

The Plasticity of some Mass Transportation Networks in the Three Dimensional Euclidean Space

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We obtain an important generalization of the inverse weighted Fermat-Torricelli problem for tetrahedra in $\mathbb{R}^3$ by assigning to the corresponding weighted Fermat-Torricelli point a remaining positive number (residual weight). As a consequence, we derive a new plasticity principle of weighted Fermat-Torricelli trees of degree five for boundary closed hexahedra in $\mathbb{R}^3$ by applying a geometric plasticity principle which lead to the plasticity of mass transportation networks of degree five in $\mathbb{R}^3$. We also derive a complete solution for an important generalization of the inverse weighted Fermat-Torricelli problem for three non-collinear points and a new plasticity principle of mass networks of degree four for boundary convex quadrilaterals in $\mathbb{R}^2$. The plasticity of mass transportation networks provides some first evidence for a creation of a new field that we may call Mathematical Botany in the future.

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1. Introduction

Let $A_1A_2A_3A_4A_5$ be a closed hexahedron in $\mathbb{R}^3$, B_i be a non-negative number (weight) which corresponds to each vertex A_i , A_0 be a point in $\mathbb{R}^3$ and a_{ij} be the Euclidean distance of the linear segment A_iA_j , for $i, j = 0, 1, 2, 3, 4, 5$ respectively. The weighted Fermat-Torricelli problem for a closed hexahedron $A_1A_2A_3A_4A_5$ in $\mathbb{R}^3$ can be stated as:

Problem 1.1. *Find a point A_0 which minimizes the sum of the lengths of the linear segments that connect every vertex A_i with A_0 multiplied by the positive weight B_i :*

$$\sum_{i=1}^5 B_i a_{0i} = \text{minimum}.$$

For $B_1 = B_2 = B_3$ and $B_4 = B_5 = 0$ we solve the classical Fermat-Torricelli problem which has been introduced by Fermat in 1643. After that Torricelli discovered the first geometrical construction in $\mathbb{R}^2$. In 1877, Engelbrecht [3] extended Torricelli's construction to the weighted case. In 2014, Uteshev [8] succeeded in finding an elegant algebraic solution of the weighted Fermat-Torricelli problem in $\mathbb{R}^2$.

A detailed history of the weighted Fermat-Torricelli problem is given in [2], [5] and [7] and a classical study for convex polyhedra in $\mathbb{R}^3$ is given in [1].

In 1997, Y. Kupitz and H. Martini gave a complete study concerning the existence, uniqueness and a characterization of the weighted Fermat-Torricelli point for n non-collinear points in $\mathbb{R}^m$ in [7] (see also in [2, Theorem 18.37, p. 250]).

Theorem 1.2. *Let n non-collinear points in $\mathbb{R}^m$ be given with corresponding positive weights $B_1, B_2, \dots, B_n$.*

- (i) *Then the weighted Fermat-Torricelli point A_0 of $\{A_1 A_2 A_3 \dots A_n\}$ exists and is unique.*
- (ii) *If $\|\sum_{j=1}^n B_j \vec{u}(A_i, A_j)\| > B_i$, $i \neq j$, for $i, j = 1, 2, \dots, n$, then*
 - (a) *the weighted Fermat-Torricelli point does not belong to $\{A_1 A_2 A_3 \dots A_n\}$ (Weighted Floating Case).*
 - (b) *$\sum_{i=1}^n B_i \vec{u}(A_0, A_i) = \vec{0}$ (Weighted Floating Case).*
- (iii) *If there is some i with $\|\sum_{j=1}^n B_j \vec{u}(A_i, A_j)\| \leq B_i$, $i \neq j$, for $i, j = 1, 2, \dots, n$, then the weighted Fermat-Torricelli point is the vertex A_i (Weighted Absorbed Case), where $\vec{u}(A_i, A_j)$ is the unit vector with direction from A_i to A_j , for $i, j = 0, 1, 2, 3, \dots, n$ and $i \neq j$.*

The inverse weighted Fermat-Torricelli problem for tetrahedra in $\mathbb{R}^3$ states that:

Problem 1.3. *Given a point A_0 which belongs to the interior of $A_1 A_2 A_3 A_4$ in $\mathbb{R}^3$, does there exist a unique set of positive weights B_i , such that*

$$B_1 + B_2 + B_3 + B_4 = c = \text{const},$$

for which A_0 minimizes $f(A_0) = \sum_{i=1}^4 B_i a_{0i}$?

By letting $B_4 = 0$ and $c = 1$ in the inverse weighted Fermat-Torricelli problem for tetrahedra we obtain the (normalized) inverse weighted Fermat-Torricelli problem for three non-collinear points in $\mathbb{R}^2$. In 2002, S. Gueron and R. Tessler introduce the normalized inverse weighted Fermat-Torricelli problem for three non-collinear points in $\mathbb{R}^2$ who also gave a positive answer in [5].

In 2009, a positive answer with respect to the inverse weighted Fermat-Torricelli problem for tetrahedra is given in [10] and recently, Uteshev also obtain a positive answer in [8] by using the Cartesian coordinates of the four non-collinear and non-coplanar fixed vertices. In 2011, a negative answer with respect to the inverse weighted Fermat-Torricelli problem for tetragonal pyramids in $\mathbb{R}^3$ is derived in [11]. This negative answer lead to an important dependence of the five variable weights, such that the corresponding weighted Fermat-Torricelli point remains the same, which we call a plasticity principle of closed hexahedra. In 2013, we prove a plasticity principle of closed hexahedra in $\mathbb{R}^3$ and a plasticity principle for convex quadrilaterals in [12] and [13], respectively. The equations of dynamic plasticity of convex quadrilaterals in $\mathbb{R}^2$ are given in [9].

In this paper, we consider an important generalization of the inverse weighted Fermat-Torricelli problem for boundary tetrahedra in $\mathbb{R}^3$ which is derived as an

application of the geometric plasticity of weighted Fermat-Torricelli trees of degree four for boundary tetrahedra in a two-way communication network (Section 3, Prop. 3). This new evolutionary approach gives a new type of plasticity of mass transportation networks of degree four for boundary tetrahedra and of degree five for boundary closed hexahedra in $\mathbb{R}^3$ (Section 3, Theorem 2, Prop. 4). As a corollary, we also derive an important generalization of the inverse weighted Fermat-Torricelli problem for three non-collinear points and a new type of plasticity for mass transportation networks of degree four for boundary weighted quadrilaterals in $\mathbb{R}^2$ (Section 4, Theorem 3, Prop. 5, Theorem 4). It is worth mentioning that this method provides a unified approach to deal with the inverse weighted Fermat-Torricelli problem for boundary triangle invented by Gueron and Tessler which also includes the weighted absorbed case (Theorem 1(iii) for $n = 3$).

2. The dependence of the angles of a weighted Fermat-Torricelli tree having degree at most four and at most five

We shall start with the definitions of a tree topology, a Fermat-Torricelli tree topology, the degree of a boundary vertex in $\mathbb{R}^3$ and the degree of the weighted Fermat-Torricelli point which is located at the interior of the convex hull of a closed hexahedron or tetrahedron, in order to describe the structure of a weighted Fermat-Torricelli tree of a boundary closed hexahedron or a boundary tetrahedron in $\mathbb{R}^3$.

Definition 2.1. [4] A *tree topology* is a connection matrix specifying which pairs of points from the list $A_1, A_2, \dots, A_m, A_{0,1}, A_{0,2}, \dots, A_{0,m-2}$ have a connecting linear segment (edge).

Definition 2.2. [6] The *degree of a vertex* corresponds to the number of connections of the vertex with linear segments.

Definition 2.3. [4] A *Fermat-Torricelli tree topology of degree at most five* is a tree topology with all boundary vertices of a closed hexahedron and one mobile vertex having at most degree five.

Definition 2.4. A tree of minimum length with a Fermat-Torricelli tree topology of degree at most five is called a *Fermat-Torricelli tree*.

Definition 2.5. A Fermat-Torricelli tree of weighted minimum length with a Fermat-Torricelli tree topology of degree at most five is called a *weighted Fermat-Torricelli tree of degree at most five*.

Definition 2.6. A Fermat-Torricelli tree of weighted minimum length of degree at most five having one zero weight is called a *weighted Fermat-Torricelli tree of degree at most four*.

Definition 2.7. A unique solution of the weighted Fermat-Torricelli problem for *closed hexahedra* is a unique weighted Fermat-Torricelli tree of degree at most five.

Definition 2.8. A unique solution of the weighted Fermat-Torricelli problem for *tetrahedra* is a unique weighted Fermat-Torricelli tree (weighted Fermat-Torricelli network) of degree at most four.

By following the methodology given in [12, Lemmas 1, 2 pp. 15–17] and [10, Solution of Problem 2, pp. 119–120], we shall show that the position of a weighted Fermat-Torricelli tree w.r. to a boundary tetrahedron is determined by five given angles.

We denote by $\alpha_{i0j} \equiv \angle A_i A_0 A_j$ and $\alpha_{i,j0k}$ the angle which is formed by the linear segment that connects A_0 with the trace of the orthogonal projection of A_i to the plane $A_j A_0 A_k$ with a_{0i} , for $i, j, k, l = 1, 2, 3, 4$, and $i \neq j \neq k \neq l$.

Proposition 2.9. *The angles $\alpha_{i,k0m}$ depend on exactly five given angles α_{102} , α_{103} , α_{104} , α_{203} and α_{204} , for $i, k, m = 1, 2, 3, 4$, and $i \neq k \neq m$.*

Proof of Lemma 2.9: We shall use the same expressions used in [10, Solution of Problem 2, pp. 119–120] for the unit vectors $\vec{a}_i$ in terms of spherical coordinates, for $i = 1, 2, 3, 4$. We denote by

$$\vec{a}_1 = (1, 0, 0)$$

$$\vec{a}_2 = (\cos(\alpha_{102}), \sin(\alpha_{102}), 0)$$

$$\vec{a}_3 = (\cos(\alpha_{3,102}) \cos(\omega_{3,102}), \cos(\alpha_{3,102}) \sin(\omega_{3,102}), \sin(\alpha_{3,102}))$$

$$\vec{a}_4 = (\cos(\alpha_{4,102}) \cos(\omega_{4,102}), \cos(\alpha_{4,102}) \sin(\omega_{4,102}), \sin(\alpha_{4,102}))$$

such that: $|\vec{a}_i| = 1$. The angles $\alpha_{3,102}$, $\alpha_{4,102}$, are calculated by the following two relations in [10, formulas (10), (11), p. 120]:

$$\cos^2(\alpha_{3,102}) = \frac{\cos^2(\alpha_{203}) + \cos^2(\alpha_{103}) - 2 \cos(\alpha_{203}) \cos(\alpha_{103}) \cos(\alpha_{102})}{\sin^2(\alpha_{102})}, \quad (1)$$

$$\cos^2(\alpha_{4,102}) = \frac{\cos^2(\alpha_{204}) + \cos^2(\alpha_{104}) - 2 \cos(\alpha_{204}) \cos(\alpha_{104}) \cos(\alpha_{102})}{\sin^2(\alpha_{102})} \quad (2)$$

$$\text{The inner product of } \vec{a}_i, \vec{a}_j \text{ is given by: } \vec{a}_i \cdot \vec{a}_j = \cos(\alpha_{i0j}). \quad (3)$$

By replacing (1) and (2) in (3), by eliminating $\omega_{3,102}$ and $\omega_{4,102}$ and by squaring both parts of the derived equation, we obtain a quadratic equation with respect to $\cos \alpha_{304}$:

$$\begin{aligned} & [-\cos \alpha_{103} \cos \alpha_{104} + \cos \alpha_{304} - (-\cos \alpha_{102} \cos \alpha_{103} + \cos \alpha_{203}) \\ & (-\cos \alpha_{102} \cos \alpha_{104} + \cos \alpha_{204}) \csc^2 \alpha_{102}]^2 = (1 - \cos^2 \alpha_{3,102})(1 - \cos^2 \alpha_{4,102}). \end{aligned} \quad (4)$$

By solving (4) with respect to $\cos \alpha_{304}$, we get:

$$\begin{aligned} \cos \alpha_{304} = & -\frac{1}{4} [2b + 4 \cos \alpha_{102} (\cos \alpha_{104} \cos \alpha_{203} + \cos \alpha_{103} \cos \alpha_{204}) - \\ & - 4 (\cos \alpha_{103} \cos \alpha_{104} + \cos \alpha_{203} \cos \alpha_{204})] \csc^2 \alpha_{102}, \quad \text{or} \end{aligned} \quad (5)$$

$$\begin{aligned} \cos \alpha_{304} = & \frac{1}{4} [4 \cos \alpha_{103} (\cos \alpha_{104} - \cos \alpha_{102} \cos \alpha_{204}) + \\ & + 2 (b + 2 \cos \alpha_{203} (-\cos \alpha_{102} \cos \alpha_{104} + \cos \alpha_{204}))] \csc^2 \alpha_{102} \end{aligned} \quad (6)$$

where

$$b \equiv \sqrt{\prod_{i=3}^4 (1 + \cos(2\alpha_{102}) + \cos(2\alpha_{10i}) + \cos(2\alpha_{20i}) - 4 \cos \alpha_{102} \cos \alpha_{10i} \cos \alpha_{20i})}.$$

Therefore, α_{304} depends exactly on α_{102} , α_{103} , α_{104} , α_{203} and α_{204} .

By projecting the vector a_i w.r. to the plane defined by $\triangle A_1 A_0 A_3$ or $\triangle A_2 A_0 A_3$ or $\triangle A_1 A_0 A_4$ or $\triangle A_2 A_0 A_4$ or $\triangle A_3 A_0 A_4$, we get:

$$\cos^2(\alpha_{i,k0m}) = \frac{\sin^2(\alpha_{k0m}) - \cos^2(\alpha_{m0i}) - \cos^2(\alpha_{k0i}) + 2 \cos(\alpha_{m0i}) \cos(\alpha_{k0i}) \cos(\alpha_{k0m})}{\sin^2(\alpha_{k0m})}.$$

Hence, taking into account this equality and (5) or (6) we derive that $\alpha_{i,k0m}$ depends on α_{102} , α_{103} , α_{104} , α_{203} and α_{204} . $\square$

Proposition 2.10. *The angles $\alpha_{i,k0m}$ depend on exactly seven given angles α_{102} , α_{103} , α_{104} , α_{105} , α_{203} , α_{204} and α_{205} , for $i, k, m = 1, 2, 3, 4, 5$ and $i \neq k \neq m$.*

Proof. We consider the directions of five unit vectors which meet a fixed point A_0 . For instance, we get:

$$\vec{a}_1 = (1, 0, 0)$$

$$\vec{a}_2 = (\cos(\alpha_{102}), \sin(\alpha_{102}), 0)$$

$$\vec{a}_3 = (\cos(\alpha_{3,102}) \cos(\omega_{3,102}), \cos(\alpha_{3,102}) \sin(\omega_{3,102}), \sin(\alpha_{3,102}))$$

$$\vec{a}_4 = (\cos(\alpha_{4,102}) \cos(\omega_{4,102}), \cos(\alpha_{4,102}) \sin(\omega_{4,102}), \sin(\alpha_{4,102}))$$

$$\vec{a}_5 = (\cos(\alpha_{5,102}) \cos(\omega_{5,102}), \cos(\alpha_{5,102}) \sin(\omega_{5,102}), \sin(\alpha_{5,102}))$$

such that: $|\vec{a}_i| = 1$. The inner product of $\vec{a}_i$, $\vec{a}_j$ is: $\vec{a}_i \cdot \vec{a}_j = \cos(\alpha_{i0j})$. (7)

By arguing in a similar way as in the proof of Proposition 2.9, we obtain that $\cos(\alpha_{304})$, $\cos(\alpha_{305})$ and $\cos(\alpha_{405})$ derived by (7) are given by the following six relations which depend on exactly seven angles α_{102} , α_{103} , α_{104} , α_{105} , α_{203} , α_{204} and α_{205} :

$$\begin{aligned} \cos \alpha_{304} = & -\frac{1}{4}[2b_{304} + 4 \cos \alpha_{102} (\cos \alpha_{104} \cos \alpha_{203} + \cos \alpha_{103} \cos \alpha_{204}) - \\ & - 4 (\cos \alpha_{103} \cos \alpha_{104} + \cos \alpha_{203} \cos \alpha_{204})] \csc^2 \alpha_{102}, \quad \text{or} \end{aligned} \quad (8)$$

$$\begin{aligned} \cos \alpha_{304} = & \frac{1}{4}[4 \cos \alpha_{103} (\cos \alpha_{104} - \cos \alpha_{102} \cos \alpha_{204}) + \\ & + 2 (b_{304} + 2 \cos \alpha_{203} (-\cos \alpha_{102} \cos \alpha_{104} + \cos \alpha_{204}))] \csc^2 \alpha_{102} \end{aligned} \quad (9)$$

where

$$\begin{aligned} b_{304} \equiv & \sqrt{\prod_{i=3}^4 (1 + \cos(2\alpha_{102}) + \cos(2\alpha_{10i}) + \cos(2\alpha_{20i}) - 4 \cos \alpha_{102} \cos \alpha_{10i} \cos \alpha_{20i}),} \\ \cos \alpha_{305} = & -\frac{1}{4}[2b_{305} + 4 \cos \alpha_{102} (\cos \alpha_{105} \cos \alpha_{203} + \cos \alpha_{103} \cos \alpha_{205}) - \\ & - 4 (\cos \alpha_{103} \cos \alpha_{105} + \cos \alpha_{203} \cos \alpha_{205})] \csc^2 \alpha_{102}, \quad \text{or} \end{aligned} \quad (10)$$

$$\begin{aligned} \cos \alpha_{305} = & \frac{1}{4}[4 \cos \alpha_{103} (\cos \alpha_{105} - \cos \alpha_{102} \cos \alpha_{205}) + \\ & + 2 (b_{305} + 2 \cos \alpha_{203} (-\cos \alpha_{102} \cos \alpha_{105} + \cos \alpha_{205}))] \csc^2 \alpha_{102}, \end{aligned} \quad (11)$$

where

$$b_{305} \equiv \sqrt{\prod_{i=3, i \neq 4}^5 (1 + \cos(2\alpha_{102}) + \cos(2\alpha_{10i}) + \cos(2\alpha_{20i}) - 4 \cos \alpha_{102} \cos \alpha_{10i} \cos \alpha_{20i})}$$

$$\text{and} \quad \cos \alpha_{405} = -\frac{1}{4}[2b_{405} + 4 \cos \alpha_{102} (\cos \alpha_{105} \cos \alpha_{204} + \cos \alpha_{104} \cos \alpha_{205}) - \\ - 4 (\cos \alpha_{104} \cos \alpha_{105} + \cos \alpha_{204} \cos \alpha_{205})] \csc^2 \alpha_{102} \quad (12)$$

$$\text{or} \quad \cos \alpha_{405} = \frac{1}{4}[4 \cos \alpha_{104} (\cos \alpha_{105} - \cos \alpha_{102} \cos \alpha_{205}) + \\ + 2 (b_{405} + 2 \cos \alpha_{204} (-\cos \alpha_{102} \cos \alpha_{105} + \cos \alpha_{205}))] \csc^2 \alpha_{102} \quad (13)$$

where

$$b_{305} \equiv \sqrt{\prod_{i=4}^5 (1 + \cos(2\alpha_{102}) + \cos(2\alpha_{10i}) + \cos(2\alpha_{20i}) - 4 \cos \alpha_{102} \cos \alpha_{10i} \cos \alpha_{20i})}. \quad \square$$

Remark 2.11. We note that the calculations of formulas of $\cos \alpha_{304}$, $\cos \alpha_{305}$, and $\cos \alpha_{405}$, which are derived in [12, Lemma 1, p. 16] are corrected and replaced by (8), (9), (10), (11), (12) and (13).

3. A generalization of the inverse weighted Fermat-Torricelli problem in $\mathbb{R}^3$.

In this section, we consider mass transportation networks which deal with weighted Fermat-Torricelli networks of degree at most four (or five), in which the weights correspond to an instantaneous collection of images of masses and satisfy some specific conditions.

We denote by $h_{0,ik}$ the length of the height of $\triangle A_0 A_i A_k$ from A_0 with respect to $A_i A_j$, by $A_{0,ij}$ the intersection of $h_{0,ij}$ with $A_i A_j$, and by $h_{0,ijk}$ the distance of A_0 from the plane defined by $\triangle A_i A_j A_k$.

We denote by α , the dihedral angle which is formed between the planes defined by $\triangle A_1 A_2 A_3$ and $\triangle A_1 A_2 A_0$, with α_{g_i} the dihedral angle which is formed by the planes defined by $\triangle A_1 A_2 A_i$ and $\triangle A_1 A_2 A_0$, and by $\alpha_{i,r0s}$ the angle which is formed by a_{0i} and the linear segment which connects A_0 with the trace from the orthogonal projection of a_{0i} to the plane defined by $\triangle A_0 A_r A_s$, for $i, k, l, m, r, s = 0, 1, 2, 3, 4, 5$.

We proceed by mentioning a fundamental result which we call a geometric plasticity principle of mass transportation networks for boundary closed hexahedra and it is proved in [11, Appendix A.II] for closed polyhedra in $\mathbb{R}^3$.

Proposition 3.1. [11, Appendix A.II] *Suppose that there is a closed polyhedron $A_1 A_2 A_3 A_4 A_5$ in $\mathbb{R}^3$ and each vertex A_i has a non-negative weight B_i for $i = 1, 2, 3, 4, 5$. Assume that the floating case of the generalized weighted Fermat-Torricelli point A_0 point is valid:*

$$\left\| \sum_{j=1}^5 B_j \vec{u}(A_i, A_j) \right\| > B_i, \quad i \neq j, \quad \text{for each } A_i \in \{A_1, A_2, A_3, A_4, A_5\}.$$

If A_0 is connected with every vertex A_i for $i = 1, 2, 3, 4, 5$, and a point A'_i is selected with a non-negative weight B_i of the line that is defined by the linear segment $A_0 A_i$ and a closed hexahedron $A'_1 A'_2 \dots A'_n$ is constructed such that:

$$\left\| \sum_{j=1}^5 B_j \vec{u}(A'_i, A'_j) \right\| > B_i, i \neq j.$$

Then the generalized weighted Fermat-Torricelli point A'_0 is identical with A_0 (geometric plasticity principle).

The geometric plasticity principle of closed hexahedra connects the weighted Fermat-Torricelli problem for closed hexahedra with the modified weighted Fermat-Torricelli problem for boundary closed hexahedra by allowing a mass flow continuity for the weights, such that the corresponding weighted Fermat-Torricelli point remains invariant in $\mathbb{R}^3$.

The modified weighted Fermat-Torricelli problem for closed hexahedra reads:

Problem 3.2. (*Modified weighted Fermat-Torricelli problem*)

Let $A_1A_2A_3A_4A_5$ be a closed hexahedron in $\mathbb{R}^3$, $\mathcal{B}_i$ be a non-negative number (weight) which corresponds to each linear segment A_0A_i , respectively. Find a point A_0 which minimizes the sum of the lengths of the linear segments that connect every vertex A_i with A_0 multiplied by the positive weight $\mathcal{B}_i$:

$$\sum_{i=1}^5 \mathcal{B}_i a_{0i} = \text{minimum}. \quad (14)$$

By letting $\mathcal{B}_i = B_i$, for $i = 1, 2, 3, 4, 5$ the weighted Fermat-Torricelli problem for closed hexahedra (Problem 1.1) and the corresponding modified weighted Fermat-Torricelli problem (Problem 3.2) are equivalent by collecting instantaneous images of the weighted Fermat-Torricelli network via the geometric plasticity principle.

We note that various generalizations of the modified Fermat-Torricelli problem for weighted minimal networks of degree at most three in the sense of the Steiner tree Problem are given in the classical work of A. Ivanov and A. Tuzhilin in [6].

We introduce a mixed weighted Fermat-Torricelli problem in $\mathbb{R}^3$ which may give some new fundamental results in molecular structures and mass transportation networks in a new field that we may call in the future Mathematical Botany and possible applications in the geometry of drug design.

We state the mixed Fermat-Torricelli problem for closed hexahedra in $\mathbb{R}^3$, considering a two way communication weighted network.

Problem 3.3. *Given a boundary closed hexahedron $A_1A_2A_3A_4A_5$ in $\mathbb{R}^3$ having one interior weighted mobile vertex A_0 with remaining positive weight $\bar{B}_0$ find a connected weighted system of linear segments of shortest total weighted length such that any two of the points of the network can be joined by a polygon consisting of linear segments:*

$$f(X) = \bar{B}_1 a_1 + \bar{B}_2 a_2 + \bar{B}_3 a_3 + \bar{B}_4 a_4 + \bar{B}_5 a_5 = \text{minimum},$$

where

$$B_i + \tilde{B}_i = \bar{B}_i$$

under the condition: $\bar{B}_i + \bar{B}_j + \bar{B}_k + \bar{B}_l = \bar{B}_0 + \bar{B}_m$

for $i, j, k, l = 1, 2, 3, 4, 5$ and $i \neq j \neq k \neq l$.

The invariance of the mixed weighted Fermat-Torricelli tree of degree at most five is obtained by the inverse mixed weighted Fermat-Torricelli problem for closed hexahedra in $\mathbb{R}^3$:

Problem 3.4. (*Inverse mixed weighted Fermat-Torricelli problem for closed hexahedra*). Given a point A_0 which belongs to the interior of $A_1A_2A_3A_4A_5$ in $\mathbb{R}^3$, does there exist a unique set of positive weights B_i , such that

$$\bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_4 + \bar{B}_5 = c = \text{const},$$

for which A_0 minimizes $f(A_0) = \sum_{i=1}^5 \bar{B}_i a_{0i}$ and $B_i + \tilde{B}_i = \bar{B}_i$ under the following condition for the weights:

$$\bar{B}_i + \bar{B}_j + \bar{B}_k + \bar{B}_l = \bar{B}_0 + \bar{B}_m$$

for $i, j, k, l, m = 1, 2, 3, 4, 5$, and $i \neq j \neq k \neq l \neq m$?

Letting $\bar{B}_5 = 0$ in Problem 3.4 we obtain the inverse mixed weighted Fermat-Torricelli problem for tetrahedra.

Theorem 3.5. Given the mixed weighted Fermat-Torricelli point A_0 to be an interior point of the tetrahedron $A_1A_2A_3A_4$ with the vertices lie on four prescribed rays that meet at A_0 and from the five given values of α_{102} , α_{103} , α_{104} , α_{203} , α_{204} , the positive real weights $\bar{B}_i$ given by the formulas

$$\bar{B}_1 = \left(\frac{\sin \alpha_{4,203}}{\sin \alpha_{1,203}} \right) \frac{c - \bar{B}_0}{2}, \quad (15)$$

$$\bar{B}_2 = \left(\frac{\sin \alpha_{4,103}}{\sin \alpha_{2,103}} \right) \frac{c - \bar{B}_0}{2}, \quad (16)$$

$$\bar{B}_3 = \left(\frac{\sin \alpha_{4,102}}{\sin \alpha_{3,102}} \right) \frac{c - \bar{B}_0}{2} \quad (17)$$

$$\text{and} \quad \bar{B}_4 = \frac{c - \bar{B}_0}{2} \quad (18)$$

give a negative answer w.r. to the inverse mixed weighted Fermat-Torricelli problem for tetrahedra for $i, j, k, m = 1, 2, 3, 4$ and $i \neq j \neq k \neq m$.

Proof. We denote by B_i a mass flow which is transferred from A_i to A_0 for $i = 1, 2, 3$ by B_0 a residual weight which remains at A_0 and by B_4 a mass flow which is transferred from A_0 to A_4 .

We denote by $\tilde{B}_i$ a mass flow which is transferred from A_0 to A_i for $i = 1, 2, 3$ by $\tilde{B}_0$ a residual weight which remains at A_0 and by $\tilde{B}_4$ a mass flow which is transferred from A_4 to A_0 .

$$\text{Hence, we get:} \quad B_1 + B_2 + B_3 = B_4 + B_0 \quad (19)$$

$$\text{and} \quad \tilde{B}_1 + \tilde{B}_2 + \tilde{B}_3 + \tilde{B}_0 = \tilde{B}_4. \quad (20)$$

By adding (19) and (20) and by letting $\bar{B}_0 = B_0 - \tilde{B}_0$ we get:

$$\bar{B}_1 + \bar{B}_2 + \bar{B}_3 = \bar{B}_4 + \bar{B}_0 \quad (21)$$

such that:

$$\bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_4 = c, \quad (22)$$

where c is a positive real number. Hence the objective function takes the form:

$$\sum_{i=1}^4 B_i a_{0i} + \sum_{i=1}^4 \tilde{B}_i a_{0i} = \text{minimum}, \quad (23)$$

which yields

$$\sum_{i=1}^4 \bar{B}_i a_{0i} = \text{minimum}. \quad (24)$$

We start by expressing the lengths a_{0i} , w.r. to a_{0j} , a_{0k} , a_{0l} .

For instance, the lengths a_{03} and a_{04} are expressed with respect to a_{01} , a_{02} and the dihedral angle α taking into account the two formulas given in [10, Formulas (2.14), (2.20) p. 116]:

$$a_{03}^2 = a_{02}^2 + a_{23}^2 - 2a_{23}[\sqrt{a_{02}^2 - h_{0,12}^2} \cos \alpha_{123} + h_{0,12} \sin \alpha_{123} \cos \alpha] \quad (25)$$

and

$$a_{04}^2 = a_{02}^2 + a_{24}^2 - 2a_{24}[\sqrt{a_{02}^2 - h_{0,12}^2} \cos \alpha_{124} + h_{0,12} \sin \alpha_{124} \cos(\alpha_{g4} - \alpha)] \quad (26)$$

By eliminating α from (25) and (26) we get:

$$\begin{aligned} a_{04}^2 = & a_{02}^2 + a_{24}^2 - 2a_{24}[\sqrt{a_{02}^2 - h_{0,12}^2} \cos \alpha_{124} \\ & + h_{0,12} \sin \alpha_{124} (\cos \alpha_{g4} \left(\frac{\left(\frac{a_{02}^2 + a_{23}^2 - a_{03}^2}{2a_{23}} \right) - \sqrt{a_{02}^2 - h_{0,12}^2} \cos \alpha_{123}}{h_{0,12} \sin \alpha_{123}} \right) + \\ & + \sin \alpha_{g4} \sin \arccos \left(\frac{\left(\frac{a_{02}^2 + a_{23}^2 - a_{03}^2}{2a_{23}} \right) - \sqrt{a_{02}^2 - h_{0,12}^2} \cos \alpha_{123}}{h_{0,12} \sin \alpha_{123}} \right))] \end{aligned} \quad (27)$$

By differentiating (27) with respect to a_{01} , a_{02} and a_{03} , we obtain:

$$\frac{\partial a_{04}}{\partial a_{01}} = -\frac{\sin \alpha_{4,203}}{\sin \alpha_{1,203}}, \quad \frac{\partial a_{04}}{\partial a_{02}} = -\frac{\sin \alpha_{4,103}}{\sin \alpha_{2,103}}, \quad \frac{\partial a_{04}}{\partial a_{03}} = -\frac{\sin \alpha_{4,102}}{\sin \alpha_{3,102}}. \quad (28)$$

By differentiating (24) with respect to a_{01} , a_{02} and a_{03} , and taking into account (28), we obtain:

$$\frac{\bar{B}_1}{\bar{B}_4} = \frac{\sin \alpha_{4,203}}{\sin \alpha_{1,203}}, \quad \frac{\bar{B}_2}{\bar{B}_4} = \frac{\sin \alpha_{4,103}}{\sin \alpha_{2,103}}, \quad \text{and} \quad \frac{\bar{B}_3}{\bar{B}_4} = \frac{\sin \alpha_{4,102}}{\sin \alpha_{3,102}}. \quad (29)$$

By following a similar process and by expressing a_{0i} as a function w.r. to a_{0j} , a_{0k} and a_{0l} , for $i, j, k, l = 1, 2, 3, 4$ and $i \neq j \neq k \neq l$, we get:

$$\frac{\bar{B}_i}{\bar{B}_j} = \frac{\sin \alpha_{j,k0l}}{\sin \alpha_{i,k0l}}. \quad (30)$$

By subtracting (21) from (22) we obtain (18).

By replacing (18) in (29) and taking into account Lemma 2.9 we derive (15), (16) and (17). Therefore, the weights $\bar{B}_i$ depend on the residual weight $\bar{B}_0$ and the five given angles α_{102} , α_{103} , α_{104} , α_{203} and α_{204} . $\square$

Corollary 3.6. *If $\alpha_{102} = \alpha_{103} = \alpha_{104} = \alpha_{203} = \alpha_{204} = \arccos(-\frac{1}{3})$, $\bar{B}_0 = \frac{1}{2}$ and $\bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_4 = 1$, then $\bar{B}_1 = \bar{B}_2 = \bar{B}_3 = \bar{B}_4 = \frac{1}{4}$.*

Proof. By letting $\alpha_{102} = \alpha_{103} = \alpha_{104} = \alpha_{203} = \alpha_{204} = \arccos(-\frac{1}{3})$ in (5), we derive that $\cos \alpha_{304} = -\frac{1}{3}$ or $\cos \alpha_{304} = 1$ which yields $\alpha_{304} = -\arccos(\frac{1}{3})$. By replacing $\bar{B}_0 = \frac{1}{2}$ in (15)–(18) we derive $\bar{B}_1 = \bar{B}_2 = \bar{B}_3 = \bar{B}_4 = \frac{1}{4}$. $\square$

Corollary 3.7. *For*

$$\bar{B}_0 = c \left(1 - \frac{2}{1 + \frac{\sin \alpha_{4,203}}{\sin \alpha_{1,203}} + \frac{\sin \alpha_{4,103}}{\sin \alpha_{2,103}} + \frac{\sin \alpha_{4,102}}{\sin \alpha_{3,102}}} \right). \quad (31)$$

we derive the unique solution

$$\bar{B}_i = \frac{c}{1 + \frac{\sin \alpha_{i,j0k}}{\sin \alpha_{l,j0k}} + \frac{\sin \alpha_{i,j0l}}{\sin \alpha_{k,j0l}} + \frac{\sin \alpha_{i,k0l}}{\sin \alpha_{j,k0l}}}. \quad (32)$$

for $i, j, k, l = 1, 2, 3, 4$ and $i \neq j \neq k \neq l$, which coincides with the unique solution of the inverse weighted Fermat-Torricelli problem for tetrahedra.

Proof. By replacing (31) in (15), (16), (17) and (18) we obtain (50), which yields a positive answer to the inverse weighted Fermat-Torricelli problem for tetrahedra in $\mathbb{R}^3$. $\square$

We proceed by generalizing the equations of (dynamic) plasticity for closed hexahedra, taking into account the residual weight $\bar{B}_0$ which exist at the knot A_0 , by following the method used in [12, Proposition 1, p. 17].

$$\text{We set } sgn_{i,j0k} = \begin{cases} +1, & \text{if } A_i \text{ is upper from the plane } A_j A_0 A_k, \\ 0, & \text{if } A_i \text{ belongs to the plane } A_j A_0 A_k, \\ -1, & \text{if } A_i \text{ is under the plane } A_j A_0 A_k, \end{cases}$$

with respect to an outward normal vector N_{j0k} for $i, j, k = 1, 2, 3, 4, 5$, $i \neq j \neq k$. We remind that the position of an arbitrary directed plane is determined by the outward normal and the distance from the weighted Fermat-Torricelli point A_0 .

Proposition 3.8. *The following equations express a new plasticity of mixed weighted closed hexahedra with respect to the non-negative variable weights $(\bar{B}_i)_{12345}$ in $\mathbb{R}^3$:*

$$\left(\frac{\bar{B}_1}{\bar{B}_4}\right)_{12345} = -\left(\frac{\text{sgn}_{4,203}}{\text{sgn}_{1,203}}\right)\left(\frac{\bar{B}_1}{\bar{B}_4}\right)_{1234}\left(1 + \frac{\text{sgn}_{5,203}}{\text{sgn}_{4,203}}\left(\frac{\bar{B}_5}{\bar{B}_4}\right)_{12345}\left(\frac{\bar{B}_4}{\bar{B}_5}\right)_{2345}\right) \quad (33)$$

$$\left(\frac{\bar{B}_2}{\bar{B}_4}\right)_{12345} = -\left(\frac{\text{sgn}_{4,103}}{\text{sgn}_{2,103}}\right)\left(\frac{\bar{B}_2}{\bar{B}_4}\right)_{1234}\left(1 + \frac{\text{sgn}_{5,103}}{\text{sgn}_{4,103}}\left(\frac{\bar{B}_5}{\bar{B}_4}\right)_{12345}\left(\frac{\bar{B}_4}{\bar{B}_5}\right)_{1345}\right) \quad (34)$$

$$\left(\frac{\bar{B}_3}{\bar{B}_4}\right)_{12345} = -\left(\frac{\text{sgn}_{4,102}}{\text{sgn}_{3,102}}\right)\left(\frac{\bar{B}_3}{\bar{B}_4}\right)_{1234}\left(1 + \frac{\text{sgn}_{5,102}}{\text{sgn}_{4,102}}\left(\frac{\bar{B}_5}{\bar{B}_4}\right)_{12345}\left(\frac{\bar{B}_4}{\bar{B}_5}\right)_{1245}\right) \quad (35)$$

$$\text{under the conditions} \quad \bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_4 + \bar{B}_5 = c = \text{constant} \quad (36)$$

$$\text{and} \quad \bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_5 = \bar{B}_0 + \bar{B}_4 \quad (37)$$

where the weight $(\bar{B}_i)_{12345}$ corresponds to the vertex that lies on the ray A_0A_i , for $i = 1, 2, 3, 4, 5$, and the weight $(\bar{B}_j)_{jklm}$ corresponds to the vertex A_j that lies in the ray A_0A_j regarding the tetrahedron $A_jA_kA_lA_m$, for $j, k, l, m = 1, 2, 3, 4, 5$ and $j \neq k \neq l \neq m$.

Proof. By eliminating $\bar{B}_1, \bar{B}_2, \bar{B}_3$ and $\bar{B}_5$ from (36) and (38) we get:

$$\bar{B}_4 = \frac{c - \bar{B}_0}{2} \quad (38)$$

We assume that the residual weight $\bar{B}_0$ can be split at the mixed weighted Fermat-Torricelli trees of degree four at A_0 , such that the residual weights $\bar{B}_{0,2345}$ and $\bar{B}_{0,1345}$, and $\bar{B}_{0,1245}$, correspond to the boundary tetrahedra $A_2A_3A_4A_5$, $A_1A_3A_4A_5$ and $A_1A_2A_4A_5$.

We select five initial (given) values $(\bar{B}_i)_{12345}(0)$ concerning the weights $(\bar{B}_i)_{12345}$ for $i = 1, 2, 3, 4, 5$ such that the mixed weighted Fermat-Torricelli point A_0 exists and it is located at the interior of $A_1A_2A_3A_4A_5$.

By applying the method used in the proof of Theorem 3.5, the length of the linear segments a_{04} , a_{05} can be expressed as functions of a_{01} , a_{02} and a_{03} for $i = 4, 5$:

$$\begin{aligned} a_{0i}^2 = & a_{02}^2 + a_{2i}^2 - 2a_{2i}[\sqrt{a_{02}^2 - h_{0,12}^2} \cos \alpha_{12i} \\ & + h_{0,12} \sin \alpha_{12i}(\cos \alpha_{gi} \left(\frac{\left(\frac{a_{02}^2 + a_{23}^2 - a_{03}^2}{2a_{23}}\right) - \sqrt{a_{02}^2 - h_{0,12}^2} \cos \alpha_{123}}{h_{0,12} \sin \alpha_{123}} \right) + \\ & + \sin \alpha_{gi} \sin \arccos \left(\frac{\left(\frac{a_{02}^2 + a_{23}^2 - a_{03}^2}{2a_{23}}\right) - \sqrt{a_{02}^2 - h_{0,12}^2} \cos \alpha_{123}}{h_{0,12} \sin \alpha_{123}} \right))] \end{aligned} \quad (39)$$

From (39), we get:

$$\bar{B}_1 a_{01} + \bar{B}_2 a_{02} + \bar{B}_3 a_{03} + \bar{B}_4 a_{04}(a_{01}, a_{02}, a_{03}) + \bar{B}_5 a_{05}(a_{01}, a_{02}, a_{03}) = \text{minimum}. \quad (40)$$

By differentiating (40) with respect to a_{01} , a_{02} and a_{03} we get:

$$\bar{B}_1 + \bar{B}_4 \frac{\partial a_{04}}{\partial a_{01}} + \bar{B}_5 \frac{\partial a_{05}}{\partial a_{01}} = 0. \quad (41)$$

$$\bar{B}_2 + \bar{B}_4 \frac{\partial a_{04}}{\partial a_{02}} + \bar{B}_5 \frac{\partial a_{05}}{\partial a_{02}} = 0. \quad (42)$$

$$\bar{B}_3 + \bar{B}_4 \frac{\partial a_{04}}{\partial a_{03}} + \bar{B}_5 \frac{\partial a_{05}}{\partial a_{03}} = 0. \quad (43)$$

By differentiating (39) with respect to a_{03} and by replacing $\frac{\partial a_{0i}}{\partial a_{03}}$ for $i = 4, 5$ in (43), we obtain:

$$\left(\frac{\bar{B}_3}{\bar{B}_4}\right)_{12345} = -\left(\frac{sgn_{4,102}}{sgn_{3,102}}\right) \frac{\sin(\alpha_{4,102})}{\sin(\alpha_{3,102})} \left(1 + \left(\frac{\bar{B}_5}{\bar{B}_4}\right)_{12345} \frac{sgn_{5,102}}{sgn_{4,102}} \frac{\sin(\alpha_{5,102})}{\sin(\alpha_{4,102})}\right). \quad (44)$$

Taking into account the solution of the inverse mixed weighted Fermat-Torricelli problem for boundary tetrahedra we derive (35). Following a similar evolutionary process, we derive (34) and (33). $\square$

From Lemma 2.9, the variable weights $(\bar{B}_1)_{12345}$, $(\bar{B}_2)_{12345}$, and $(\bar{B}_3)_{12345}$, depend on the weight $(\bar{B}_5)_{12345}$ the residual weight $\bar{B}_0$ and the seven given angles α_{102} , α_{103} , α_{104} , α_{105} , α_{203} , α_{204} and α_{205} .

Remark 3.9. We note that numerical examples of the plasticity of tetragonal pyramids are given in [11, Examples 3.4, 3.7, p. 844–847].

4. A generalization of the inverse weighted Fermat-Torricelli problem in $\mathbb{R}^2$

The inverse mixed weighted Fermat-Torricelli problem for three non-collinear points in $\mathbb{R}^2$ states that:

Problem 4.1. (*Inverse mixed weighted Fermat-Torricelli problem for three non-collinear points*) Given a point A_0 which belongs to the interior of $\triangle A_1 A_2 A_3$ in $\mathbb{R}^2$, does there exist a unique set of positive weights $\bar{B}_i$, such that

$$\bar{B}_1 + \bar{B}_2 + \bar{B}_3 = c = \text{const}, \quad (45)$$

for which A_0 minimizes $f(A_0) = \sum_{i=1}^3 \bar{B}_i a_{0i}$, and $B_i + \tilde{B}_i = \bar{B}_i$ under the condition for the weights:

$$\bar{B}_i + \bar{B}_j = \bar{B}_0 + \bar{B}_k \quad (46)$$

for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$?

Theorem 4.2. *Given the mixed weighted Fermat-Torricelli point A_0 to be an interior point of the triangle $\triangle A_1A_2A_3$ with the vertices lie on three prescribed rays that meet at A_0 and from the two given values of α_{102} , α_{103} , the positive real weights $\bar{B}_i$ given by the formulas*

$$\bar{B}_1 = - \left(\frac{\sin(\alpha_{103} + \alpha_{102})}{\sin \alpha_{102}} \right) \frac{c - \bar{B}_0}{2}, \quad \bar{B}_2 = \left(\frac{\sin \alpha_{103}}{\sin \alpha_{102}} \right) \frac{c - \bar{B}_0}{2}, \quad (47)$$

$$\text{and} \quad \bar{B}_3 = (c - \bar{B}_0)/2 \quad (48)$$

give a negative answer with respect to the inverse mixed weighted Fermat-Torricelli problem for three non-collinear points in $\mathbb{R}^2$.

Proof. Eliminating $\bar{B}_1$ and $\bar{B}_2$ from (45) and (46) we get (48). By setting $\alpha_{g_i} = \alpha$ in (25), we derive that $a_{03} = a_{03}(a_{01}, a_{02})$. By differentiating $a_{03} = a_{03}(a_{01}, a_{02})$ w.r. to a_{0i} and by replacing $\frac{\partial a_{03}}{\partial a_{0i}}$ for $i = 1, 2$ and setting $B_4 = 0$ in (24) we obtain (47). $\square$

Corollary 4.3. *If $\alpha_{102} = \alpha_{103} = 120^\circ$, $\bar{B}_0 = \frac{1}{3}$ and $\bar{B}_1 + \bar{B}_2 + \bar{B}_3 = 1$, then $\bar{B}_1 = \bar{B}_2 = \bar{B}_3 = \bar{B}_0 = \frac{1}{3}$.*

Proof. By letting $\alpha_{102} = \alpha_{103} = 120^\circ$ we have: $\alpha_{203} = 2\pi - \alpha_{102} - \alpha_{103} = 120^\circ$. By replacing $\bar{B}_0 = \frac{1}{3}$ and $c = 1$ in (47)–(48) we derive $\bar{B}_1 = \bar{B}_2 = \bar{B}_3 = \frac{1}{3}$. $\square$

Corollary 4.4. *For*

$$\bar{B}_0 = c \left(1 - \frac{2}{1 - \left(\frac{\sin(\alpha_{103} + \alpha_{102})}{\sin \alpha_{102}} \right) + \left(\frac{\sin \alpha_{103}}{\sin \alpha_{102}} \right)} \right). \quad (49)$$

$$\text{we derive the unique solution} \quad \bar{B}_i = \frac{c}{1 + \frac{\sin \alpha_{j0i}}{\sin \alpha_{j0k}} + \frac{\sin \alpha_{k0i}}{\sin \alpha_{j0k}}}. \quad (50)$$

for $i, j, k = 1, 2, 3$, and $i \neq j \neq k$, which coincides with the unique solution of the inverse weighted Fermat-Torricelli problem for three non-collinear points.

Proof. By replacing (49) in (47)–(48) we obtain (50), which yields a positive answer to the inverse weighted Fermat-Torricelli problem for three non-collinear points in $\mathbb{R}^2$. $\square$

Proposition 4.5. *Let $\triangle A_1A_2A_3$ be a triangle in $\mathbb{R}^2$. If*

$$\|\bar{B}_1 \vec{u}(A_3, A_1) + \bar{B}_2 \vec{u}(A_3, A_2)\| \leq \bar{B}_3 \quad (51)$$

and $\bar{B}_1 + \bar{B}_2 = \bar{B}_3 + \bar{B}_0$ holds, then the solution with respect to the inverse mixed weighted Fermat-Torricelli problem for three non-collinear points in $\mathbb{R}^2$ for the weighted absorbed case is not unique.

Proof. Suppose that we choose three initial weights $\bar{B}_i(0) \equiv \bar{B}_i$, such that (51) holds. From Theorem 1.2 the weighted absorbed case occurs and the mixed weighted Fermat-Torricelli point $A_0 \equiv A_3$. Hence, if we select a new weight $\bar{B}_3 + \bar{B}_0$ which remains at the knot A_3 , then (51) also holds and the corresponding mixed weighted Fermat-Torricelli point remains the same $A'_0 \equiv A_3$. $\square$

Remark 4.6. Proposition 4.5 generalizes the inverse weighted Fermat-Torricelli problem for three non-collinear points in the weighted absorbed case. $\square$

Setting a condition with respect to the specific dihedral angles $\alpha_{g_3} = \alpha_{g_4} = \alpha$ we obtain quadrilaterals as a limiting case of tetrahedra on the plane defined by $\triangle A_1 A_0 A_2$. These equations are important, in order to derive a new plasticity for weighted quadrilaterals in $\mathbb{R}^2$, where the weighted floating case of Theorem 1.2 occurs.

Theorem 4.7. *If $\alpha_{g_3} = \alpha_{g_4} = \alpha$ then the following four equations point out the mixed dynamic plasticity of convex quadrilaterals in $\mathbb{R}^2$:*

$$\left(\frac{\bar{B}_2}{\bar{B}_1}\right)_{1234} = \left(\frac{\bar{B}_2}{\bar{B}_1}\right)_{123} \left[1 - \left(\frac{\bar{B}_4}{\bar{B}_1}\right)_{1234} \left(\frac{\bar{B}_1}{\bar{B}_4}\right)_{134}\right], \quad (52)$$

$$\left(\frac{\bar{B}_3}{\bar{B}_1}\right)_{1234} = \left(\frac{\bar{B}_3}{\bar{B}_1}\right)_{123} \left[1 - \left(\frac{\bar{B}_4}{\bar{B}_1}\right)_{1234} \left(\frac{\bar{B}_1}{\bar{B}_4}\right)_{124}\right], \quad (53)$$

$$(\bar{B}_1)_{1234} + (\bar{B}_2)_{1234} + (\bar{B}_3)_{1234} + (\bar{B}_4)_{1234} = c = \text{constant} \quad (54)$$

and $(\bar{B}_1)_{1234} + (\bar{B}_2)_{1234} + (\bar{B}_3)_{1234} = (\bar{B}_4)_{1234} + (\bar{B}_0)_{1234}. \quad (55)$

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