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Characterizations of a Slow Growth Assumption in Variational Calculus and Gale-Klee Continuity of Epigraphs

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The paper is devoted to the geometric and topological characterizations of a slow growth assumption in variational calculus introduced by Cellina. This assumption plays an important role in studying relaxation, Lipschitzianity of solutions, and nonoccurrence of the Lavrentiev phenomenon in variational problems. It is shown that the slow growth assumption is equivalent to the continuity of the epigraph (as defined by Gale and Klee, 1959) of the second Legendre-Fenchel conjugate f^{**} of the integrand with respect to the derivative variables. It is also shown that this assumption is equivalent to the openness of the domain of the Legendre-Fenchel conjugate with respect to the derivative variables plus continuity of f^* on the whole space. As a spin-off of the tools developed in the paper we obtain a criterion for the openness of the domain of the conjugate of a convex function in terms of the continuity of the epigraph of this function. We use these tools also to establish a necessary and sufficient condition for the inf-compactness of lower semicontinuous functions and some related results.

Keywords: Variational calculus, slow growth assumption, Legendre-Fenchel conjugate, continuous set, subdifferential.

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1. Introduction

The slow growth assumption on the integrand $f(t, x, \dot{x})$ of a variational problem introduced by Cellina [3] and its modifications play prominent roles in studying relaxation, Lipschitzianity of solutions, and nonoccurrence of the Lavrentiev phenomenon in variational problems (see [3, 4, 5, 6, 11]). Here we show that the slow growth assumption, which is somewhat intricate, is equivalent to the continuity of the epigraph (as defined by Gale and Klee [8]) of the second Legendre-Fenchel conjugate f^{**} of the integrand with respect to the derivative variables. Gale and Klee defined the continuity of closed convex sets in Euclidean spaces as the absence of both boundary rays and asymptotes. Thus, the slow growth assumption can be formulated as a clear geometric property of the epigraph of the function $f^{**}(t, x, \cdot)$ for each fixed pair (t, x).

We also show that this assumption is equivalent to the openness of the domain of the Legendre-Fenchel conjugate with respect to the derivative variables plus

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continuity of f^* on the whole space. As a spin-off of the methods developed in the paper we obtain a criterion for the openness of the domain of the conjugate f^* in terms of the continuity of the epigraph of function f. In the appendix we use tools developed in the paper for studying the inf-compactness of functions.

Let $f(x, \dot{x})$, where $x, \dot{x} \in R^n$, be the Lagrangian of an autonomous basic variational problem. Let $f^*(x, \dot{x})$ be the (Legendre-Fenchel) conjugate of f with respect to its second variable, i.e.

$$f^*(x,p) = \sup_{\xi \in R^n} [p \cdot \xi - f(x,\xi)],$$

and $f^{**}(x, \xi)$ be the biconjugate of f with respect to its second variable.

Slow Growth Assumption (SGA): The *slow growth assumption* is satisfied at point x if for every selection $\xi \to p(x,\xi)$ from the subdifferential correspondence $\xi \to \partial_\xi f^{**}(x,\xi)$ we have $\lim_{||\xi|| \to \infty} f^*(x, p(x,\xi)) = +\infty$.

Note that the (SGA) is a pointwise assumption with respect to x. In the special case when the biconjugate f^{**} is differentiable in ξ the growth assumption at x states that

$$\lim_{||\xi|| \to \infty} f^*(x, \nabla_\xi f^{**}(x,\xi)) = +\infty.$$

Since the (SGA) assumption is a pointwise (with respect to the first variable) assumption we will supress the first variable.

The following examples are given in Cellina [3].

Example 1.1. The function $f\colon R \to R$ defined as $f(t) = |t| - \sqrt{|t|}$ has linear growth. Although f is not convex it coincides with its second biconjugate f^{**} on $(-\infty, -\frac{1}{4}) \cup (\frac{1}{4}, +\infty)$. It is easy to check that

$$\lim_{|t| \to \infty} [tf'(t) - f(t)] = \lim_{|t| \to \infty} \frac{1}{2}\sqrt{|t|} = +\infty.$$

That is function f satisfies the (SGA).

Example 1.2. The convex function $g\colon R \to R$ defined as $g(t) = \sqrt{1+t^2}$ has also linear growth and satisfies $\lim_{|t| \to \infty} [tg'(t) - g(t)] = 0$. That is g does not satisfy the (SGA).

Although on the basis of examples given in [3], the (SGA) is interpreted as a *slow growth* assumption, it does not necessarily imply that an integrand f satisfying it goes to infinity when ξ goes to infinity. Indeed, observe that if function f satisfies the (SGA), then for any vector $a \in R^n$ the function $f_a(\xi) = f(\xi) + a \cdot \xi$ also satisfies this assumption. Now as function f in Example 1.1 satisfies the (SGA) it follows that the function $h(t) = f(t) - t$ also satisfies it. However, $\lim_{t \to +\infty} h(t) = -\infty$.

The concept of continuity of convex sets along with the related tools from convex analysis plays a prominent role in modern optimization theory and variational inequalities (see [1]).

The author plans to study the growth assumptions discussed by Mariconda and Treu [11], which are closely related to (SGA), and their relationship to Convex Analysis.

2. Definitions and notations

Recall some basic concepts and notations. A *ray* in R^n is a (closed or open) half-line in R^n. The *domain* of function $f\colon R^n \to R \cup \{+\infty\}$ is defined as

$$\operatorname{dom} f = \{x \in R^n \ :\ f(x) < +\infty\}.$$

Given a nonempty set A in R^n, the function $\sigma_A\colon R^n \to R \cup \{+\infty\}$ defined as

$$\sigma_A(d) = \sup\{d \cdot x \ :\ x \in A\}$$

is called the *support function of* A. The domain of the support function σ_A, which is obviously a convex cone, is called the *barrier cone* of A and is denoted as $b(A)$.

Definition 2.1. Let A be a nonempty closed set in R^n. A ray in R^n is a *boundary ray* of A if it is contained in the boundary of A. A ray ρ in R^n is an *asymptote* of A if it does not intersect A and its distance from A is zero:

$$\rho \cap A = \emptyset \ \text{ and } \ \inf\{||x - y|| \ :\ x \in \rho,\ y \in A\} = 0.$$

A *k-asymptote* of A is a k-dimensional affine subspace X disjoint with A, such that

$$\inf\{||x - y|| \ :\ x \in X,\ y \in A\} = 0.$$

So, a 1-asymptote of A is its asymptote.

$B_\delta(x)$ will denote the open ball in R^n centered at x and of radius δ.

Definition 2.2. A function $g\colon R^n \to R \cup \{+\infty\}$ is *continuous* at $x_0 \in R^n$, where $f(x_0) = +\infty$, if for an arbitrary positive number $M > 0$ there exists a positive number $\delta > 0$ such that $f(x) > M$ for all $x \in B_\delta(x_0)$.

Definition 2.3. A nonempty convex closed subset A of R^n is *continuous* if the restriction of the support function σ_A to the unit sphere of R^n is continuous.

Gale and Klee [8] proved that a nonempty convex closed subset A of R^n is continuous if and only if A admits no boundary ray or asymptote. Proceeding from this characterization of continuous sets we call a convex closed set A of R^n *semicontinuous* if it has no boundary ray. It is an easy exercise to show that for a convex function $f\colon R^n \to R$ the epigraph, epi f, is a semicontinuous set if and only if f satisfies the Bounded Intersection Property (see Cellina [3]): for each point $x \in R^n$ the intersection of the epigraph with the support hyperplane to epi f at point $(x, f(x))$ is a bounded set.

Denote by $\partial f(R^n)$ the range of the subdifferential correspondence $x \mapsto \partial f(x)$. For a set $A \subset R^n$, ∂A will denote its boundary. For a function f the graph is denoted as $G(f)$, and the epigraph as epi f. For a function $f\colon R^n \to R$ and point $a \in R^n$ denote $f_a(x) = f(x) + a \cdot x$.

Definition 2.4. A convex lower semi continuous function $f\colon R^n \to (-\infty,\infty]$ is *inf-compact for slope a* if for every number c, the c-lower contour set $L(f_a,c) = \{x \in R^n \ : f_a(x) \le c\}$ is compact (possibly empty) (see [2]). The function f is *inf-compact* if it is inf-compact for slope 0.

3. Characterizations of the Slow Growth Assumption

The following theorem contains the main results of the present paper.

Theorem 3.1. *For an arbitrary function $f\colon R^n \to R$ the following three properties are equivalent:*

(a) *The slow growth assumption (SGA) is satisfied.*

(b) *The domain of function f, dom f^*, is an open set.*

(c) *The epigraph of the Legendre-Fenchel biconjugate f^{**} of f is a continuous set.*

We present a proof of the theorem after a series of preparatory lemmas and claims.

Lemma 3.2. *A convex function $f\colon R^n \to R$ attains its infimum on a compact set, that is* $\arg\min f$ *is compact, if and only if there exists $\delta > 0$ such that for each point $a \in B_\delta(0)$ function $f_a(x)$ attains its infimum on R^n.*
Moreover, multi-valued function $a \mapsto$ arg min f_a is upper semicontinuous on the set $\{a \in R^n \ : \text{arg min}\, f_a$ is compact$\}$.

Proof. Assume that f attains its infimum on a compact set K. Without loss of generality, assume $0 \in K$ and $f(0) = 0$. Set $K_1 = \{x \in R^n : f(x) \leqslant 1\}$. Obviously K_1 is a closed convex set including K. We claim that K_1 is bounded. Indeed, if not, then there would exist a ray $r \subset K_1$ with $0 \in r$. Now as f is a convex and bounded function on r it follows that $f(x) = 0$ for all $x \in r$. Hence $r \subset K$ which contradicts compactness of K. So K_1 is compact and $K \subset \text{int}\, K_1$.

Denote $R := \max\{||x|| : \ x \in K_1\} > 0$. The convexity of the function f and $f(0) = 0$ imply that

$$f(x) > \tfrac{1}{R}||x|| \ \text{ for all } x \ \text{ such that } ||x|| > R. \tag{1}$$

Now for each $a \in B_{\frac{1}{R}}(0)$ we have $\ \frac{1}{R}||x|| \geqslant -a \cdot x\ $ for all $\ x \in R^n$.

The last two inequalities imply

$$f_a(x) > 0 \ \text{ for all } x \text{ with } ||x|| > R \ \text{ and } \ a \in B_{\frac{1}{R}}(0).$$

It follows that $f_a(x)$ attains its infimum on R^n. Moreover, as $f_a(0) = 0$ we have

$$\arg\min(f_a) \subset B_R(0) \text{ for each } a \in B_{\frac{1}{R}}(0).$$

Thus, the multi-valued function $a \mapsto$ arg minf_a is upper semicontinuous on the set $\{a\in R^n$: arg min f_a is compact$\}$. Now let f attain its infimum on an unbounded set C in R^n. As f is convex, C is convex, closed, and unbounded. Let the half-line $\rho = x_0 + R_+ x_1$ be contained in C. Then for any positive number $\alpha \in (-1,0)$, $\inf\{f_{\alpha x_1}(x) : x \in \rho\} = -\infty$, which implies $\sup\{f_{\alpha x_1}(x)\} = -\infty$. So the function $f_a(x)$ does not attain an infimum for vectors $a = \alpha x_1$ arbitrarily close to 0. $\square$

Remark 3.3. In the proof of the necessity in Lemma 3.2 we showed that if a convex function f on R^n attains its infimum on a compact set K, then there exists a compact set K_1 with $K \subset \operatorname{int} K_1$ and a positive number δ such that for every vector $a \in B_\delta(0)$ function $f(x) + a \cdot x$ attains its infimum on a compact set $K_a \subset K_1$.

Corollary 3.4. *Let $f\colon R^n \to R$ be a convex function. Then point $x^* \in R^n$ is an interior point of $\partial f(R^n)$ if and only if function $f(x) - x^* \cdot x$ attains its infimum on a compact set.*

Proof. Applying Lemma 3.2 to function $f(x) - x^* \cdot x$ we obtain that it attains its infimum on a compact set if and only if there exists a positive number δ such that function $f(x) - x_1^* \cdot x$ attains its infimum for each x_1^* with $||x_1^* - x^*|| < \delta$. Clearly, each such point x_1^* belongs to $\partial f(x')$ for some $x' \in R^n$. □

The next lemma gives a necessary and sufficient condition for the openness of the range of the subdifferential correspondence. It turns out, that for the openness of the range $\partial f(R^n)$ a weaker assumption than the continuity of epi f is needed.

Lemma 3.5. *Given a convex function $f\colon R^n \to R$, the range of the subdifferential correspondence $\partial f\colon R^n \rightrightarrows R^n$ is open if and only if the epigraph of f has no boundary ray.*

Proof. Let $x^* \in \partial f(x_0)$ for $x_0 \in R^n$, that is

$$f(x) - f(x_0) \geq x^* \cdot (x - x_0) \ \text{ for all } \ x \in R^n.$$

So the convex function $F(x) = f(x) - f(x_0) - x^* \cdot (x - x_0)$ is nonnegative and attains its infimum at point x_0.

As the epi f has no boundary ray, the epigraph of function F, which is a 'shift' of the epi f, has no boundary ray. Therefore, the $\arg\min F$ is a nonempty compact set. By Lemma 3.2 there exists $\delta > 0$ such that for each point $a \in B_\delta(0)$ function

$$f(x) - (x^* + a) \cdot x - f(x_0) - x^* \cdot x_0$$

attains its infimum. That is for some $x_1 \in R^n$

$$f(x_1) - (x^* + a) \cdot x_1 - f(x_0) + x^* \cdot x_0 \leq f(x) - (x^* + a) \cdot x - f(x_0) + x^* \cdot x_0$$

for all $x \in R^n$. Equivalently

$$f(x) - f(x_1) \geq (x^* + a) \cdot (x - x_1) \ \text{ for all } x \in R^n.$$

That is $x^* + a \in \partial f(x_1)$. So we have proved that

$$x^* + a \in \partial f(R^n) \ \text{ for all } \ a \in B_\delta(0),$$

and hence x^* is an interior point of $\partial f(R^n)$.

Now let $\rho = z_0 + R_+^1 z_1$ be a boundary ray of the epigraph of f. Let x_0 and x_1 be projections of points z_0 and z_1, respectively. Let H be a hyperplane through the point $z = z_0 + z_1 \in \partial(\text{epi}\, f)$ support to epi f. As H is not vertical it is the graph of an affine function $h(x) = x^* \cdot (x - \bar{x}) + f(\bar{x})$. That is $H = \{(x, h(x)) \,:\, x \in R^n\}$. Show that x^* is a boundary point of $\partial f(R^n)$. Denote $x^*(s) = x^* + s\bar{x}$ for $s > 0$. Then any hyperplane $H(x^*(s), c) = \{(x, x^*(s) \cdot x + c) \,:\, x \in R^n\}$ $(s > 0)$ intersects ray ρ and hence $x^*(s)$ is not a subgradient of function f at any point. That is $x^*(s) \notin \partial f(R^n)$. As point $x^*(s)$ is arbitrarily close to x^*, x^* is a boundary point of $\partial f(R^n)$. Hence $\partial f(R^n)$ is not open. □

Lemma 3.6. *For an arbitrary function $f \colon R^n \to R \cup \{+\infty\}$ the following inclusions hold*

$$\mathit{int}\,(\mathit{dom}\, f^*) \subset \partial f(R^n) \subset \mathit{dom}\, f^*.$$

Hence $\partial f(R^n) = \mathit{dom}\, f^*$ whenever $\mathit{dom}\, f^*$ is open.

Proof. The second inclusion follows from the equivalence of the conditions (a) and (b) of Theorem 23.5 in Rockafellar [12] which is a direct consequence of the definition of the subgradient.

We prove the first one. Let $x^* \in \text{int}\,(\text{dom}\, f^*)$. Assume, without loss of generality, that $x^* = 0$. We have that there is a number $r > 0$ such that $B_r(0) \subset \text{int}\,(\text{dom}\, f^*)$. Denote by e_i the i^{th} unit vector in R^n, and set $x_i^* = re_i$ and $x_{n+i}^* = -re_i$ for $i = 1, \cdots n$. As $x_k^* \in \text{dom}\, f^*$ there exists $c \in R$ such that

$$x_k^* \cdot x - f(x) < c \ \text{ for all } x \in R^n \text{ and } k = 1, \cdots, 2n$$

or

$$f(x) > x_k^* \cdot x - c \ \text{ for all } x \in R^n \text{ and } k = 1, \cdots, 2n.$$

So f majorizes each of the affine functions $x_k^* \cdot x - c$, $k = 1, \cdots, 2n$. Therefore, denoting $l(x) = \max\{x_k^* \cdot x - c \,:\, k = 1, \cdots, 2n\}$ we will have

$$f(x) > l(x) \text{ for all } x \in R^n.$$

As $0 \in \text{int}(\text{co}\{x_1^*, \ldots, x_{2n}^*\})$ it follows that $\lim_{x \to \infty} l(x) = +\infty$. This and the above inequality imply that $\lim_{x \to \infty} f(x) = +\infty$. This in turn implies that f attains its infimum on R^n. Let $x_0 \in R^n$ be a minimum point. Then $x^* = 0 \in \partial f(x_0)$. □

Lemma 3.7. *If the epigraph* epi f *of a convex function $f \colon R^n \to R$ is a continuous set, then $\partial f(R^n) = \mathit{dom}\, f^*$ and these sets are open.*

Proof. By Lemma 3.6 it suffices to show that under the assumptions of the lemma dom $f^* \subset \partial f(R^n)$. Suppose that $x^* \in \text{dom}\, f^*$. That is

$$c = \sup\{x^* \cdot x - f(x) \,:\, x \in R^n\} < +\infty. \tag{2}$$

Now, if the supremum above is not attained, i.e., if $x^* \cdot x - f(x) < c$ for all $x \in R^n$, then the graph Γ of the affine function $x^* \cdot x - c$ is an n-asymptote of epi f. By Klee [8, Proposition 2] then epi f admits a boundary ray or asymptote. Hence epi f is not continuous, which contradicts the assumption. Thus the supremum in (2) is attained. That is $x^* \cdot x - f(\bar{x}) = c$ for some $\bar{x} \in R^n$. This and inequality (2) imply that $x^* \in \partial f(\bar{x}) \subset \partial f(R^n)$. □

Remark 3.8. For the function $g(t) = \sqrt{1 + t^2}$, considered in Example 1.2, $\text{dom}\, g^* = [-1, 1]$ but $\partial g(R^n) = (-1, 1)$. So, this example shows that the openness assumption (dom f^* is open) in the last statement of Lemma 3.6 is important.

This example also shows that the absence of a boundary ray in the epi f alone is not sufficient for the coincidence of the range $\partial f(R^n)$ and the domain dom f^*.

Lemma 3.9. *If for a convex function $f : R^n \to R$, its conjugate's domain dom f^* is open, then the epigraph of f, epi f, is a convex closed continuous set.*

Proof. Assume on the contrary, that the convex closed epi f is not continuous. Then epi f contains a boundary ray or an asymptote. In the case epi f contains a boundary ray ρ using the supporting hyperplane theorem applied for a relative interior point of ρ, and arguing as in the proof of Lemma 3.5 we obtain that dom f^* includes a boundary point. In the case epi f^* contains an asymptote ρ applying the Separation Theorem we obtain a hyperplane $H = \{(x, h(x)) : x \in R^n\}$, where $h(x) = x^* \cdot x + c$ is an affine function, including ρ and disjoint from epi f^*. Now it is easy to show that vector x^* is a boundary point of dom f^*. □

The following theorem follows from Lemmas 3.7 and 3.9. Although it follows from Theorem 3.1 we formulate it as a separate statement because of its importance to Convex Analysis.

Theorem 3.10. *Let f be a convex function defined on R^n. Then the dom f^* is open if and only if the epigraph of the function f, epi f, is a convex closed continuous set.*

Lemma 3.11. *If dom f^* is open, then f^* is continuous and coercive.*

Proof. It suffices to show that

(i) $$\lim_{x^* \in \,\mathrm{dom}\, f^*;\ ||x^*|| \to \infty} f^*(x^*) = +\infty, \quad \text{and}$$

(ii) $$\lim_{x^* \in \,\mathrm{dom}\, f^*;\ x^* \to x_0^*} f^*(x^*) = +\infty \text{ for } x_0^* \in \partial(\mathrm{dom}\ f^*).$$

Condition (i) is simple and we omit it. We show (ii). Let for $x_0^* \in \partial(\mathrm{dom}\ f^*)$, and M be an arbitrary positive number. We have

$$\sup\{x_0^* \cdot x - f(x) \ : \ x \in R^n\} = +\infty.$$

Obviously there exists $x_0 \in R^n$ such that $x_0^* \cdot x_0 - f(x_0) > M + 1$. This implies that there is $\delta > 0$ such that

$$x^* \cdot x_0 - f(x_0) > M \ \text{ for all } \ x^* \in B_\delta(x_0^*).$$

Therefore $f^*(x^*) > M$ for all $x^* \in B_\delta(x_0^*)$, which proves (ii). □

Proof of Theorem 3.1: The equivalence of the properties (b) and (c) follows from Theorem 3.10. We will show that property (a) implies (c) and that (b) implies (a).

(a) ⇒ (c): Assume on the contrary that (SGA) is satisfied, but epi f has a boundary ray or an asymptote. The case when epi f has a boundary ray is simple, and can be omitted. Let epi f have an asymptote ρ. Let

$$H = \{(x, \alpha) \in R^n \times R \ : \ \alpha = x^* \cdot x + c\}$$

for some $x^* \in R^n$ and $c \in R$, be a hyperplane separating ρ and epi f. Since $\rho \subset H$ and ρ is an asymptote of epi f we have that if $(\text{epi}\, f) \cap H \neq \emptyset$, then for each point $x \in (\text{epi}\, f) \cap H$, $\rho' = (x - x_0) + \rho \subset \partial(\text{epi}\, f)$, where x_0 is the starting point of ρ. Then ρ' is a boundary ray of epi f, and we are in the case "epi f has a boundary ray". So assume $(\text{epi}\, f) \cap H = \emptyset$. Then

$$F(x) = f(x) - (x^* \cdot x + c) > 0 \text{ for all } x \in R^n \text{ and } \inf\{F(x) : x \in R^n\} = 0.$$

Now for each $k \in N$ there exists a point $x_k \in R^n$ such that $||x_k|| \geq k$ and $F(x_k) < \frac{1}{k}$. By the Ekeland's Variational Principle [7] there exists a sequence $y_k \in B_1(x_k)$ such that

$$F(y_k) \leq F(x_k) \text{ and } F(w) > F(y_k) - \tfrac{1}{k}||w - y_k||, \text{ for all } w \neq y_k.$$

It follows that $\partial F(y_k) \subset B_{\frac{1}{k}}(0)$, $k \in N$, or equivalently

$$\partial f(y_k) \subset B_{\frac{1}{k}}(x^*) \text{ for } k \in N.$$

Thus, we have a sequence $\{y_k\} \subset R^n$ with $y_k \to \infty$ and $x_k^* \in \partial f(y_k)$, $k \in N$, such that $x_k^* \to x^*$. By Lemma 3.6, $x_k^* \in \text{dom}\, f^*$, and by the definition of x^*, $x^* \in \text{dom}\, f^*$. Therefore

$$\lim_k f^*(x_k^*) = f^*(x^*) \neq +\infty,$$

that is, (SGA) is violated.

(b) ⇒ (a): Let $||x_k|| \to \infty$ in R^n and $x_k^* \in \partial f(x_k)$ for $k \in N$. Assume, without loss of generality, that $||x_k||$ strictly increases. By Theorems 24.4 and 24.7 in Rockafellar [12], $A_k = \partial f(B_{||x_k||}(0))$ is a nonempty compact set for each $k \in N$ So $\{A_k\}$ is an increasing sequence of compact sets. Now because of the implication $(b) \to (c)$ the epigraph of f is a continuous set. Therefore, by Lemma 3.7 we have $\partial f(R^n) = \cup A_k = \text{dom}\, f^*$.

As $x_k^* \in \partial f(x_k)$, by Remark 3.3 there exist a number $k_0 \in N$, and a sequence $j_k\, (k = k_0, k_0+1, \ldots)$ with $j_k < k$, $j_k \to \infty$ such that $x_k^* \notin A_{j_k}$ $(k = k_0, k_0+1, \ldots)$. By Lemma 3.2, $x_k^* \to \partial(\text{dom}\, f^*)$ or $||x_k^*|| \to \infty$. Lemma 3.11 implies that in both cases $f^*(x_k^*) \to +\infty$. That is the (SGA) is satisfied. □

4. Appendix: Continuity of the epigraph and inf-compactness of functions

In this section using the tools developed for the proof of the main results we study the continuity of the epigraph of lower semicontinuous functions and their inf-compactness. For the basic properties of inf-compact functions see Castaing and Valadier [2].

Theorem 4.1. *A lower semicontinuous function $f : R^n \to R$ is inf-compact for slope $x^* \in R^n$ if and only if $x^* \in int\,(dom\, f^*)$. Moreover, multivalued function $x^* \mapsto arg\min f_{-x^*}$ is upper semicontinuous on int $(dom\, f^*)$.*

Proof. Let $x^* \in \text{int}\,(\text{dom}\, f^*)$. Then by Lemma 3.6 there is $x_0 \in R^n$ such that

$$f(x) - f(x_0) \geq x^* \cdot (x - x_0) \;\text{ for all }\; x \in R^n$$

or
$$f_{-x^*}(x) \geq f_{-x^*}(x_0) \;\text{ for all }\; x \in R^n.$$

This implies
$$f^{**}_{-x^*}(x) \geq f^{**}_{-x^*}(x_0) \;\text{ for all }\; x \in R^n.$$

So we have that the convex function $f^*_{-x^*}$ attains its infimum for each x^* in an open set int (dom f^*). By Lemma 3.2 it follows that $f^{**}_{-x^*}$ attains its infimum on a compact set. Again by Lemma 3.2 multi-valued function $x^* \mapsto \arg\min f^{**}_{x^*}$ is upper semicontinuous on inf (dom f^*).

Now let function f be inf-compact for slope $x_0^* \in R^n$. That is, let function $f_{-x_0^*}$ attain its infimum on a compact set $K \subset R^n$. As $f^{**}_{-x_0^*} = (f_{-x_0^*})^{**}$ and the function $f_{-x_0^*}$ is lower semicontinuous it follows that $f^{**}_{-x_0^*}$ attains its infimum on a compact set co K. By Lemma 3.2 we have that there exists $\delta > 0$ such that $f^{**}_{-x_0^*}$ attains its infimum on a compact set for each slope x^* in $B_\delta(x_0^*)$. Since f is lower semicontinuous this implies that function f_{-x^*} attains its infimum on a compact set for each slope x^* in $B_\delta(x_0^*)$. Thus x_0^* belongs to int (dom f^*). □

Corollary 4.2. *A lower semicontinuous function $f\colon R^N \to R$ whose second conjugate's epigraph, epi f^{**}, is semicontinuous (in particular continuous), is inf-compact for slope x^* if and only if*

$$x^* \in \text{dom}\, f^* = \partial f(R^n).$$

The arguments in the proof of Lemma 3.2 show that a function f is inf-compact if and only if it attains its infimum on a compact set. Therefore f is inf-compact for every slope if and only if it has the Bounded Intersection Property.

The following proposition provides a necessary and sufficient condition for the continuity of the epigraph of the convexification of a lower semicontinuous function.

Proposition 4.3. *Let f be a lower semicontinuos function defined on R^n. Then the epigraph of the biconjugate of f, epi f^{**}, is a continuous set if and only if*

$$\partial f(R^N) = \partial f^{**}(R^n) = \text{dom}\, f^*.$$

Proof. Let the epigraph of function f^{**}, epi f^{**}, be a continuous set. By Lemma 3.7 we have $\partial f^{**}(R^n) = \text{dom}\,(f^{**})^* = \text{dom}\, f^*$.

It remains to show that $\partial f(R^n) = \partial f^{**}(R^n)$. Obviously $\partial f(R^n) \subset \partial f^{**}(R^n)$. We show the reverse inclusion $\partial f^{**}(R^n) \subset \partial f(R^n)$. Let x^* belong to $\partial f^{**}(R^n)$. Then $x^* \in \partial f^{**}(x_0)$ for some $x_0 \in R^n$. As the epigraph, epi f^{**}, is a continuous set the intersection of the hyperplane

$$H = \{(x, a) \in R^n \;: a = f^{**}(x_0) + x^* \cdot (x - x_0)\}$$

with the epigraph of f^{**}, $H \cap (\text{epi } f^{**})$, is compact. Indeed, if this intersection is unbounded then the boundary of epi f^{**} would contain a ray. Obviously

$$(x_0,\ f^{**}(x_0)) \in H \cap (\text{epi } f^{**}).$$

As f is a lower semicontinuous function there exist points $x_1, \dots, x_{n+1}$ in R^n and the coefficients of a convex combination $\alpha_1, \dots, \alpha_{n+1}$ such that

$$(x_0, f^{**}(x_0)) = (\sum_{k=1}^{n+1} \alpha_k x_k,\ \sum_{k=1}^{n+1} \alpha_k f(x_k)).$$

Now it is easily seen that $x^* \in \partial f(x_k)$ for each $k = 1, \dots, n+1$.
Therefore $x^* \in \partial f(R^n)$. □

The following theorem generalizes Theorem 1 from Cellina [3] which together with its parametric form (Theorem 2 in Cellina [3]) are main tools in his study of variational problems.

Theorem 4.4. *Let $f\colon R^n \to R$ be a lower semicontinuous, affinely minorized function such that its second conjugate f^{**} satifies BIP. Then, for every point $x_0 \in R^n$ there exist $x^* \in \partial f^{**}(x_0)$, at most $m \le n+1$ points $x_k \in R^n$ and coefficients of a convex combination α_k such that*

$$\sum_1^m \alpha_k x_k = x_0 \ \text{ and } \ \sum_1^m \alpha_k f(x_k) = f^{**}(x_0) \tag{3}$$

and $$f(x_k) = f^{**}(x_k) = f^{**}(x_0) + x^* \cdot (x_k - x_0) \quad \text{for} \quad k = 1, \dots, m. \tag{4}$$

Proof. By BIP for each slope $x^* \in \partial f^{**}(R^n)$ the convex function $f^{**}_{-x^*}$ attains its infimum on a compact set. Then, by inequality (1) in the proof of Lemma 3.2 there is a real R such that

$$f^{**}_{-x^*} \ge \frac{1}{R}||x|| \ \text{ for all } \ x \ \text{ such that } \ ||x|| > R. \tag{5}$$

This implies that epi $(f^{**}_{-x^*})$ contains no line. Now by the assumption epi f_{-x^*} is a closed set whose convex hull co (epi f_{-x^*}) $\subset$ epi $f^{**}_{-x^*}$ contains no line. Inequality (5) implies that the convex closure of epi $f^{**}_{-x^*}$, $\bar{\text{co}}$ (epi f_{-x^*}) has no extreme ray. By Theorem 2.1 from Husseinov [9], co(epi $f^{**}_{-x^*}$) is closed. Let x_0 be an arbitrary point in R^n. Then, by assumption BIP there is a subgradient $x^* \in f^{**}(x_0)$ such that the intersection

$$S = (\text{epi } (f^{**}_{-x^*})) \cap H(x_0, x^*)$$

where $H(x_0, x^*)$ is the hyperplane in R^{n+1} through point x_0 and with slope x^*, is compact. Now S is a convex compact set with $S = \text{co } (G(f) \cap H(x_0, x^*))$, where $G(f)$ denotes the graph of function f. Therefore, there are points $x_1, \dots, x_m$ $(m \le n+1)$ in R^n and coefficients of a convex combination $\alpha_1, \dots, \alpha_m$ satisfying equations (3) and (4). □

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References

[1] A. Auslender, M. Teboulle: *Asymptotic Cones and Functions in Optimization and Variational Inequalities*, Springer, New York (2003).

[2] C. Castaing, M. Valadier: *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics 580, Springer, Berlin (1977).

[3] A. Cellina: *The classical problem of the calculus of variations in the autonomous case: relaxation and Lipschitzianity of solutions*, Trans. Amer. Math. Soc. 356 (2004) 415–426.

[4] A. Cellina, A. Ferrero: *Existence of Lipschitzian solutions to the classical problem of the calculus of variations in the autonomous case*, Annal. Inst. H. Poincaré C, Anal. Non Lineaire 20 (2003) 911–919.

[5] C. Cellina, V. Staicu: *The existence of solutions to variational problems of slow growth*, J. Differential Equations 260 (2016) 5834–5846.

[6] A. Cellina, S. Treu, G. Zagatti: *On the minimum problem for a class of non-coercive functionals*, J. Differential Equations 127 (1996) 225–262.

[7] I. Ekeland: *On the variational principle*, J. Math. Anal. Appl.47 (1974) 324–353.

[8] D. Gale, V. Klee: *Continuous convex sets*, Math. Scand. 7 (1959) 379–391.

[9] F. Husseinov: *A note on the closedness of the convex hull and its applications*, J. Convex Analysis 6 (1999) 387–393.

[10] V. Klee: *Asymptotes and projections of convex sets*, Mat. Scand. 8 (1960) 356–362.

[11] C. Mariconda, T. Giulia: *Lipschitz regularity of the minimizers of autonomus integral functionals with discontinuous non-convex integrands of slow growth*, Calc. Var. Parial Differential Equations 29 (2007) 99–117.

[12] R. T. Rockafellar: *Convex Analysis*, Princeton University Press, Princeton (1970).