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A General Result About Inner Regularization of Sets

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We generalize a recent result of C. Nour, H. Saoud and J. Takche [*Regularization via sets satisfying the interior sphere condition*, J. Convex Analysis 25(1) (2018) 1–19], where the authors provided an inner approximation of a set $S \subset \mathbb{R}^n$ by sets satisfying the interior sphere condition. The main idea is to replace the convex cones used there by curved ones which leads us to weaken the assumptions and cover more sets S.

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1. Introduction

In control theory, the regularity of a set $S \subset \mathbb{R}^n$ plays an important role in the study of the regularity of the *minimal time function* associated with S (called *target set*). In fact, if S satisfies the *uniform interior sphere condition* then the minimal time function associated with S becomes *semiconcave* as Cannarsa and Sinestrari proved in [3], see also [4]. Recall that S satisfies the uniform interior sphere condition if there exists a positive constant ρ such that for any boundary point s of S, one can find a closed ball in S of radius ρ containing s. The results of [3] are generalized by Cannarsa and Frankowska in [2] and Sinestrari in [22] for the cases where the uniform interior sphere condition is satisfied by the dynamic and not by the target set S. More general results in this direction can be also found in Colombo, Marigonda and Wolenski [9] and Colombo and Nguyen [10]. Another important use of the interior sphere condition is the regularity study of functions with hypographs satisfying this property. These functions turned out to enjoy some properties of concave functions, in particular a.e. twice differentiability. For more information, we invite the reader to see [11, 14]. See also [13, 16] where the

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equivalence between the uniform interior sphere condition and the *union of closed balls property* is studied.

In [12], Nour, Saoud and Takche provided for a given nonempty and closed subset S of $\mathbb{R}^n$, conditions under which S can be approximated by sets $S_\varepsilon \subset S$ satisfying the interior sphere condition with *continuous variable radius* ρ. The fact that their approximation sets satisfy an interior sphere condition with variable radius, allows them to approach any corner and to use the Pompeiu-Hausdorff convergence even if the set S is unbounded. To guarantee the existence of such approximation, Nour, Saoud and Takche assumed the nonemptiness of the interior of S^*, the star of S (see [12, Theorem 3.1]). We recall that the set S^* is defined by

$$S^* := \{s \in S : \text{the interval } [s, x] \subset S \text{ for all } x \in S\}.$$

This assumption can be seen as the following: There is a fixed closed ball B_0 in S such that each point in S is the vertex of a circular convex cone in S having B_0 as base. Nour, Saoud and Takche generalized this result in [12, Theorem 3.3] where the closed balls bases are allowed to be different but are moving tangentially on the boundary of a set $A \subset S$. The existence of such balls is guaranteed by the φ_0-*convexity* (or *proximal smoothness*) assumption of A.

The goal of this paper is to generalize the main result of [12], [12, Theorem 3.3], to cover more sets S. The main idea is to replace the circular convex cones used in [12] by *curved* ones. More precisely, we will replace the existence of the φ_0-convex set A with the properties mentioned above by the existence of a family of curves that cover the set S in a special geometric and topological way. Our new assumptions will be more general compared to the assumptions of [12], as we will prove in Section 3.

The layout of this paper is as follows. In the next section we present our notations, some definitions and basic results from nonsmooth analysis. We present our new hypotheses and compare them to the hypotheses of [12] in Section 3. Section 4 is devoted to our main result.

2. Preliminaries

We begin this section by providing our notations. We denote by $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, B, $\overline{B}$ and S^n, the Euclidean norm, the usual inner product, the open unit ball, the closed unit ball and the unit sphere, respectively. For $\rho > 0$ and $x \in \mathbb{R}^n$, we set $B(x;\rho) := x + \rho B$ and $\overline{B}(x;\rho) := x + \rho\overline{B}$. The segment joining two points x and y in $\mathbb{R}^n$ is denoted by $[x, y]$, that is,

$$[x, y] := \{(1-\lambda)x + \lambda y : \lambda \in [0, 1]\}.$$

We also denote, for $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$, by $[A, B]$ the set of all segments joining a point in A to a point in B, that is,

$$[A, B] := \{[a, b] : a \in A \text{ and } b \in B\}.$$

For a set $S \subset \mathbb{R}^n$, S^c, $\operatorname{int} S$, ∂S and $\operatorname{cl} S$ are the complement (with respect to $\mathbb{R}^n$), the interior, the boundary and the closure of S, respectively. The distance from a point x to a set S is denoted by $d_S(x)$. We also denote by $\operatorname{proj}_S(x)$ the set of closest points in S to x, that is, the set of points $s \in S$ which satisfy $d_S(x) = \|s - x\|$. The Pompeiu-Hausdorff distance between two sets A and B will be denoted by $d_H(A, B)$.

We proceed to give some definitions and results from nonsmooth analysis. Our general reference for these constructs is Clarke et al. [5]; see also Penot [18] and Rockafellar and Wets [21]. Let S be a nonempty and closed subset of $\mathbb{R}^n$. For $x \in S$, a vector $\zeta \in \mathbb{R}^n$ is said to be *proximal normal to S at x* provided that there exists $\sigma = \sigma(x, \zeta) \geq 0$ such that

$$\langle \zeta, s - x \rangle \leq \sigma \|s - x\|^2 \quad \forall s \in S. \tag{1}$$

The relation (1) is commonly referred to as the *proximal normal inequality*. No nonzero ζ satisfying (1) exists if $x \in \operatorname{int} S$, but this may also occur for $x \in \partial S$, as is the case when S is the epigraph of the function $f(z) = -|z|$ and $x = (0, 0)$. For such a point, the only proximal normal is $\zeta = 0$. In view of (1), the set of all proximal normals to S at x is a convex cone, and we denote it by $N_S^P(x)$. Now let $x \in \partial S$, $0 \neq \zeta \in \mathbb{R}^n$ and $r > 0$ be such that

$$B\left(x + r\frac{\zeta}{\|\zeta\|}; r\right) \cap S = \emptyset. \tag{2}$$

Then ζ is a proximal normal to S at x and we say that ζ is *realized by an r-sphere*. Note that ζ is then also realized by an r'-sphere for any $0 < r' < r$. One can show that ζ being realized by an r-sphere is equivalent to the proximal normal inequality holding with $\sigma = 1/2r$; that is,

$$\left\langle \frac{\zeta}{\|\zeta\|}, s - x \right\rangle \leq \frac{1}{2r}\|s - x\|^2 \quad \forall s \in S. \tag{3}$$

We terminate this section by introducing a well-known regularity property for a nonempty and closed subset S of $\mathbb{R}^n$: the θ-interior sphere condition. For more information about this property, we invite the reader to see the papers [13, 14, 15, 16].

Definition 2.1. A nonempty and closed set $S \subset \mathbb{R}^n$ is said to satisfy the *θ-interior sphere condition* if and only if there exists a continuous function $\theta \colon \partial S \longrightarrow [0, +\infty[$ such that for all $x \in \partial S$ one can find a point $y_x \in S$ satisfying:

(i) $x \in \overline{B}\left(y_x; \frac{1}{2\theta(x)}\right) \subset S$, if $\theta(x) > 0$.

(ii) $x \in \overline{B}(x + t(y_x - x); t) \subset S$ for all $t > 0$, if $\theta(x) = 0$.

If $\theta(\cdot) = \theta_0$ a positive constant, then the θ_0-interior sphere condition coincides with the well known condition in control theory, the uniform interior sphere condition. This condition is important in deriving regularity properties of the minimal time function; see e.g. Cannarsa and Frankowska [2], Cannarsa and Sinestrari [2, 3] and Sinestrari [22]. In [15, Theorem 3.1], Nour, Stern and Takche proved that there is an equivalence between the θ-interior sphere condition and the ψ-*union of closed ball property.* Let us recall the definition of this property.

Definition 2.2. A nonempty and closed set $S \subset \mathbb{R}^n$ is said to be the ψ-*union of closed balls* if and only if there exists a function $\psi \colon S \longrightarrow [0, +\infty[$ such that:

(i) ψ is upper semicontinuous on S and continuous on ∂S.

(ii) For all $x \in S$ there exists $y_x \in S$ such that:

(a) $x \in \overline{B}\left(y_x; \frac{1}{2\psi(x)}\right) \subset S$, if $\psi(x) > 0$.

(b) $x \in \overline{B}(x + t(y_x - x); t) \subset S$ for all $t > 0$, if $\psi(x) = 0$.

If $\psi \equiv \psi_0$ is a positive constant then the ψ_0-union of closed balls property coincides with the well-know union of closed balls property (with radius $\frac{1}{2\psi_0}$).

3. Hypotheses

The goal of this section it to present the hypotheses that will be used in the next section to prove our main result. We will also discuss the relation between these new hypotheses and the hypotheses used in [12]. More precisely, we will prove that our new hypotheses are in fact weaker. For a family of smooth and natural (or parametrized by arc length) curves $(\gamma_i)_{i \in I}$ defined on $[0, \infty[$, we denote by C the set of the initial points of $(\gamma_i)_{i \in I}$, that is, $C := \{\gamma_i(0) : i \in I\}$.

Definition 3.1. Let $S \subset \mathbb{R}^n$ be a nonempty and closed set. We say that S has an *LCC-cover* (*Lipschitz curved cone cover*) if there exist $K > 0$, $\rho > 0$ and a family of smooth and natural curves $(\gamma_i)_{i \in I}$ defined on $[0, \infty[$ such that

(H1) For all $x \in S \setminus C$, there is a unique $i_x \in I$ and a unique $t_x > 0$ such that $x = \gamma_{i_x}(t_x)$ and $\gamma_{i_x}(t) \in \operatorname{int} S \setminus C$ for all $t \in]0, t_x[$.

(H2) For all $x, y \in S \setminus C$, if $x \longrightarrow y$ then $\gamma_{i_x}(t) \longrightarrow \gamma_{i_y}(t)$ for all $t \geq 0$.

(H3) For all x, $y \in S$ we have $|t_x - t_y| \leq K\|x - y\|$.[1]

(H4) We have $\overline{B}(\gamma_i(0); \rho) \subset S$ for all $i \in I$ and

$$\overline{B}\left(x; \rho\left(1 - \frac{t_x}{T_{i_x}}\right)\right) \subset S \quad \forall x \in S \setminus C, \tag{4}$$

where $T_i := \min\{t > 0 : \gamma_i(t) \in \partial S\}$.[2]

The following proposition gives a sufficient condition for a set S to have a Lipschitz curved cone cover. This sufficient condition is the hypothesis under which Nour, Saoud and Takche proved in [12, Theorem 3.1] the existence of an inner approximation of S by sets satisfying the interior sphere condition.

[1] $t_x := 0$ for $x \in C$.

[2] If the curve γ_i does not intersect the boundary of S, then T_i is considered to be ∞. In this case the condition (4) becomes $\overline{B}(x; \rho) \subset S$.

Proposition 3.2. *Let S be a nonempty and closed subset of $\mathbb{R}^n$ and assume that* $\operatorname{int} S^* \neq \emptyset$. *Then S has an LCC-cover.*

Proof. Since $\operatorname{int} S^* \neq \emptyset$ there exist $\rho > 0$ and $s_0 \in S^*$ such that $\overline{B}(s_0; \rho) \subset S^*$. For each unit vector u, we define the family of curves γ_u on $[0, \infty[$ by

$$\gamma_u(t) := s_0 + tu \quad \forall t \in [0, \infty[.$$

We claim that conditions (H1)–(H4) are satisfied by the family $(\gamma_u)_{u \in S^n}$ for the constant ρ and $K = 1$. Indeed, for (H1) it is sufficient to take $t_x = \|x - s_0\|$ and $u_x = \frac{x - s_0}{\|x - s_0\|}$ and remark (using $\overline{B}(s_0; \rho) \subset S^*$) that

$$\overline{B}\left(\gamma_{u_x}(t); \rho\left(1 - \frac{t}{t_x}\right)\right) \subset S \quad \forall t \in [0, t_x[.$$

Condition (H2) follows directly from the definition of $\gamma_{u_x}(\cdot)$. For (H3), remark that $t_x = \|x - s_0\|$ and $t_y = \|y - s_0\|$ and then

$$|t_x - t_y| = |\|x - s_0\| - \|y - s_0\|| \leq \|x - y\|.$$

To prove (H4), let $s_0 \neq x \in S$ and let $u_x = \frac{x - s_0}{\|x - s_0\|}$.

Case 1: The curve γ_{u_x} intersects ∂S.
Then T_{u_x} is finite and $s_0 + T_{u_x} u_x \in \partial S$. Since $\overline{B}(s_0; \rho) \subset S^*$ we get that

$$\overline{B}\left(s_0 + t u_x; \rho\left(1 - \frac{t}{T_{u_x}}\right)\right) \subset S \quad \forall t \in [0, T_{u_x}].$$

Clearly $t_x \leq T_{u_x}$ (since $\gamma_u(t) \in \operatorname{int} S$ for all $t \in [0, t_x[$). Then for $t = t_x$ in the preceding inclusion, we get that

$$\overline{B}\left(x; \rho\left(1 - \frac{t_x}{T_{u_x}}\right)\right) \subset S.$$

Case 2: The curve γ_{u_x} does not intersect ∂S.
Then $T_{u_x} = \infty$ and $s_0 + t u_x \in S$ for all $t \geq 0$. Hence, $\overline{B}(s_0; \rho) \subset S^*$ implies

$$\overline{B}\left(s_0 + t_x u_x; \rho\left(1 - \frac{t_x}{t}\right)\right) \subset S \quad \forall t \geq t_x.$$

Taking $t \longrightarrow \infty$ in the preceding inclusion we get that $\overline{B}(x; \rho) \subset S$. This terminates the proof of the proposition. □

We proceed to study the relation between our new hypothesis, the existence of a Lipschitz curved cone cover, and the hypothesis used by Nour, Saoud and Takche to prove the general main result of [12] in which, the existence of an inner approximation of S by sets satisfying the interior sphere condition is given, see [12, Theorem 3.3]. First we introduce a well-known regularity property for a nonempty and closed subset S of $\mathbb{R}^n$: the φ-convexity. For more information about this property, we invite the reader to see [1, 8, 14].

Definition 3.3. A nonempty and closed set $S \subset \mathbb{R}^n$ is said to be *φ-convex* if and only if there exists a continuous function $\varphi\colon \partial S \longrightarrow [0,+\infty[$ such that for all $x \in \partial S$ and for all $0 \neq \zeta \in N_S^P(x)$ we have

(i) ζ is realized by a $\frac{1}{2\varphi(x)}$-sphere, if $\varphi(x) \neq 0$,

(ii) ζ is realized by an r-sphere for all $r > 0$, if $\varphi(x) = 0$.

If $\varphi(\cdot) = \varphi_0$ a positive constant, then the φ_0-convexity coincides with the *positive reach* property and the *proximal smoothness* property defined in Federer [7] and Clarke et al. [6], respectively. More precisely, a set S is φ_0-convex if and only if it is $\frac{1}{2\varphi_0}$-proximal smooth (or has positive reach with radius $\frac{1}{2\varphi_0}$). We also refer the reader to Poliquin and Rockafellar [19], Poliquin, Rockafellar and Thibault [20], Rockafellar and Wets [21] for investigations and applications of related properties such as *prox-regularity*. Another needed geometric property is the following one introduced in [12] and studied in depth in [17].

Definition 3.4. Let A and S be two nonempty sets in $\mathbb{R}^n$ such that A is closed and $A \subset S$. We say that A is *S-convex* if and only if

$$s - a \notin N_A^P(a) \ \text{ or } \ s - a' \notin N_A^P(a') \ \text{ for all } s \in S \cap A^c \text{ and } a \neq a' \in A.$$

Remark 3.5. One can easily prove that if A is S-convex then each point in S has a unique projection on A. The converse is not necessarily true as it is proved in [12, Remark 3.4]. □

In the following proposition we will show that under the hypothesis of [12, Theorem 3.3], a special subset of S has a Lipschitz curved cone cover. This proposition will be used in the next section to prove that [12, Theorem 3.3] is a direct consequence of the main result of this paper. Note that for a point $x \in \mathbb{R}^n$ and a vector $\zeta \in \mathbb{R}^n$, the *directional distance* from x to a closed set $S \subset \mathbb{R}^n$ in the direction ζ, denoted by $d_S(x,\zeta)$, is defined by

$$d_S(x,\zeta) := \min\{t \geq 0 : x + t\zeta \in S\},$$

(if $x + t\zeta \notin S$ for all $t \geq 0$ then $d_S(x,\zeta)$ is taken to be ∞). We also denote, for $A \subset \mathbb{R}^n$ a closed set and $r > 0$, by A_r the set

$$A_r := \{x \in \mathbb{R}^n : d(x,A) < r\} = A + B(0;r).$$

Proposition 3.6. *Let $S \subset \mathbb{R}^n$ be a nonempty and closed set satisfying the following: there exists a closed set $A \subset \operatorname{int} S$ such that*

(1) *A is S-convex,*

(2) *A is φ_0-convex,*

(3) *For $s \in A^c \cap S$, $a \in \operatorname{proj}_A(s)$ and $\zeta := \frac{s-a}{\|s-a\|}$, we have*

(i) $s \in \left[\overline{B}\left(a + \frac{1}{2\varphi_0}\zeta; \frac{1}{2\varphi_0}\right),\, a + d_{\partial S}(a,\zeta)\,\zeta\right] \subset S$, *if* $d_{\partial S}(a,\zeta) < \infty$.

(ii) $s \in \left[\overline{B}\left(a + \frac{1}{2\varphi_0}\zeta; \frac{1}{2\varphi_0}\right),\, a + t\zeta\right] \subset S$ *for all* $t \geq 0$, *if* $d_{\partial S}(a,\zeta) = \infty$.

Then $S^A := S \setminus A_{\frac{1}{4\varphi_0}}$ has an LCC-cover.

Proof. Since A is φ_0-convex, we have that for all $a \in \partial A$ and for all $0 \neq \zeta \in N_A^P(a)$, ζ is realized by a $\frac{1}{2\varphi_0}$-sphere, that is,

$$B\left(a + \frac{1}{2\varphi_0}\frac{\zeta}{\|\zeta\|}; \frac{1}{2\varphi_0}\right) \cap A = \emptyset.$$

Then
$$B\left(a + \frac{1}{2\varphi_0}\frac{\zeta}{\|\zeta\|}; \frac{1}{4\varphi_0}\right) \cap A_{\frac{1}{4\varphi_0}} = \emptyset. \tag{5}$$

We define the set $\mathcal{N}_A$ by $\mathcal{N}_A := \{(a,\zeta) : a \in \partial A \text{ and } \zeta \in N_A^P(a) \text{ is unit}\}$.

Since A is φ_0-convex, each point in ∂A is a projection of at least one point in $A^c \cap S$. Then by (5) and 3(i)–3(ii) we have

$$B\left(a + \frac{1}{2\varphi_0}\zeta; \frac{1}{4\varphi_0}\right) \subset \operatorname{int} S^A \quad \forall (a,\zeta) \in \mathcal{N}_A. \tag{6}$$

For each $(a,\zeta) \in \mathcal{N}_A$, we consider the following two smooth curves:

- $\gamma_{a,\zeta} \colon [0,\infty[\longrightarrow \mathbb{R}^n$ by $\gamma_{a,\zeta}(t) = a + \left(\frac{1}{2\varphi_0} + t\right)\zeta$ for all $t \geq 0$.
- $\overline{\gamma}_{a,\zeta} \colon [0,\infty[\longrightarrow \mathbb{R}^n$ by $\overline{\gamma}_{a,\zeta}(t) = a + \left(\frac{1}{2\varphi_0} - t\right)\zeta$ for all $t \geq 0$.

We claim that the conditions (H1)–(H4) are satisfied for $S^A := S \setminus A_{\frac{1}{4\varphi_0}}$ by the family of curves $(\gamma_{a,\zeta})_{(a,\zeta)\in\mathcal{N}_A} \cup (\overline{\gamma}_{a,\zeta})_{(a,\zeta)\in\mathcal{N}_A}$ for $\rho = \frac{1}{4\varphi_0}$ and $K = 1$. We begin by (H1). Let $x \in S^A \setminus C$. Since $C = \left\{a + \frac{1}{2\varphi_0}\zeta : (a,\zeta) \in \mathcal{N}_A\right\}$, two cases need to be considered.

Case 1: $d_A(x) > \dfrac{1}{2\varphi_0}$.

Let $a_x \in \operatorname{proj}_A(x)$, $\zeta_x = \frac{x-a_x}{\|x-a_x\|}$ and $t_x = \|a_x - x\| - \frac{1}{2\varphi_0}$. Clearly we have $\gamma_{a_x,\zeta_x}(t_x) = x$. Moreover, the S-convexity of A and conditions 3(i)–3(ii) give that γ_{a_x,ζ_x} and t_x are unique and that $\gamma_{a_x,\zeta_x}(t) \in \operatorname{int} S^A \setminus C$ for all $t \in]0, t_x[$.

Case 2: $\dfrac{1}{4\varphi_0} \leq d_A(x) < \dfrac{1}{2\varphi_0}$.

Let $a_x \in \operatorname{proj}_A(x)$, $\zeta_x = \frac{x-a_x}{\|x-a_x\|}$ and $t_x = \frac{1}{2\varphi_0} - \|a_x - x\|$. Again we have $\overline{\gamma}_{a_x,\zeta_x}(t_x) = x$. Moreover, the S-convexity of A and (6) give that $\overline{\gamma}_{a_x,\zeta_x}$ and t_x are unique and that $\overline{\gamma}_{a_x,\zeta_x}(t) \in \operatorname{int} S^A \setminus C$ for all $t \in]0, t_x[$. So (H1) follows. Condition (H2) follows directly from the continuity of the function $x \mapsto a_x$ on S^A. For (H3), let x, $y \in S^A$.

We need to prove that $|t_x - t_y| \leq \|x - y\|$. We can easily prove this inequality if $d_A(x)$ and $d_A(y)$ are bothgreater than or equal or both less than or equal to $\frac{1}{2\varphi_0}$. If $d_A(x) < \frac{1}{2\varphi_0}$ and $d_A(y) > \frac{1}{2\varphi_0}$ then

$$|t_x - t_y| \le t_x + t_y = \frac{1}{2\varphi_0} - \|a_x - x\| + \|a_y - y\| - \frac{1}{2\varphi_0}$$
$$\le |\|a_x - x\| - \|a_y - y\|| = |d_A(x) - d_A(y)| \le \|x - y\|.$$

We proceed to prove (H4). Using (6) we deduce that

$$\bar{B}\left(a + \frac{1}{2\varphi_0}\zeta; \frac{1}{4\varphi_0}\right) \subset S^A \quad \forall (a,\zeta) \in \mathcal{N}_A.$$

Now let $x \in S^A \setminus C$.

Case 1: $d_A(x) > \frac{1}{2\varphi_0}$.

By the S-convexity of A, we have that $T_{(a_x,\zeta_x)} = d_{\partial S}(a_x,\zeta_x) - \frac{1}{2\varphi_0}$ (we assume for instance that $d_{\partial S}(a_x,\zeta_x)$ is finite). Then

$$\overline{B}\Big(x; \rho\Big(1 - \frac{t_x}{T_{(a_x,\zeta_x)}}\Big)\Big) = \overline{B}\Big(x; \frac{1}{4\varphi_0}\Big(1 - \frac{\|a_x - x\| - \frac{1}{2\varphi_0}}{d_{\partial S}(a_x,\zeta_x) - \frac{1}{2\varphi_0}}\Big)\Big)$$
$$\subset \Big[\bar{B}\Big(a_x + \frac{1}{2\varphi_0}\zeta_x; \frac{1}{4\varphi_0}\Big), a_x + d_{\partial S}(a_x,\zeta_x)\,\zeta_x\Big] \subset S.$$

Since $d_A(x) > \frac{1}{2\varphi_0}$ we have $\overline{B}\big(x; \rho\big(1 - \frac{t_x}{T_{(a_x,\zeta_x)}}\big)\big) \cap A_{\frac{1}{4\varphi_0}} = \emptyset$ and then

$$\overline{B}\Big(x; \rho\Big(1 - \frac{t_x}{T_{(a_x,\zeta_x)}}\Big)\Big) \subset S^A.$$

If $d_{\partial S}(a_x,\zeta_x) = \infty$ then by 3(ii) and as above, we have that

$$\overline{B}\Big(x; \rho\Big(1 - \frac{t_x}{t}\Big)\Big) \subset S,$$

for all t sufficiently large. Taking $t \longrightarrow \infty$ we conclude that $\overline{B}\,(x;\rho) \subset S$. Since $d_A(x) > \frac{1}{2\varphi_0}$ we have that $\overline{B}\,(x;\rho) \cap A_{\frac{1}{4\varphi_0}} = \emptyset$ and this gives us $\overline{B}\,(x;\rho) \subset S^A$.

Case 2: $\frac{1}{4\varphi_0} \le d_A(x) < \frac{1}{2\varphi_0}$.

Then $T_{(a_x,\zeta_x)} = \frac{1}{4\varphi_0}$ (follows from the φ_0-convexity of A). Hence

$$\overline{B}\Big(x; \rho\Big(1 - \frac{t_x}{T_{(a_x,\zeta_x)}}\Big)\Big) = \overline{B}\Big(x; \frac{1}{4\varphi_0}\Big(1 - \frac{\frac{1}{2\varphi_0} - \|a_x - x\|}{\frac{1}{4\varphi_0}}\Big)\Big)$$
$$= \overline{B}\Big(x; \|a_x - x\| - \frac{1}{4\varphi_0}\Big) \subset \bar{B}\Big(a_x + \frac{1}{2\varphi_0}\zeta_x; \frac{1}{4\varphi_0}\Big) \subset S^A.$$

This terminates the proof of (H4) and then the proof of Proposition 3.6. $\square$

4. Main result

The goal of this section is to present our main result in which we prove that any set with a Lipschitz curved cone cover can be approximated from inside by sets satisfying the interior sphere condition. We also prove, using the results of Section 3, that the main results of [12] are direct consequences of our new main result. The following is our main result.

Theorem 4.1. *Let $S \subset \mathbb{R}^n$ be a nonempty and closed set and assume that S has an LCC-cover. Then for all $\varepsilon > 0$ small, there exists a nonempty and closed set $S_\varepsilon \subset S$ such that*

(1) *S_ε is the ψ-union of closed balls;* (2) *$d_H(S_\varepsilon, S) \leq \varepsilon$.*

Proof. Since S has a Lipschitz curved cone cover, there exist $K > 0$, $\rho > 0$ and a family of smooth and natural curves $(\gamma_i)_{i\in I}$ defined on $[0, \infty[$ such that

(H1) For all $x \in S \setminus C$, there is a unique $i_x \in I$ and a unique $t_x > 0$ such that $x = \gamma_{i_x}(t_x)$ and $\gamma_{i_x}(t) \in \operatorname{int} S \setminus C$ for all $t \in]0, t_x[$.

(H2) For all $x, y \in S \setminus C$, if $x \longrightarrow y$ then $\gamma_{i_x}(t) \longrightarrow \gamma_{i_y}(t)$ for all $t \geq 0$.

(H3) For all x, $y \in S$ we have $|t_x - t_y| \leq K\|x - y\|$.

(H4) We have $\overline{B}(\gamma_i(0); \rho) \subset S$ for all $i \in I$ and for all $x \in S \setminus C$,

$$\overline{B}\Big(x; \rho\Big(1 - \frac{t_x}{T_{i_x}}\Big)\Big) \subset S, \quad \text{where } T_i := \min\{t > 0 : \gamma_i(t) \in \partial S\}.$$

We define over S the function $r(\cdot)$ by

$$r(x) := \begin{cases} \dfrac{\rho}{2} & x \in C, \\ \dfrac{\rho}{2}\Big(1 - \dfrac{t_x}{T_{i_x}}\Big) & x \notin C. \end{cases}$$

Claim 1: The function $r(\cdot)$ is continuous on S.

Before giving the proof of this claim, we note that since $\overline{B}(\gamma_i(0); \rho) \subset S$ for all $i \in I$, we notice $T_i \geq \rho$ for all $i \in I$. Let $x_m \in S$ be a sequence that converges to a point $x_0 \in S$. We need to prove that $r(x_m) \longrightarrow r(x_0)$.

Case 1: $r(x_0) = \frac{\rho}{2}$. Hence for all m we have

$$|r(x_m) - r(x_0)| = \Big|r(x_m) - \frac{\rho}{2}\Big| = \begin{cases} 0 & x_m \in C, \\ \dfrac{\rho}{2}\dfrac{t_{x_m}}{T_{i_{x_m}}} & x_m \notin C. \end{cases} \tag{7}$$

Case 1.1: $x_0 \in C$. Then $t_{x_0} = 0$ and, if $x_m \notin C$ we have $T_{i_{x_m}} \geq \rho$, hence

$$|r(x_m) - r(x_0)| \leq \frac{t_{x_m}}{2} = \frac{|t_{x_m} - t_{x_0}|}{2} \leq \frac{K\|x_m - x_0\|}{2} \quad \forall m.$$

This gives us $r(x_m) \longrightarrow r(x_0)$.

Case 1.2: $x_0 \notin C$ (that is, $T_{i_{x_0}} = \infty$).

By (7) it is sufficient to prove that $\frac{1}{T_{i_{x_m}}} \longrightarrow 0$ for $x_m \notin C$. By contradiction, assume that $\frac{1}{T_{i_{x_m}}} \not\longrightarrow 0$ then there exist $\varepsilon_0 > 0$ and a subsequence of $x_m \notin C$ (we dot not relabel) such that

$$\frac{1}{T_{i_{x_m}}} \geq \varepsilon_0.$$

Let $s_m := \gamma_{i_{x_m}}(T_{i_{x_m}}) \in \partial S$. We have

$$\|s_m - x_0\| \leq \|s_m - x_m\| + \|x_m - x_0\| \leq T_{i_{x_m}} + \|x_m - x_0\| \leq \frac{1}{\varepsilon_0} + \|x_m - x_0\|.$$

Since $x_m \longrightarrow x_0$, we get that s_m is bounded and then it can be assumed to be convergent. Let $s_0 \in \partial S$ be its limit. We have

$$s_0 = \gamma_{i_{s_0}}(T_{i_{s_0}}) = \lim_{m \longrightarrow \infty} \gamma_{i_{s_m}}(T_{i_{s_0}}) = \lim_{m \longrightarrow \infty} \gamma_{i_{x_m}}(T_{i_{s_0}}) = \gamma_{i_{x_0}}(T_{i_{s_0}}).$$

This gives that $\gamma_{i_{x_0}}(T_{i_{s_0}}) \in \partial S$ and then the desired contradiction.

Case 2: $r(x_0) = \frac{\rho}{2}\left(1 - \frac{t_{x_0}}{T_{i_{x_0}}}\right) \neq \frac{\rho}{2}$ (that is, $x_0 \notin C$ and $T_{i_{x_0}} < \infty$).

Since $t_{x_m} \longrightarrow t_{x_0}$ we can assume that $x_m \notin C$ for all m (if not, then $t_{x_0} = 0$ and then $r(x_0) = \frac{\rho}{2}$). Then we have

$$r(x_m) = \frac{\rho}{2}\left(1 - \frac{t_{x_m}}{T_{i_{x_m}}}\right) \quad \forall m.$$

Since $t_{x_m} \longrightarrow t_{x_0}$ it is sufficient to prove that $T_{i_{x_m}} \longrightarrow T_{i_{x_0}}$. Assume by contradiction that $T_{i_{x_m}} \not\longrightarrow T_{i_{x_0}}$. Then, there exist $\varepsilon_0 > 0$ and a subsequence of x_m (we do not relabel) such that $|T_{i_{x_m}} - T_{i_{x_0}}| > \varepsilon_0$ for all m.

Case 2.1: There exists a subsequence of x_m (we do not relabel) such that

$$T_{i_{x_m}} - T_{i_{x_0}} > \varepsilon_0 \quad \forall m \;\; \text{(see Figure 4.1)}.$$

Then $T_{i_{x_0}} < T_{i_{x_m}}$ which gives that $\gamma_{i_{x_m}}(T_{i_{x_0}}) \in \operatorname{int} S$. Hence

$$\overline{B}\Big(\gamma_{i_{x_m}}(T_{i_{x_0}}); \rho\Big(1 - \frac{T_{i_{x_0}}}{T_{i_{x_m}}}\Big)\Big) \subset S.$$

But $\gamma_{i_{x_0}}(T_{i_{x_0}}) \in \partial S$, then $\|\gamma_{i_{x_0}}(T_{i_{x_0}}) - \gamma_{i_{x_m}}(T_{i_{x_0}})\| \geq \rho\left(1 - \frac{T_{i_{x_0}}}{T_{i_{x_m}}}\right)$.

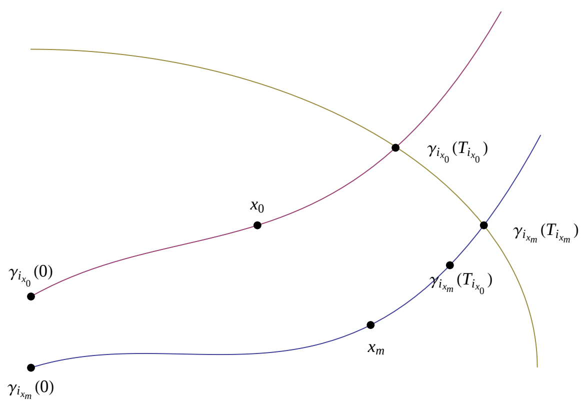

Figure 4.1: Case 2.1

Using the fact that $T_{i_{x_m}} > T_{i_{x_0}} + \varepsilon_0$, we conclude that

$$\|\gamma_{i_{x_0}}(T_{i_{x_0}}) - \gamma_{i_{x_m}}(T_{i_{x_0}})\| > \rho\Big(1 - \frac{T_{i_{x_0}}}{T_{i_{x_0}} + \varepsilon_0}\Big) = \frac{\rho\varepsilon_0}{T_{i_{x_0}} + \varepsilon_0},$$

which contradicts the convergence of $\gamma_{i_{x_m}}(T_{i_{x_0}})$ to $\gamma_{i_{x_0}}(T_{i_{x_0}})$.

Case 2.2: There exists a subsequence of x_m (we do not relabel) such that

$$T_{i_{x_0}} - T_{i_{x_m}} > \varepsilon_0 \quad \forall m.$$

Then $T_{i_{x_m}} < T_{i_{x_0}}$ which gives that $\gamma_{i_{x_0}}(T_{i_{x_m}}) \in \operatorname{int} S$. Hence

$$\overline{B}\left(\gamma_{i_{x_0}}(T_{i_{x_m}}); \rho\left(1 - \frac{T_{i_{x_m}}}{T_{i_{x_0}}}\right)\right) \subset S.$$

Proceeding as above, we get that

$$\|\gamma_{i_{x_m}}(T_{i_{x_m}}) - \gamma_{i_{x_0}}(T_{i_{x_m}})\| \geq \rho\left(1 - \frac{T_{i_{x_m}}}{T_{i_{x_0}}}\right) = \rho\left(\frac{T_{i_{x_0}} - T_{i_{x_m}}}{T_{i_{x_0}}}\right) > \frac{\rho\varepsilon_0}{T_{i_{x_0}}}. \tag{8}$$

Now let $s_m := \gamma_{i_{x_m}}(T_{i_{x_m}}) \in \partial S$. We have

$$i_{s_m} = i_{x_m} \text{ and } T_{i_{x_m}} = T_{i_{s_m}} = t_{s_m}.$$

On the other hand,

$$\|s_m - x_0\| \leq \|s_m - x_m\| + \|x_m - x_0\| \leq T_{i_{x_m}} + \|x_m - x_0\| \leq T_{i_{x_0}} + \|x_m - x_0\|.$$

Since $x_m \longrightarrow x_0$, we get that s_m is bounded and then it can be assumed to be convergent. Let $s_0 \in \partial S$ be its limit. We have $s_0 = \gamma_{i_{s_0}}(T_{i_{s_0}}) = \gamma_{i_{s_0}}(t_{s_0})$ and for all $t \geq 0$

$$\gamma_{i_{s_0}}(t) = \lim_{m\longrightarrow\infty} \gamma_{i_{s_m}}(t) = \lim_{m\longrightarrow\infty} \gamma_{i_{x_m}}(t) = \gamma_{i_{x_0}}(t).$$

Hence $i_{s_0} = i_{x_0}$. Using (8) we get

$$\begin{aligned}\frac{\rho\varepsilon_0}{T_{i_{x_0}}} &\leq \lim_{m\longrightarrow\infty} \|\gamma_{i_{x_m}}(T_{i_{x_m}}) - \gamma_{i_{x_0}}(T_{i_{x_m}})\| = \lim_{m\longrightarrow\infty} \|s_m - \gamma_{i_{s_0}}(T_{i_{s_m}})\| \\ &= \lim_{m\longrightarrow\infty} \|s_m - \gamma_{i_{s_0}}(t_{s_m})\| = \|s_0 - \gamma_{i_{s_0}}(t_{s_0})\| = 0.\end{aligned}$$

Contradiction. This terminates the proof of Claim 1.

Now let $0 < \varepsilon < \frac{\rho}{2(K+1)}$. We denote by

- $I_1 := \{i \in I : \gamma_i(\cdot) \text{ does not intersect } \partial S\}$,
- $I_2 := I \setminus I_1$.

We also define the set

$$S_\varepsilon := \bigcup_{i\in I_1,\, t\geq 0} \overline{B}\left(\gamma_i(t); \frac{\rho}{2}\right) \cup \bigcup_{i\in I_2,\, t\in[0,T_i-\varepsilon]} \overline{B}(\gamma_i(t); r(\gamma_i(t))).$$

Claim 2: $S_\varepsilon \subset S$ and $d_H(S_\varepsilon, S) \leq \varepsilon$.
By (H4) and the definition of $r(\cdot)$ we have $S_\varepsilon \subset S$. To prove that $d_H(S_\varepsilon, S) \leq \varepsilon$ it is sufficient to prove that $d(x, S_\varepsilon) \leq \varepsilon$ for all $x \in S$.

Let $x \in S \setminus S_\varepsilon$ (if $x \in S_\varepsilon$ then nothing to prove). Since $x \notin S_\varepsilon$ we have that $i_x \in I_2$ and $T_{i_x} - \varepsilon < t_x \leq T_{i_x}$. Then

$$\|x - \gamma_{i_x}(T_{i_x} - \varepsilon)\| \leq t_x - (T_{i_x} - \varepsilon) = (t_x - T_{i_x}) + \varepsilon \leq \varepsilon.$$

But $\gamma_{i_x}(T_{i_x} - \varepsilon) \in S_\varepsilon$, therefore $d(x, S_\varepsilon) \leq \varepsilon$. The Claim 2 follows.

Claim 3: S_ε is the $\frac{1}{2r(\cdot)}$-union of closed balls.

Let $x \in S_\varepsilon$. We need to find $y_x \in S_\varepsilon$ such that

$$x \in \overline{B}(y_x; r(x)) \subset S_\varepsilon. \tag{9}$$

Case 1: $T_{i_x} = \infty$.
Then $i_x \in I_1$ and $r(x) = \frac{\rho}{2}$. For $y_x = x$ we have

$$x \in \overline{B}(y_x; r(x)) = \overline{B}\left(\gamma_{i_x}(t_x); \frac{\rho}{2}\right) \subset S_\varepsilon.$$

Case 2: $T_{i_x} < \infty$ and $x \in \overline{B}(\gamma_{i_0}(t_0); \frac{\rho}{2})$ with $i_0 \in I_1$.
By the definition of $r(\cdot)$, we have $r(x) \leq \frac{\rho}{2}$. Then the existence of $y_x \in S_\varepsilon$ satisfying (9) follows directly from

$$x \in \overline{B}\left(\gamma_{i_0}(t_0); \frac{\rho}{2}\right) \subset S_\varepsilon.$$

Case 3: $T_{i_x} < \infty$ and $x \in \overline{B}(\gamma_{i_0}(t_0); r(\gamma_{i_0}(t_0)))$ with $i_0 \in I_2$ and $t_0 \in [0, T_{i_0} - \varepsilon]$. Three subcases need to be considered.

Case 3.1: $t_x \leq T_{i_x} - \varepsilon$.
Then for $y_x = x$ we have that $x \in \overline{B}(y_x, r(x)) = \overline{B}(\gamma_{i_x}(t_x), r(\gamma_{i_x}(t_x)) \subset S_\varepsilon$.

Case 3.2: $t_x > T_{i_x} - \varepsilon$ and $x \in \overline{B}\left(\gamma_{i_x}(T_{i_x} - \varepsilon); r(\gamma_{i_x}(T_{i_x} - \varepsilon)\right)$.
Since $t_x > T_{i_x} - \varepsilon$ we get that $r(x) < r(\gamma_{i_x}(T_{i_x} - \varepsilon))$. Then the existence of $y_x \in S_\varepsilon$ satisfying (9) follows directly from

$$x \in \overline{B}\left(\gamma_{i_x}(T_{i_x} - \varepsilon); r(\gamma_{i_x}(T_{i_x} - \varepsilon)\right) \subset S_\varepsilon.$$

Case 3.3: $t_x > T_{i_x} - \varepsilon$ and $x \notin \overline{B}\left(\gamma_{i_x}(T_{i_x} - \varepsilon); r(\gamma_{i_x}(T_{i_x} - \varepsilon)\right)$ (See Figure 4.2).

We claim that $r(x) \leq r(\gamma_{i_0}(t_0))$. If not, then

$$\frac{\rho}{2}\left(1 - \frac{t_0}{T_{i_0}}\right) < \frac{\rho}{2}\left(1 - \frac{t_x}{T_{i_x}}\right).$$

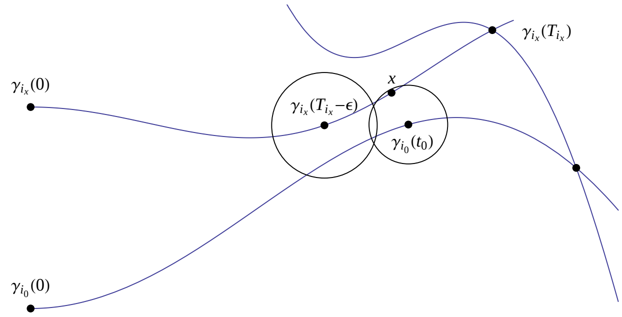

Figure 4.2: Case 3.3

Hence $T_{i_0}t_x < T_{i_x}t_0$. But $T_{i_0} - t_0 \geq \varepsilon$ and $T_{i_x} - t_x \leq \varepsilon - r(\gamma_{i_x}(T_{i_x} - \varepsilon))$.

Then $$(t_0 + \varepsilon)t_x \leq T_{i_0}t_x < T_{i_x}t_0 \leq (t_x + \varepsilon - r(\gamma_{i_x}(T_{i_x} - \varepsilon)))t_0,$$

which leads to $\varepsilon t_x < \varepsilon t_0 - r(\gamma_{i_x}(T_{i_x} - \varepsilon))t_0$, and hence

$$\begin{aligned} r(\gamma_{i_x}(T_{i_x} - \varepsilon))t_0 &< \varepsilon(t_0 - t_x) \leq \varepsilon K\|\gamma_{i_0}(t_0) - x\| \quad \text{(due to (H3))} \\ &\leq \varepsilon K r(\gamma_{i_0}(t_0)) \quad (\text{since } x \in \overline{B}(\gamma_{i_0}(t_0); r(\gamma_{i_0}(t_0)))) \\ &< \varepsilon K r(x) \quad \text{(by the contradiction assumption)} \\ &< \varepsilon K r(\gamma_{i_x}(T_{i_x} - \varepsilon)). \end{aligned}$$

Therefore $t_0 \leq \varepsilon K$ and then

$$\begin{aligned} \|\gamma_{i_0}(0) - \gamma_{i_x}(T_{i_x})\| &\leq \|\gamma_{i_0}(0) - \gamma_{i_0}(t_0)\| + \|\gamma_{i_0}(t_0) - x\| + \|x - \gamma_{i_x}(T_{i_x})\| \\ &\leq t_0 + r(\gamma_{i_0}(t_0)) + (T_{i_x} - t_x) < \varepsilon K + r(x) + \varepsilon \\ &= \varepsilon(K+1) + \frac{\rho}{2} < \frac{\rho}{2} + \frac{\rho}{2} = \rho. \end{aligned}$$

But this gives that $\gamma_{i_x}(T_{i_x}) \in B(\gamma_{i_0}(0); \rho) \subset \operatorname{int} S$ (due to (H4)) which contradicts the fact that $\gamma_{i_x}(T_{i_x}) \in \partial S$. Therefore $r(x) \leq r(\gamma_{i_0}(t_0))$. Now the existence of $y_x \in S_\varepsilon$ satisfying (9) follows directly from

$$x \in \overline{B}(\gamma_{i_0}(t_0); r(\gamma_{i_0}(t_0))) \subset S_\varepsilon.$$

The proof of our main result is complete. $\square$

Remark 4.2. The Lipschitz assumption (H3) cannot be replaced by a continuity one, that is, by

$$\lim_{x \xrightarrow[S]{} y} t_x = t_y \quad \forall y \in S. \tag{10}$$

Indeed, in Figure 4.3 let S be the hypograph of the upper curve $f(s)$ and let $(\gamma_i)_{i\in I}$ be the family of vertical semi-straight lines starting from a point on the lower curve $g(s)$ (which is the set C). One can easily verify that (H1), (H2) and (H4) are satisfied by S and $(\gamma_i)_{i\in I}$. Moreover, (10) is also satisfied but (H3) does not hold. In fact, we have

- For all x below the curve $g(s)$, $T_{i_x} = \infty$ and $r(x) = \frac{\rho}{2}$.
- For all $x = (s,t) \in S$ above the curve $g(s)$, $T_{i_x} = f(s) - g(s)$, $t_x = t - g(s)$ and
$$r(x) = \frac{\rho}{2}\left(1 - \frac{t_x}{T_{i_x}}\right).$$

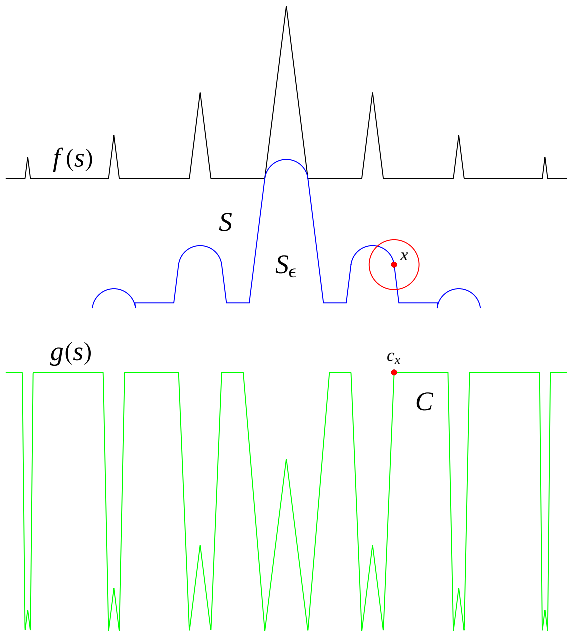

Figure 4.3: Remark 4.2

The set S_ε is the hypograph of the blue curve. The center x of the red circle has the following properties

- The point x is in S_ε since it belongs to the blue circle whose center is at a distance ε from ∂S.
- The radius $r(x)$ of the red circle is large because x is close to the starting point $c_x = \gamma_{i_x}(0)$.
- We cannot find in S_ε a ball of radius $r(x)$ containing x.

Note that for any smaller ε, the blue curve moves up, and the point x with the previous properties moves right to a smaller blue circle. □

In the following two corollaries, we will prove that the main results of [12] are direct consequences of Theorem 4.1.

Corollary 4.3. ([12, Theorem 3.1]) *Let S be a nonempty and closed subset of $\mathbb{R}^n$ and assume that $\operatorname{int} S^* \neq \emptyset$. Then for all $\varepsilon > 0$ small, there exists a nonempty and closed set $S_\varepsilon \subset S$ such that*

- *S_ε satisfies the θ-interior sphere condition,*
- *$d_H(S_\varepsilon, S) \le \varepsilon$.*

Proof. Follows directly from Proposition 3.2 and Theorem 4.1. □

Corollary 4.4. ([12, Theorem 3.3]) *Let $S \subset \mathbb{R}^n$ be a nonempty and closed set satisfying the following: there exists a closed set $A \subset \operatorname{int} S$ such that*

(1) *A is S-convex,*

(2) *A is φ_0-convex,*

(3) *For $s \in S \setminus A$, $a \in \operatorname{proj}_A(s)$ and $\zeta := \frac{s-a}{\|s-a\|}$, we have*

(i) $s \in \left[\overline{B}\left(a + \frac{1}{2\varphi_0}\zeta; \frac{1}{2\varphi_0}\right), a + d_{\partial S}(a,\zeta)\,\zeta\right] \subset S$, *if* $d_{\partial S}(a,\zeta) < \infty$.

(ii) $s \in \left[\overline{B}\left(a + \frac{1}{2\varphi_0}\zeta; \frac{1}{2\varphi_0}\right), a + t\zeta\right] \subset S$ *for all* $t \geq 0$, *if* $d_{\partial S}(a,\zeta) = \infty$.

Then for all $\varepsilon > 0$ small, there exists a nonempty and closed set $S_\varepsilon \subset S$ such that

- *S_ε satisfies the θ-interior sphere condition,*
- *$d_H(S_\varepsilon, S) \leq \varepsilon$.*

Proof. By Proposition 3.6 we get that $S^A = S \setminus A_{\frac{1}{4\varphi_0}}$ has an LCC-cover. Then for ε small (in fact less than $\frac{1}{16\varphi_0}$) and by Theorem 4.1 there exists a nonempty and closed set $S_\varepsilon^A \subset S_A$ such that

- S_ε^A is the ψ-union of closed balls,
- $d_H(S_\varepsilon^A, S) \leq \varepsilon$.

Now taking $S_\varepsilon := S_\varepsilon^A \cup \left(A + \overline{B}\left(0; \varepsilon + \frac{1}{4\varphi_0}\right)\right)$, one can easily prove that

- S_ε satisfies the θ-interior sphere condition,
- $d_H(S_\varepsilon, S) \leq \varepsilon$. □

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