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Yosida Approximation Methods for Generalized Equilibrium Problems

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We present new iteration methods, which are called *Yosida approximation methods*, for finding a critical point of generalized equilibrium problems. The methods are based on the idea of the DC (Difference of convex functions) decomposition method and Yosida regularization method. We show that the iterative sequences generated by the methods converge to a critical point under mild assumptions on parameters. Application to the Cournot-Nash oligopolistic market model with concave cost functions is reported.

Keywords: Equilibrium problems, pseudomonotonicity, Yosida regularization, critical point, projection method.

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1. Introduction

Equilibrium problems were first intrduced in [8] (shortly (EPs)). Various classes of the optimization problems, the variational inequality, the saddle point problem, Cournot-Nash oligopolistic market model, and the Kakutani fixed point problem; see [1, 12], can be formulated in the form of (EPs). There are myriad of literature related to Problem (EPs) and its applications in electricity market, transportation, economics and network; see [10, 21, 24]. The papers on existence, stability and solution methods when the cost bifunctions are convex are obtained in many researchers; see [5, 10, 13, 28] and the references quoted therein. However, when the bifunctions are nonconvex, these results might be no longer preserved.

There are several solution approaches that have been developed for (EPs), among them the proximal point method is one of fundamental methods, which was introduced first by Martinet in [19] for variational inequality and extended by Rockafellar in [30] for finding the zero point of a maximal monotone operator.

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Konnov in [13, 14] further extended the proximal point method to (EPs) with monotone bifunctions. Noor in [23] used the auxiliary problem principle to develop iterative methods for solving (EPs), where the bifunctions are supposed to be strongly monotone. In [20], Mastroeni extended the auxiliary problem principle to Problem (EPs) involving strongly monotone bifunctions satisfying some Lipschitz-type continuous conditions. In a recent article, Nguyen et al. in [22] developed a bundle method for solving (EPs), where the bifunctions satisfy a certain cocoercivity condition. Very recently, Pappalardo et al. in [24] used gap functions, developed in the literature for variational inequalities, to study a bridge between optimization and equilibria. Based on this method concerning with the resolvent of a bifunction, many mathematical works have been devoted to presenting numerical algorithms for finding approximate solutions of (EPs) with various types of conditions such as the gap function, the extragradient methods, the subgradient method and other; see [3, 10, 13, 31].

It is well-known that the DC decomposition technique is a powerfull tool for analyzing and solving optimization problems. Using DC decomposition, Algorithm DCA was introduced first by Tao and An in a preliminary form in 1985 and extensively developed to many large scale nonconvex programs in various domains of applied sciences such as solving linear complementary problems [25], the trust-region subproblem [27], a class of nonlinear bilevel programs [15], the inertial proximal method of Maingé and Moudafi [17], optimizing over the efficient set [16], and many other applications in [26].

In fact, the cost bifunction of the Cournot-Nash oligopolistic market model is not always linear or convex, the bifunction also may be concave when amount of production increasing. To motivate this fact, we further apply the techniques of DC decomposition method to the generalized equilibrium problems by using DC decomposition of the cost function $g - h$ and the Yosida regularization method. This allows us to develop proximal schemes to find a critical point of (EPs). We show that the schemes are useful when g is a convex function such that a convex subproblem is easy to minimize, and h is differentiable on C.

The paper is organized as follows. In Section 2, some preliminary results are recalled. The convergence of the sequence obtained by using DC decomposition and Yosida regularization techniques is discussed in Section 3. Section 4 gives a modified Yosida regularization method. In Section 5, the particular case of mixed variational inequalities is examined. Finally, a Cournot-Nash oligopolistic market model illustrates our consideration in Section 6.

2. Preliminaries

We consider the generalized equilibrium problems, which are formulated as follows:

$$\text{Find } \ x^* \in C \text{ such that } \ f(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C,$$

where C is a nonempty, closed, and convex subset of $\mathcal{R}^s$, $f\colon C \times C \to \mathcal{R}$ is a *bifunction* such that $f(x, x) = 0$ for all $x \in C$, $\varphi\colon C \to \mathcal{R}$ a DC function and is not necessarily convex. We suppose that $f(x, \cdot)$ is convex on C for all $x \in C$,

f is continuous on $C \times C$, and the solution set of Problem (EPs) is nonempty. When $f(x,y) = \langle F(x), y-x \rangle$ for every $x, y \in C$, and $\varphi = 0$, where $F\colon C \to \mathcal{R}^s$, the equilibrium problems reduce to the variational inequality problem (shortly, $VI(F,C)$), which is to find a point $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C.$$

The bifunction f is called *strongly monotone* on C with constant $\beta > 0$ iff

$$f(x,y) + f(y,x) \leq -\beta \|x-y\|^2, \quad \forall x, y \in C;$$

monotone on C iff $\qquad f(x,y) + f(y,x) \leq 0, \quad \forall x, y \in C;$

Lipschitz-type continuous on C with constants $c_1 > 0$ and $c_2 > 0$ in the sense of Mastroeni in [20] iff

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2, \quad \forall x, y, z \in C.$$

When $f(x,y) = \langle F(x), y-x \rangle$ with $F\colon C \to \mathcal{R}^s$,

$$f(x,y) + f(y,z) - f(x,z) = \langle F(x) - F(y), y - z \rangle \quad \text{for all } x, y, z \in C,$$

and it is easy to see that if F is Lipschitz continuous on C with constant $L > 0$, i.e., $\|F(x) - F(y)\| \leq L\|x-y\|$ for all $x, y \in C$, then

$$|\langle F(x) - F(y), y - z \rangle| \leq L\|x-y\| \, \|y-z\| \leq \tfrac{L}{2}\big(\|x-y\|^2 + \|y-z\|^2\big),$$

and thus, f satisfies Lipschitz-type continuous condition with $c_1 = c_2 = L/2$. Furthermore, when $z = x$, this condition becomes

$$f(x,y) + f(y,x) \geq -(c_1 + c_2)\|y-x\|^2, \quad \forall x, y \in C.$$

This gives a lower bound on $f(x,y) + f(y,x)$ while the strong monotonicity gives an upper bound on $f(x,y) + f(y,x)$. A multivalued mapping $F\colon C \to 2^{\mathcal{R}^s}$ is called *β-strongly monotone* on C iff

$$\langle w_x - w_y, x - y \rangle \geq \beta\|x-y\|^2, \quad \forall x, y \in C, w_x \in F(x), w_y \in F(y);$$

monotone on C iff

$$\langle w_x - w_y, x - y \rangle \geq 0, \quad \forall x, y \in C, w_x \in F(x), w_y \in F(y);$$

co-coercive with constant $\beta > 0$ on C iff

$$\langle w_x - w_y, x - y \rangle \geq \beta\|w_x - w_y\|^2, \quad \forall x, y \in C, w_x \in F(x), w_y \in F(y).$$

It is easy to see that if F is co-coersive with constant $\beta > 0$ on C, then F is Lipschitz-type continuous with constant $L := \frac{1}{\beta}$ on C. A popular example of a strongly monotone multivalued mapping is $F := \partial g$, where $g\colon C \to \mathcal{R}$ is strongly convex with constant $\sigma > 0$, i.e., for all $\lambda \in [0,1]$ and $x, y \in C$:

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y) - \frac{\sigma}{2}\lambda(1-\lambda)\|x-y\|^2.$$

Then, F is σ-strongly monotone on C.

By using the above definitions, we get the following lemma, which will be used in our analysis.

Lemma 2.1. *Let C be a nonempty, closed, and convex subset of $\mathcal{R}^s$, $\beta > 0$. Suppose that $f\colon C \times C \to \mathcal{R}$ is monotone, for each $x \in C$, $f(x,\cdot)$ is subdifferentiable, and $f(x,x) = 0$. Then, the function $\bar{f}(x) := \partial f(x,\cdot)(x) + \beta x$ is β-strongly monotone on C.*

Proof. From the definition of the subdifferential, we have

$$f(x,y) - f(x,x) \geq \langle w_x, y - x\rangle, \quad \forall x, y \in C, w_x \in \partial f(x,\cdot)(x),$$

and
$$f(y,x) - f(y,y) \geq \langle w_y, x - y\rangle, \quad \forall x, y \in C, w_y \in \partial f(y,\cdot)(y).$$

Adding the inequalities, we get for all $x, y \in C$, $w_y \in \partial f(y,\cdot)(y)$, $w_x \in \partial f(x,\cdot)(x)$:

$$f(y,x) + f(x,y) \geq \langle w_y - w_x, x - y\rangle.$$

Combining this and the monotonicity of f, we get that $\partial f(x,\cdot)(x)$ is monotone on C. And then the function $\bar{f}(\cdot)$ also is β-strongly monotone on C. □

Let f and φ be defined in Problem (EPs). For each open subset U of $\mathcal{R}^s$, we denote a solution mapping $S\colon C \to 2^C$, and a gap function $\bar{g}\colon C \to \mathcal{R}$, which are defined by

$$S(x) := \{\bar{y} : \ \bar{y} \in \operatorname{argmin}\{f(x,y) + \varphi(y) : \ y \in C \cap U\}\},$$

$$\bar{g}(x) = \min\{f(x,y) + \varphi(y) - \varphi(x) : \ y \in C \cap U\},$$

respectively. Then, the following lemma will show the relation beween the local solution of Problem (EPs) and the mappings $S, \bar{g}$.

Lemma 2.2. *Suppose that the solution set of Problem (EPs) is nonempty on $C \cap U$. Then the following statements are equivalent:*

(a) $\bar{x} \in S(\bar{x})$.

(b) $\bar{x}$ *is a solution to Problem (EPs) on $C \cap U$.*

(c) $\bar{g}(\bar{x}) = 0$.

Proof. Suppose that $\bar{x} \in S(\bar{x})$. Then, we have

$$f(\bar{x},y) + \varphi(y) - \varphi(\bar{x}) \geq f(\bar{x},\bar{x}) + \varphi(\bar{x}) - \varphi(\bar{x}) = 0, \quad \forall y \in C \cap U.$$

This means that $\bar{x}$ is a local solution to Problem (EPs). It is easy to see that if $\bar{x} \in Sol(EPs)$ then $\bar{x} \in S(\bar{x})$. So, (a) is equivalent to (b). Otherwise, from $f(x,x) + \varphi(x) - \varphi(x) = 0$, it follows that $\bar{g}(\bar{x}) = 0$ if and only if $f(\bar{x},x) + \varphi(x) - \varphi(\bar{x}) \geq 0$ for all $x \in C \cap U$. It means that (b) is equivalent to (c). □

We denote Ω the set of proper lower semi-continuously subdifferentiable convex functions on C. Since $g(y) - h(y) = \big(g(y) + k(y)\big) - \big(h(y) + k(y)\big)$, where $g, h \in \Omega$, for every arbitrary convex function $k \in \Omega$, we can assume that the functions g and h are strongly convex and subdifferentiable, and $\varphi(y) = g(y) - h(y) + \frac{\beta}{2}\|y\|^2$ for $\beta > 0$. Combining this and Lemma 2.2 that $\bar{x}$ is a solution to Problem (EPs) if and only if it solves the optimization problem:

$$\min\left\{f(\bar{x},y) + \frac{\beta}{2}\|y\|^2 + g(y) - h(y) : \ y \in C\right\},$$

we have the following definition of the concept of critical point from optimization to Problem (EPs).

Definition 2.3. A point $x^* \in C$ is called a critical point to Problem (EPs) iff

$$0 \in \partial f(x^*, \cdot)(x^*) + \beta x^* + \partial g(x^*) - \nabla h(x^*) + N_C(x^*),$$

where $\beta > 0$, and $N_C(x^*)$ denotes the outward normal cone of C at x^*.

The following lemma shows a necessary and sufficient condition for a critical point of Problem (EPs).

Lemma 2.4. *The point $x^* \in C$ is a critical point of Problem (EPs) iff*

$$x^* \in \Big(I + \lambda\partial g_1\Big)^{-1}\big(x^* + \lambda\nabla h(x^*) - \lambda\partial f(x^*, \cdot)(x^*)\big), \tag{1}$$

where $\lambda > 0, \delta_C(x)$ is the indicator function of C at x, and $g_1 := g + \delta_C$.

Proof. Note that g_1 is proper and convex on C. The inverse mapping $(I+\lambda\partial g_1)^{-1}$ is single-valued, nonexpansive, and its domain is $\mathcal{R}^n$; see [10]. By the definition of a critical point x^*, the subdifferentiality of g and h, we have

$$0 \in \lambda\partial f(x^*, \cdot)(x^*) + \lambda\beta x^* + \lambda\partial g(x^*) - \lambda\nabla h(x^*) + \partial\delta_C(x^*),$$

where $\lambda > 0$. This means that there exist $v^* \in \nabla h(x^*)$ and $w_f \in \partial f(x^*, \cdot)(x^*) + \beta x^*$ such that

$$x^* - \lambda w_f + \lambda v^* \in x^* + \lambda\partial\left[g(x^*) + \delta_C(x^*)\right].$$

The inclusion is equivalent to

$$x^* - \lambda w_f + \lambda v^* \in \left[I + \lambda\partial g_1\right](x^*),$$

where $g_1 := g + \delta_C$, which implies (1). □

Remark 2.5. Let g be proper, semi-continuous and subdifferentiable on C, let h is differentiable on C. By Minty in [30], $g_1 := g + \delta_C$ is maximal monotone on C. In a special case $\partial f(x, \cdot)(x) = \{\nabla h(x)\}$ for each $x \in C$, the form (1) becomes the well-known proximal point scheme for finding a particular solution x^* to a zero of g_1 as follows

$$x^* = \Big(I + \lambda\partial g_1\Big)^{-1}(x^*).$$

3. Yosida regularization method

Lemma 2.4 shows that DC optimization method can be applied to the generalized equilibrium problems (EPs) for finding a critical point which is defined by Definition 5.1. Then, the iteration scheme is presented as follows:

$$\begin{cases} x^0 \in C, \\ x^{k+1} := \Big(I + \lambda_k\partial g_1\Big)^{-1}\big(x^k + \lambda_k v^k - \lambda_k w^k\big), \end{cases} \tag{2}$$

where $\{\lambda_k\}$ is a sequence of positive real numbers, $\beta > 0$, $v^k \in \nabla h(x^k)$, and $w^k \in \partial f_1(x^k, \cdot)(x^k) = \partial f(x^k, \cdot)(x^k) + \beta x^k$.

Note that if we denote $y^k := x^k + \lambda_k v^k - \lambda_k w^k$, then by the Yosida regularization in [33], x^{k+1} is the unique solution to the strongly convex problem:

$$\min\left\{g(x) + \frac{1}{2\lambda_k}\|x - y^k\|^2 : \ x \in C\right\}.$$

So, the sequences of Scheme (2) can be defined as follows:

$$\begin{cases} \text{Choose arbitrary } x^0 \in C, \ v^k \in \nabla h(x^k), \\ \qquad\qquad\qquad w^k \in \partial f_1(x^k, \cdot)(x^k) = \partial f(x^k, \cdot)(x^k) + \beta x^k. \\ \text{Set } y^k := x^k + \lambda_k v^k - \lambda_k w^k. \\ \text{Find } x^{k+1} := \text{argmin}\left\{g(x) + \frac{1}{2\lambda_k}\|x - y^k\|^2 : \ x \in C\right\}. \end{cases}$$

Theorem 3.1. *Suppose that the bifunction f be monotone on C, h is differentiable and the gradient ∇h is Lipschitz with constant $L > 0$ on C, that is, $\|\nabla h(x) - \nabla h(y)\| \le L\|x - y\|$ for all $x, y \in C$, and g is σ-strongly convex on C. Then, the sequence $\{x^k\}$ generalized by Scheme (2) satisfies the inequality:*

$$\beta_k\|x^{k+1} - x^*\|^2 \le \delta_k\|x^k - x^*\|^2 - (1 - \lambda_k)\|x^{k+1} - x^k\|^2 + \frac{\lambda_k^2}{2}\|w^k - w^*\|^2. \quad (3)$$

where $\beta_k := 1 + \lambda_k\sigma - \frac{L\lambda_k}{t}$, $\delta_k := 1 - \lambda_k\beta + \lambda_k Lt$, x^* belongs to the set of critical points (shortly, S^*) of Problem (EPs), and t is an arbitrary positive real number.

Proof. From Scheme (2), it follows that

$$x^k + \lambda_k v^k - \lambda_k w^k \in \Big(I + \lambda_k \partial g_1\Big)(x^{k+1}),$$

where $v^k = \nabla h(x^k)$. This means that

$$x^{k+1} = x^k - \lambda_k w^k + \lambda_k v^k - \lambda_k z^{k+1}, \quad (4)$$

where $z^{k+1} \in \partial g_1(x^{k+1})$. Then, we have

$$\begin{aligned} \|x^{k+1} - x^k\|^2 =& \lambda_k^2\|w^k - w^*\|^2 + \lambda_k^2\|v^k - z^{k+1} + z^* - v^*\|^2 \\ &- 2\lambda^2\langle w^k - w^*, v^k - z^{k+1} + z^* - v^*\rangle, \end{aligned} \quad (5)$$

where $w^* \in \partial f_1(x^*, \cdot)(x^*)$ and $v^* = \nabla h(x^*)$. By Lemma 2.2, $\partial f_1(x, \cdot)(x)$ is strongly monotone with constant $\beta > 0$ on C, i.e.,

$$\langle w^k - w^*, x^k - x^*\rangle \ge \beta\|x^k - x^*\|^2, \quad \forall w^* \in \partial f_1(x^*, \cdot)(x^*),$$

and we have

$$\begin{aligned} 0 &\le \langle w^k - w^*, x^k - x^*\rangle - \beta\|x^k - x^*\|^2 \\ &= \langle w^k - w^*, x^k - x^* - \lambda_k(w^k - w^*)\rangle - \beta\|x^k - x^*\|^2 + \lambda_k\|w^k - w^*\|^2. \end{aligned} \quad (6)$$

It follows from $x^* \in S^*$ that

$$x^* \in \operatorname{argmin}\{f_1(x^*, y) + g_1(y) - h(y) : \ y \in \mathcal{R}^s\}.$$

Note that g_1 and h are continuous, h is differentiable, $f_1(x, \cdot)$ is convex for each $x \in C$. By the well-known Moreau-Rockafellar theorem ([29], Theorem 23.8), we have

$$0 = w^* + z^* - v^*, \tag{7}$$

where $w^* \in \partial f_1(x^*, \cdot)(x^*), v^* = \nabla h(x^*)$, and $z^* \in \partial g_1(x^*)$. Combining (4), (6), and (7), we get

$$\begin{aligned} 0 \ &\leq \langle w^k - w^*, x^k - \lambda_k w^k + \lambda_k v^k - \lambda_k z^{k+1} - x^*\rangle - \beta\|x^k - x^*\|^2 \\ &\quad + \lambda_k\|w^k - w^*\|^2 - \lambda_k\langle w^k - w^*, v^k - v^* - z^{k+1} + z^*\rangle \\ &= \langle w^k - w^*, x^{k+1} - x^*\rangle - \beta\|x^k - x^*\|^2 + \lambda_k\|w^k - w^*\|^2 \\ &\quad - \lambda_k\langle w^k - w^*, v^k - v^* - z^{k+1} + z^*\rangle. \end{aligned} \tag{8}$$

Since g is strongly convex with constant $\sigma > 0$ on C, the multivalued mapping $\partial g_1 = \partial g + N_C$ is strongly monotone with constants σ on C. Then, we have

$$\langle z^{k+1} - z^*, x^{k+1} - x^*\rangle \geq \sigma\|x^{k+1} - x^*\|^2. \tag{9}$$

Combining (4), (7), (8), and (9), we obtain

$$\begin{aligned} 0 \ &\leq \langle w^k - w^*, x^{k+1} - x^*\rangle - \beta\|x^k - x^*\|^2 + \lambda_k\|w^k - w^*\|^2 \\ &\quad - \lambda_k\langle w^k - w^*, v^k - v^* - z^{k+1} + z^*\rangle \\ &\leq \langle w^k - w^* + z^{k+1} - z^*, x^{k+1} - x^*\rangle - \beta\|x^k - x^*\|^2 + \lambda_k\|w^k - w^*\|^2 \\ &\quad - \sigma\|x^{k+1} - x^*\|^2 - \lambda_k\langle w^k - w^*, v^k - v^* - z^{k+1} + z^*\rangle \\ &= \left\langle \frac{x^k - x^{k+1}}{\lambda_k} + v^k - v^*, x^{k+1} - x^* \right\rangle - \beta\|x^k - x^*\|^2 + \lambda_k\|w^k - w^*\|^2 \\ &\quad - \lambda_k\langle w^k - w^*, v^k - v^* - z^{k+1} + z^*\rangle - \sigma\|x^{k+1} - x^*\|^2. \end{aligned}$$

Then, we have

$$\begin{aligned} \langle x^{k+1} - x^*, x^{k+1} - x^k\rangle &\leq \lambda_k\langle x^{k+1} - x^*, v^k - v^*\rangle - \lambda_k\beta\|x^k - x^*\|^2 \\ -\lambda_k^2\langle w^k - w^*, v^k - z^{k+1} + z^* - v^*\rangle &+ \lambda_k^2\|w^k - w^*\|^2 - \lambda_k\sigma\|x^{k+1} - x^*\|^2. \end{aligned} \tag{10}$$

Applying (4), (7), and the equality

$$2\langle x - y, x - z\rangle = \|x - y\|^2 + \|x - z\|^2 - \|y - z\|^2,$$

we have

$$\begin{aligned} &2\langle w^k - w^*, v^k - z^{k+1} + z^* - v^*\rangle \\ &= \|w^k - w^*\|^2 + \|v^k - z^{k+1} + z^* - v^*\|^2 - \|v^k - z^{k+1} + z^* - v^* - w^k + w^*\|^2 \\ &= \|w^k - w^*\|^2 + \|v^k - z^{k+1} + z^* - v^*\|^2 - \frac{1}{\lambda_k}\|x^k - x^{k+1}\|^2. \end{aligned} \tag{11}$$

Using (5), (10), and (11), we obtain

$$\begin{aligned}
&\langle x^{k+1}-x^*, x^{k+1}-x^k\rangle = \|x^{k+1}-x^*\|^2 + \|x^{k+1}-x^k\|^2 - \|x^k-x^*\|^2\\
&\le \lambda_k\langle x^{k+1}-x^*, v^k-v^*\rangle - \lambda_k\beta\|x^k-x^*\|^2 - \lambda_k^2\langle w^k-w^*, v^k-z^{k+1}+z^*-v^*\rangle\\
&\qquad + \lambda_k^2\|w^k-w^*\|^2 - \lambda_k\sigma\|x^{k+1}-x^*\|^2\\
&= \lambda_k\langle x^{k+1}-x^*, v^k-v^*\rangle - \lambda_k\beta\|x^k-x^*\|^2 - \lambda_k^2\|v^k-z^{k+1}+z^*-v^*\|^2\\
&\qquad + \frac{\lambda_k^2}{2}\|w^k-w^*\|^2 - \lambda_k\sigma\|x^{k+1}-x^*\|^2 + \lambda_k\|x^{k+1}-x^k\|^2\\
&= \lambda_k\langle x^{k+1}-x^*, v^k-v^*\rangle - \lambda_k\beta\|x^k-x^*\|^2 + \frac{\lambda_k^2}{2}\|w^k-w^*\|^2\\
&\qquad - \lambda_k\sigma\|x^{k+1}-x^*\|^2 + \lambda_k\|x^{k+1}-x^k\|^2.
\end{aligned}$$

Then, we have

$$\begin{aligned}
(1+\lambda_k\sigma)\|x^{k+1}-x^*\|^2 &\le (1-\lambda_k\beta)\|x^k-x^*\|^2 - (1-\lambda_k)\|x^{k+1}-x^k\|^2\\
&\quad + \frac{\lambda_k^2}{2}\|w^k-w^*\|^2 + \lambda_k\langle x^{k+1}-x^*, v^k-v^*\rangle. \qquad (12)
\end{aligned}$$

From the Lipschitz-type continuous assumption of ∇h, this implies

$$\|v^k-v^*\| \le L\|x^k-x^*\|,$$

and hence $\langle x^{k+1}-x^*, v^k-v^*\rangle \le \|x^{k+1}-x^*\|\|v^k-v^*\|$

$$\le L\|x^{k+1}-x^*\|\|x^k-x^*\| \le L\left(t\|x^k-x^*\|^2 + \frac{1}{t}\|x^{k+1}-x^*\|^2\right).$$

Then, it follows from (12) that

$$\left(1+\lambda_k\sigma - \frac{L\lambda_k}{t}\right)\|x^{k+1}-x^*\|^2 \le$$

$$\le (1-\lambda_k\beta+\lambda_k Lt)\|x^k-x^*\|^2 - (1-\lambda_k)\|x^{k+1}-x^k\|^2 + \frac{\lambda_k^2}{2}\|w^k-w^*\|^2,$$

and the theorem is proved. □

To investigate the convergence of Scheme (2), we recall the following technical lemmas which will be used in the sequel.

Lemma 3.2. ([32]) *Assume $\{a_k\}$ is a sequence of nonnegative real numbers such that $a_{k+1} \le (1-\alpha_k)a_k + \beta_k$, where $\{\alpha_k\}$ is a sequence in $]0,1[$, and $\{\beta_k\}$ is a sequence in $\mathcal{R}$ such that*

(i) $\sum_{i=1}^{\infty} \alpha_k = +\infty$, *and* (ii) $\limsup_{k\to\infty} \frac{\beta_k}{\alpha_k} \le 0$ *or* $\sum_{i=1}^{\infty} |\beta_k| < +\infty$.

Then $\lim_{k\to\infty} a_k = 0$.

The following corollary proves the convergence of proximal sequence $\{x^k\}$ by using estimation (2).

Lemma 3.3. *Let the parameters t, β, and the sequence $\{\lambda_k\}$ satisfy the following conditions:*

$$\begin{cases} t\sigma < L, \\ \beta > 1 + Lt, \\ \lambda_k \in]0, 1[, \sum\limits_{k=1}^{\infty} \lambda_k = \infty, \sum\limits_{k=1}^{\infty} \lambda_k^2 < \infty. \end{cases} \tag{13}$$

Suppose that the sequence $\{\|w^k - w^*\|^2\}$ is bounded by $M > 0$. Then, under the assumptions of Theorem 3.1, the sequence $\{x^k\}$ converges a critical point to Problem (EPs).

Proof. From (13) we get that $\alpha_k = 1+\lambda_k\sigma - \frac{L\lambda_k}{t} > 1$ and $\frac{1}{\alpha_k}(\sigma+\beta-Lt-\frac{L}{t}) > 1$. Set $\gamma_k := 1 - \frac{\delta_k}{\alpha_k}$, then

$$\gamma_k = \frac{\lambda_k}{\alpha_k}(\sigma + \beta - Lt - \frac{L}{t}) \in (\lambda_k, 1).$$

Combining this, Theorem 3.1, and the sequence $\{\|w^k - w^*\|^2\}$ is bounded by $M > 0$, we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 \leq & \frac{\delta_k}{\alpha_k}\|x^k - x^*\|^2 - \frac{1-\lambda_k}{\alpha_k}\|x^{k+1} - x^k\|^2 + \frac{\lambda_k^2}{2\alpha_k}\|w^k - w^*\|^2 \\ \leq & (1-\gamma_k)\|x^k - x^*\|^2 + \frac{\lambda_k^2 M}{2}. \end{aligned} \tag{14}$$

Applying Lemma 3.2 for the sequence $a_k := \|x^k - x^*\|^2$, we have

$$\lim_{k\to\infty} \|x^k - x^*\|^2 = 0.$$

Thus, the sequence $\{x^k\}$ converges to the critical point x^* of Problem (EPs). □

The following remark shows some conditions of the boundedness of the sequence $\{\|w^k - x^*\|^2\}$.

Remark 3.4. (i) If $\partial f(x, \cdot)(x)$ is upper semicontinuous on the set C and the sequence $\{x^k\} \subset C$ is bounded, then the sequence $\{\|w^k - w^*\|^2\}$ is bounded, where $w^k \in \partial f(x^k, \cdot)(x^k) + \beta x^k$.

(ii) If there exists a sequence $\{\lambda_k\}$ such that

$$\{\lambda_k\} \subset]0, 1[, \quad \sum_{k=1}^{\infty} \lambda_k = \infty, \quad \sum_{i=1}^{\infty} \lambda_k^2 \|w^k - w^*\|^2 < \infty,$$

then by Lemma 3.2, and (14), the sequence $\{x^k\}$ converges to a critical point of Problem (EPs).

(iii) Condition (13) shows that the convergence of Scheme (2) does not depend on the Lipschitz-type constant L of the mapping ∇h. Then, the condition $\beta > 1 + Lt$ is equivalent to $\beta > 1$ with an enough small parameter $t > 0$.

4. Modified Yosida regularization method

The DC iteration method considered in the previous section was proven to be convergent under the conditions on the parameters $\lambda_k (k = 1, 2, \cdots)$, and computing x^{k+1} in (2). The scheme (2) and the nonexpansive property of the mapping $(I + \lambda\partial g_1)^{-1}$ show that the scheme definitely may be replaced by a well-known classical iteration scheme of Mann in [18] for approximating a fixed point of a nonexpansive mapping T in a real Hilbert space $\mathcal{H}$, i.e.,

$$x^0 \in C, x^{k+1} = \alpha_k x^k + (1 - \alpha_k)T(x^k), \quad k \geq 0,$$

where C is a nonempty, closed, and convex subset of $\mathcal{H}$, and $\{\alpha_k\} \subset]0, 1[$. Then, the sequence $\{x^k\}$ converges weakly to a fixed point of T.

More precisely, in this section, we suppose that the positive parameters t, β, and the sequences $\{\lambda_k\}, \{\epsilon_k\}, \{\rho_k\}$ satisfy the following conditions:

$$\begin{cases} t\sigma < L, \ \beta > 1 + Lt, \\ \epsilon_k > 0, \ \sum\limits_{k=1}^{\infty} \epsilon_k < \infty, \\ \rho_k \in]0, 1[, \ \sum\limits_{k=1}^{\infty} \rho_k < \infty, \\ \lambda_k \in]0, 1[, \ \sum\limits_{k=1}^{\infty} \lambda_k \|w^k - w^*\| < \infty. \end{cases} \tag{15}$$

Note that at the iteration step k, for $w^k \in \partial f_1(x^k, \cdot)(x^k)$, it is very easy to choose the number λ_k such that $\lambda_k \in]0, 1[$, and $\sum\limits_{k=1}^{\infty} \lambda_k \|w^k - w^*\| < \infty$. Then, the sequence generated by Scheme (2) can be modified into the following form:

$$\begin{cases} x^0 \in C, \ y^k := x^k + \lambda_k v^k - \lambda_k w^k, \\ \left\| z^k - \big(I + \lambda_k \partial g_1\big)^{-1}(y^k) \right\| \leq \epsilon_k, \\ x^{k+1} := (1 - \rho_k)x^k + \rho_k z^k, \end{cases} \tag{16}$$

where $v^k \in \nabla h(x^k)$, $\{\lambda_k\}$ is a sequence of positive real numbers, $\beta > 0$, and $w^k \in \partial f_1(x^k, \cdot)(x^k) = \partial f(x^k, \cdot)(x^k) + \beta x^k$.

The following theorem is the main result of the subsection, and fully describes the convergence of the modified DC iteration scheme (16) for finding a critical point of the generalized equilibrium problems (EPs).

Theorem 4.1. *Suppose that the function f be monotone on C, h is differentiable, and its differential is L-Lipschitz on C, g is σ-strongly convex on C, and conditions* (15) *are satisfied. Then, the sequence $\{x^k\}$ generalized by Scheme* (16) *satisfies the inequality:*

$$\|x^{k+1} - x^*\| \leq (1 - \rho_k) \|x^k - x^*\| + \frac{1}{\sqrt{2}} \rho_k \lambda_k \|w^k - w^*\| + \rho_k \epsilon_k. \tag{17}$$

where $\beta_k := 1 + \lambda_k\sigma - \frac{L\lambda_k}{t}$, $\delta_k := 1 - \lambda_k\beta + \lambda_k Lt$, x^* *is belong to the set of critical points (shortly,* S^**) of Problem* (EPs)*, and* t *is an arbitrary positive real number. Consequently, the sequence* $\{x^k\}$ *converges to the critical* $x^* \in S^*$.

Proof. Set $\hat{x}^{k+1} := \big(I + \lambda_k \partial g_1\big)^{-1}(y^k), \bar{x}^{k+1} := (1-\rho_k)x^k + \rho_k\hat{x}^{k+1}$.
From this and Scheme (16), it follows that

$$\begin{aligned}\|x^{k+1} - x^*\| \leq& \|x^{k+1} - \bar{x}^{k+1}\| + \|\bar{x}^{k+1} - x^*\|\\ =&\rho_k\epsilon_k + \|(1-\rho_k)x^k + \rho_k\hat{x}^{k+1} - x^*\|\\ \leq&\rho_k\epsilon_k + (1-\rho_k)\|x^k - x^*\| + \rho_k\|\hat{x}^{k+1} - x^*\|. \end{aligned}\tag{18}$$

Applying Theorem 3.1 for $x^{k+1} := \hat{x}^{k+1}$, we get

$$\beta_k\|\hat{x}^{k+1} - x^*\|^2 \leq \delta_k\|x^k - x^*\|^2 + \frac{\lambda_k^2}{2}\|w^k - w^*\|^2.$$

where $\beta_k := 1 + \lambda_k\sigma - \frac{L\lambda_k}{t}$, $\delta_k := 1 - \lambda_k\beta + \lambda_k Lt$, x^* is belong to the set of critical points (shortly, S^*) of Problem (EPs), and t is an arbitrary positive real number. Then, from (14), it follows that

$$\|\hat{x}^{k+1} - x^*\| \leq \sqrt{\frac{\delta_k}{\beta_k}}\|x^k - x^*\| + \frac{1}{\sqrt{2}}\lambda_k\|w^k - w^*\|.$$

Combinating this, and (18), we have

$$\begin{aligned}\|x^{k+1} - x^*\| \leq& \left[1 - \rho_k\left(1 - \sqrt{\frac{\delta_k}{\beta_k}}\right)\right]\|x^k - x^*\| + \frac{1}{\sqrt{2}}\rho_k\lambda_k\|w^k - w^*\| + \rho_k\epsilon_k\\ \leq& (1-\rho_k)\|x^k - x^*\| + \frac{1}{\sqrt{2}}\rho_k\lambda_k\|w^k - w^*\| + \rho_k\epsilon_k.\end{aligned}\tag{19}$$

This implies (17). Applying Lemma 3.2 for (19) with the assumptions (15) on the parameters, we have

$$\lim_{k\to\infty}\|x^k - x^*\| = 0.$$

The proof is complete. $\square$

5. Application to mixed variational inequalities

Let C be a nonempty, closed, and convex subset of $\mathcal{R}^s$, F be a mapping from C to $\mathcal{R}^s$, and φ be a real-valued (not necessarily convex) function defined on C. In this section, we consider the following *mixed variational inequality problem* (shortly, (MVI)):

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), x - x^*\rangle + \varphi(x) - \varphi(x^*) \geq 0 \quad \text{for all } x \in C.$$

If the bifunction f is defined by $f(x,y) := \langle F(x), y-x\rangle$ for all $x,y \in C$, then Problem (EPs) can be formulated as the mixed variational inequality problem (MVI).

Since Ω is the set of proper lower semi-continuously subdifferentiable convex functions on C, and $\varphi(y) = g(y)-h(y) = \big(g(y)+k(y)\big)-\big(h(y)+k(y)\big)$, where $g,h \in \Omega$, for every arbitrary convex function $k \in \Omega$, we can assume that the functions g and h are strongly convex, and $\varphi(y) = g(y)-h(y)+\frac{\beta}{2}\|y\|^2$ for $\beta > 0$. We suppose that h is subdifferentiable, and ∇h is Lipschitz-type continuous with constant $L > 0$ on C. With this assumptions, we usually call Problem (MVI) to be *DC mixed variational inequality problem*, which is denoted by $(DCVI)$. By the same way as the generalized equilibrium problems, we have the following definition of the concept of critical points from optimization to Problem $(DCVI)$.

Definition 5.1. A point $x^* \in C$ is called a *critical point* to Problem $(DCVI)$ iff

$$0 \in F(x^*) + \beta x^* + \partial g(x^*) - h(x^*) + N_C(x^*),$$

where $\beta > 0, v^* \in \nabla h(x^*)$, $N_C(x^*)$ is the normal cone of C at x^*, $\partial g(x^*)$ is the subgradient of g at x^*, and $\nabla h(x^*)$ is the gradient of h at x^*.

Applying Lemma 2.4 for the DC mixed variational inequality problem, the point $x \in C$ is a critical point of Problem $(DCVI)$ iff

$$x \in \Big(I + \lambda \partial g_1\Big)^{-1}\big(x + \lambda \nabla h(x) - \lambda F(x)\big),$$

where $\lambda > 0, \delta_C$ is indicator function of C at x, and $g_1 := g + \delta_C$. Then, the iteration scheme for finding a critical point to Problem $(DCVI)$ is presented as follows:

$$\begin{cases} x^0 \in C, \\ x^{k+1} := \Big(I + \lambda_k \partial g_1\Big)^{-1}\big(x^k + \lambda_k v^k - \lambda_k w^k\big), \end{cases} \tag{20}$$

where $v^k = \nabla h(x^k)$, $\{\lambda_k\}$ is a positive real sequence, $\beta > 0$ and $w^k := F(x^k)+\beta x^k$. Denote $y^k := x^k + \lambda_k v^k - \lambda_k w^k$, then the sequence of Scheme (20) can be defined as the the following:

$$\begin{cases} \text{Choose arbitrary } x^0 \in C, \\ \text{Set } y^k := x^k + \lambda_k v^k - \lambda_k F(x^k) - \lambda_k \beta x^k, \\ \text{Find } x^{k+1} := \operatorname{argmin}\{g(x) + \frac{1}{2\lambda_k}\|x-y^k\|^2 :\ x \in C\}. \end{cases}$$

From Theorem 3.1 and Lemma 3.3, it follows the convergence of the sequence generated by (20) as follows.

Theorem 5.2. *Suppose that the function $F\colon C \to \mathcal{R}^s$ be monotone, h is subdifferentiable and the gradient ∇h is Lipschitz continuous with constant $L > 0$ on C, and g is σ-strongly convex on C. Then, the sequence $\{x^k\}$ generalized by Scheme* (20) *satisfies the inequality:*

$$\beta_k\|x^{k+1}-x^*\|^2 \leq \delta_k\|x^k-x^*\|^2-(1-\lambda_k)\|x^{k+1}-x^k\|^2+\frac{\lambda_k^2}{2}\|w^k-w^*\|^2.$$

where $\beta_k := 1+\lambda_k\sigma-\frac{L\lambda_k}{t}$, $\delta_k := 1-\lambda_k\beta+\lambda_k Lt$, x^* *is a critical point of Problem* $(DCVI)$, *and* t *is an arbitrary positive real number. Moreover, if the parameters* t,β, *the sequence* $\{\lambda_k\}$ *satisfy the following conditions:*

$$\begin{cases} t\sigma < L, \ \ \beta > 1+Lt, \\ \lambda_k \in]0,1[, \ \ \sum\limits_{k=1}^{\infty}\lambda_k=\infty, \ \ \sum\limits_{k=1}^{\infty}\lambda_k^2<\infty, \end{cases}$$

and the sequence $\{F(x^k)\}$ *is bounded, then the sequence* $\{x^k\}$ *converges a critical point to Problem* $(DCVI)$.

Theorem 5.2 shows the convergence of Scheme (20) which requires that $F(x^k)$ is bounded and computing $(I+\lambda\partial g_1)^{-1}(y^k)$. In order to avoid this requirements, in Section 4 we modify Scheme (2) for Problem (EPs) by using Mann's fixed point technique. The technique has been used widely in the fixed point theory and applications as well as for the fixed point of nonexpansive mappings. Applying Scheme (16) for finding a critical point of Problem $(DCVI)$, we get the follows.

$$\begin{cases} x^0 \in C, \ \ y^k := x^k+\lambda_k v^k-\lambda_k(F(x^k)+\beta x^k), \\ \left\| z^k-\left(I+\lambda_k\partial g_1\right)^{-1}(y^k)\right\| \leq \epsilon_k, \\ x^{k+1} := (1-\rho_k)x^k+\rho_k z^k, \end{cases} \tag{21}$$

where $\{\lambda_k\}$ is a positive real sequence and $\beta > 0$. By using Theorem 4.1, we have the convergent results of Scheme (21) as follows.

Theorem 5.3. *Suppose that the function* F *be monotone on* C, h *is subdifferentiable, and its differential is Lipschitz-type continuous with constant* $L > 0$ *on* C, g *is* σ*-strongly convex on* C, *and conditions* (15) *are satisfied. Then, the sequence* $\{x^k\}$ *generalized by Scheme* (21) *satisfies the inequality:*

$$\|x^{k+1}-x^*\| \leq (1-\rho_k)\|x^k-x^*\|+\frac{1}{\sqrt{2}}\rho_k\lambda_k\|w^k-w^*\|+\rho_k\epsilon_k. \tag{22}$$

where $\beta_k := 1+\lambda_k\sigma-\frac{L\lambda_k}{t}$, $\delta_k := 1-\lambda_k\beta+\lambda_k Lt$, $w^k = F(x^k)+\beta x^k, w^* = F(x^*)-\beta x^*$, x^* *is belong to the set of critical points of Problem* $(DCVI)$, *and* t *is an arbitrary positive real number. Consequently, the sequence* $\{x^k\}$ *converges to the critical point* x^*.

6. Numerical illustration

In this section, we consider Cournot-Nash oligopolistic market equilibrium models in [13, 21, 28] as some examples for the generalized equilibrium problems (EPs).

In this model, it is assumed that there are n-firms producing a common homogenous commodity, and that the price p_i of firm i depends on the total quantity $\sigma := \sum_{i=1}^{n} x_i$ of the commodity. Let $h_i(x_i)$ denote the cost of the firm i when its production level is x_i . Suppose that the profit of firm i is given by

$$f_i(x_1, \cdots, x_n) := x_i p_i \left(\sum_{i=1}^{p} x_i \right) - h_i(x_i), \quad \forall i = 1, \cdots, n. \tag{23}$$

where h_i is the cost function of firm i that is assumed to be dependent only on its production level.

Let $C_i \subset \mathcal{R}$ for all $i = 1, \cdots, n$ denote the strategy set of the firm i. Each firm seeks to maximize its own profit by choosing the corresponding production level under the presumption that the production of the other firms are parametric input. In this context, a Nash equilibrium is a production pattern in which no firm can increase its profit by changing its controlled variables. Thus under this equilibrium concept, each firm determines its best response given other firm's actions. Mathematically, a point $x^* := (x_1^*, \cdots, x_n^*) \in C := C_1 \times \cdots C_n$ is said to be a *Nash equilibrium* if

$$f_i(x_1^*, \cdots, x_{i-1}^*, x_i, x_{i+1}^*, \cdots, x_n^*) \leq f_i(x_1^*, \cdots, x_n^*), \quad \forall x_i \in C_i, i = 1, \cdot, n.$$

When h_i is affine, this market problem can be formulated as a special Nash equilibrium problem in the n-person nonco-operative game theory.

Set $$\phi(x,y) := -\sum_{i=1}^{n} f_i(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n), \tag{24}$$

and $$f(x,y) := \phi(x,y) - \phi(x,x). \tag{25}$$

Then it has been proved in [13] that x^* is an equilibrium point of this model if and only if it is a solution to Problem (EPs).

In classical Cournot-Nash models [13], the price and the cost functions for each firm are assumed to be affine of the forms

$$\begin{cases} p_i(\sigma) := \alpha_0 - \chi\sigma, \ \alpha_i \geq 0, \ \chi > 0, \ \sigma = \sum_{i=1}^{n} x_i, \\ h_i(x_i) = \mu_i x_i + \xi_i, \ \mu_i \geq 0, \ \xi_i \geq 0, \quad \forall i = 1, \cdots, n. \end{cases} \tag{26}$$

Substitute p_i, h_j defined by (26) into (25) and (24), we get

$$f(x,y) = \langle Ax + \mu - \alpha, y - x \rangle + \langle y, By \rangle - \langle x, Bx \rangle,$$

where

$$A = \begin{pmatrix} 0 & \chi & \chi & \cdots & \chi \\ \chi & 0 & \chi & \cdots & \chi \\ \chi & \chi & 0 & \cdots & \chi \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \chi & \chi & \cdot & \cdots & 0 \end{pmatrix}_{n \times n}, \quad B = \begin{pmatrix} \chi & 0 & 0 & \cdots & 0 \\ 0 & \chi & 0 & \cdots & 0 \\ 0 & 0 & \chi & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & \chi \end{pmatrix}_{n \times n},$$

and $$\mu = (\mu_1, \cdots, \mu_n)^T, \alpha = (\alpha_0, \cdots, \alpha_0)^T.$$

This problem is formulated as Problem (EPs) with assumptions that the price and the cost functions for each firm are assumed to be affine.

In general, the cost depends linearly on the quantity of the commodity is not practical, since usually the cost per a unit of the action does decrease when the quantity of the commodity exceeds a certain amount. Recently, Cournot-Nash oligopolistic equilibrium market models with piecewise concave cost functions have been considered in [21] on the strategy box C. The authors gave conditions for existence of global equilibrium points and proposed an algorithm for finding a global equilibrium point or for detecting that the problem is unsolvable.

Now we suppose that the strategy set C of the firms is nonempty, closed, convex, and the cost functions $h_i(\cdot)$ for all $i = 1, \cdots, n$ are concave (not necessarily picewise as in [21]) and that the price function $p(\sigma)$ can change from firm by firm and has the following form:

$$p_i(\sigma) = \alpha_i - \chi_i \sigma, \ \alpha_i \geq 0, \ \chi_i \geq 0, \quad \forall i = 1, \cdots, n.$$

It is easy to see that finding a Nash equilibrium point of this model can be formulated as solving the following generalized equilibrium problems:

$$\text{Find } x^* \in C \text{ such that } \ f(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C,$$

where $$f(x, y) := \langle Dx - \alpha, y - x \rangle, \quad \varphi(x) := \langle x, Ex \rangle + \sum_{i=1}^n h_i(x_i), \tag{27}$$

and $\alpha = (\alpha_1, \cdots, \alpha_n)^T$,

$$D = \begin{pmatrix} 0 & \chi_1 & \chi_1 & \cdots & \chi_1 \\ \chi_2 & 0 & \chi_2 & \cdots & \chi_2 \\ \chi_3 & \chi_3 & 0 & \cdots & \chi_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \chi_n & \chi_n & \cdot & \cdots & 0 \end{pmatrix}_{n \times n}, \quad E = \begin{pmatrix} \chi_1 & 0 & 0 & \cdots & 0 \\ 0 & \chi_2 & 0 & \cdots & 0 \\ 0 & 0 & \chi_3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & \chi_n \end{pmatrix}_{n \times n}.$$

With the assumptions $\chi_i > 0$ for all $i = 1, \cdots, n$ and the concavity of h, this problem is a form of Problem (EPs).

To illustrate the schemes (2), we consider to numerical examples involving concave cost functions. We perform in Matlab R2008a running on a PC Desktop Intel(R) Core(TM) 2 Duo CPU T5750@2GHz, 1.66GHz, 1.99GB of RAM. The tolerance is taken by $\epsilon = 10^{-9}$.

Example 6.1. In the Nash-Cournot model (27), we suppose now that $n = 6$, $\chi_i = 2i + 1$ $(i = 1, \cdots, 6)$, the cost function $h_i(x_i)$ of the firm i is given as $h_i(x_i) = a_i + b_i \ln(1 + c_i x_i)$, where $a_i = 3, b_i = 5, \alpha_i = i, c_i$ is randomly chosen in $(2, 3)$ for all $i = 1, \cdots, 6$. The strategy set of the firms is

$$C = \begin{cases} x \in \mathcal{R}^6_+, \\ x_1 + x_2 \geq 1.5, \\ x_1 + 1.3x_2 + 1.2x_3 + 2.1x_4 + 2.7x_5 + 3x_6 \geq 9, \\ 3x_1 + 2x_2 + x_3 + 1.3x_4 + 4x_5 + 5x_6 \leq 17. \end{cases}$$

Then, we have $\nabla h_i(x_i) = \frac{b_i c_i}{1+c_i x_i}$, $\nabla^2 h_i(x_i) = -\frac{b_i c_i^2}{(1+c_i x_i)^2} < 0, \quad \forall x \in C$. The function h is formulated by

$$\hat{h}(x) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \ln(1 + c_i x_i),$$

is concave on C and its gradient is computed by $\nabla \hat{h}(x) = \left(\frac{b_1 c_1}{1+c_1 x_1}, \cdots, \frac{b_n c_n}{1+c_n x_n}\right)^T$, which is L-Lipschitz on C, where

$$L = \max\{|\nabla^2 h_i(x_i)| : \ x \in C\} \leq \max\{b_i c_i^2 : \ i = 1, \cdots, n\} < 45.$$

In the schemes (2), $h := -\hat{h}$ and g is defined by

$$g(x) := \langle x, Ex\rangle - \frac{\beta}{2}\|x\|^2, \quad \forall x \in C.$$

It is easy to see that g is σ-strongly convex on C, where $\sigma := \min\{\chi_1, \cdots, \chi_n\} - \frac{\beta}{2} > 0$. The bifunction f is subdifferentiable on C and $w^k \in \partial f(x^k, \cdot)(x^k) + \beta x^k$ is computed by $w^k = Dx^k - \alpha + \beta x^k$. The parameters are chosen in the scheme as the follows:

$$\beta = 2, \ \sigma = 2, \ \rho_k = \frac{1}{k^2}, \ \lambda_k = \frac{1}{k}.$$

Then the conditions (13) are satisfied.

The computations are performed by Matlab R2008a running on a PC Desktop Intel(R) Core(TM)i5 650@3.2GHz 3.33GHz 4Gb RAM.

When c_i is randomly chosen in $(2, 3)$ for all $i = 1, \cdots, 6$, we obtained the results in Table 1.

Problem	Iter.	CPU-Times/sec
1	40	6.0625
2	38	6.0631
3	38	5.8906
4	41	6.2656
5	38	6.0781
6	37	5.9219
7	38	5.7344
8	40	6.1563
9	38	5.8125
10	38	5.9688

Table 1. The tolerance is $\epsilon = 10^{-4}$.

Example 6.2. In Problem (EPs), we consider a monotone bifunction in [2], which is given as

$$f(x, y) = \langle d, \text{arctg } (x - y)\rangle + \langle q, x - y\rangle,$$

where $n = 6$, $q = (2,4,6,8,1,3)^T$, $\operatorname{arctg}(x) = \big(\operatorname{arctg}(x_1), \cdots, \operatorname{arctg}(x_6)\big)^T$, and $d = (1,3,5,7,9,11)^T$. In this example, the function φ is defined as in Example 6.1, we choose the cost function h_i as $h_i(x_i) = a_i - b_i e^{-c_i x_i}$, where $a_i = 3i^2 + 1, b_i = 2i$, $c_i = 3i - 1$, and the feasible domain is as follows

$$C = \begin{cases} x \in \mathcal{R}^6_+, \\ x_5 + x_6 \geq 1, \\ x_1 + x_2 \geq 1.5, \\ x_1 + x_2 + x_3 + 2x_4 + x_5 + x_6 \geq 4, \\ 3x_1 + 2x_2 + x_3 + 3x_4 + 4x_5 + 2x_6 \leq 17. \end{cases}$$

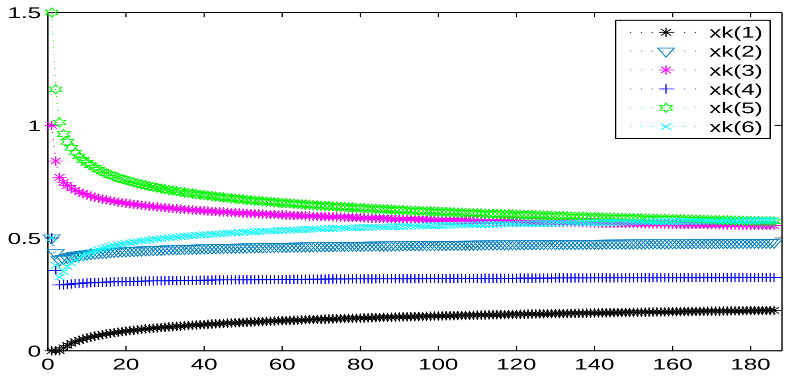

Figure 6.1: The convergent graph of the iteration sequence $x^k = \big(xk(1), \cdots, xk(6)\big)^T$.

It is easy to see that $h_i''(x_i) = -b_i c_i^2 e^{-c_i x_i} < 0$ for all $x \in C, i = 1, \cdots, 6$. So, for each $i = 1, \cdots, 6$, h_i is concave on C. The gradient of h in Scheme (2) is defined by $\nabla h(x) = (-b_1 c_1 e^{-c_1 x_1}, \cdots, -b_6 c_6 e^{-c_6 x_6})^T$, which is Lipschitz continuous on C with constant $L = \max\{b_i c_i : \ i = 1, \cdots, 6\} = 4$. Note that the vector $w^k \in \partial f(x^k, \cdot)(x^k)$ is evaluated by $w^k = \beta x^k - d - q$. Then, we obtain the numerical results of Scheme (2) with the starting-point $x^0 = (1,1,1,1,1,1)^T$ and tolerance $\epsilon = 10^{-6}$ in Figure 6.2. The parameters are chosen by $\beta = 2$ and $\lambda_k = \frac{1}{k+1}$.

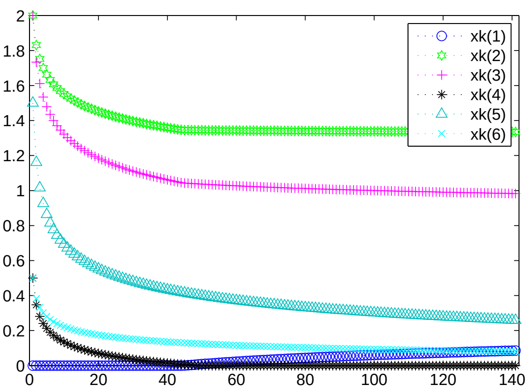

Figure 6.2: The convergent graph of the iteration sequence $x^k = \big(x1(k), \cdots, x6(k)\big)^T$.

7. Conclusion

We presented new iterative algorithms for finding a critical point of the generalized equilibrium problems, which are based on the idea of the DC decomposition method. By choosing the suitable regular parameters, we show that the sequences generated by the algorithms converge to the critical point. This paper deals with numerical computations for Nash-Cournot equilibrium models involving nonconvex cost bifunctions. Numerical examples are implemented to illustrate the convergence behavior of the proposed algorithms.

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