Journal of Convex Analysis
Volume 27 (2020), No. 3, 923–934

# The Lipschitz Condition of the Metric Projection in the Pliś Metric

**Maxim V. Balashov**
*V.A.Trapeznikov Institute of Control Sciences, Russian Academy of Sciences
Moscow 117997, Russia
balashov73@mail.ru*

Received: December 28, 2018
Revised manuscript received: June 30, 2019
Accepted: July 7, 2019

We consider the class of Banach spaces where the metric projection on a convex closed bounded set is Lipschitz continuous in the Pliś metric with respect to the set. We obtain some geometrical and analytical conditions for this property. In fact, the space should has a Hilbertlike structure.

*Keywords:* Uniform convexity, uniform smoothness, Banach space, Hilbert space, metric projection, Pliś metric, moduli of convexity and smoothness.

*2010 Mathematics Subject Classification:* Primary: 49J53, 52A07, 46C05, 26B25; secondary: 46B25, 46B20.

## 1. Introduction and main notations

Suppose that $E$ is a real reflexive Banach space and $\mathbb{F} = \mathbb{F}(E)$ is the set of all nonempty closed convex bounded subsets of the space $E$.

The *Pliś metric* ([5, 8]) is defined as follows

$$\rho(A_1, A_2) = \sup_{\|p\|_*=1} h(A_1(p), A_2(p)), \qquad \forall A_1, A_2 \in \mathbb{F}, \tag{1}$$

where
$$\begin{aligned} h(A_1, A_2) &= \max\left\{ \sup_{a\in A_1} \operatorname{dist}(a, A_2),\ \sup_{b\in A_2} \operatorname{dist}(b, A_1) \right\} \\ &= \inf\{r > 0 \mid A_1 \subset A_2 + B_r(0),\ A_2 \subset A_1 + B_r(0)\} \end{aligned}$$

is the standard Hausdorff metric and $\operatorname{dist}(x, A) = \inf_{a\in A} \|x - a\|$ for $x \in E$. For a subset $A \in \mathbb{F}$ and $p \in E^*$ with $\|p\|_* = 1$

$$A(p) = \left\{ x \in A \;\middle|\; (p, x) \geq \sup_{a\in A}(p, a) \right\}.$$

The Pliś metric is important for extremal problems and set-valued analysis. It realizes the Lipschitz stability for solutions of a wide class of extremal problems with respect to the set in the case of a real Hilbert space [1, 2]. The Lipschitz stability of the metric projection of a point on elements from $\mathbb{F}(\mathcal{H})$ is the base for

ISSN 0944-6532 / $ 2.50 

the mentioned result. Namely, we have the estimate for metric projections $P_{A_i}x$, $i = 1, 2$: for the real Hilbert space $\mathcal{H}$, for any subsets $A_1, A_2 \in \mathbb{F}(\mathcal{H})$ and any point $x \in \mathcal{H}$

$$\|P_{A_1}x - P_{A_2}x\| \le C\rho(A_1, A_2), \tag{2}$$

where $C = 1$. We want to admit, that in the case of the Hausdorff metric the metric projection is not Lipschitz continuous with respect to the set even on the plane $\mathbb{R}^2$. Moreover, there is no Lipschitz (or even uniformly continuous) selector $f\colon \mathbb{F}(\mathcal{H}) \to \mathcal{H}$ in the Hilbert space. The latter means that there is no such a Lipschitz (or uniformly) continuous function satisfying $f(A) \in A$ for any $A \in \mathbb{F}(\mathcal{H})$, see [7, Corollary 5]. It was proved in [1, 2] that the Pliś metric is the unique (to a certain degree) metric in $\mathcal{H}$ with the Lipschitz property of the metric projection in the sense of Formula (2).

In the present paper we want to clarify the situation with the Lipschitz property of the metric projection on $\mathbb{F}(E)$ in a real Banach space $E$. The question under consideration is the following: when does formula (2) take place in a Banach space with some positive constant $C$?

Let $\mathcal{H}$ be a real Hilbert space with the inner product $(\cdot, \cdot)$, $E$ be a real Banach space and $E^*$ its dual. For $x \in E$ and $p \in E^*$ denote by $(p, x)$ the value of functional $p$ at the point $x$. Put $B_r(a) = \{x \in E \mid \|x - a\| \le r\}$. We denote by $\mathbb{F} = \mathbb{F}(E)$ the set of all nonempty closed convex bounded subsets from $E$. The distance from the point $x \in E$ to the set $A \subset E$ is defined by the formula $\operatorname{dist}(x, A) = \inf\limits_{a \in A} \|x - a\|$. Denote the *metric projection* of the point $x \in E$ on the set $A \subset E$ as

$$P_A(x) = \{a \in A \mid \ \|x - a\| = \operatorname{dist}(x, A)\}.$$

By the separation theorem for any closed convex bounded subsets $A_1, A_2 \subset E$ we get

$$h(A_1, A_2) = \sup_{\|p\|_* = 1} |s(p, A_1) - s(p, A_2)|, \tag{3}$$

where $s(p, A) = \sup_{x \in A}(p, x)$ is the supporting function of the set $A$. From Formula (3) it is easy to see that $h(A_1, A_2) \le \rho(A_1, A_2)$ for any $A_1, A_2 \in \mathbb{F}(E)$.

For any vector $p \in \mathcal{H}$ and any $A \in \mathbb{F}$ denote by $A(p) = \{x \in A \mid (p, x) = s(p, A)\}$ the subdifferential of the supporting function $s(\cdot, A)$ at the point $p$.

For a set $A \in \mathbb{F}$ and a point $a \in A$ define the normal cone as follows

$$N(A, a) = \{p \in E \mid (p, a) \ge s(p, A)\}.$$

Put $\operatorname{diam} A = \sup_{x,y \in A} \|x - y\|$. Denote the segment with the endpoints $a, b \in E$ by $[a, b]$.

For the points $a, b \in E$ we shall define by $\overrightarrow{[a, b)} = \{a + t(b - a) \mid t \ge 0\}$ the ray with the vertex $a$ and the directing vector $b - a$.

We shall denote the *convex hull* of the set $A \subset E$ by $\operatorname{co} A$.

We define by $\operatorname{int} A$, $\partial A$ the *interior* and the *boundary* of the set $A$, respectively.

The function $\delta\colon [0,2] \to [0,1]$

$$\delta(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| \;\middle|\; \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\right\}$$

is called the *modulus of convexity* for the space $E$ ([6]). The space $E$ is called *uniformly convex* if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$.

The function $\varrho\colon [0,+\infty) \to [0,+\infty)$

$$\varrho(\tau) = \sup\left\{\frac{1}{2}\|x-y\| + \frac{1}{2}\|x+y\| - 1 \;\middle|\; \|x\| = 1,\ \|y\| = \tau\right\}$$

is called the *modulus of smoothness* for the space $E$ ([6]). The space $E$ is called *uniformly smooth* if $\lim\limits_{\tau\to+0} \frac{\varrho(\tau)}{\tau} = 0$.

Several times we will need the next Kadeč theorem [6, Remark, p. 58]. Let $\varepsilon = \varepsilon(d)$ be the least upper bound of the diameters of sets cut off from the unit sphere $\partial B_1(0) \subset E$ by hyperplanes determined by norm 1 functionals at distance $1-d$ from the origin. Then the function inverse to $\varepsilon(d)$ is the modulus of convexity $\delta(\varepsilon)$ of $E$.

## 2. The property of three perpendiculars (P3P property)

We shall call a closed subspace of codimension 1 *hyperplane*.

**Definition 2.1.** Let $E$ be a strictly convex Banach space. Suppose that there exists such $C > 0$ that for any hyperplane $H \subset E$ and any hyperplane $H_0 \subset H$ (i.e. codimension of $H_0$ in E equals 2) for any $x \notin H$, $z = P_H x$, $a = P_{H_0} x$, $b = P_{H_0} z$ we have the inequality

$$\|z-a\| \leq C\|z-b\|.$$

Then we shall say that the space $E$ has the *property of three perpendiculars*, or, simply, the *P3P property*.

Note that the Hilbert space has the P3P property with constant $C = 1$ and $a = b$ (in notations of Definition 2.1).

**Theorem 2.2.** *Let $E$ be a uniformly convex Banach space with the P3P property. Then the metric projection operator of a point $x \in E$ on a set from the class $\mathbb{F}(E)$ is Lipschitz continuous with respect to the set in the Pliś metric.*

**Proof.** Without loss of generality assume that $x = 0$. Let $A_i \in \mathbb{F}(E)$, $a_i = P_{A_i}0$, here and below $i = 1,2$. Define $r_i = \|a_i\|$ and assume that $r_1 \leq r_2$. Let $\|p_i\|_* = 1$ be such vectors that $(p_i, a_i) = \|a_i\|$. Put $H_i = \{x \in E \mid (p_i, x) = (p_i, a_i)\}$.

If $\operatorname{dist}(a_2, H_1) \geq r_1$, then, by (3), from the estimate $\operatorname{dist}(a_2, H_1) \leq h = h(A_1, A_2)$ we get $r_1 \leq h$ and by the inequality $r_2 - r_1 \leq h$

$$\|a_1 - a_2\| \leq \|a_1\| + \|a_2\| = r_1 + r_2 \leq 3h \leq 3\rho(A_1, A_2). \tag{4}$$

In the case $a_1 = 0$ we have $r_2 \le h$ and formula (4) is valid.

Further consider the situation when $\operatorname{dist}(a_2, H_1) < r_1$ and $a_1 \neq 0$.

Denote $H = \{x \in E \,|\, (p_1, x) = (p_1, a_2)\}$. Let $H_0 = H \cap H_2$. $h(H_1, H) \le h$; $z = \{\lambda a_1 \,|\, \lambda \ge 0\} \cap H$. It is easy to see that $z = P_H 0$.

By (3) we have the inequality $\operatorname{dist}(a_2, H_1) \le h$ and $\|a_1 - z\| \le h$.

From similarity arguments we may assume that $\|a_2\| = 1$. Let $u = \overrightarrow{[0, z)} \cap \partial B_1(0)$, note that $(p_1, u) = 1$.

For $d \in [0, 1]$ define $\overline{H}_d = \{x \in E \mid (p_1, x) = (1 - d)(p_1, u)\}$, $S_d = B_1(0) \cap \overline{H}_d$, in particular $\overline{H}_0 = \{x \in E \,|\, (p_1, x) = (p_1, u) = 1\}$.

Due to the continuity of $d \to S_d$ ([3, Theorem 3.1]) we can choose $d_0 > 0$ such that $\operatorname{diam} S_d \le \frac{1}{4}$ for all $d \in [0, d_0]$. Note that $d_0$ does not depend on the position of the point $u$ on the unit sphere.

We shall associate $d \in [0, 1]$ with the distance $\|z - u\|$. Otherwise, points $\{z, b, a_2\}$ belong to $\overline{H}_d$.

We prove that for all $d \in [d_0, 1]$ (note that $\{z, a_2\} \subset \overline{H}_d$, $d = \|z - u\|$) we have the Lipschitz property of the metric projection in the Pliś metric.

Choose $y \in A_2(-p_1)$ (note that $a_1 \in A_1(-p_1)$) such that,

$$\|a_1 - y\| \le \rho = \rho(A_1, A_2) \tag{5}$$

We want to admit that

$$y \in \{x \in E \,|\, (p_1, x) \ge (p_1, a_2)\} \cap \{x \in E \mid (p_2, x) \ge (p_2, a_2)\}.$$

From the triangle $Ozy$

$$\|z - y\| \ge \|y\| - \|z\| \ge 1 - (1 - d) \ge d_0.$$

By the Kadeč theorem $\|z - a_2\| \le \delta^{-1}(d)$ and

$$\|a_1 - a_2\| \le \|a_1 - z\| + \|z - a_2\| \le \|u - z\| + \|z - a_2\| \le d + \delta^{-1}(d). \tag{6}$$

We have
$$\rho(A_1, A_2) \ge \|a_1 - y\| \ge \|z - y\| - \|z - a_1\|. \tag{7}$$

Note that $\|z - a_1\| \le h$.

If $h \ge \frac{d_0}{2}$, then $\rho \ge h \ge \frac{d_0}{2}$ and

$$d + \delta^{-1}(d) \le \frac{2}{d_0}\left(d + \delta^{-1}(d)\right)\rho \le \frac{2}{d_0}\left(1 + \delta^{-1}(1)\right)\rho(A_1, A_2). \tag{8}$$

If $h < \frac{d_0}{2}$, then by (7)

$$\rho(A_1, A_2) \ge \|z - y\| - \|z - a_1\| \ge d_0 - h \ge \frac{d_0}{2}$$

and we again obtain estimate (8).

Thus, by (6) $$\|a_1 - a_2\| \le \frac{2}{d_0}\left(1+\delta^{-1}(1)\right)\rho(A_1, A_2).$$

Now consider the case $d \in [0, d_0]$. Analogously with the point $a_2$ (by (3)) $\operatorname{dist}(y, H_1) \le h$ and hence $\operatorname{dist}(y, H) \le \operatorname{dist}(H, H_1) + \operatorname{dist}(y, H_1) \le 2h$.

Put $y' = P_H y$ and $b = P_{H_0} z$.

Consider in $S_0 = \overline{H}_0 \cap B_1(0)$ radius $[0, a_3]$: $\|a_3\| = 1$ and the segment $[0, a_3]$ is parallel to the segment $[z, a_2]$. The polygon $0a_3a_2z$ is the trapeze with the large base $[0, a_3]$, $\|a_3\| = 1$, and the small base $[z, a_2]$, $\|z - a_2\| \le \frac{1}{4}$.

Let $a_4 \in [0, a_3]$ be such that $\|a_4\| = \|z - a_2\|$. Then $0a_4a_3z$ is a parallelogram and the segment $[0, z]$ is parallel to the segment $[a_2, a_4]$. Hence $a_2 = P_H a_4$.

We claim that $\|z - b\| \le \|z - y'\|$. The last inequality takes place because the points $y'$ and $z$ in $H$ are separated by the plane $H_0$. Indeed,

$$\{z, [a_2, a_4]\} \subset B_1(0) \subset \{x \in E \mid (p_2, x) \le (p_2, a_2)\}, \qquad a_2 \in H_0,$$

$a_2, y' \in H$, vectors $a_4 - a_2$, $y - y'$ are aligned and $y \in \{x \in E \mid (p_2, x) \ge (p_2, a_2)\}$. Thus $[y, y'] \subset \{x \in E \mid (p_2, x) \ge (p_2, a_2)\}$.

We have $$\|a_1 - a_2\| \le \|a_1 - z\| + \|z - a_2\| \le h + \|z - a_2\|. \tag{9}$$

By the P3P property we get

$$\|z - a_2\| \le C\|z - b\| \le C\|z - y'\|. \tag{10}$$

From Formulae (9, 10) we have

$$\|a_1 - a_2\| \le h + C\|z - y'\|. \tag{11}$$

From the previous estimates we obtain that

$$\|z - y'\| \le \|z - a_1\| + \|a_1 - y\| + \|y - y'\|,$$

where $\|z - a_1\| \le h$, $\|a_1 - y\| \le \rho$, $\|y - y'\| \le 2h$. Thus

$$\|a_1 - a_2\| \le h + C\rho + 3Ch \le (4C+1)\rho(A_1, A_2). \qquad \square$$

**Theorem 2.3.** *Let $E$ be a uniformly smooth and strictly convex Banach space without the P3P property. Then the metric projection operator of a point $x \in E$ on sets from the class $\mathbb{F}(E)$ is not Lipschitz continuous with respect to the set in the Pliś metric.*

**Proof.** Fix arbitrarily $C > 3$. Suppose that there exist a hyperplane $H$ and a hyperplane (in $H$) $H_0$, $\operatorname{co\,dim} H_0 = 2$, such that for $a_2 = P_{H_0} 0$, $z = P_H 0$, $b = P_{H_0} z$ we have

$$\|z - a_2\| > C\|z - b\|. \tag{12}$$

Note that $\|z\| < \|a_2\| < \|b\|$. Let $a_1 \in \overrightarrow{[0, z)}$ and $\|a_1\| = \|a_2\|$.

By similarity arguments we may assume that $\|a_1\| = 1$. Define the sets $A_2 = \mathrm{co}\{b, a_2\}$ and $A_1 = \mathrm{co}\{a_1, w\}$, where $w = a_1 + (a_2 - b)$. It's easy to see (12) that $2\|a_1\| = 2 \geq \|z - a_2\| > C\|z - b\|$, thus $\|z - b\| < 2/C$ and $\|z - b\| \to 0$ as $C \to +\infty$. We claim that

$$\|z - a_1\| \leq \|z - b\| \tag{13}$$

for sufficiently large $C$. Indeed, put $c = a_1 + (b - z)$. Consider the function $f$ from segment of length 2 from the affine hull $\mathrm{aff}\{a_1, c\}$ with centerpoint $a_1$ to $\mathbb{R} = \mathrm{aff}\{z, a_1\}$ (positive direction of $\mathbb{R}$ along vector $z - a_1$) with the graph $\partial B_{\|a_1\|}(0) \cap \{(\alpha, \beta) \mid \alpha \in \mathrm{aff}\{a_1, c\},\ \beta \in \mathbb{R},\ \beta \leq 0\}$, $f(a_1) = a_1$. From the uniform smoothness of the space $E$ we have $f'(a_1) = 0$ and

$$f(c) = f(a_1) + o(\|a_1 - c\|) = f(a_1) + o(\|z - b\|),$$

i.e.
$$\|z - a_1\| = f(c) - f(a_1) = o(\|z - b\|)$$

(here $o(t)$ is o-small function with $\lim\limits_{t\to+0} \frac{o(t)}{t} = 0$). We want to admit that $o(\|z-b\|)$ does not depend on position of the point $a_1$ (and positions of other points) by uniform smoothness. Thus (13) takes place.

We have the next estimates

$$\|a_1 - b\| \leq \|a_1 - z\| + \|z - b\| \leq 2\|z - b\|, \tag{14}$$

$$\|a_2 - b\| \geq \|a_2 - z\| - \|z - b\| \geq (C - 1)\|z - b\|. \tag{15}$$

By Formula (15)

$$\begin{aligned} \|a_1 - a_2\| &\geq \|a_2 - b\| - \|a_1 - b\| \\ &\geq (C - 1)\|z - b\| - \|z - b\| - \|a_1 - z\| \\ &= (C - 2)\|z - b\| - \|a_1 - z\| \geq (C - 3)\|z - b\|. \end{aligned} \tag{16}$$

By Formula (14) the distance between parallel segments $A_1$ and $A_2$ in the Hausdorff=Pliś sense equals $\|a_1 - b\| \leq 2\|z - b\|$. From (16) we see that the distance between the metric projections of zero on the sets $A_1$ and $A_2$ is greater than $(C - 3)\|z - b\|$. Hence

$$\frac{\|a_1 - a_2\|}{\rho(A_1, A_2)} = \frac{\|a_1 - a_2\|}{\|a_1 - b\|} \geq \frac{C - 3}{2}.$$

The Lipschitz condition for the metric projection does not hold. □

So, the Lipschitz property for the metric projection in the Pliś metric is equivalent to the P3P property for a uniformly smooth and uniformly convex space.

## 3. Uniform skewness (US) of the ball and regular spaces

Consider a real uniformly convex and uniformly smooth Banach space $E$. Let $\|e\| = 1$ and $\|p\|_* = 1$ be such that $(p, e) = 1$. For $d \in [0, 1)$ put $H_d = H_d(e) = \{x \in E \mid (p, x) = 1 - d\}$ and $S_d = S_d(e) = H_d \cap B_1(0)$. Note that $S_0 = \{e\}$. Let $z = H_d(e) \cap [0, e]$.

**Definition 3.1.** If the value

$$\lim_{d\to+0}\sup_{\|e\|=1}\frac{\sup\{\|z-a\|\mid\|a\|=1,\ a\in H_d(e)\}}{\inf\{\|z-w\|\mid\|w\|=1,\ w\in H_d(e)\}}$$

is finite, then we shall say that the ball in the space $E$ has *uniform skewness*, or the ball is uniformly skewed.

Note that if $d\geq d_0>0$, then $\sup\limits_{\|e\|=1}\ldots$ from Definition 3.1 is bound from above.

Indeed, by the Kadeč theorem $\operatorname{diam}S_d(e)\leq\delta^{-1}(d)$ and hence (in notations of Definition 3.1) $\|z-a\|\leq\delta^{-1}(d)$. From the triangle $0zw$ $\|z-w\|\geq\|w\|-\|z\|=d$.

$$\frac{\sup\{\|z-a\|\mid\|a\|=1,\ a\in H_d(e)\}}{\inf\{\|z-w\|\mid\|w\|=1,\ w\in H_d(e)\}}\leq\frac{\delta^{-1}(d)}{d}\leq\frac{\delta^{-1}(d_0)}{d_0}$$

(the function $\frac{\delta^{-1}(d)}{d}$ decreases, see [3, Lemma 2.1]).

From the definition of uniform skewness we have the P3P property. If the converse implication (US$\Leftarrow$ P3P) holds we shall say that the space $E$ is *regular.*

**Lemma 3.2.** *Let $A\subset\mathbb{R}^2$ be a convex compact set with $C^1$-smooth boundary $\partial A$ and $0\in\operatorname{int}A$. Consider two points $a,w\in\partial A$, $r_1=\operatorname{dist}(0,w)$, $r_2=\operatorname{dist}(0,a)$. Then we can find a point $b\in\partial A$ with*

$$\operatorname{tg}\alpha\geq\frac{1}{2\pi}\ln\frac{r_2}{r_1}$$

*where $\alpha$ is the angle between vector $b$ and the outer normal to the curve $\partial A$ at the point $b$.*

**Proof.** For the polar angle $\varphi\in[0,2\pi)$ define the ray $R_\varphi$ with the directing vector $(\cos\varphi,\sin\varphi)$. The ray $R_\varphi$ has exactly one intersection with $\partial A$ at the point $(x(\varphi),y(\varphi))$. Denote $r(\varphi)=\sqrt{x^2(\varphi)+y^2(\varphi)}$. Note that $(x'(\varphi),y'(\varphi))$ is the tangent vector to the curve $\partial A$ at the point $(x(\varphi),y(\varphi))$. The points $w$ and $a$ have polar coordinates $(r_1,\varphi_1)$ and $(r_2,\varphi_2)$ respectively.

Let $(\cos\psi(\varphi),\sin\psi(\varphi))$ be the outer normal vector to the curve $\partial A$ at the point $(x(\varphi),y(\varphi))$. Then

$$x'\cos\psi+y'\sin\psi=0. \tag{17}$$

The angle between tangent line and normal line to the curve $\partial A$ at the point $(x(\varphi),y(\varphi))$ equals $\alpha(\varphi)=\varphi-\psi(\varphi)$. From the equalities

$$x'=r'\cos\varphi-r\sin\varphi,\qquad y'=r'\sin\varphi+r\cos\varphi$$

and Formula (17) we obtain that $r'\cos(\varphi-\psi)-r\sin(\varphi-\psi)=0$, and hence

$$\operatorname{tg}\alpha(\varphi)=(\ln r(\varphi))'.$$

By the mean value theorem there exists $\varphi_0 \in [\varphi_1, \varphi_2]$ such that

$$(\ln r)'\,|_{\varphi=\varphi_0} = \frac{\ln r_2 - \ln r_1}{\varphi_2 - \varphi_1} \geq \frac{1}{2\pi} \ln \frac{r_2}{r_1}. \qquad \square$$

It follows from Lemma 3.2 that uniformly convex and uniformly smooth finite dimensional space with the P3P property has US ball and thus regular. The result follows from equivalence of norms in a finite dimensional space.

## 4. Local moduli of convexity and smoothness

Consider a real uniformly convex and uniformly smooth Banach space $E$. By the Kadeč theorem (in notations of paragraph 3)

$$\sup_{\|a\|=1} \operatorname{diam} S_d(a) = \delta^{-1}(d)$$

where $\delta^{-1}(\cdot)$ is the inverse function for the modulus of convexity $\delta(\cdot)$, that means, $\delta^{-1}(\delta(\varepsilon)) = \varepsilon$ for $\varepsilon \in [0, 2)$.

For $\|a\| = 1$ consider the function

$$\varepsilon_a(d) = \operatorname{diam} S_d(a).$$

Note that $\varepsilon_a(d) \leq \varepsilon(d)$ for all $\|a\| = 1$. Let's show that the function $\varepsilon_a(\cdot)$ is continuous and strictly monotone. The continuity of $\varepsilon_a(\cdot)$ follows from the definition, the equality $S_d(a) = H_d(a) \cap B_1(0)$, the continuity of the mapping $d \to H_d(a)$ and [3, Theorem 3.1].

Suppose that $0 < d_1 < d_2 < 1$, $b_i \in H_{d_1}(a) \cap \partial B_1(0)$, here and below $i = 1, 2$. Then there exist points $c_i \in H_1(a) \cap \partial B_1(0)$ with $[c_1, c_2] || [b_1, b_2]$ and $\|c_1 - c_2\| = 2$.

Define $\tilde{a}_i = H_{d_2}(a) \cap [c_i, b_i]$. The polygon $b_1c_1c_2b_2$ is a trapeze with larger base $[c_1, c_2]$ and thus

$$\|\tilde{a}_1 - \tilde{a}_2\| > \|b_1 - b_2\|.$$

Continue the segment $[\tilde{a}_1, \tilde{a}_2]$ to the segment $[a_1, a_2]$ where $a_i \in H_{d_2}(a) \cap \partial B_1(0)$.

Then

$$\|a_1 - a_2\| > \|b_1 - b_2\|$$

and moreover $\|c_i - \tilde{a}_i\| \leq 2 - (d_2 - d_1)$, $\|\tilde{a}_i - b_i\| \geq d_2 - d_1$. Then by [4, Lemma 1.1] there exists $\alpha = \alpha(d_1, d_2) > 0$ ($\alpha$ does not depend on $a_1, a_2$) such that $B_\alpha(\tilde{a}_i) \subset B_1(0)$ or in other words

$$\|a_1 - a_2\| \geq 2\alpha + \|b_1 - b_2\|.$$

This means that $\varepsilon_a(d_2) \geq 2\alpha + \varepsilon_a(d_1)$, i.e. $\varepsilon_a$ is a strictly monotone function.

The function $\varepsilon_a(\cdot)$ has the inverse function $\delta_a(\cdot)$.

**Definition 4.1.** We shall call the function $\delta_a$ the *local modulus of convexity at the point $a \in \partial B_1(0)$.*

**Definition 4.2.** For a point $a \in \partial B_1(0)$ the *local modulus of smoothness* $\varrho_a$ is defined as follows

$$\varrho_a(\tau) = \sup\left\{ \frac{\|a+y\|}{2} + \frac{\|a-y\|}{2} - 1 \;\middle|\; \|y\| = \tau \right\}.$$

We claim that for any $0 < t < s$:
$$\frac{\varrho_a(t)}{t} \le \frac{\varrho_a(s)}{s}. \tag{18}$$

The proof is analogous to the proof for the modulus of smoothness. Choose $\|a\| = \|y\| = 1$. Define $\|p\|_* = \|q\|_* = 1$ with $(p, a+ty) = \|a+ty\|$ and $(q, a-ty) = \|a-ty\|$. Then

$$\frac{\|a+ty\| + \|a-ty\| - 2}{2t} = \frac{(p+q,a)-2}{2t} + \frac{(p-q,y)}{2} = \Lambda.$$

From the inequality $(p+q,a) \le 2$ we have

$$\frac{(p+q,a)-2}{2t} \le \frac{(p+q,a)-2}{2s}$$

and
$$\Lambda \le \frac{(p+q,a)-2}{2s} + \frac{(p-q,y)}{2} = \frac{(p,a+sy)+(q,a-sy)-2}{2s}$$
$$\le \frac{\|a+sy\| + \|a-sy\| - 2}{2s}.$$

From the last estimate we obtain Formula (18).

Note that $\varrho(\tau) = \sup\limits_{\|a\|=1} \varrho_a(\tau)$ and hence $\varrho^{-1}(d) \le \varrho_a^{-1}(d)$ for all $\|a\| = 1$ and $d \ge 0$.

## 5. Conditions for the Lipschitz continuity of the metric projection in the Pliś metric

**Theorem 5.1.** *Let $E$ be a real uniformly convex and uniformly smooth Banach space. Denote*
$$f(d) = \inf_{\|a\|=1} \frac{\varrho_a^{-1}(d)}{\varepsilon_a(d)}. \tag{19}$$

(1) *If $\liminf\limits_{d\to+0} f(d) = \Lambda > 0$ then the ball in $E$ is US and the space $E$ is regular.*

(2) *If $\liminf\limits_{d\to+0} f(d) = 0$ then the ball in $E$ is not US and the space $E$ is not regular.*

**Proof.** Fix $\|a\| = 1$, $d \in [0,1)$. Let $z = H_d(a) \cap [0,a]$.

Fix $a_2 \in H_d(a)$, $\|a_2\| = 1$. By Definition 4.1 $\|z - a_2\| \le \varepsilon_a(d)$. Let also $a_2$ be such a point that $\frac{1}{2}\varepsilon_a(d) \le \|a_2 - z\|$. The reason for the last restriction is that $\sup_{\|x\|=1,\ x\in H_d} \|x - z\|$ is realized in such points $a_2$. Finally, we consider $a_2 \in H_d(a) \cap \partial B_1(0)$ such that

$$\tfrac{1}{2}\varepsilon_a(d) \le \|a_2 - z\| \le \varepsilon_a(d). \tag{20}$$

Consider $\|w\|=1$, $w \in H_d(a)$ and the point $u = \overrightarrow{[w,z)} \cap \partial B_1(0)$, $u \neq w$. Without loss of generality we may assume that $\|u-z\| \geq \|z-w\|$, otherwise we replace $u \leftrightarrow w$. By Definition 4.2 we have $\varrho_a(\|z-w\|) \leq d$ because

$$\varrho_a(\|z-w\|) \leq \frac{1}{2}\mathrm{dist}\,(w, H_0(a)) + \frac{1}{2}\mathrm{dist}\,(u, H_0(a)).$$

Let $v = \overrightarrow{[0,w)} \cap H_0(a)$. From the similarity of the triangles $0zw$ and $0av$ we obtain

$$\varrho_a(\|a-v\|) = \varrho_a\left(\frac{\|z-w\|}{1-d}\right) \geq \frac{1}{2}d,$$

i.e.

$$(1-d)\varrho_a^{-1}\left(\frac{1}{2}d\right) \leq \|z-w\| \leq \varrho_a^{-1}(d)\,. \tag{21}$$

(1) Let $\liminf_{d\to+0} f(d) = \Lambda > 0$. Note that we should consider sufficiently small $d>0$, say $d \in (0, \frac{1}{2})$. By the right inequality in (20) and the left inequality in (21) we get

$$\frac{\|z-a_2\|}{\|z-w\|} \leq \frac{\varepsilon_a(d)}{(1-d)\varrho_a^{-1}\left(\frac{1}{2}d\right)}.$$

Repeating speculations about the monotonicity of the function $\varepsilon_a(\cdot)$ we obtain that $\varepsilon_a(d) \leq 2\varepsilon_a\left(\frac{1}{2}d\right)$ and

$$\frac{\|z-a_2\|}{\|z-w\|} \leq \frac{2\varepsilon_a\left(\frac{1}{2}d\right)}{(1-d)\varrho_a^{-1}\left(\frac{1}{2}d\right)}.$$

The last fraction is bounded from above due to condition 1). Thus the ball is US and the space $E$ is regular.

(2) Let $\liminf\limits_{d\to+0} f(d) = 0$. Then there exist sequences $\{a_k\} \subset \partial B_1(0)$ and $d_k \to 0$ with

$$\lim_{k\to\infty} \frac{\varrho_{a_k}^{-1}(d_k)}{\varepsilon_{a_k}(d_k)} = 0.$$

Choosing $a_2^k \in H_{a_k}(d_k) \cap \partial B_1(0)$ that satisfies the left inequality in (20) and $w_k \in H_{a_k}(d_k) \cap \partial B_1(0)$ that satisfies the right inequality in (21) we have

$$\frac{\|z-w_k\|}{\|z-a_2^k\|} \leq \frac{\varrho_{a_k}^{-1}(d_k)}{\frac{1}{2}\varepsilon_{a_k}(d_k)} \to 0, \quad k \to \infty.$$

Hence the ball is not US and the space $E$ is not regular. □

**Corollary 5.2.** *Let $E$ be a uniformly convex and uniformly smooth Banach space. If $\dim E < \infty$ then conditions* (1) *and* (2) *of the above theorem give conditions for the fulfillment of the P3P property. Namely, in the case* (1) *the P3P property is fulfilled, whereas in the case* (2) *it is not.*

The proof of Corollary 5.2 follows from Lemma 3.2 and Theorem 5.1.

**Corollary 5.3.** *Let $E$ be a uniformly convex and uniformly smooth Banach space. If moduli of uniform convexity and uniform smoothness are of the second order (i.e. $\delta(t) \asymp t^2$ and $\varrho(t) \asymp t^2$ as $t \to +0$) then the P3P property is fulfilled.*

**Proof.** For any $\|a\| = 1$ we have the inequalities $\varepsilon_a(d) \le \delta^{-1}(d)$, $\varrho_a^{-1}(d) \ge \varrho^{-1}(d)$.

Then
$$\frac{\varrho_a^{-1}(d)}{\varepsilon_a(d)} \ge \frac{\varrho^{-1}(d)}{\delta^{-1}(d)} \asymp \frac{C_1\sqrt{d}}{C_2\sqrt{d}} = C_3 > 0, \quad d \to 0.$$

By Theorem 5.1,(1), the space $E$ is regular and the P3P property is valid. □

We consider some examples.

**Example 5.4.** (1) Suppose that the ball in $\mathbb{R}^3$ coincides with the graph of the function $f(x_1, x_2) = x_1^2 + x_2^{\frac{4}{3}}$ in the vicinity of the origin. The Lebesgue set $\{(x_1, x_2) \,|\, f(x_1, x_2) \le t\}$ is ellipse-like with points $a_2(t, 0)$ and $w(t^{\frac{3}{2}}, 0)$ on its boundary. By Corollary 5.2 the P3P property does not fulfill in 3-dimensional space with this ball.

(2) Suppose that the ball in $\mathbb{R}^3$ coincides with the graph of the function $f(x_1, x_2) = |x_1|^\alpha + |x_2|^\alpha$, $\alpha > 2$, in the vicinity of the origin (ball of $\alpha$-norm). Similar to (1) this ball has not the P3P property. Indeed, take a point $(t, 0, 0)$, $t > 0$. Then

$$(t + x_1)^\alpha - t^\alpha \sim \alpha t^{\alpha-1} x_1, \quad x_1 \to +0, \quad x_2^\alpha + t^\alpha - t^\alpha = x_2^\alpha.$$

Thus, our shift from the point $(t, 0, 0)$ along the vector $(0, 0, 1)$ on the distance $h > 0$ leads to the shift $\asymp h$ along the axis $x_1$ and $\asymp h^{1/\alpha}$ along the axis $x_2$. By Corollary 5.2 the P3P property does not valid in 3-dimensional space with this ball. □

What will be in a uniformly convex and uniformly smooth space with moduli of different order: $\delta(t) = o(\varrho(t))$, $t \to 0$? We have not clear answer in this case, because we have only estimate from below of local moduli via global as in Corollary 5.3. We have the next

**Conjecture 5.5.** *If in a real uniformly convex and uniformly smooth Banach space $E$ $\delta(t) = o(\varrho(t))$, $t \to 0$, then the P3P property does not hold in $E$.*

**Acknowledgements.** The author is greatful to G. E. Ivanov for the proposed proof of Lemma 3.2. The work was supported by RFBR grant 18-01-00209.

## References

[1] M. V. Balashov: *The Plis metric and Lipschitz stability of minimization problems*, Sbornik: Mathematics 210/7 (2019) 911–927.

[2] M. V. Balashov: *Lipschitz stability of extremal problems with a strongly convex set*, J. Convex Analysis 27/1 (2020) 103–116.

[3] M. V. Balashov, D. Repovš: *Uniform convexity and the splitting problem for selections*, J. Math. Anal. Appl. 360 (2009) 307–316.

[4] M. V. Balashov, D. Repovš: *Weakly convex sets and modulus of nonconvexity*, J. Math. Anal. Appl. 371/1 (2010) 113–127.

[5] M. V. Balashov, D. Repovš: *On Pliś metric on the space of strictly convex compacta*, J. Convex Analysis 19/1 (2012) 171–183.

[6] J. Diestel: *Geometry of Banach Spaces – Selected Topics*, Lecture Notes in Mathematics 485, Springer, Berlin (1975).

[7] J. Lindenstrauss: *On nonlinear projections in Banach spaces*, Michigan Math. J. 11/3 (1964) 263–287.

[8] A. Pliś: *Uniqueness of optimal trajectories for non-linear control system*, Ann. Polom. Math. 29 (1975) 397–401.