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On the Structure Topology on the Set of all Extreme Points of the Closed Unit Ball of the Dual of a Banach Space

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Let X be a real Banach space, and let E_{X^*} stand for the set of all extreme points of the closed unit ball of X^*, endowed with the Alfsen-Effros structure topology [see E. M. Alfsen and E. G. Effros, *Structure in real Banach spaces I, II*, Annals of Math. 96 (1972) 98–128; ibid. 96 (1972) 129–73]. The fact that, for a given $s^* \in E_{X^*}$, the set $\{\pm s^*\}$ is structurally open can be characterized in many apparently different ways, whenever X is nice. (We recall that X is said to be nice if every extreme operator from any Banach space to X is a nice operator, i.e. its adjoint preserves extreme points.) As a consequence, we obtain new characterizations (as well as new proofs of known characterizations) of those nice Banach spaces which are isometrically isomorphic to $c_0(I)$ for some set I.

Keywords: Banach space, extreme operator, nice operator, structure topology.

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1. Introduction

Throughout this paper, X will stand for a real Banach space, $\mathbb{B}_X$ and $\mathbb{S}_X$ will denote the closed unit ball and the unit sphere of X. X^* will denote the dual of X, and E_X will stand for the set of all extreme points of $\mathbb{B}_X$. As usual, for any subspace Y of X, we denote by Y° the polar of Y in X^*.

A subspace Y of X is said to be an *L-summand* if it is the range of a linear projection P on X such that $\|x\| = \|P(x)\| + \|x - P(x)\|$ for every $x \in X$, and is called an *M-ideal* if it is closed in X and Y° is an L-summand of X^*.

According to [13, Proposition I.1.11 and Definition I.3.11], the collection of sets $E_{X^*} \cap Y^\circ$, where Y runs through the family of all M-ideals in X, is the collection of closed sets for some topology on E_{X^*}, called *the structure topology.* This topology is never Hausdorff, since, for $p \in E_{X^*}$, p and $-p$ cannot be separated. For this reason we shall also consider the quotient space $E_{X^*}/\sim$ (where $p \sim q$ if and only

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if $p = \pm q$) equipped with the corresponding quotient topology, which will be denoted by $(E_{X^*})_\sigma$.

In [7, Proposition 2] we proved the following.

Theorem 1.1. *X is isometrically isomorphic to $c_0(I)$, for some set I, if and only if $(E_{X^*})_\sigma$ is discrete.*

Definition 1.2. Let Y be any real Banach space. We denote by $BL(Y,X)$ the Banach space of all bounded linear operators from Y to X. According to [21], a bounded linear operator $T : Y \to X$ is said to be *nice* if $T^*(E_{X^*}) \subseteq E_{Y^*}$ (where $T^* : X^* \to Y^*$ denotes the adjoint of T), and is said to be *extreme* if it lies in $E_{BL(Y,X)}$.

It is easy to check that nice bounded linear operators are extreme operators. Since their birth in [21], nice operators have been widely studied in the literature (see [2, 3, 4, 11, 12, 14, 15, 19, 20, 22, 23, 24, 25, 26]).

Definition 1.3. According to [5], the Banach space X is said to be *nice* if, for every real Banach space Y, every extreme bounded linear operator from Y to X is nice.

In [5, 6, 7, 8, 9] we characterized those nice classical Banach spaces which are isomorphic to $c_0(I)$ for some set I. To derive these characterizations from Theorem 1.1 it is noteworthy that, for arbitrary X, the topological space $(E_{X^*})_\sigma$ is discrete if and only if, for every $s^* \in E_{X^*}$, the set $\{\pm s^*\}$ is structurally open in E_{X^*}. In this paper, we provide a 'local' approach to the above reasoning. Indeed, in Theorem 2.8 we prove that, for s^* in the dual of a nice Banach space X, the fact that $\{\pm s^*\}$ is a structurally open subset of E_{X^*} can be characterized in many apparently different ways. These characterizations involve numerical range techniques, as well as the relationship between these techniques and the theory of semi-L-summands and semi-M-ideals (see Propositions 2.2 and 2.5), and allow us to obtain proofs of previously known results in a unified way (see Remark 2.9).

2. The results

Let u be in $\mathbb{S}_X$. By a *state of X relative to u* we mean an element $f \in \mathbb{B}_{X^*}$ satisfying $f(u) = 1$. We denote by $D(X,u)$ the set of all states of X relative to u, and remark that, by the Hahn-Banach and Banach-Alaoglu theorems, $D(X,u)$ becomes a non-empty w^*-compact convex subset of X^*. Given $x \in X$, we define the *numerical range of x relative to u* (denoted by $V(X,u,x)$) as the non-empty compact convex subset of $\mathbb{R}$ given by

$$V(X,u,x) := \{f(x) : f \in D(X,u)\},$$

and the *numerical radius of x relative to u* (denoted by $v(X,u,x)$) as the non-negative real number given by

$$v(X,u,x) := \max\{|z| : z \in V(X,u,x)\}.$$

The *numerical index, $n(X,u)$, of X at u*, is defined as the maximum non-negative real number k satisfying $k\|x\| \leq v(X,u,x)$ for every $x \in X$. We note that $0 \leq n(X,u) \leq 1$.

Let Y be a closed subspace of X. We say that Y is a *semi-L-summand* of X if, for each $x \in X$, there exists a unique $y \in Y$ such that $\|x-y\| = \|x+Y\|$, and moreover this y satisfies $\|x\| = \|y\| + \|x-y\|$. It is straightforward that *L-summands are semi-L-summands.* As the next example shows, the converse is not true.

Example 2.1. As usual, let $\mathbb{R}^3_\infty$ denote the real Banach consisting of the vector space $\mathbb{R}^3$ and the sup norm. Then, according to [16, Theorem 5.14], $\mathbb{R}(1,1,1)$ is a semi-L-summand of $\mathbb{R}^3_\infty$, but is not an L-summand.

Proposition 2.2. *For $u \in \mathbb{S}_X$, the following conditions are equivalent:*

(i) $\mathbb{R}u$ *is a semi-L-summand of X.*

(ii) $n(X,u) = 1$.

(iii) $\mathbb{B}_{X^*} = \mathrm{co}(D(X,u) - D(X,u))$, *where* $\mathrm{co}(\cdot)$ *means convex hull.*

(iv) $|f(u)| = 1$ *for every* $f \in E_{X^*}$.

Moreover, if the above conditions are fulfilled, then

(v) u *lies in* E_X.

(vi) $\|e + \mathbb{R}u\| = 1$ *for every* $e \in E_X \setminus \{\pm u\}$.

Proof. The equivalence (i)$\Leftrightarrow$(ii) follows from [10, Fact 2.9.65 and § 2.9.66].

(ii)$\Rightarrow$(iii) By [10, Theorem 2.1.17(iii)].

(iii)$\Rightarrow$(iv) By the assumption (iii), for each $f \in \mathbb{B}_{X^*}$ there are $\alpha \in [0,1]$ and $f_1, f_2 \in D(X,u)$ such that

$$f = \alpha f_1 - (1-\alpha) f_2.$$

Therefore, if in fact f belongs to E_{X^*}, then either $f_1 = f$ or $f_2 = -f$, which implies $f(u) = \pm 1$.

(iv)$\Rightarrow$(ii) Let x be in $\mathbb{S}_X$. Take an extreme point f of the non-empty w^*-compact convex set $D(X,x)$. Since $D(X,x)$ is a face of $\mathbb{B}_{X^*}$, we realize that f belongs to E_{X^*}. Therefore, by the assumption (iv), we have that $|f(u)| = 1$ (equivalently, $\pm f \in D(X,u)$), which implies that $v(X,u,x) = 1$. Thus, norm-one elements in X have numerical radius relative to u equal to one, i.e., $n(X,u) = 1$.

Now that we know that conditions (i) to (iv) are equivalent, let us prove that any of them implies conditions (v) and (vi).

(ii)$\Rightarrow$(v) Suppose that condition (ii) is fulfilled. Then it follows from [10, Definition 2.1.16, Theorem 2.1.17(i), and Lemma 2.1.25] that u lies in E_X.

(i)$\Rightarrow$(vi) Suppose that condition (i) is fulfilled, so that, for each $x \in X$, there is a unique $\pi(x) \in \mathbb{R}u$ such that

$$\|x - \pi(x)\| = \|x + \mathbb{R}u\|, \tag{1}$$

and the equality
$$\|x\| = \|\pi(x)\| + \|x - \pi(x)\| \tag{2}$$
holds. Let e be in $E_X \setminus \{\pm u\}$. It is clear that $(E_X \setminus \{\pm u\}) \cap \mathbb{R}u = \emptyset$. Therefore $\pi(e) \neq e$. Now assume that $\pi(e) \neq 0$. Then, since $\pi(e) \neq e$, we can write

$$e = \|\pi(e)\| \frac{\pi(e)}{\|\pi(e)\|} + \|e - \pi(e)\| \frac{e - \pi(e)}{\|e - \pi(e)\|},$$

so that, since $1 = \|\pi(e)\| + \|e - \pi(e)\|$ (a consequence of (2)), we realize that e is a non-trivial convex combination of two different points of $\mathbb{B}_X$, contradicting $e \in E_X$. Therefore $\pi(e) = 0$, and then it follows from (1) that $\|e + \mathbb{R}u\| = \|e\| = 1$. □

Remark 2.3. (a) In the above proof, we have derived (v) from (ii) by simply referring to [10]. Actually, (v) can also be derived from (iv) as follows. Write $u = \frac{1}{2}(v + w)$ with $v, w \in \mathbb{B}_X$, and let f be in E_{X^*}. If condition (iv) is fulfilled, then we have that $\pm 1 = f(u) = \frac{1}{2}(f(v) + f(w))$ with $f(v), f(w) \in [-1, 1]$, and hence $f(v) = f(w) = \pm 1 = f(u)$, which implies (in view of the arbitrariness of $f \in E_{X^*}$) that $u = v = w$ because $\mathbb{B}_{X^*}$ is the w^*-closure of the convex hull of E_{X^*}. Thus u lies in E_X.

(b) In the particular case that u lies in E_X, the equivalence (i)⇔(iv) in Proposition 2.2 was proved by Lima [17, Theorem 3.1].

(c) The implication (i)⇒(vi) in Proposition 2.2 can be seen as a particular non-linear version of [13, Lemma I.1.5]. □

By a *semi-M-ideal* of X we mean any closed subspace Y of X such that Y° is a semi-L-summand of X^*. A subspace Y of X is said to be an *M-summand* if it is the range of a linear projection P on X such that $\|x\| = \max\{\|P(x)\|, \|x - P(x)\|\}$ for every $x \in X$. It is straightforward that *M-summands are M-ideals* and that *M-ideals are semi-M-ideals.*

Given $s^* \in \mathbb{S}_{X^*}$, we denote by $F(s^*)$ the set of all w^*-continuous states of X^* relative to s^*, namely $F(s^*) := D(X^*, s^*) \cap X = \{x \in \mathbb{S}_X : s^*(x) = 1\}$. If $F(s^*)$ is nonempty, it is clear that $F(s^*)$ is a proper closed face of B_X.

The next lemma is pointed out in [18].

Lemma 2.4. *Maximal proper faces of B_X are of the form $F(e^*)$ for some $e^* \in E_{X^*}$.*

We will denote by $\mathrm{mex}(\mathbb{B}_{X^*})$ the set of elements $e^* \in E_{X^*}$ such that $F(e^*)$ is a maximal proper face of $\mathbb{B}_X$.

Proposition 2.5. *For $s^* \in \mathbb{S}_{\mathbb{X}^*}$, consider the following conditions:*

(1) $\{\pm s^*\}$ *is a structurally open subset of E_{X^*} (equivalently, the class $[s^*]$ is an isolated point of the topological space $(E_{X^*})_\sigma$).*

(2) $\ker(s^*)$ *is an M-summand of X.*

(3) $\mathbb{R}s^*$ *is an L-summand of X^* (equivalently,* $\ker(s^*)$ *is an M-ideal of X).*

(4) $\mathbb{R}s^*$ *is a semi-L-summand of X^* (equivalently,* $\ker(s^*)$ *is a semi-M-ideal of X).*

(5) $n(X^*, s^*) = 1$.

(6) $\mathbb{B}_{X^{**}} = \mathrm{co}(D(X^*, s^*) - D(X^*, s^*))$.

(7) $|e^{**}(s^*)| = 1$ *for every* $e^{**} \in E_{X^{**}}$.

(8) $\mathbb{B}_X = \overline{\mathrm{co}}(F(s^*) \cup -F(s^*))$, *where* $\overline{\mathrm{co}}(\cdot)$ *means closed convex hull.*

(9) s^* *lies in* $\mathrm{mex}(\mathbb{B}_{X^*})$ *and* $\|e^* + \mathbb{R}s^*\| = 1$ *for every* $e^* \in E_{X^*} \setminus \{\pm s^*\}$.

(10) s^* *lies in* E_{X^*} *and* $\|e^* + \mathbb{R}s^*\| = 1$ *for every* $e^* \in E_{X^*} \setminus \{\pm s^*\}$.

(11) $\mathbb{B}_X = \mathrm{co}(F(s^*) \cup -F(s^*))$.

Then

$$\begin{tikzcd}
 & & (1) \arrow[d, Leftrightarrow] & & \\
 & & (2) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & (3) \arrow[dl, Rightarrow] & & (11) \arrow[dr, Rightarrow] & \\
(4) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & (5) \arrow[r, Leftrightarrow] & (6) \arrow[r, Leftrightarrow] & (7) \arrow[r, Leftrightarrow] & (8) \\
(9) \arrow[d, Rightarrow] & & & & \\
(10). & & & &
\end{tikzcd}$$

Proof. (1)$\Rightarrow$(2): By the assumption (1), there exists an M-ideal Y of X such that $E_{X^*} \cap Y^\circ = E_{X^*} \setminus \{\pm s^*\}$. Then, by the definition of an M-ideal, there exists a (closed) subspace Z of X^* such that $X^* = Y^\circ \oplus_1 Z$. It follows from [13, Lemma I.1.5] that $E_Z = E_{X^*} \cap Z = \{\pm s^*\}$. Since Z is a dual Banach space (in fact it is isometrically isomorphic to $X^*/Y^\circ \equiv Y^*$), it follows from the Krein-Milman theorem that $Z = \mathbb{R}s^*$. Therefore

$$X^* = Y^\circ \oplus_1 \mathbb{R}s^* = Y^\circ \oplus_1 (\ker(s^*))^\circ,$$

so $X = Y \oplus_\infty \ker(s^*)$, and $\ker(s^*)$ is indeed an M-summand of X.

(2)$\Rightarrow$(1): The assumption (2) reads as that $X = \ker(s^*) \oplus_\infty Y$ for some (closed) subspace Y of X. Therefore $X^* = (\ker(s^*))^\circ \oplus_1 Y^\circ = \mathbb{R}s^* \oplus_1 Y^\circ$, and hence, since $E_{\mathbb{R}s^*} = \{\pm s^*\}$, it follows from [13, Lemma I.1.5] that s^* lies in E_{X^*} and that $E_{X^*} \cap Y^\circ = E_{X^*} \setminus \{\pm s^*\}$. Since M-summands are M-ideals, and Y is an M-summand of X, the above implies that $\{\pm s^*\}$ is a structurally open subset of E_{X^*}.

The implications (2)$\Rightarrow$(3)$\Rightarrow$(4), (9)$\Rightarrow$(10) and (11)$\Rightarrow$(8) are clear.

The equivalences (4)$\Leftrightarrow$(5)$\Leftrightarrow$(6)$\Leftrightarrow$(7) follow from Proposition 2.2.

The implication (5)$\Rightarrow$(8) is the non-trivial part of the desired equivalence (5)$\Leftrightarrow$(8), and follows from [10, Corollary 2.9.29].

(8)$\Rightarrow$(5): Let x^* be in X^*. Then, since $F(s^*) \subseteq D(X^*, s^*)$, we have

$$\begin{aligned} v(X^*, s^*, x^*) &\geq \sup\{|x^*(x)| : x \in F(s^*)\} \\ &= \sup\{|x^*(x)| : x \in \overline{\text{co}}(F(s^*) \cup -F(s^*))\}. \end{aligned}$$

Now assume that condition (8) is fulfilled. Then the above reads as

$$v(X^*, s^*, x^*) \geq \|x^*\|.$$

It follows from the arbitrariness of $x^* \in X^*$ that $n(X^*, s^*) = 1$.

(4)$\Rightarrow$(9): Assume that the condition (4) is fulfilled. Then, by Proposition 2.2, $s^* \in E_{X^*}$ and $\|e^* + \mathbb{R}s^*\| = 1$ for every $e^* \in E_{X^*} \setminus \{\pm s^*\}$. On the other hand, by [1, Corollary 2.2], there exists a maximal face F of B_X such that $F(s^*) \subseteq F$, and, by Lemma 2.4, we have $F = F(e^*)$ for some $e^* \in E_{X^*}$. Let x be in F. Since (4) $\Rightarrow$(8), for each $n \in \mathbb{N}$ there exist $\alpha_n \in [0, 1]$ and $x_n, y_n \in F(s^*)$ such that $\{\alpha_n x_n - (1 - \alpha_n)y_n\}_{n\in\mathbb{N}} \to x$, hence

$$\{2\alpha_n - 1\}_{n\in\mathbb{N}} = \{e^*(\alpha_n x_n - (1 - \alpha_n)y_n)\}_{n\in\mathbb{N}} \to e^*(x) = 1,$$

i.e. $\{\alpha_n\}_{n\in\mathbb{N}} \to 1$. This implies that $\{x_n\}_{n\in\mathbb{N}} \to x$ and, as a consequence, we have $x \in \overline{F(s^*)} = F(s^*)$. Since x is arbitrary in F, and $F(s^*) \subseteq F$, we conclude that $F = F(s^*)$ and $s^* \in \text{mex}(\mathbb{B}_{X^*})$.

(2)$\Rightarrow$(11): By assumption (2), there exists $s \in \mathbb{S}_X$ such that

$$X = \ker(s^*) \oplus_\infty \mathbb{R}s.$$

Then we have that $\mathbb{B}_X = \mathbb{B}_{\ker(s^*)} + [-1, 1]s$, and hence

$$1 = \|s^*\| = \sup\{|s^*(x)| : x \in \mathbb{B}_X\} = \sup\{|\lambda||s^*(s)| : \lambda \in [-1, 1]\} = |s^*(s)|.$$

Therefore, replacing s with $-s$ if necessary, we may suppose that $s^*(s) = 1$. Let x be in $\mathbb{B}_X$. Then there are $y \in \mathbb{B}_{\ker(s^*)}$ and $\lambda \in [-1, 1]$ such that $x = y + \lambda s$. Therefore, since $x = \frac{1}{2}(1+\lambda)(y+s)+\frac{1}{2}(1-\lambda)(y-s)$, we realize that x is a convex combination of $y + s \in F(s^*)$ and $y - s \in -F(s^*)$. In view of the arbitrariness of $x \in \mathbb{B}_X$, we conclude that $\mathbb{B}_X = \text{co}(F(s^*) \cup -F(s^*))$. □

For later reference, we insert here the following.

Lemma 2.6. [18, p. 108] $\mathbb{B}_{X^*}$ *is the* w^**-closed convex hull of* $\text{mex}(\mathbb{B}_{X^*})$.

Remark 2.7. (a) Suppose that $X = Y^*$ for some real Banach space Y, and let $u \in \mathbb{S}_Y$ be such that $\mathbb{R}u$ is a semi-L-summand of Y. Then, by writing

$$s^* := u \in \mathbb{S}_Y \subseteq \mathbb{S}_{X^*},$$

we clearly have that $F(s^*) \, (= D(X^*, s^*) \cap X) = D(Y, u)$, and hence, by the implication (i)$\Rightarrow$(iii) in Proposition 2.2 (with Y instead of X), we realize that

$\mathbb{B}_X = \mathrm{co}(F(s^*) \cup -F(s^*))$, i.e., X fulfils condition (11) in Proposition 2.5. If $\mathbb{R}s^* = \mathbb{R}u$ were an L-sumand of $X^* = Y^{**}$, then $\mathbb{R}u$ would be an L-sumand of Y, a fact that, as shown by Example 2.1, is not always true under the requirements done on the couple (Y, u). Therefore, the implication (11)⇒(3) is not fulfilled. As a result, the implications (2)⇒(11) and (3)⇒(4) in Proposition 2.5 are not reversible.

(b) According to [27] (see also [10, Example 2.9.67]), the implication (3)⇒(11) is not fulfilled. As a consequence, the implications (2)⇒(3) and (11)⇒(8) in Proposition 2.5 are not reversible.

(c) Take X equal to the real Banach space whose dual X^* consists of the vector space $\mathbb{R}^2$ and the norm

$$\|(\lambda, \mu)\| := \max\{|\lambda| + \frac{1}{2}|\mu|, |\mu|\},$$

and write $s^* := (1, 0) \in \mathbb{S}_{X^*}$. Then $E_{X^*} = \{\pm s^*, \pm(\frac{1}{2}, 1), \pm(-\frac{1}{2}, 1)\}$, so that we can check that X fulfils condition (9) in Proposition 2.5. Nevertheless, $\mathbb{R}s^*$ cannot be a semi-L-summand of X^* because the closed unit ball of any two-dimensional real Banach space containing a one-dimensional semi-L-summand is a square, whereas the closed unit ball of X^* is a hexagon. Therefore, the implication (4)⇒(9) in Proposition 2.5 is not reversible.

(d) We do not know whether the implication (9)⇒(10) is reversible. If it were not, the space X in the appropriate counter-example must be infinite-dimensional. Indeed, condition (10) implies that s^* is an isolated point of E_{X^*}, and, as a consequence of Lemma 2.6 and the 'reversed' Krein-Milman theorem, $\mathrm{mex}(\mathbb{B}_{X^*})$ is w^*-dense in E_{X^*}.

(e) The equivalence (4)⇔(8) in Proposition 2.5 was proved by Lima [17, Theorem 3.4] under the assumption that s^* lies in $\mathrm{mex}(\mathbb{B}_{X^*})$. Actually, the crucial implication (4)⇒(8) is proved in [17, Theorem 3.5] under the unnecessary additional assumption that s^* lies in E_{X^*} (cf. the implication (i)⇒(v) in Proposition 2.2). □

Now we can conclude the proof of the main result in this paper, namely the following.

Theorem 2.8. *Suppose that X is nice, and let s^* be in $\mathbb{S}_{X^*}$. Then conditions* (1) *to* (11) *in Proposition* 2.5 *are equivalent.*

Proof. In view of Proposition 2.5, we only need to prove that condition (10) in that proposition implies condition (1). Assume that condition (10) is fulfilled. Then, since X is nice, it follows from [7, Theorem 1 and Proposition 1] that X^* is not equal to the w^*-closed linear hull of $E_{X^*} \setminus \{\pm s^*\}$. Therefore, by [9, Proposition 3.3], $\{\pm s^*\}$ is structurally open in E_{X^*}. □

Remark 2.9. (a) In [8, Theorem 1] we proved that, *if X is nice, and if X^* is equal to the w^*-closed linear hull of the set*

$$\{s^* \in E_{X^*} : \|e^* + \mathbb{R}s^*\| = 1 \text{ for every } e^* \in E_{X^*} \setminus \{\pm s^*\}\},$$

then X is isometrically isomorphic to $c_0(I)$ for some set I. Keeping in mind the equivalence (3)⇔(10) in Theorem 2.8, we realize that the result just quoted is equivalent to the one shown in [8, Corollary 1] asserting that, *if X is nice, and if X^* is equal to the w^*-closed linear hull of the set*

$$\{s^* \in \mathbb{S}_{X^*} : \mathbb{R}s^* \text{ is an } L\text{-summand of } X^*\},$$

then X is isometrically isomorphic to $c_0(I)$ for some set I. Of course, replacing (3)⇔(10) with (n)⇔(10), for $n = 1, 2, 4, 5, 6, 7, 8, 9, 11$, we are provided with nine additional reformulations of [8, Theorem 1].

(b) The case $n = 8$ in the above discussion shows that, *if X is nice, and if X^* is equal to the w^*-closed linear hull of the set*

$$\{s^* \in \mathbb{S}_{X^*} : \mathbb{B}_X = \overline{\mathrm{co}}(F(s^*) \cup -F(s^*))\},$$

then X is isometrically isomorphic to $c_0(I)$ for some set I.

(c) Following [17, p. 44], the real Banach space X is said to be an *almost CL-space* if $\mathbb{B}_X = \overline{\mathrm{co}}(F \cup -F)$ for every proper maximal face F of $\mathbb{B}_X$. Keeping in mind Lemma 2.4, we realize that, if X is an almost CL-space, then $\mathrm{mex}(\mathbb{B}_{X^*})$ is contained in the set

$$B := \{s^* \in \mathbb{S}_{X^*} : \mathbb{B}_X = \overline{\mathrm{co}}(F(s^*) \cup -F(s^*))\},$$

and hence, by Lemma 2.6, X^* is equal to the w^*-closed linear hull of B. It follows from part (b) of the present remark that, as shown in [9, Theorem 4.1], *if X is a nice almost CL-space, then X is isometrically isomorphic to $c_0(I)$ for some set I.*

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