Journal of Convex Analysis
Volume 27 (2020), No. 3, 935–942

Describing Multiplicative Convex Functions*

Pablo Jiménez-Rodríguez
Dept. of Mathematical Sciences, Kent State University, Kent, OH 44242, U.S.A.
pjimene1@kent.edu

Maria E. Martínez-Gómez
Dep. de Análisis Matemático y Matemática Aplicada, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain
mariae24@ucm.es

Gustavo A. Muñoz-Fernández
IMI, Dep. de Análisis Matemático y Matemática Aplicada, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain
gustavo_fernandez@mat.ucm.es

Juan B. Seoane-Sepúlveda
IMI, Dep. de Análisis Matemático y Matemática Aplicada, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain
jseoane@ucm.es

Received: March 27, 2019
Accepted: July 7, 2019

Multiplicative convex functions have been mimicking the behavior of convex functions, but focusing on geometric average, instead of arithmetic average. In the present note we propose a different definition which focuses on the arithmetic operations taking part and we study the resulting functions. A characterization is also provided.

Keywords: Convexity, convex function, multiplicative convex.

2010 Mathematics Subject Classification: 26A51, 30C45.

1. Introduction

The theory of convex functions keeps playing a central role in operator theory, in real analysis and in some realms of applied mathematics, such as management science or optimization theory (we refer the interested reader to [4, 2, 1] for applications of the property of convexity). Since the beginning of their study by Jensen, they have been thoroughly described and many of their properties have been unveiled.

We recall that a function $f\colon V \to \mathbb{R}$ (where V is a vector space over $\mathbb{R}$) is called *convex* if, whenever $x, y \in V$ and $0 \leq \lambda \leq 1$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

*Supported by Grant PGC2018-097286-B-I00.

ISSN 0944-6532 / $ 2.50

The study of convex functions has extended to the consideration of other inequalities. In this direction, Niculescu proposed the following definition (see [5]):

Let $I \subseteq (0,\infty)$ be an interval. A function $f\colon I \to (0,\infty)$ is called *multiplicative convex* if, for every $x, y \in I$ and $\lambda \in [0,1]$, we have

$$f(x^{1-\lambda}y^{\lambda}) \leq f(x)^{1-\lambda}f(y)^{\lambda}. \tag{1}$$

Functions satisfying inequality (1) are also known as *GG-convex functions*, since they are supposed to substitute the arithmetic mean in the inequality that defines the convex functions by the geometric mean. Yet, the definition of the multiplicative convex functions could be regarded as a way of upgrading the operations that take part in the definition of convex functions. In this direction, we see that the addition operation turns into multiplication operation, and the multiplication operation turns into power operation.

If we adopt this point of view, the upgrade is yet not complete. λ and $\mu = 1-\lambda$ are related by $\lambda + \mu = 1$. If we keep in mind that addition turns into multiplication, then the exponents λ and μ should be related by $\lambda \cdot \mu = 1$. Following that idea, we propose the following variation for the definition of multiplicative convex functions:

Definition 1.1. Let $f\colon (0,\infty) \to [0,\infty)$ be such that $f(1) = 1$. We will say that f is *multiplicative convex* if, for every $\mu > 0$ and $x, y \geq 0$ we have

$$f(x^{\mu}y^{1/\mu}) \leq f(x)^{\mu}f(y)^{1/\mu}. \tag{2}$$

In particular, by setting $\mu = 1$, we obtain $f(xy) \leq f(x)f(y)$ for every $x, y \geq 0$.

The aim of this paper is to study the properties that the multiplicative convex functions, in the setting of definition 1.1, verify. In fact, we will be able to completely describe what those functions look like.

2. Characterizing multiplicative convex functions

We will start by studying some of the features that the inequality introduced in Definition 1.1 enjoys. The first result tells us about the increasing/decreasing behavior of such functions.

Theorem 2.1. *Let f be a multiplicative convex function. Then, f is either monotone or it decreases until $x = 1$ and then increases (for simplicity, we will call the functions of the third kind increasing-decreasing functions).*

Proof. Assume there is $0 < \theta < 1$ so that $f(\theta) \leq 1$. Let us show that in that case $f(\psi) \leq 1$ for every $0 < \psi < 1$. Having shown that, we will have also proved that if there is $0 < \theta < 1$ so that $f(\theta) > 1$, then $f(\psi) > 1$ for every $0 < \psi < 1$ (otherwise it contradicts the previous statement). Indeed, let $0 < \psi \leq \theta^2$. Then, we can find $1 \leq \mu$ so that $\psi = \theta^{\mu+1/\mu}$ and, hence,

$$f(\psi) = f(\theta^{\mu}\theta^{1/\mu}) \leq f(\theta)^{\mu}f(\theta)^{1/\mu} \leq 1.$$

Next, let $\theta^2 < \psi < \theta$. We can find $1 \leq \mu < 2$ with $\psi = \theta^\mu$ and, therefore,

$$f(\psi) = f(\theta^\mu 1^{1/\mu}) \leq f(\theta)^\mu f(1)^{1/\mu} \leq 1.$$

Finally, if $\theta < \psi < 1$, then there is $\mu \geq 1$ so that $\psi = \theta^{1/\mu}$ and hence

$$f(\psi) = f(1^\mu \theta^{1/\mu}) \leq f(\theta)^{1/\mu} \leq 1.$$

Assume then that $x < y$. We can write

$$f(x) = f\left(xy\frac{1}{y}\right) \leq f(y) f\left(\frac{x}{y}\right) \leq f(y)$$

and hence f is increasing.

Let us define $g(x) = f(1/x)$. Then, g is also a multiplicative convex function. Furthermore, if there is $x > 1$ so that $f(x) \leq 1$, then $g(1/x) \leq 1$ with $\frac{1}{x} < 1$ and hence g is increasing (and therefore f is decreasing).

If, on the contrary, there is $x > 1$ with $f(x) > 1$, then we would have that $f(x) > 1$ for every $x > 1$ (again, using the function g, which has to be higher than 1 over $(0, 1)$). Hence, if $1 < x < y$, then we can find $0 < \mu < 1$ so that $x = y^\mu$ and therefore

$$f(x) = f(y^\mu 1^{1/\mu}) \leq f(y)^\mu \leq f(y),$$

so that f is increasing on $(1, \infty)$. Via another argument involving the function $g(x)$, we would conclude that f is decreasing on $(0, 1)$. $\square$

The following Lemma will be crucial in the theorems to come.

Lemma 2.2. *Let f be a multiplicative convex function and $q \in \mathbb{Q}^+$. Then, $f(x^q) = f(x)^q$*

Proof. Assume, first, that $q \in \mathbb{N}$.

Since $f(xy) \leq f(x)f(y)$, we obtain $f(x^q) \leq f(x)^q$. On the other hand,

$$f(x) = f((x^q)^{1/q}) \leq f(x^q)^{1/q} f(1)^q = f(x^q)^{1/q},$$

so that $f(x)^q \leq f(x^q) \leq f(x)^q$.

If now q is of the form $\frac{1}{n}$, with $n \in \mathbb{N}$, we obtain, similarly,

$$f(x^{1/n}) \leq f(x)^{1/n} f(1)^n = f(x)^{1/n},$$

being f multiplicative convex. Conversely, $f(x) = f((x^{1/n})^n) \leq f(x^{1/n})^n$, from which $f(x^{1/n}) \geq f(x)^{1/n}$.

For the general case, $f(x^q) = f(x^{n/m}) = f(x^{1/m})^n = f(x)^{n/m}$. $\square$

Theorem 2.3. *Let f be a multiplicative convex function. Then, f is continuous.*

Proof. We claim first that f is continuous at $x = 1$. Indeed, otherwise we can find a sequence $\{x_n\}_{n=1}^{\infty}$ and a number $M > 0$ so that either $x_n \uparrow 1$ or $x_n \downarrow 1$, and $|f(x_n) - 1| > M$ for every $n \in \mathbb{N}$.

Assume that f is increasing. If we are in the case $x_n \uparrow 1$, then $M < 1 - f(x_n) < 1$, so that $f(x_n) < 1 - M < 1$.

Let $n \in \mathbb{N}$. Then, we can find $m_n \in \mathbb{N}$ so that $x_1 \leq x_{m_n}^n$. Then,

$$f(x_1) \leq f(x_{m_n}^n) = f(x_{m_n})^n \to 0 \quad \text{as } n \to \infty$$

and hence $f(x_1) = 0$. On the other hand,

$$1 = f(x_1 \frac{1}{x_1}) \leq f(x_1) f(\frac{1}{x_1}) = 0,$$

and we reach a contradiction.

If we are in the case where $x_n \downarrow 1$, then $f(x_n) - 1 > M$ for every n, so that we have $f(x_n) > M + 1 > 1$. Since f is increasing, we can also deduce that $f(y) > M + 1$ for every $y > 1$.

On the other hand, if $y > 1$, we can find $\mu \geq 1$ so that $f(y)^{1/\mu} < M + 1$. Let $x < 1$ so that $x^\mu y^{1/\mu} > 1$. Then,

$$M + 1 < f(x^\mu y^{1/\mu}) \leq f(x)^\mu f(y)^{1/\mu} < M + 1$$

(let us recall that $f(z) < 1$ for every $z < 1$ if f is an increasing multiplicative convex function), reaching a contradiction.

Assume next that f is decreasing. Then, we just need to apply the previous study to the function $g(x) = f(\frac{1}{x})$.

Assume finally that f is increasing-decreasing. Since, in that case, g is also an increasing-decreasing multiplicative convex function, we may assume, w.l.o.g., that we can find a sequence $x_n \uparrow 1$ and $M > 0$ so that $f(x_n) > M + 1$, for every $n \in \mathbb{N}$. Again, for $n \in \mathbb{N}$, we can find $m_n \in \mathbb{N}$ so that $x_{m_n}^n \geq x_1$. Then,

$$f(x_1) \geq f(x_{m_n}^n) = f(x_{m_n})^n > (M + 1)^n \to \infty \quad \text{as } n \to \infty,$$

and the claim is proved also for this case.

We will prove continuity in the general case. Let $x \in (0, \infty)$ and $\epsilon > 0$. Since f is continuous at 1, we can find $0 < \delta < 1$ so that, if $|1 - \zeta| < \delta$, then $|1 - f(\zeta)| < \frac{\epsilon}{f(x)}$.

Let us assume first that f is increasing (which will also cover the case of f being decreasing, via the use of the function $g(x) = f(\frac{1}{x})$) and let $y > 0$ such that we have $|x - y| < \frac{\delta x}{2} < \frac{x}{2}$. If $y < x$, then

$$\frac{x}{y} - 1 = \frac{x - y}{y} < \frac{\frac{\delta x}{2}}{\frac{x}{2}} = \delta,$$

so that $0 < f\left(\frac{x}{y}\right) - 1 < \frac{\epsilon}{f(x)}$. On the other hand,

$$\begin{aligned}0 < f(x) - f(y) &= f\left(y\frac{x}{y}\right) - f(y)\\ &\leq f(y)\left[f\left(\frac{x}{y}\right) - 1\right] \leq f(x)\left[f\left(\frac{x}{y}\right) - 1\right] < \epsilon.\end{aligned}$$

In the case where f is decreasing-increasing, we may reproduce the argument for f increasing and $x < y$ to show that f is right continuous. We can conclude the situation for $y < x$ using the function $g(x) = f(\frac{1}{x})$. □

With Theorem 2.3, we can extend Lemma 2.2 to positive exponents:

Lemma 2.4. *Let f be a multiplicative convex function. Then, $f(x^t) = f(x)^t$ for every $t > 0$.*

Proof. If $t > 0$, we can find a sequence $\{q_n\}_{n=1}^{\infty} \subseteq \mathbb{Q}^+$ so that $q_n \to t$ as $n \to \infty$. Using the fact that f is continuous,

$$f(x^t) = f\left(x^{\lim_{n\to\infty} q_n}\right) = \lim_{n\to\infty} f(x^{q_n}) = \lim_{n\to\infty} f(x)^{q_n} = f(x)^t.$$ □

We are now ready to describe the way the multiplicative convex functions are, as in Definition 1.1.

Theorem 2.5. *Let $f\colon (0,\infty) \to [0,\infty)$. Then, f is a multiplicative convex function if and only if it can be written in the form*

$$f(x) = \begin{cases} b^{\log_a(x)} & \text{if } 0 < x \leq 1, \\ b'^{\log_{a'}(x)} & \text{if } x > 1, \end{cases} \tag{3}$$

where a, b, a' and b' satisfy the following conditions:

(1) $0 < a < 1$ *and* $a' > 1$.

(2) *If* $b < 1$, *then* $\log_b(b') \leq \log_a(a') < 0$ (*which, in particular, implies* $b' > 1$).

(3) *If* $b > 1$, *then* $\log_b(b') \geq \log_a(a')$.

Proof. Assume first f is multiplicative convex. Let us fix $0 < x_0 < 1$ and $x_1 > 1$.

If $0 < x < 1$, we can find $\mu > 0$ so that $x = x_0^{\mu}$. More specifically, we can write $\mu = \log_{x_0}(x) > 0$. Therefore,

$$f(x) = f(x_0^{\mu}) = f(x_0)^{\mu} = f(x_0)^{\log_{x_0}(x)}.$$

Similarly, writing $f(x) = f(x_1)^{\log_{x_1}(x)}$ for $x > 1$, we can identify $a = x_0 \in (0,1)$, $b = f(x_0)$, $a' = x_1 > 1$, $b' = f(x_1)$.

Notice now that we can find $\mu \geq 1$ so that $x_0^{\mu} x_1^{1/\mu} < 1$. Then,

$$\begin{aligned}f(x_0^{\mu} x_1^{1/\mu}) = f(x_0)^{\log_{x_0}(x_0^{\mu} x_1^{1/\mu})} &= \left[f(x_0)^{\log_{x_0}(x_0)}\right]^{\mu} \left[f(x_0)^{\log_{x_0}(x_1)}\right]^{1/\mu}\\ &= f(x_0)^{\mu} \left[f(x_0)^{\log_{x_0}(x_1)}\right]^{1/\mu}.\end{aligned}$$

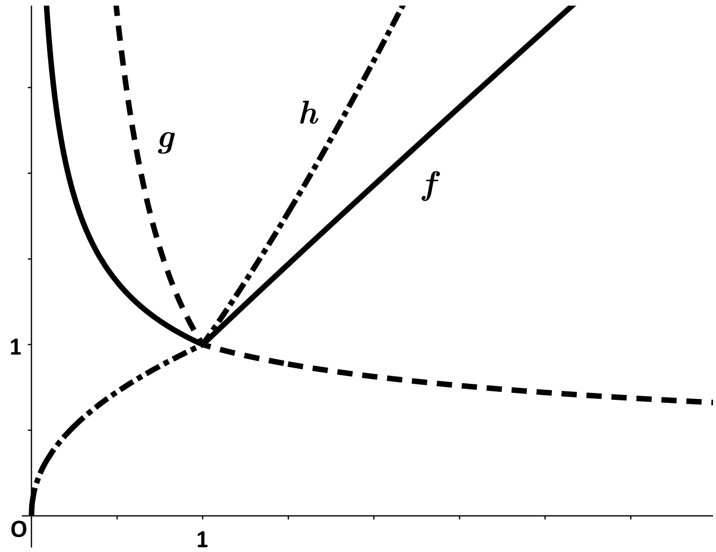

Figure 2.1: The functions are defined as in Equation (2) by using the following constants: $a_f = 0.27$, $b_f = 1.8$, $a'_f = 2.35$, $b'_f = 2.25$, $a_g = 0.45$, $b_g = 3.45$, $a'_g = 2.62$, $b'_g = 0.75$, $a_h = 0.18$, $b_h = 0.45$, $a'_h = 1.86$, and $b'_h = 2.4$

On the other hand, $f(x_0^\mu x_1^{1/\mu}) \leq f(x_0)^\mu f(x_1)^{1/\mu}$. Therefore, $f(x_0)^{\log_{x_0}(x_1)} \leq f(x_1)$ and hence

$$\log_{x_0}(x_1) \begin{cases} \leq \log_{f(x_0)}(f(x_1)) & \text{if } f(x_0) > 1, \\ \geq \log_{f(x_0)}(f(x_1)) & \text{if } f(x_0) < 1. \end{cases}$$

Conversely, let $f(x)$ be defined as Equation (2), with a, a', b and b' satisfying the corresponding conditions. Assume first $b < 1$ (so that $\log_b(b') \leq \log_a(a')$).

Let $x, y > 0$ and $\mu > 0$ so that $x^\mu y^{1/\mu} > 1$ (we may assume, w.l.o.g., that $y > 1$). Then,

$$f(x^\mu y^{1/\mu}) = b'^{\log_{a'}(x^\mu y^{1/\mu})} = \left[b'^{\log_{a'}(x)}\right]^\mu \left[b'^{\log_{a'}(y)}\right]^{1/\mu}.$$

If $x > 1$, we obtain trivially $f(x^\mu y^{1/\mu}) \leq f(x)^\mu f(y)^{1/\mu}$.

If $x < 1$, then we need to show that $b'^{\log_{a'}(x)} \leq b^{\log_a(x)}$, which is equivalent to show

$$\log_{a'}(x) \log_b(b') \geq \log_a(x), \tag{4}$$

since $b < 1$. Now, having that $a' > 1$, we obtain that $\log_{a'}(x) < 0$ and therefore

$$\log_{a'}(x) \log_b(b') = \frac{\log_a(x)}{\log_a(a')} \log_b(b') \geq \log_a(x),$$

as desired. Assume next that we are in the situation of $x^\mu y^{1/\mu} < 1$ (so that, w.l.o.g., $x < 1$). Then,

$$f(x^\mu y^{1/\mu}) = \left[b^{\log_a(x)}\right]^\mu \left[b^{\log_a(y)}\right]^{1/\mu}$$

and we only need to show that, if $y > 1$, then $b^{\log_a(y)} \leq b'^{\log_{a'}(y)}$, which is equivalent to show that

$$\log_a(y) \geq \log_{a'}(y) \log_b(b').$$

On the other hand, notice that $a, b < 1$, so $\log_a(y)$, $\log_a(a') < 0$ and hence

$$\log_a(y) \geq \log_a(y) \frac{\log_b(b')}{\log_a(a')} = \log_{a'}(y) \log_b(b').$$

If now $b > 1$ (so $\log_b(b') \geq \log_a(a')$) we have that the conditions 1,2,3 imply

$$\frac{\log_a(x)}{\log_a(a')} \log_b(b') = \log_{a'}(x) \log_b(b') \leq \log_a(x) \quad \text{for every } 0 < x < 1,$$
$$\frac{\log_a(y)}{\log_a(a')} \log_b(b') = \log_{a'}(y) \log_b(b') \geq \log_a(y) \quad \text{for every } y > 1.$$

Notice again that $\log_a(a') < 0$ and therefore $\dfrac{\log_b(b')}{\log_a(a')} \leq 1.$ □

The previous theorem characterizes the multiplicative convex functions and allows us, through the use of appropriate values of the constants, to graphically show different types of multiplicative convex functions (see Figure 2.1).

In Definition 1.1 we assumed $f(1) = 1$. But since $f(1) = f(1 \cdot 1) \leq f(1)^2$, we can deduce that $f(1) \geq 1$. On the other hand, if $C > 1$ and f is multiplicative convex, then Cf satisfies (2). Then, we wonder if every function that verifies (2) can be set as Cf with C a constant higher than 1 and f a function as in Theorem 2.5. That is, if a function f satisfies (2), would we have that $g(x) = f(x)/f(1)$ verifies (2)? We can state, by a negative answer, that the condition $f(1) = 1$ is essential in Theorem 2.5.

Remark 2.6. The function $f(x) = x + 1$ is multiplicative convex (condition (2)) but $g(x) = f(x)/2$ is not. Indeed, we have to see that for every $\mu > 0$ and $x, y \geq 0$ we have

$$x^\mu y^{1/\mu} + 1 \leq (x+1)^\mu (y+1)^{1/\mu}.$$

If $\mu = 1$, then $(x+1)(y+1) = xy + x + y + 1 \geq xy + 1$. Next, if $\mu > 1$ (it will be analogous for $\mu < 1$) we will have that

$$(x+1)^\mu (y+1)^{1/\mu} \geq (x^\mu + 1)(y+1)^{1/\mu}.$$

Then, if $y \geq 1$ we have that $1 \leq y^{1/\mu} \leq (y+1)^{1/\mu}$ so,

$$(x^\mu + 1)(y+1)^{1/\mu} \geq (x^\mu + 1) y^{1/\mu} = x^\mu y^{1/\mu} + y^{1/\mu} \geq x^\mu y^{1/\mu} + 1$$

And, if $y < 1$, we have that $y^{1/\mu} \leq 1 \leq (y+1)^{1/\mu}$, so

$$(x^\mu + 1)(y+1)^{1/\mu} \geq (x^\mu + 1)1 = x^\mu + 1 \geq x^\mu y^{1/\mu} + 1$$

Therefore $f(x) = x + 1$ satisfies condition (2).

Next, let us see that $g(x)$ does not satisfy (2). Taking $x = y = \frac{1}{2}$ and $\mu = 10$, we have that, on the one hand,

$$\frac{x^\mu y^{1/\mu} + 1}{2} = \frac{\left(\frac{1}{2}\right)^{10+1/10} + 1}{2} \approx 0.50045,$$

and, on the other hand,

$$\left(\frac{x+1}{2}\right)^\mu \left(\frac{y+1}{2}\right)^{1/\mu} = \left(\frac{3}{4}\right)^{10+1/10} \approx 0.05471,$$

and we are done, since we have just showed that, for this previous choice of x, y, and μ we actually have

$$g(x^\mu y^{1/\mu}) > g(x)^\mu g(y)^{1/\mu}.$$

References

[1] G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen: *Generalized convexity and inequalities*, J. Math. Anal. Appl. 335(2) (2007) 1294–1308.

[2] E. F. Bechenbach: *Convex functions*, Bull. Amer. Math. Soc. 54 (1948) 439–460.

[3] L. Bernal-González, P. Jiménez-Rodríguez, G. A. Muñoz-Fernández, J. B. Seoane-Sepúlveda: *On inequalities for convex functions*, J. Convex Analysis 26(2) (2019) 437–448.

[4] C. T. Ng: *On midconvex functions with midconcave bounds*, Proc. Amer. Math. Soc. 102 (1988) 538–540.

[5] C. P. Niculescu: *Convexity according to the geometric mean*, Math. Inequal. Appl. 2 (2000) 155–167.