, 1)$. Since $x \perp_B (-1, 0)$, we must have $(-1, 0) \in G(x, \epsilon)$. But $(-1, 0) \notin K \cup (-K)$. This contradiction proves that there does not exist any $x \in S_X$ and $\epsilon \in [0, 1)$ such that $G(x, \epsilon) = K \cup (-K)$.

In our next theorem, we show that even if X is a two-dimensional smooth Banach space which is not strictly convex, then also the converse of Theorem 2.4 may not hold.

Theorem 2.8. *Let X be a two-dimensional Banach space which is not strictly convex. Then there exists a normal cone K in X such that for no $x \in S_X$ and $\epsilon \in [0, 1)$, $G(x, \epsilon) = K \cup (-K)$.*

Proof. Since X is not strictly convex, there exist v_1 and $v_2 \in S_X$ such that $v_1 \neq \pm v_2$ and $((1-t)v_1 + tv_2) \in S_X$ for all $t \in [0, 1]$. Let K be the normal cone determined by v_1 and $-v_2$. If possible, let there exist $x \in S_X$ and $\epsilon \in [0, 1)$ such that $G(x, \epsilon) = K \cup (-K)$. Since $x \notin G(x, \epsilon)$, without loss of generality, we may assume that $x = ((1-t)v_1 + tv_2)$ for some $t \in (0, 1)$. Then x is a smooth point and so $J(x) = \{\phi\}$, for some $\phi \in S_{X^*}$. We first claim that $\epsilon \neq 0$. If $\epsilon = 0$, then by Proposition 2.2, $\phi(v_1) = \phi(-v_2) = 0$. Now, $\phi(x)$=1 implies that $(1-t)\phi(v_1) + t\phi(v_2) = 1$. But then $0 = 1$, which is a contradiction. Now, if $\epsilon \in (0, 1)$, then by Proposition 2.2, $|\phi(v_1)| = |\phi(-v_2)| = \epsilon$ and $\phi(v_1)\ \phi(-v_2) < 0$. Let $\phi(v_1) = \epsilon$ and $\phi(-v_2) = -\epsilon$, i.e., $\phi(v_2) = \epsilon$. Then $\phi(x) = 1$ implies that $(1-t)\phi(v_1) + t\phi(v_2) = 1$. So we get $\epsilon = 1$, a contradiction. Similarly, $\phi(v_1) = -\epsilon$ and $\phi(-v_2) = \epsilon$ gives a contradiction. Hence, there does not exist any $x \in S_X$ and $\epsilon \in [0, 1)$ such that $G(x, \epsilon) = K \cup (-K)$. This completes the proof of the theorem. □

Next, we prove that if the Banach space X is two-dimensional, smooth and strictly convex, then for any given normal cone K in X, there exist $x \in S_X$ and $\epsilon \in [0,1)$ such that $G(x,\epsilon) = K \cup (-K)$.

Theorem 2.9. *Let X be a two-dimensional strictly convex and smooth Banach space. Let K be a normal cone determined by $v_1, v_2 \in S_X$. Then there exist $x \in S_X$ and $\epsilon \in [0,1)$ such that $G(x,\epsilon) = K \cup (-K)$.*

Proof. Clearly, $v_1 \neq -v_2$ as K is a normal cone. Otherwise $-v_2 \in K \cap (-K)$, a contradiction. If $v_1 = v_2$, then choose $x \in S_X$ such that $x \perp_B v_1$. Then for $\epsilon = 0$, $G(x,\epsilon) = G(x,0) = \{y \in X : x \perp_B y\} = \{\alpha v_1 : \alpha \in \mathbb{R}\} = K \cup (-K)$. Let $v_1 \neq \pm v_2$. Let $w = v_1 + v_2$. Then by [5, Th. 2.3] there exists $x \in S_X$ such that $x \perp_B w$. Therefore, for $\phi \in J(x)$, $\phi(w) = 0$ which shows $\phi(v_1) = -\phi(v_2)$. Let $|\phi(v_1)| = |\phi(v_2)| = \epsilon$. Then $0 \leq \epsilon \leq 1$. If $\epsilon = 1$, then $|\phi((1-t)v_1 + tv_2)| = 1$ for all $t \in [0,1]$. Thus, $(1-t)v_1 + tv_2 \in S_X$ for all $t \in [0,1]$, contradicting that X is strictly convex. Therefore, $\epsilon < 1$. If $\epsilon = 0$, then $\phi(v_1) = \phi(v_2) = 0$. So $x \perp_B v_1$ and $x \perp_B v_2$. Thus, Birkhoff-James orthogonality is not right unique. Thus, by [5, Th. 5.1], X is not smooth, a contradiction. Therefore, $\epsilon \in (0,1)$. Clearly, $\phi(v_1)\phi(v_2) < 0$. Hence, by Proposition 2.3, $G(x,\epsilon) = K \cup (-K)$. This completes the proof of the theorem. $\square$

Now, we study some uniqueness property of the set $G(x,\epsilon)$, where $x \in S_X$ and $\epsilon \in [0,1)$. We study whether $G(x_1,\epsilon_1) = G(x_2,\epsilon_2)$ for $x_1, x_2 \in S_X$ and $\epsilon_1, \epsilon_2 \in [0,1)$ implies that $x_1 = \pm x_2$ and $\epsilon_1 = \epsilon_2$. The following example shows that this is not true in general.

Example 2.10. Consider the space $X = l^2_\infty$. Let $x_1 = (1, \frac{1}{2})$ and $x_2 = (1, -\frac{1}{2})$. Choose $\epsilon_1 = \epsilon_2 = \frac{1}{2}$. Then it is easy to see that

$$G(x_1,\epsilon_1) \cap S_X = \{t(\tfrac{1}{2},1) + (1-t)(-\tfrac{1}{2},1) : t \in [0,1]\} \cup$$
$$\cup \{t(-\tfrac{1}{2},-1) + (1-t)(\tfrac{1}{2},-1) : t \in [0,1]\} = G(x_2,\epsilon_2) \cap S_X.$$

But $x_1 \neq \pm x_2$.

In the following theorem, we show that some restriction on the space gives the desired uniqueness of the set $G(x,\epsilon)$.

Theorem 2.11. *Let X be a two-dimensional strictly convex and smooth Banach space. Suppose there exist $x_1, x_2 \in S_X$ and $\epsilon_1, \epsilon_2 \in [0,1)$ such that $G(x_1,\epsilon_1) = G(x_2,\epsilon_2)$. Then $x_1 = \pm\, x_2$ and $\epsilon_1 = \epsilon_2$.*

Proof. Let $G(x_1,\epsilon_1) = G(x_2,\epsilon_2) = K \cup (-K)$, where K is the normal cone determined by $v_1, v_2 \in S_X$. Let $w = v_1 + v_2$. If $\epsilon_1 = 0$, then $x_1 \perp_B v_1$ and $x_1 \perp_B v_2$. Since X is smooth, by [5, Th. 4.2], $x_1 \perp_B (v_1 + v_2)$, i.e., $x_1 \perp_B w$. If $\epsilon_1 \in (0,1)$, then by Proposition 2.2, for $\phi \in J(x_1)$, $|\phi(v_1)| = |\phi(v_2)| = \epsilon_1$ and $\phi(v_1)\phi(v_2) < 0$. Then clearly, $\phi(w) = 0$. So $x_1 \perp_B w$. Similarly, we can show that $x_2 \perp_B w$. Since X is strictly convex, by [5, Th. 4.3], Birkhoff-James orthogonality is left unique. Therefore, $x_2 = a x_1$ for some scalar a. Again, $x_1, x_2 \in S_X$ implies that $x_2 = \pm x_1$. If $x_1 = x_2$, then $\phi \in J(x_2)$. Hence, by Proposition 2.2, $|\phi(v_1)| = |\phi(v_2)| = \epsilon_2$. Thus, $\epsilon_1 = \epsilon_2$. Similarly, $x_1 = -x_2$ implies that $\epsilon_1 = \epsilon_2$. This completes the proof of the theorem. $\square$

Using Theorem 2.6 and Theorem 2.11, we now obtain the following obvious uniqueness theorem for $G(x, \epsilon)$ in arbitrary strictly convex and smooth Banach space.

Theorem 2.12. *Let X be an arbitrary strictly convex and smooth Banach space. Suppose there exist $x_1, x_2 \in S_X$ and $\epsilon_1, \epsilon_2 \in [0, 1)$ such that $G(x_1, \epsilon_1) = G(x_2, \epsilon_2)$. Then $x_1 = \pm\, x_2$ and $\epsilon_1 = \epsilon_2$.*

If we only assume that the two-dimensional Banach space X is smooth, then we obtain the following uniqueness theorem for $G(x, \epsilon)$.

Theorem 2.13. *Let X be a two-dimensional smooth Banach space. Let $x \in S_X$ and $\epsilon_1, \epsilon_2 \in [0, 1)$ such that $G(x, \epsilon_1)=G(x, \epsilon_2)$. Then $\epsilon_1 = \epsilon_2$.*

Proof. Let $G(x, \epsilon_1)=G(x, \epsilon_2)=K\cup(-K)$, where K is the normal cone determined by $v_1, v_2 \in S_X$. Then by Proposition 2.2, for $\phi \in J(x)$, $|\phi(v_1)|=\epsilon_1=\epsilon_2$, hence the theorem. $\square$

Combining Theorem 2.4, Theorem 2.9 and Theorem 2.11, we obtain the following characterization of normal cones in a two-dimensional strictly convex and smooth Banach space.

Theorem 2.14. *Let X be a two-dimensional strictly convex and smooth Banach space. Then K is a normal cone in X if and only if there exist a unique $x \in S_X$ (upto multiplication ± 1) and a unique $\epsilon \in [0, 1)$ such that $G(x, \epsilon) = K \cup (-K)$.*

For $T \in B(X)$, let M_T denote the set all points of S_X where T attains norm, i.e., $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$. Now, we study the relation between $T(G(x, \epsilon))$ and $G(Tx, \epsilon)$, where $T \in B(X)$, $\|T\| = 1$ and $x \in M_T$. For this we first prove the following easy proposition.

Proposition 2.15. *Let X be a Banach space. Let $T \in B(X)$ and $x \in M_T$. Then for $\epsilon \in [0, 1)$ and $y \in X$, $Tx \perp_B^\epsilon Ty \Rightarrow x \perp_B^\epsilon y$.*

Proof. Since $Tx \perp_B^\epsilon Ty$, for any $\lambda \in \mathbb{R}$, $\|Tx + \lambda Ty\|^2 \geq \|Tx\|^2 - 2\epsilon\|Tx\|\|\lambda Ty\|$.

Hence,
$$\begin{aligned}\|T\|^2\|x + \lambda y\|^2 &\geq \|Tx + \lambda Ty\|^2 \geq \|Tx\|^2 - 2\epsilon\|Tx\|\|\lambda Ty\| \\ &\geq \|T\|^2\|x\|^2 - 2\epsilon\|T\|\|x\||\lambda|\|T\|\|y\|\end{aligned}$$

for any $\lambda \in \mathbb{R}$. Thus, $\|x + \lambda y\|^2 \geq \|x\|^2 - 2\epsilon\|x\|\|\lambda y\|$ for any $\lambda \in \mathbb{R}$. Therefore, $x \perp_B^\epsilon y$. This completes the proof of the proposition. $\square$

Corollary 2.16. *Let X be a Banach space. Let $T \in B(X)$ be such that T is onto, $\|T\| = 1$ and $x \in M_T$. Then for $\epsilon \in [0, 1)$, $G(Tx, \epsilon) \subseteq T(G(x, \epsilon))$.*

Proof. Let $z \in G(Tx, \epsilon)$. Then $Tx \perp_B^\epsilon z$. Since T is onto, there exists $y \in X$ such that $Ty = z$. Thus, $Tx \perp_B^\epsilon Ty$. Hence, by Proposition 2.15, $x \perp_B^\epsilon y$. So $y \in G(x, \epsilon) \Rightarrow z = Ty \in T(G(x, \epsilon))$. Thus, $G(Tx, \epsilon) \subseteq T(G(x, \epsilon))$. $\square$

The following example shows that the converse of the Proposition 2.15 is not true in general.

Example 2.17. Let $X = \ell_2^2$. Define $T \in B(\ell_2^2)$ by $T(a,b) = (a,0)$ for all $(a,b) \in \ell_2^2$. Then clearly, $(1,0) \in M_T$. Let $\epsilon = \frac{\sqrt{3}}{2}$, $x = (1,0)$ and $y = (-\frac{\sqrt{3}}{2}, 1)$. Then

$$|\langle x, y\rangle| = |\langle (1,0), (-\tfrac{\sqrt{3}}{2}, 1)\rangle| = \tfrac{\sqrt{3}}{2} < \epsilon \|x\|\|y\|.$$

Thus, $x \perp^{\epsilon} y$. But

$$|\langle Tx, Ty\rangle| = |\langle (1,0), (-\tfrac{\sqrt{3}}{2}, 0)\rangle| = \|Tx\|\|Ty\| > \epsilon \|Tx\|\|Ty\|.$$

Thus, $Tx \not\perp^{\epsilon} Ty$. □

Next, we obtain a relation between the sets $T(G(x,\epsilon))$ and $G(Tx,\delta)$ for a norm one linear operator T defined on a two-dimensional strictly convex and smooth Banach space, where $x \in M_T$.

Theorem 2.18. *Let X be a two-dimensional strictly convex and smooth Banach space. Let $T \in B(X)$ be such that T is onto, $\|T\| = 1$ and $x \in M_T$. Let $\epsilon \in (0,1)$. Suppose that $G(x,\epsilon) = K \cup (-K)$, where K is the normal cone determined by $v_1, v_2 \in S_X$. Then $T(G(x,\epsilon)) = G(Tx,\delta)$ for some $\delta \in [0,1)$ if and only if $\|Tv_1\| = \|Tv_2\| = k$ and $\delta = \frac{\epsilon}{k}$.*

Proof. First suppose that $T(G(x,\epsilon)) = G(Tx,\delta)$ for some $\delta \in [0,1)$. Now, $T(G(x,\epsilon)) = T(K \cup (-K)) = T(K) \cup (-T(K))$. It is easy to observe that $T(K)$ is a normal cone in X determined by $\frac{Tv_1}{\|Tv_1\|}$ and $\frac{Tv_2}{\|Tv_2\|}$. Now, by Proposition 2.2, for $\psi \in J(Tx)$, we have

$$|\psi(\tfrac{Tv_1}{\|Tv_1\|})| = |\psi(\tfrac{Tv_1}{\|Tv_1\|})| = \delta.$$

Hence, $|\psi \circ T(v_1)| = \delta\|Tv_1\|$ and $|\psi \circ T(v_2)| = \delta\|Tv_2\|$. Again $\psi \in J(Tx) \Rightarrow \psi(Tx) = \|Tx\| = 1$. Thus, $\psi \circ T \in J(x)$. Since $G(x,\epsilon) = K \cup (-K)$, by Proposition 2.2, for $\psi \circ T \in J(x)$, we have $|\psi \circ T(v_1)| = |\psi \circ T(v_2)| = \epsilon$. Therefore, $\epsilon = \delta\|Tv_1\| = \delta\|Tv_2\|$. This implies that $\|Tv_1\| = \|Tv_2\| = k$ (say) and $\delta = \frac{\epsilon}{k}$.

Conversely, suppose that $\|Tv_1\| = \|Tv_2\| = k$. Since $G(x,\epsilon) = K \cup (-K)$, by Proposition 2.2, for $\phi \in J(x)$, we have $|\phi(v_1)| = |\phi(v_2)| = \epsilon$ and $\phi(v_1)\phi(v_2) < 0$. Clearly, $\phi(v_1 + v_2) = 0$. Thus, $x \perp_B (v_1 + v_2)$. Hence, by [8, Lemma 2.1], $Tx \perp_B (Tv_1 + Tv_2)$. Now, if $\psi \in J(Tx)$, then $\psi(Tv_1 + Tv_2) = 0 \Rightarrow \psi(\frac{Tv_1 + Tv_2}{k}) = 0$.

Let $|\psi(\frac{Tv_1}{k})| = |\psi(\frac{Tv_2}{k})| = \delta$. Then proceeding similarly as Theorem 2.9, it can be shown that $\delta \in (0,1)$ and $G(Tx,\delta) = T(K) \cup (-T(K)) = T(G(x,\epsilon))$. Now, $\psi \in J(Tx) \Rightarrow \psi(Tx) = \|Tx\| = 1$. This implies that $\psi \circ T \in J(x)$. Since x is smooth, $\psi \circ T = \phi$. Thus, $\epsilon = |\phi(v_1)| = |\psi \circ T(v_1)| = \delta k$. Hence, $\delta = \frac{\epsilon}{k}$. This completes the proof of the theorem. □

In [9, Th. 2.1], Sain et al. proved that in a two-dimensional Banach space X, $F(x,\epsilon) = K \cup (-K)$, where K is a normal cone in X. In the following theorem, we obtain a connection between the set $F(x,\epsilon)$ and norm one linear functionals on X.

Theorem 2.19. *Let X be a two-dimensional smooth Banach space. Let $\epsilon \in [0,1)$ and $x \in S_X$. Then the following conditions are equivalent.*

(i) $F(x,\epsilon) = K \cup (-K)$, *where K is the normal cone determined by $u_1, u_2 \in S_X$ and $u_1 \neq \pm u_2$.*

(ii) *There exist $\phi_i \in S_{X^*}$ such that $\phi_i(x) = \sqrt{1-\epsilon^2}$, $\phi_i(u_i) = 0$ for $i = 1,2$ and $\phi_1(u_2)\phi_2(u_1) < 0$.*

Proof. (i) $\Rightarrow$ (ii). Since the normal cone K is determined by $u_1, u_2 \in S_X$, by [9, Th. 2.4], we have, $\inf_{\lambda\in\mathbb{R}} \| x + \lambda u_i \| = \sqrt{1-\epsilon^2}$, for $i = 1,2$. Now, as $x \perp_D^\epsilon u_i$, by Theorem 1.2, for $i = 1,2$, there exist $\phi_i \in S_{X^*}$ such that $\phi_i(u_i) = 0$ and $\phi_i(x) \geq \sqrt{1-\epsilon^2}$. Again, $\phi_i(x) = \inf_{\lambda\in\mathbb{R}} \phi_i(x+\lambda u_i) \leq \inf_{\lambda\in\mathbb{R}} \| x + \lambda u_i \| = \sqrt{1-\epsilon^2}$. Thus, $\phi_i(x) = \sqrt{1-\epsilon^2}$, for $i = 1,2$.

Now, we show that $\phi_1(u_2)\phi_2(u_1) < 0$. Clearly, $x \notin F(x,\epsilon)$. Without loss of generality we may assume that $x = \frac{(1-t)u_1 - tu_2}{\|(1-t)u_1 - tu_2\|}$ for some $t \in (0,1)$. Now,

$$(1-\epsilon^2) = \phi_1(x)\phi_2(x) = -\tfrac{t(1-t)}{\|(1-t)u_1 - tu_2\|^2}\phi_1(u_2)\phi_2(u_1).$$

Thus, $\phi_1(u_2)\phi_2(u_1) = -\frac{(1-\epsilon^2)\|(1-t)u_1 - tu_2\|^2}{t(1-t)} < 0.$

(ii) $\Rightarrow$ (i). Since $\phi_1(u_2)\phi_2(u_1) < 0$ and $\phi_i(u_i) = 0$ for $i = 1,2$, $u_1 \neq \pm u_2$. Let K be the normal cone determined by u_1 and $u_2 \in S_X$. We first show that $K \cup (-K) \subseteq F(x,\epsilon)$. Let $w \in K \cup (-K)$. If $w = 0$, then clearly $w \in F(x,\epsilon)$. So let $w = a\{tu_1 + (1-t)u_2\}$, where $a \in \mathbb{R} \setminus \{0\}$ and $t \in [0,1]$. If $t = 0$, then $w = au_2$ and hence by (ii), $x \perp_D^\epsilon w$. Thus, $w \in F(x,\epsilon)$. Similarly, $t = 1$ implies that $w \in F(x,\epsilon)$. So let $t \in (0,1)$. Now, $\phi_1(w)\phi_2(w) = a^2 t(1-t)\phi_1(u_2)\phi_2(u_1) < 0$.

So there exists $s \in [0,1]$ such that $(s\phi_1 + (1-s)\phi_2)(w) = 0$. Let $\psi = \frac{s\phi_1 + (1-s)\phi_2}{\|s\phi_1 + (1-s)\phi_2\|}$. Then $\psi \in S_{X^*}$. Clearly, $\psi(w) = 0$. Now, $\psi(x) = \frac{\sqrt{1-\epsilon^2}}{\|s\phi_1 + (1-s)\phi_2\|} \geq \sqrt{1-\epsilon^2}$. Therefore, by Theorem 1.2, we have, $x \perp_D^\epsilon w$. Hence, $w \in F(x,\epsilon)$. Thus, we have $K \cup (-K) \subseteq F(x,\epsilon)$.

Now, we show that $F(x,\epsilon) \subseteq K \cup (-K)$. Let $y \notin K \cup (-K)$. If possible, let $y \in F(x,\epsilon)$. Then by Theorem 1.2, there exists $\phi \in S_{X^*}$ such that $\phi(x) \geq \sqrt{1-\epsilon^2}$ and $\phi(y) = 0$. Without loss of generality we may assume that $y = b\{(1-t)u_1 - tu_2\}$ where $b > 0$ and $t \in (0,1)$. Now, $-\phi_1(y)\phi_2(y) = b^2 t(1-t)\phi_1(u_2)\phi_2(u_1) < 0$, as $\phi_1(u_2)\phi_2(u_1) < 0$. Therefore, we can say that there exists $s \in (0,1)$ such that $(s\phi_2 - (1-s)\phi_1)(y) = 0$. Since X is two-dimensional, there exists unique $\phi \in S_{X^*}$ (upto multiplication by ± 1) such that $\phi(y) = 0$. Therefore, without loss of generality we may assume that $\phi = \frac{s\phi_2 - (1-s)\phi_1}{\|s\phi_2 - (1-s)\phi_1\|}$. Since X is smooth, X^* is strictly convex. Therefore, $\|s\phi_2 - (1-s)\phi_1\| > ||s| - |(1-s)|| = |2s-1|$. Hence, $|\phi(x)| < \frac{|2s-1|\sqrt{1-\epsilon^2}}{|2s-1|} = \sqrt{1-\epsilon^2}$, a contradiction. Thus, $y \notin K \cup (-K)$ implies that $y \notin F(x,\epsilon)$. Therefore, $F(x,\epsilon) \subseteq K \cup (-K)$. This completes the proof of the theorem. $\square$

We end this section with a relation between the sets $T(F(x,\epsilon))$ and $F(Tx,\delta)$ for a norm one linear operator T defined on a two-dimensional smooth Banach space, where $x \in M_T$.

Theorem 2.20. *Let X be a two-dimensional smooth Banach space. Let $T \in B(X)$ be bijective. Suppose $\|T\| = 1$, $x \in M_T$ and $\epsilon \in [0,1)$. Suppose the following conditions hold.*

(i) $F(x,\epsilon) = K \cup (-K)$, *where K is the normal cone determined by u_1, $u_2 \in S_X$ and $u_1 \neq u_2$.*

(ii) $\|\psi_1 \circ T\| = \|\psi_2 \circ T\| = k$, *where $\psi_1, \psi_2 \in S_{X^*}$ be such that $\psi_1(T(u_1)) = \psi_2(T(u_2)) = 0$.*

Then $T(F(x,\epsilon)) = F(Tx,\delta)$, where $\delta = \sqrt{1-k^2(1-\epsilon^2)}$.

Proof. Since $F(x,\epsilon) = K \cup (-K)$, where K is the normal cone determined by u_1, $u_2 \in S_X$, by Theorem 2.19, there exist $\phi_i \in S_{X^*}$ with $\phi_i(u_i) = 0$, $\phi_i(x) = \sqrt{1-\epsilon^2}$ for $i = 1,2$ and $\phi_1(u_2)\phi_2(u_1) < 0$. Now, $T(F(x,\epsilon)) = T(K) \cup T(-K)$.

Clearly, $T(K)$ is the normal cone determined by $w_1 = \frac{T(u_1)}{\|T(u_1)\|}$ and $w_2 = \frac{T(u_2)}{\|T(u_2)\|}$. Since $\psi_i(T(u_i)) = 0$ and X is two-dimensional, without loss of generality we may assume that $\frac{\psi_i \circ T}{\|\psi_i \circ T\|} = \phi_i$ for $i = 1,2$. Clearly,

$$\psi_1(w_2)\psi_2(w_1) = \frac{k^2\phi_1(u_2)\phi_2(u_1)}{\|Tu_2\|\|Tu_1\|} < 0.$$

Now, for $i = 1,2$,

$$\psi_i(T(x)) = \|\psi_i \circ T\|\phi_i(x) = k\phi_i(x) = k\sqrt{1-\epsilon^2} = \sqrt{1-\delta^2},$$

say. Then $\delta = \sqrt{1-k^2(1-\epsilon^2)} \in [0,1)$. Thus, by Theorem 2.19, we have, $F(Tx,\delta) = T(K) \cup (-T(K))$. Therefore, $T(F(x,\epsilon)) = F(Tx,\delta)$. This completes the proof of the theorem. □

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