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What is the Best Viscous Approximation to a Rate-Independent Process?

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Viscous approximations of a rate-independent process with regulated inputs are considered with a general viscosity operator. It is shown that the limit as the viscosity coefficient tends to zero defines a continuous rate-independent input-output mapping with respect to the uniform topology in the space of regulated functions, the limits are, however, in general different for different viscosity operators. Examples show that if the viscosity operator is chosen independently of the energy potential, the limit jump trajectories may violate both the normality rule and the maximal dissipation principle.

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1. Introduction

Rate-independent problems arise typically in systems with remanent irreversible memory. In the Prandtl-Reuss theory of plasticity, for example, it is assumed that the stress tensor is constrained to a given convex domain K^*. In the interior of K^*, the strain-stress relation is linearly elastic. As soon as the stress value reaches the boundary of K^*, plastic yielding occurs, and the plastic strain rate points in the direction of the outward normal cone to K^*. Physically, this corresponds to the *maximal dissipation principle,* see [20]. The stress response is indeed independent of the time scale of the input strain path, and this is why we talk about rate-independence.

A geometric view was proposed by Moreau [27] who introduced the concept of *sweeping process,* where the evolution of a system is driven by a moving convex set in a Hilbert space. Moreau solved the problem by an implicit time discretization scheme called the *catching up algorithm* which consists in projecting at each time step the input value orthogonally in the sense of outward normal cone to the actual position of the convex constraint. He also developed a measure-theoretical framework for the convergence of the discrete approximations.

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Another natural way of approximating a rate-independent system is the so-called *vanishing viscosity method.* It introduces a higher order 'viscous' dissipation mechanism which stabilizes the process with the hope that letting the viscosity coefficient tend to zero, some compactness will be preserved and a possible limit can be declared to be the desired solution of the original rate-independent problem. However, the viscous approximations are typically continuous in time, and if the limit process is expected to exhibit discontinuities, it is necessary to design a suitable convergence concept in order to identify the jumps in the limit.

Discontinuities may occur in systems involving energy functionals which are not convex, and this is the case for example of material models for complex solids. The idea of combining vanishing viscosity analysis and reparametrization techniques goes back to [6], and was further developed in [22, 24], including the extension to abstract settings [21, 23]. This vanishing viscosity approach has a wide range of applications, e. g., in crack propagation [8, 13], large strain plasticity [4, 5], damage [9, 10, 11], and optimal control [1, 7, 12].

Another type of problems appear in the analysis of genuinely discontinuous processes, and the extension of hysteresis operators to discontinuous inputs is therefore a question of comparable importance. In [28], such an extension is proposed in a very natural way, and it is based on the observation that the solution to a rate-independent system follows the same trajectory independently of the input speed. This idea is extrapolated to a virtually infinite speed and, to handle the discontinuities, [28] makes use of a reparametrization approach which consists in 'filling in' the jumps of the input with segments which represent the 'preferred way' (shortest in a sense, for example). See also [29, 30].

An alternative approach ([16, 17]) to discontinuous hysteresis phenomena relies on reformulating a rate-independent evolution system as a Kurzweil integral variational inequality in the space of regulated functions. Recall that a function f of variable $t \in [a, b]$ with values in a Banach space is said to be *regulated* if it only admits discontinuities of the first kind. Such functions naturally appear, for example, in problems of optimal control of nonsmooth constrained processes and in models in economics, see [2, 15].

It is interesting to note that in the particular case of quadratic potentials and a suitably chosen viscosity operator, the vanishing viscosity limit was shown to coincide with the Kurzweil solution of the problem, see [17, Theorem 2.4]. Moreover, the limit energy balance contains a quadratic jump dissipation term which, on the one hand, can be interpreted as residual of the viscous dissipation as the viscosity coefficient tends to zero and, on the other hand, follows from the Kurzweil integration by parts formula.

In this text, we compare vanishing viscosity approximations for rate-independent variational inequalities with regulated inputs for different choices of the viscosity operator. We prove that the vanishing viscosity limit always exists and defines a strongly continuous operator in the space of regulated functions. We also present an example showing that the limit solution can be expected to coincide with the Kurzweil solution only if the energy potential and the viscous dissipation potential are coordinated in a way. In this 2D example, if the viscosity matrix does not coincide with the elasticity matrix, the residual viscous dissipation is smaller than the Kurzweil dissipation, the jump trajectory does not belong to the normal cone to the admissible output domain, and also the maximum dissipation principle is violated.

The paper is organized as follows. In Section 2 we give a precise formulation of the problem of A-viscous regularization via symmetric positive operators A. Auxiliary estimates as a first step toward the convergence argument are derived in Section 3. In Section 4, we study the special case of piecewise constant inputs. Main results on the convergence of the viscous solutions are stated and proved in Section 5. Finally, in Section 6, we exhibit an example showing that the viscous limit may yield a non-variational solution to the rate-independent system.

2. Motivation

We discuss some issues related to viscous approximations to rate-independent processes by considering a simple differential inclusion in a Hilbert space X

$$u(t) - \xi(t) \in \partial M_K(\dot{\xi}(t)), \tag{1}$$

associated with a bounded convex closed set $K \subset X$ containing 0, where M_K is the Minkowski functional of K, and ∂M_K is its subdifferential. Here, $u\colon [0,T] \to X$ is given and $\xi\colon [0,T] \to X$ is the unknown of the problem. The dot denotes derivative with respect to t, which has to be given a proper meaning.

Note that the mapping which with a given u associates the solution ξ is called the *vectorial play operator* and the terminology was introduced in [14], although the connection of (1) with the mechanical play has been already mentioned in 1974 in [26]. Still formally, (1) admits an equivalent 'sweeping process' formulation

$$\dot{\xi}(t) \in \partial I_{K^*}(u(t) - \xi(t)), \tag{2}$$

where $$K^* = \{y \in X : \langle y, z\rangle \le 1, \forall z \in K\}$$

is the *polar set* of K, I_{K^*} is the *indicator function* of K^*, and ∂I_{K^*} is its *subdifferential*. Note that from the boundedness of K it follows that there exists $\rho > 0$ such that the implication

$$|x| \le \rho \implies x \in K^* \tag{3}$$

holds, that is, the ball $B_\rho(0)$ of radius ρ centered at 0 is contained in K^*. The inclusions (1) or (2) have to be complemented with a prescribed initial condition $x_0 \in K^*$ such that

$$\xi(0) = u(0) - x_0. \tag{4}$$

Consider now a symmetric positive definite mapping $A \in \mathcal{L}(X)$, that is, there exists $c > 0$ such that

$$\langle Av, v\rangle \ge c|v|^2 \quad \text{for every} \quad v \in X.$$

In the sequel, such operators will be called *viscosity operators*.

We call the inclusion

$$u(t) - \xi^\varepsilon(t) - \varepsilon A^2 \dot{\xi}^\varepsilon(t) \in \partial M_K(\dot{\xi}^\varepsilon(t)), \quad \xi^\varepsilon(0) = u(0) - x_0 \tag{5}$$

A-viscous regularization of (1) with viscosity parameter $\varepsilon > 0$. It can be represented by a variational inequality

$$\begin{aligned} &u(t) - \xi^\varepsilon(t) - \varepsilon A^2 \dot{\xi}^\varepsilon(t) \in K^* && \text{for a. e. } t \in]0,T[, \\ &\left\langle \dot{\xi}^\varepsilon(t), u(t) - \xi^\varepsilon(t) - \varepsilon A^2 \dot{\xi}^\varepsilon(t) - z \right\rangle \geq 0 && \text{for a. e. } t \in]0,T[,\ \forall z \in K^*, \end{aligned} \tag{6}$$

or, equivalently,

$$A^{-1}(u(t) - \xi^\varepsilon(t)) - \varepsilon A \dot{\xi}^\varepsilon(t) \in A^{-1} K^* \qquad \text{for a. e. } t \in]0,T[, \tag{7}$$

$$\left\langle \varepsilon A \dot{\xi}^\varepsilon(t), A^{-1}\big(u(t) - \xi^\varepsilon(t)\big) - \varepsilon A \dot{\xi}^\varepsilon(t) - z \right\rangle \geq 0 \quad \text{for a. e. } t \in]0,T[,\ \forall z \in A^{-1} K^*.$$

Let us recall here the notion of orthogonal projection $Q_Z \colon X \to Z$ of X onto a convex closed set $Z \subset X$ and its complement $P_Z = I - Q_Z$, where I is the identity operator on X. The relation $y = P_Z(v)$ admits a variational characterization

$$v - y \in Z\,,\ \langle y, v - y - z \rangle \geq 0 \ \ \forall z \in Z. \tag{8}$$

Using (8) we see that (7) can be reformulated as an ODE of the form

$$\varepsilon A \dot{\xi}^\varepsilon(t) = P_{A^{-1}K^*}(A^{-1}(u(t) - \xi^\varepsilon(t))) \ \text{ for a. e. } t \in]0,T[,\ \xi^\varepsilon(0) = u(0) - x_0. \tag{9}$$

The projection is Lipschitz continuous. Hence, Eq. (9) admits a unique absolutely continuous solution ξ^ε for every $u \in L^1(0,T;X)$ and every $\varepsilon > 0$.

It is worth mentioning that in the case when A is the identity operator, the regularization above corresponds to the Yosida approximation considered in [25]; therein the limiting solution is obtained by means of a graph convergence.

The aim of this note is to point out that passage to the limit as $\varepsilon \to 0+$ in (5) may exhibit some unexpected behavior when the input has some jump discontinuities. If the input u belongs to the space $C(0,T;X)$ of continuous functions, then there is a general agreement in the community on the fact (proved in [3] under additional compactness assumptions for inputs $u \in H^1(0,T;X)$, and found in [25] in a more general context of graph convergence for $A = I$) that the sequence ξ^ε converges uniformly to a continuous function ξ with bounded variation satisfying the variational inequality

$$\begin{cases} x(t) := u(t) - \xi(t) \in K^* & \forall t \in [0,T], \\ \xi(0) = u(0) - x_0, \\ \int_0^T \langle u(t) - \xi(t) - y(t)\,,\ \mathrm{d}\xi(t) \rangle \geq 0, & \forall y \in C(0,T;X),\ y(t) \in K^* \ \forall t \in [0,T], \end{cases} \tag{10}$$

where the integral is to be interpreted in the Riemann-Stieltjes sense. The proof of this statement will also be given below in Corollary 5.2 as a by-product of our computations.

The situation is substantially different if u is not continuous. The first natural step is to assume that u is regulated. We denote by $G(0,T;X)$ the space of regulated functions with values in X, $G_L(0,T;X)$ will be the subspace of left continuous regulated functions. We also denote by $BV(0,T;X)$ the space of functions of bounded variation and $BV_L(0,T;X) = BV(0,T;X) \cap G_L(0,T;X)$. It was shown in Problem 2.2 of [17] that the limit $\xi = \lim_{\varepsilon\to 0+} \xi^\varepsilon \in BV_L(0,T;X)$ exists for $u \in G_L(0,T;X)$ provided $A = I$. In this case, ξ satisfies the variational inequality

$$\begin{cases} x(t) := u(t) - \xi(t) \in K^* & \forall t \in [0,T], \\ \xi(0) = u(0) - x_0, & \\ \int_0^T \langle u(t+) - \xi(t+) - y(t), \mathrm{d}\xi(t)\rangle \geq 0 & \forall y \in G(0,T;X),\ y(t) \in K^*\ \forall t \in [0,T], \end{cases} \tag{11}$$

where the integral is understood as the Kurzweil-Stieltjes integral. The argument in [17] uses in a substantial way special properties of the projection Q_{K^*}. Here, we show that the situation is not the same if $A \neq I$. The limit $\xi = \lim_{\varepsilon\to 0+} \xi^\varepsilon \in BV_L(0,T;X)$ still exists, the associated input-output operator is still continuous in the strong topology of $G(0,T;X)$, but examples show that at discontinuity points, the jump trajectories of the limit ξ can be different for different viscosity operators A. This has interesting consequences also for the energetic interpretation of the limit system. We believe that this observation justifies the question mark in the title.

3. Estimates

Let us consider a fixed input $u \in G(0,T;X)$, and a sequence of functions $u_n \in G(0,T;X)$ which converge uniformly to u. In other words

$$\lim_{n\to\infty} \|u_n - u\|_{[0,T]} = 0, \tag{12}$$

where $\|\cdot\|_{[0,T]}$ stands for the sup-norm in $G(0,T;X)$. Given a symmetric positive definite operator $A \in \mathcal{L}(X)$, for each $n \in \mathbb{N}$ and each $\varepsilon > 0$ we consider (9) with u replaced by u_n with the same initial condition x_0, and denote by ξ_n^ε the associated solution, that is, ξ_n^ε satisfies

$$\begin{aligned} & u_n(t) - \xi_n^\varepsilon(t) - \varepsilon A^2 \dot\xi_n^\varepsilon(t) \in K^* && \text{for a.e. } t \in]0,T[\,, \\ & \left\langle \dot\xi_n^\varepsilon(t), u_n(t) - \xi_n^\varepsilon(t) - \varepsilon A^2 \dot\xi_n^\varepsilon(t) - z \right\rangle \geq 0 && \text{for a.e. } t \in]0,T[,\ \forall z \in K^*, \\ & \xi_n^\varepsilon(0) = u_n(0) - x_0. \end{aligned} \tag{13}$$

Here, we prove the following statement using a technique proposed in [14].

Lemma 3.1. *The solutions ξ_n^ε to* (13) *satisfy the estimate*

$$\operatorname*{Var}_{[0,T]} \xi_n^\varepsilon \leq C$$

with a constant C depending only on K, x_0 and u, while independent of n and ε.

Proof. Assuming that $\rho > 0$ is a number with the property (3), let $0 = t_0 < t_1 < \cdots < t_N = T$ be a division such that

$$|u(t) - u(t_{j-1}+)| < \tfrac{\rho}{6}, \quad t \in]t_{j-1}, t_j[,$$

and choose $n_0 \in \mathbb{N}$ such that $\|u_n - u\|_{[0,T]} < \rho/6$ for $n \geq n_0$. Fix arbitrary $n \geq n_0$ and $\varepsilon > 0$. For $t \in]t_{j-1}, t_j[$ put

$$z(t) = u_n(t) - u_n(t_{j-1}+) + \tfrac{\rho}{2} y(t)$$

with $|y(t)| \leq 1$ such that $\langle y(t), \dot{\xi}_n^\varepsilon(t)\rangle = |\dot{\xi}_n^\varepsilon(t)|$ a.e. Clearly $|z(t)| < \rho$ a.e. in $]t_{j-1}, t_j[$, hence it is an admissible choice in (13) and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|\xi_n^\varepsilon(t) - u_n(t_{j-1}+)|^2 + \rho|\dot{\xi}_n^\varepsilon(t)| \leq 0 \quad \text{for a. e. } t \in]t_{j-1}, t_j[.$$

By integrating over the interval $]t_{j-1}, t_j[$ we obtain (note that ξ_n^ε is absolutely continuous)

$$\rho \int_{t_{j-1}}^{t_j} |\dot{\xi}_n^\varepsilon(t)| \mathrm{d}t + |\xi_n^\varepsilon(t_j) - u_n(t_{j-1}+)|^2 \leq |\xi_n^\varepsilon(t_{j-1}) - u_n(t_{j-1}+)|^2. \tag{14}$$

By virtue of (14) and of the triangle inequality we have for each $j = 1, \ldots, N$ that

$$|\xi_n^\varepsilon(t_j) - u_n(t_j+)| \leq |\xi_n^\varepsilon(t_{j-1}) - u_n(t_{j-1}+)| + |u(t_j+) - u(t_j-)| + \rho$$

with the convention $u(T+) = u(T)$. Hence,

$$\max_j |\xi_n^\varepsilon(t_j) - u_n(t_j+)| \leq \rho N + |x_0| + \sum_{j=1}^N |u(t_j+) - u(t_j-)| =: \Gamma.$$

Altogether

$$\operatorname*{Var}_{[0,T]} \xi_n^\varepsilon = \int_0^T |\dot{\xi}_n^\varepsilon(t)| \mathrm{d}t = \sum_{j=1}^N \int_{t_{j-1}}^{t_j} |\dot{\xi}_n^\varepsilon(t)| \mathrm{d}t \leq \frac{N\Gamma^2}{\rho} \quad \text{for every } \varepsilon > 0,\ n \geq n_0,$$

and the assertion follows. □

4. Viscous limit for processes with piecewise constant inputs

It is well-known that the set of piecewise constant (step) functions is dense in the space of regulated functions. Therefore, as a first step towards our main goal, in this section we derive a formula for the limit as $\varepsilon \to 0+$ in (5) in the case when the input function u is piecewise constant. To this end, let us consider an auxiliary ODE with independent variable $s \in [0, \infty[$

$$Ay'(s) = P_{A^{-1}K^*}(A^{-1}(w - y(s))), \quad s \in]0, \infty[, \quad y(0) = y_0, \tag{15}$$

with a fixed element $w \in X$ and an initial condition $y_0 \in X$, where the prime denotes the derivative with respect to s. The mapping $P_{A^{-1}K^*}$ is Lipschitz continuous. Hence, a unique global continuously differentiable solution $y\colon [0,\infty[\to X$ of (15) exists and satisfies the variational inequality

$$\begin{aligned} & w - y(s) - A^2 y'(s) \in K^* && \text{for all } s \in]0,\infty[, \\ & \langle y'(s), w - y(s) - A^2 y'(s) - z\rangle \geq 0 && \text{for all } s \in]0,\infty[,\ \forall z \in K^*, \\ & y(0) = y_0. \end{aligned} \tag{16}$$

Note that the derivative $y'(s)$ is absolutely continuous in $[0,\infty[$. Testing (16) by $z = w - y(s \pm h) - A^2 y'(s \pm h)$ for s in the set of Lebesgue points of y'', with $h > 0$, and letting $h \to 0^+$ we obtain

$$|y'(s)|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|Ay'(s)|^2 = \big\langle y'(s), y'(s) + A^2 y''(s)\big\rangle = 0 \quad \text{a.e.}$$

This implies that $y'(s)$ decays exponentially in the sense that there exists $a > 0$ depending only on the mapping A such that

$$|y'(s)| \leq |y'(0)|\,\mathrm{e}^{-as} \leq C|w - y_0|\,\mathrm{e}^{-as} \quad \text{for} \quad s \geq 0, \tag{17}$$

with a constant $C > 0$ independent of w and y_0. Hence, $\int_0^\infty |y'(s)|\mathrm{d}s < \infty$ and the limit $\bar{y} = \lim_{s\to\infty} y(s)$ exists. Furthermore, it follows from (17) that

$$|P_{A^{-1}K^*}(A^{-1}(w - y(s)))| \leq C|w - y_0|\mathrm{e}^{-as}, \quad s \in [0,\infty),$$

with possibly a different constant C, still independent of w and y_0. This together with the continuity of $P_{A^{-1}K^*}$ yields $P_{A^{-1}K^*}(A^{-1}(w - \bar{y})) = 0$, and we conclude that $w - \bar{y} \in K^*$.

We consider now a step function $u \in G(0,T;X)$ of the form

$$u(t) = \sum_{j=0}^{m} \hat{u}_j \chi_{\{t_j\}}(t) + \sum_{j=1}^{m} u_j \chi_{]t_{j-1},t_j[}(t) \tag{18}$$

associated with a division $0 = t_0 < \cdots < t_m = T$ of $[0,T]$ and with constant elements $\hat{u}_0,\ldots,\hat{u}_m, u_1,\ldots u_m$ of the space X, where χ_S denotes the characteristic function of the set $S \subset [0,T]$, that is, $\chi_S(t) = 1$ if $t \in S$, $\chi_S(t) = 0$ if $t \notin S$. With an input function u as in (18), problem (6) is in each interval $]t_{j-1},t_j[$ of the form

$$\begin{aligned} & u_j - \xi^\varepsilon(t) - \varepsilon A^2 \dot{\xi}^\varepsilon(t) \in K^* && \text{for a.e. } t \in]t_{j-1},t_j[, \\ & \Big\langle \dot{\xi}^\varepsilon(t), u_j - \xi^\varepsilon(t) - \varepsilon A^2 \dot{\xi}^\varepsilon(t) - z \Big\rangle \geq 0 && \text{for a.e. } t \in]t_{j-1},t_j[,\ \forall z \in K^*. \end{aligned} \tag{19}$$

For $j = 1,\ldots,m$, we associate with (19) an auxiliary system of the form (16), namely

$$\begin{aligned} & u_j - \hat{\xi}_j(s) - A^2 \hat{\xi}_j'(s) \in K^* && \text{for all } s \in]0,\infty[, \\ & \Big\langle \hat{\xi}_j'(s), u_j - \hat{\xi}_j(s) - A^2 \hat{\xi}_j'(s) - z \Big\rangle \geq 0 && \text{for all } s \in]0,\infty[, \forall z \in K^*, \\ & \hat{\xi}_j(0) = \xi_{j-1}, \end{aligned} \tag{20}$$

with initial conditions defined by the recurrent formula

$$\xi_j = \lim_{s\to\infty} \hat{\xi}_j(s), \ \xi_0 = \hat{u}_0 - x_0. \tag{21}$$

The convergence result for step functions reads as follows.

Proposition 4.1. *Let* $u\colon [0,T] \to X$ *be a step function as in* (18). *Given a symmetric positive definite operator* $A \in \mathcal{L}(X)$, *let* $\xi^\varepsilon\colon [0,T] \to X$ *denote the solutions to* (6) *for* $\varepsilon > 0$. *Then*

$$\lim_{\varepsilon\to 0+} \xi^\varepsilon(t) = \xi(t) \quad \text{for } \ t \in [0,T],$$

where $\xi\colon [0,T] \to X$ *is the left-continuous step function given by*

$$\xi(t) = \xi_0 \chi_{\{0\}}(t) + \sum_{j=1}^{m} \xi_j \chi_{]t_{j-1},t_j]}(t). \tag{22}$$

with ξ_j *as in* (21), $j = 0, 1, \ldots, m$.

Proof. For $j = 1, \ldots, m$, put $\hat{\xi}^\varepsilon_j(s) = \xi^\varepsilon(\varepsilon s + t_{j-1})$ for $s \in [0, (t_j - t_{j-1})/\varepsilon]$. Then $\hat{\xi}^\varepsilon_j$ is the solution of a problem similar to (20), more specifically,

$$\begin{aligned}
&u_j - \hat{\xi}^\varepsilon_j(s) - A^2(\hat{\xi}^\varepsilon_j)'(s) \in K^* && \text{for a. e. } s \in]0, (t_j - t_{j-1})/\varepsilon[,\\
&\Big\langle (\hat{\xi}^\varepsilon_j)'(s), u_j - \hat{\xi}^\varepsilon_j(s) - A^2(\hat{\xi}^\varepsilon_j)'(s) - z\Big\rangle \ge 0 && \text{for a. e. } s \in]0, (t_j - t_{j-1})/\varepsilon[, \ \forall z \in K^*,\\
&\hat{\xi}^\varepsilon_j(0) = \xi^\varepsilon(t_{j-1}).
\end{aligned} \tag{23}$$

Taking $z = u_j - \hat{\xi}^\varepsilon_j(s) - A^2(\hat{\xi}^\varepsilon_j)'(s)$ in (20), $z = u_j - \hat{\xi}_j(s) - A^2\hat{\xi}'_j(s)$ in (23), and summing up the resulting inequalities, we obtain

$$\Big\langle (\hat{\xi}^\varepsilon_j)'(s) - \hat{\xi}'_j(s), \hat{\xi}^\varepsilon_j(s) - \hat{\xi}_j(s)\Big\rangle \le 0 \ \text{ for a. e. } s \in]0, (t_j - t_{j-1})/\varepsilon[,$$

hence

$$|\xi^\varepsilon(t_j) - \hat{\xi}_j((t_j - t_{j-1})/\varepsilon)| \le |\xi^\varepsilon(t_{j-1}) - \xi_{j-1}|. \tag{24}$$

Furthermore, by virtue of formula (17) related to the auxiliary problem (15), we have that

$$|\xi_j - \hat{\xi}_j((t_j - t_{j-1})/\varepsilon)| \le \int_{(t_j - t_{j-1})/\varepsilon}^{\infty} |\hat{\xi}'_j(s)|\,\mathrm{d}s \le C|u_j - \xi_{j-1}|\mathrm{e}^{-(a/\varepsilon)(t_j - t_{j-1})} \tag{25}$$

with a constant C independent of ε. Note that the choice $z = 0$ in (20) yields

$$|\xi_j - u_j| \le |\xi_{j-1} - u_j| \le |\xi_{j-1} - u_{j-1}| + |u_{j-1} - u_j|, \tag{26}$$

and it follows from (25)–(26) that

$$|\xi_j - \hat{\xi}_j((t_j - t_{j-1})/\varepsilon)| \le C\big(|x_0| + \operatorname*{Var}_{[0,T]} u\big)\mathrm{e}^{-(a/\varepsilon)(t_j - t_{j-1})}.$$

The inequality above together with (24) implies that for each $k = 1, \dots, m$

$$|\xi^\varepsilon(t_k) - \xi_k| \le C\big(|x_0| + \operatorname*{Var}_{[0,T]} u\big) \sum_{j=1}^{k} \mathrm{e}^{-(a/\varepsilon)(t_j - t_{j-1})}, \tag{27}$$

and consequently $\lim_{\varepsilon\to 0+} \xi^\varepsilon(t_k) = \xi_k$. It remains to show that $\xi(t) = \xi_k$ in $]t_{k-1}, t_k[$. Note that for $t \in]t_{k-1}, t_k[$, we have

$$|\xi^\varepsilon(t) - \xi_k| = |\hat{\xi}_k^\varepsilon((t - t_{k-1})/\varepsilon) - \xi_k| \le \int_{(t-t_{k-1})/\varepsilon}^{(t_k - t_{k-1})/\varepsilon} |(\hat{\xi}_k^\varepsilon)'(s)| \mathrm{d}s + |\xi^\varepsilon(t_k) - \xi_k|.$$

By (17), the function $|(\hat{\xi}_k^\varepsilon)'(s)|$ decays exponentially at infinity. Furthermore, using (27), we can therefore pass to the limit as $\varepsilon \to 0+$ in the inequality above and conclude that $\lim_{\varepsilon\to 0+} \xi^\varepsilon(t) = \xi_k$. □

Remark 4.2. For left-continuous piecewise constant inputs the solution of the Kurzweil variational inequality (11) is also described by a step function of the form (22). The difference is that in the "Kurzweil" case, the elements ξ_j are given explicitly in terms of the dual projection P_{K^*} by the recurrent relation

$$\xi_j - \xi_{j-1} = P_{K^*}(u_j - \xi_{j-1}). \tag{28}$$

Note that this is exactly the discrete 'catching up algorithm' proposed by Moreau in [27].

We now prove an auxiliary inequality related to the limiting function described in Proposition 4.1.

Proposition 4.3. *Let $u, v \colon [0,T] \to X$ be step functions of the form* (18) *associated with sequences $\hat{u}_0, \dots, \hat{u}_m, u_1, \dots, u_m$ and $\hat{v}_0, \dots, \hat{v}_m, v_1, \dots, v_m$, respectively. Let us denote by $\xi_0, \xi_1, \dots, \xi_m$, and $\eta_0, \eta_1, \dots, \eta_m$ the output sequences corresponding to u and v, respectively, according to the recipe* (20)–(21)*. Then, for every $j = 1, \dots m$ we have*

$$|\xi_j - \eta_j|^2 \le |\xi_{j-1} - \eta_{j-1}|^2 + 2\left(|\xi_j - \xi_{j-1}| + |\eta_j - \eta_{j-1}|\right)|u_j - v_j|. \tag{29}$$

In particular, $\|\xi - \eta\|^2_{[0,T]} \le |\xi_0 - \eta_0|^2 + 2(\operatorname*{Var}_{[0,T]} \xi + \operatorname*{Var}_{[0,T]} \eta)\, \|u - v\|_{[0,T]}$.

Proof. For $j = 1, \dots, m$ and $s > 0$, taking $z = v_j - \hat{\eta}_j(s) - A^2\hat{\eta}_j'(s)$ in inequality (20) for $\hat{\xi}_j$ and $z = u_j - \hat{\xi}_j(s) - A^2\hat{\xi}_j'(s)$ in the corresponding inequality for $\hat{\eta}_j$, we obtain

$$\Big\langle \hat{\xi}_j'(s), u_j - v_j + \hat{\eta}_j(s) - \hat{\xi}_j(s) - A^2(\hat{\xi}_j'(s) - \hat{\eta}_j'(s)) \Big\rangle \ge 0,$$
$$\Big\langle \hat{\eta}_j'(s), v_j - u_j + \hat{\xi}_j(s) - \hat{\eta}_j(s) + A^2(\hat{\xi}_j'(s) - \hat{\eta}_j'(s)) \Big\rangle \ge 0.$$

Summing up the two inequalities, we obtain

$$|A(\hat{\xi}_j'(s) - \hat{\eta}_j'(s))|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|\hat{\xi}_j(s) - \hat{\eta}_j(s)|^2 \le \frac{\mathrm{d}}{\mathrm{d}s}\left\langle \hat{\xi}_j(s) - \hat{\eta}_j(s), u_j - v_j \right\rangle,$$

and the statement now easily follows from the integration of the above inequality from 0 to ∞. □

5. Main results

In this section, we analyze the convergence of the viscous solutions for processes with regulated inputs. Our strategy in passing to the limit as $\varepsilon \to 0+$ in (9) is to first approximate the input u by a sequence u_n of step functions and then, having in mind the results from Section 4, pass to the limit as $n \to \infty$.

Theorem 5.1. *Let $u \in G(0,T;X)$. Given a symmetric positive definite operator $A \in \mathcal{L}(X)$, let $\xi^\varepsilon\colon [0,T] \to X$ denote the solutions to (6) for $\varepsilon > 0$. Then there exists $\xi \in BV_L(0,T;X)$ such that*

$$\lim_{\varepsilon\to 0+} \xi^\varepsilon(t) = \xi(t) \quad \forall t \in [0,T]. \tag{30}$$

Furthermore, the mapping which with $u \in G(0,T;X)$ and $x_0 \in K^*$ associates the viscous limit $\xi \in BV_L(0,T;X)$ is strongly continuous in $G(0,T;X)$.

Proof. Let $\{u_n\}$ be a sequence of step functions such that $\|u_n - u\|_{[0,T]} \to 0$ as $n \to \infty$. For $\varepsilon > 0$, let ξ^ε be the viscous solution to (6), and for each $n \in \mathbb{N}$, let ξ_n^ε denote the solution to (13). By taking $z = u_n(t) - \xi_n^\varepsilon(t) - \varepsilon A^2\dot{\xi}_n^\varepsilon(t)$ in (6) and $z = u(t) - \xi^\varepsilon(t) - \varepsilon A^2\dot{\xi}^\varepsilon(t)$ in (13), and summing up the two inequalities we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\xi^\varepsilon(t) - \xi_n^\varepsilon(t)|^2 \le \langle \dot{\xi}^\varepsilon(t) - \dot{\xi}_n^\varepsilon(t), u(t) - u_n(t)\rangle \le \|u - u_n\|_{[0,T]}\big(|\dot{\xi}^\varepsilon(t)| + |\dot{\xi}_n^\varepsilon(t)|\big).$$

Hence for every $t \in [0,T]$

$$|\xi^\varepsilon(t) - \xi_n^\varepsilon(t)|^2 \le |u(0) - u_n(0)|^2 + 2\|u - u_n\|_{[0,T]} \int_0^T \big(|\dot{\xi}^\varepsilon(s)| + |\dot{\xi}_n^\varepsilon(s)|\big)\,\mathrm{d}s. \tag{31}$$

By Lemma 3.1, we thus have

$$\|\xi^\varepsilon - \xi_n^\varepsilon\|_{[0,T]} \le C\,\|u - u_n\|_{[0,T]}^{1/2} \tag{32}$$

with a constant $C > 0$ independent of n and ε.

On the other hand, by Proposition 4.1, for each $n \in \mathbb{N}$ there exists a left-continuous step function $\xi_n\colon [0,T] \to X$ such that

$$\lim_{\varepsilon\to 0+} \xi_n^\varepsilon(t) = \xi_n(t) \quad \forall t \in [0,T]. \tag{33}$$

Moreover, (33) together with Lemma 3.1 ensures that $\operatorname{Var}_{[0,T]} \xi_n \leq C$ for $n \in \mathbb{N}$. Thus, applying Proposition 4.3 with $u = u_n$, $v = u_m$, we get

$$\|\xi_n - \xi_m\|_{[0,T]}^2 \leq |u_n(0) - u_m(0)|^2 + 2C \|u_n - u_m\|_{[0,T]}, \quad n, m \in \mathbb{N}, \tag{34}$$

that is, $\{\xi_n\}$ is a Cauchy sequence in $G_L(0,T;X)$, thus it converges uniformly to a function $\xi \in G_L(0,T;X)$. It remains to show that ξ is the desired limit in (30).

By virtue of (32) and (34) given an arbitrary $\gamma > 0$, there exists $n_1 \in \mathbb{N}$, such that

$$\|\xi_n - \xi\|_{[0,T]} < \frac{\gamma}{3}, \quad \sup_{\varepsilon>0} \|\xi_n^\varepsilon - \xi^\varepsilon\|_{[0,T]} < \frac{\gamma}{3} \quad \text{for } n \geq n_1.$$

For each $t \in [0,T]$, by (33) we can find $\varepsilon_0 > 0$ such that

$$|\xi_{n_1}^\varepsilon(t) - \xi_{n_1}(t)| < \frac{\gamma}{3} \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

Hence, for $0 < \varepsilon < \varepsilon_0$ we have

$$|\xi^\varepsilon(t) - \xi(t)| \leq |\xi^\varepsilon(t) - \xi_{n_1}^\varepsilon(t)| + |\xi_{n_1}^\varepsilon(t) - \xi_{n_1}(t)| + |\xi_{n_1}(t) - \xi(t)| < \gamma,$$

which proves (30).

To prove the continuity of the mapping $(u, x_0) \mapsto \xi$, we consider two sequences $\{u_n\}$ in $G(0,T;X)$ and $\{x_0^n\}$ in K^* such that $\|u - u_n\|_{[0,T]} \to 0$ and $|x_0^n \to x_0| \to 0$ as $n \to \infty$. Inequality (31) still holds in a slightly modified form accounting for the different initial conditions

$$\begin{aligned} &|\xi^\varepsilon(t) - \xi_n^\varepsilon(t)|^2 \leq \\ &\leq |u(0) - u_n(0) - x_0 + x_0^n|^2 + 2 \|u - u_n\|_{[0,T]} \int_0^T \left(|\dot{\xi}^\varepsilon(s)| + |\dot{\xi}_n^\varepsilon(s)|\right) \mathrm{d}s. \end{aligned} \tag{35}$$

For every $t \in [0,T]$ we thus have by virtue of (30) and Lemma 3.1 that

$$|\xi(t) - \xi_n(t)|^2 \leq |u(0) - u_n(0) - x_0 + x_0^n|^2 + C \|u - u_n\|_{[0,T]} \tag{36}$$

with a constant $C > 0$ independent of n, and the proof is complete. □

Note that in Theorem 5.1, unlike in [17], the inputs are assumed to be just regulated with no additional requirement of left-continuity. It is worth recalling that the vanishing viscosity analysis carried out in [17] deals with the case when A is the identity operator on X. As shown in [17, Theorem 2.4], the viscous limit coincides with the solution of the Kurzweil variational inequality. Notably, it will be shown in Section 6 that this is not the typical outcome when more general viscosity operators A are taken into account. However, if the input u is continuous, as mentioned in Section 2, the viscous approximations converge uniformly and the limit is the solution of the Riemann-Stieltjes variational inequality (10). For the sake of completeness we prove this result here.

Corollary 5.2. *Let $u \in C([0,T];X)$ be given and let ξ^ε be the solutions of* (6) *for $\varepsilon > 0$ with an arbitrary viscosity operator A. Let $\xi \in C([0,T];X) \cap BV(0,T;X)$ be the solution of* (10). *Then*

$$\lim_{\varepsilon \to 0+} \|\xi^\varepsilon - \xi\|_{[0,T]} = 0. \tag{37}$$

Proof. We choose a sequence of functions $u_n \in W^{1,2}(0,T;X)$ with the property that $\|u_n - u\|_{[0,T]} \to 0$ as $n \to \infty$, and denote by ξ_n^ε the solution to (13). We further define $\xi_n \in W^{1,2}(0,T;X)$ as the solution of the variational inequality

$$\begin{cases} u_n(t) - \xi_n(t) \in K^* & \forall t \in [0,T], \\ \left\langle \dot\xi_n(t), u_n(t) - \xi_n(t) - z \right\rangle \geq 0 & \text{for a.e. } t \in]0,T[,\ \forall z \in K^*, \\ \xi_n(0) = u_n(0) - x_0. \end{cases} \tag{38}$$

Putting in (38) $z = u_n(t \pm h) - \xi_n(t \pm h)$ for $h > 0$, dividing the inequality by h and letting $h \to 0$ we obtain

$$\left\langle \dot\xi_n(t), \dot u_n(t) - \dot\xi_n(t) \right\rangle = 0 \quad \text{for a.e. } t \in]0,T[.$$

Hence,
$$|\dot\xi_n(t)|^2 \leq |\dot u_n(t)|^2 \quad \text{for a.e. } t \in]0,T[. \tag{39}$$

Choosing $z = u_n(t) - \xi_n^\varepsilon(t) - \varepsilon A^2 \dot\xi_n^\varepsilon(t)$ in (38) and $z = u_n(t) - \xi_n(t)$ in (13), and summing up the two resulting inequalities we obtain

$$\left\langle \dot\xi_n^\varepsilon(t) - \dot\xi_n(t), \varepsilon A^2 \dot\xi_n^\varepsilon(t) + \xi_n^\varepsilon(t) - \xi_n(t) \right\rangle \leq 0 \quad \text{a.e.}$$

Therefore

$$\varepsilon |A\dot\xi_n^\varepsilon(t)|^2 + \frac{\mathrm{d}}{\mathrm{d}t} |\xi_n^\varepsilon(t) - \xi_n(t)|^2 \leq \varepsilon |A\dot\xi_n(t)|^2 \leq C\varepsilon |\dot u_n(t)|^2 \quad \text{a.e.},$$

and we conclude that
$$\|\xi_n^\varepsilon - \xi_n\|_{[0,T]} \leq C_n \varepsilon \tag{40}$$
with a constant C_n depending possibly on n and independent of ε. The estimate (32) and its counterpart $\|\xi - \xi_n\|_{[0,T]} \leq C \, \|u - u_n\|_{[0,T]}^{1/2}$ with constants C independent of ε and n are obtained as in the proof of Theorem 5.1. In conclusion, given an arbitrary $\gamma > 0$, there exists $n_1 \in \mathbb{N}$, such that $\|u - u_n\|_{[0,T]} \leq \gamma^2$ for $n \geq n_1$, and consequently

$$\|\xi^\varepsilon - \xi\|_{[0,T]} \leq 2C \, \|u - u_{n_1}\|_{[0,T]}^{1/2} + \left\|\xi_{n_1}^\varepsilon - \xi_{n_1}\right\|_{[0,T]} \leq 2C\gamma + C_{n_1}\varepsilon,$$

which becomes small provided $C_{n_1}\varepsilon < \gamma$. $\square$

6. Example

In this section, we show that unlike what has been observed for continuous inputs, the viscous limit cannot be represented in typical cases by the Kurzweil variational inequality (11) if $A \neq I$.

Consider $X = \mathbb{R}^2$, $u(t) := \chi_{]0,T]}(t)\bar{u}$ for $t \in [0,T]$. Given $\lambda > 0$, $\lambda \neq 1$, let

$$K = \left\{(x,y) \in \mathbb{R}^2 : \lambda^2 x^2 + y^2 \leq 1\right\},$$

that is, K is an ellipse centered at the origin, with axes $1/\lambda$ and 1. Note that the polar set K^* is again an ellipse, more precisely

$$K^* = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{\lambda^2} + y^2 \leq 1\right\}.$$

For $x_0 = (0,0)$, and $\bar{u} = (p,q)$ with $p^2 + \lambda^2 q^2 > \lambda^2$, it follows from (28) that the Kurzweil solution $\xi_{\mathcal{K}}\colon [0,T] \to \mathbb{R}^2$ of the problem (11) is given by $\xi_{\mathcal{K}}(t) := \chi_{]0,T]}(t)\bar{\xi}$, $t \in [0,T]$, where $\bar{\xi} = P_{K^*}(\bar{u})$. In other words, $\bar{z} = \bar{u} - \bar{\xi}$ is the projection of $\bar{u}$ onto the ellipse K^*, therefore $\bar{z} = (\bar{x}, \bar{y})$ satisfies

$$\frac{\lambda^2 p}{\bar{x}} - \lambda^2 = \frac{q}{\bar{y}} - 1, \quad \frac{\bar{x}^2}{\lambda^2} + \bar{y}^2 = 1.$$

Let us consider the viscosity matrix to be

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}.$$

With the corresponding viscous problem (9) we can associate the ODE

$$A\hat{\xi}'(s) = P_{A^{-1}K^*}(A^{-1}(\bar{u} - \hat{\xi}(s))), \quad s \in]0,\infty[\,, \quad \hat{\xi}(0) = (0,0). \tag{41}$$

By Proposition 4.1, the viscous limit solution is of the form $\xi_{\mathcal{V}}(t) := \chi_{]0,T]}(t)\xi_\infty$ with $\hat{\xi}_\infty = \lim_{s\to\infty} \hat{\xi}(s)$. Alternatively, taking $\hat{z} = A^{-1}(\bar{u} - \hat{\xi})$, we consider

$$A^2 \hat{z}'(s) + P_{A^{-1}K^*}(\hat{z}(s)) = 0, \quad s \in]0,\infty[\,, \quad \hat{z}(0) = A^{-1}\bar{u}. \tag{42}$$

Clearly, $A^{-1}K^*$ corresponds to the unit ball $B_1 \subset \mathbb{R}^2$, and consequently $P_{A^{-1}K^*}$ is the complement of the orthogonal projection onto the unit sphere, $Q_{A^{-1}K^*}$. Denoting $\hat{z} = (\hat{x}, \hat{y})$, from (42) we obtain the following system of equations

$$\lambda^2 \hat{x}'(s) + \hat{x}(s)\left(1 - \frac{1}{\|\hat{z}(s)\|}\right) = 0, \qquad \hat{x}(0) = \frac{p}{\lambda},$$

$$\hat{y}'(s) + \hat{y}(s)\left(1 - \frac{1}{\|\hat{z}(s)\|}\right) = 0, \qquad \hat{y}(0) = q,$$

hence,
$$\frac{\lambda^2 \hat{x}'(s)}{\hat{x}(s)} = \frac{\hat{y}'(s)}{\hat{y}(s)}, \ s \in]0,\infty[\,, \quad \hat{x}(0) = \frac{p}{\lambda}, \ \hat{y}(0) = q.$$

Assuming, without loss of generality, that p and q are positive we have

$$\left(\frac{\lambda \hat{x}(s)}{p}\right)^{\lambda^2} = \frac{\hat{y}(s)}{q}, \ \ s \in]0, \infty[\,. \tag{43}$$

Let $z_\infty = (x_\infty, y_\infty)$ denote the limit of $\hat{z}(s)$ when $s \to \infty$. We know that $Az_\infty \in K^*$, and (43) implies

$$\left(\frac{\lambda x_\infty}{p}\right)^{\lambda^2} = \frac{y_\infty}{q}. \tag{44}$$

We claim that Az_∞ and $\bar{z}$ do not coincide. Recall that both p and q are assumed to be positive. The point $\bar{z}$ is defined as the intersection of the hyperbola

$$\frac{q}{y} - 1 = \lambda^2 \left(\frac{p}{x} - 1\right) \tag{45}$$

with ∂K^*, while Az_∞ is the intersection point of the “parabola”

$$\frac{q}{y} = \left(\frac{p}{x}\right)^{\lambda^2} \tag{46}$$

with ∂K^*. We call the curves given by (45) and (46) jump trajectories.

Put $a = \lambda^2$ and $w = p/x$. For $a > 1$, the function $f(w) = w^a$ is strictly convex, thus

$$w^a - 1 = f(w) - f(1) > f'(1)(w-1) = a(w-1) \ \text{ for } \ w > 1, \tag{47}$$

showing that the trajectories (45) and (46) have no intersection for $0 < x < p$. This is illustrated in Figure 6.1, where the bold curve connecting the point $\bar{u}$ with the point $\bar{z}$ on the boundary ∂K^* of K^* represents the Kurzweil jump trajectory and the dashed curve from $\bar{u}$ to Az_∞ represents the A-viscous jump trajectory. For $a < 1$ we get the same conclusion using the fact that the function $f(w) = w^a$ in this case is strictly concave. Altogether, Az_∞ and $\bar{z}$ do not coincide, which implies that the Kurzweil solution $\xi_{\mathcal{K}}$ differs from the A-viscous limit $\xi_{\mathcal{V}}$.

This example admits an energetic interpretation. The Kurzweil formula $\bar{\xi} = P_{K^*}(\bar{u})$ can be equivalently written as $\left\langle \bar{\xi}, \bar{u} - \bar{\xi} \right\rangle = M_K(\bar{\xi})$, that is,

$$\frac{1}{2}|\bar{u}|^2 = \frac{1}{2}|\bar{u} - \bar{\xi}|^2 + \frac{1}{2}|\bar{\xi}|^2 + M_K(\bar{\xi}). \tag{48}$$

On the other hand, the viscous limit $\xi_\infty = \lim_{s\to\infty} \hat{\xi}(s)$ is based on the solution $\hat{\xi}$ of the auxiliary problem (41) that can be equivalently reformulated as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|\bar{u} - \hat{\xi}(s)|^2 + |A\hat{\xi}'(s)|^2 + M_K(\hat{\xi}'(s)) = 0. \tag{49}$$

Integrating from 0 to ∞ we thus obtain

$$\frac{1}{2}|\bar{u}|^2 = \frac{1}{2}|\bar{u} - \xi_\infty|^2 + \int_0^\infty |A\hat{\xi}'(s)|^2 \mathrm{d}s + \int_0^\infty M_K(\hat{\xi}'(s))\mathrm{d}s. \tag{50}$$

The left-hand side of (48) and (50) can be interpreted as the power supply to the system, the first term on the right-hand side is the stored energy, the second term is the limit viscous dissipation, and the third term is the rate-independent dissipation.

Both $\bar{z} = \bar{u} - \bar{\xi}$ and $Az_\infty = \bar{u} - \xi_\infty$ lie on ∂K^*. According to (47), we have $|\bar{z}| < |Az_\infty|$, see Fig. 6.1, so that in this particular example, the stored energy corresponding to the anisotropic viscosity matrix is higher (and therefore the dissipation is lower) in the vanishing viscosity limit than in the isotropic (Kurzweil) case.

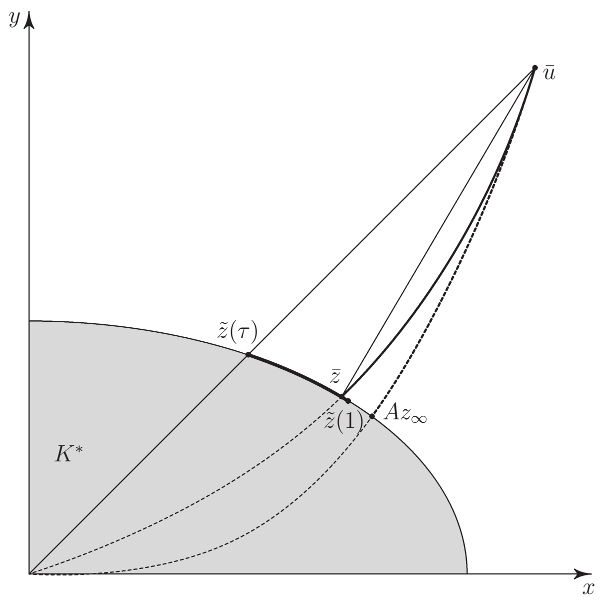

Figure 6.1: Different jump trajectories for $\lambda = \sqrt{3}$.

Another notion of solution for discontinuous processes was suggested by V. Recupero in [28]. Relying on reparametrization techniques, the method consists in choosing a Lipschitz continuous function $\tilde{u}\colon [0,T] \to X$ and a reparametrization $\ell_u\colon [0,T] \to [0,T]$ in such a way that the input u is described by the identity $u = \tilde{u} \circ \ell_u$. The reparametrized solution of (1) in the sense of [28] is then given by $\xi_{\mathcal{R}} = \tilde{\xi} \circ \ell_u$, where $\tilde{\xi}\colon [0,T] \to X$ corresponds to the absolutely continuous solution of the problem

$$\tilde{u}(t) - \tilde{\xi}(t) \in \partial M_K(\dot{\tilde{\xi}}(t)),\ \tilde{\xi}(0) = \tilde{u}(0) - x_0.$$

Assuming $T = 1$, for the particular choice of input u in this example we associate functions $\tilde{u}\colon [0,1] \to \mathbb{R}^2$ and $\ell_u\colon [0,1] \to [0,1]$ given by

$$\tilde{u}(t) = (pt, qt)\,, \quad \ell_u(t) = \chi_{]0,1]}(t).$$

Denote $\tilde{z} = \tilde{u} - \tilde{\xi}$ and put $\tau := \max\{s \in]0,1] : \tilde{u}(s) \in K^*\}$. Clearly, $\tilde{z}(t) = \tilde{u}(t)$ if $t \in [0,\tau]$. Further, for $t \in]\tau, 1]$, $\dot{\tilde{z}}(t)$ is the tangential component of $\dot{\tilde{u}}(t)$ at the point $\tilde{z}(t) = (\tilde{x}(t), \tilde{y}(t))$, that is,

$$\dot{\tilde{z}}(t) = \dot{\tilde{u}}(t) - \frac{\vec{n}(t)}{\|\vec{n}(t)\|^2}\langle \vec{n}(t), \dot{\tilde{u}}(t)\rangle, \quad \text{where } \vec{n}(t) = \left(\frac{\tilde{x}(t)}{\lambda^2}, \tilde{y}(t)\right).$$

To compute $\tilde{z}$ we can consider $\tilde{x}(t) = \lambda \cos\rho(t)$ and $\tilde{y}(t) = \sin\rho(t)$, with $\rho(t)$ satisfying the initial value problem

$$\dot{\rho}(t) = \frac{q\cos\rho(t) - p\lambda\sin\rho(t)}{\lambda^2\sin^2\rho(t) + \cos^2\rho(t)}, \quad \rho(\tau) = \tan^{-1}(\lambda q/p). \tag{51}$$

Therefore, the reparametrized solution to (1) is given by

$$\xi_{\mathcal{R}}(t) = \tilde{\xi}(1)\chi_{]0,1]}(t) = \begin{cases} 0 & \text{for} \quad t = 0\,, \\ (p - \lambda\cos\rho(1), q - \sin\rho(1)) & \text{for} \quad t \in]0,1]\,, \end{cases}$$

and the corresponding trajectory $\{\tilde{z}(t); t \in [\tau, 1]\}$ along ∂K^* is represented in Figure 6.1 by the bold curve connecting the points $\tilde{z}(\tau)$ and $\tilde{z}(1)$.

According to [18], in the finite dimensional case, a necessary and sufficient condition for the Kurzweil solution and the reparametrized solution of (1) to coincide is that K^* is a non-obtuse polyhedron (see also [19]). This is obviously not the case of the set K^* chosen in this example. We therefore have $\xi_{\mathcal{K}} \neq \xi_{\mathcal{R}}$. It remains to compare the reparametrized solution with the viscous solution. To this end, for a parameter $\alpha > 0$, we consider the solutions $\xi_{\mathcal{R}}^\alpha$ and $\xi_{\mathcal{V}}^\alpha$ of the problem (1) with the input function $u_\alpha(t) := \chi_{]0,1]}(t)\alpha\bar{u}$, and we analyze their asymptotic behavior when $\alpha \to \infty$.

From the calculation above we know that $\xi_{\mathcal{V}}^\alpha(t) = \alpha\bar{u} - \tilde{z}_\alpha(1)$ for $t \in]0,1]$, where $\tilde{z}_\alpha$ is related to the solution ρ_α of a problem like (51). Therefore, in the limit when $\alpha \to \infty$, we get

$$\frac{q\cos\rho_\infty(1) - p\lambda\sin\rho_\infty(1)}{1 + (\lambda^2 - 1)\sin^2\rho_\infty(1)} = 0,$$

which after appropriate reparametrization yields

$$\frac{q\tilde{x}_\infty}{\lambda} - p\lambda\tilde{y}_\infty = 0, \tag{52}$$

with $(\tilde{x}_\infty, \tilde{y}_\infty)$ denoting the corresponding limiting point on ∂K^*. We see that this limit coincides with the limit as $\alpha \to \infty$ on the Kurzweil jump trajectory (45) which is the expected natural value. On the other hand, the A-viscous limit solution is given by $\xi_{\mathcal{V}}^\alpha(t) = \alpha\bar{u} - Az_{\infty,\alpha}$ for $t \in]0,1]$, where $z_{\infty,\alpha} = (x_{\infty,\alpha}, y_{\infty,\alpha})$ satisfies the corresponding identity in (44), or equivalently

$$y_{\infty,\alpha} = \left(\frac{\lambda x_{\infty,\alpha}}{p}\right)^{\lambda^2} \frac{q}{\alpha^{\lambda^2-1}}.$$

Thus, for $\lambda > 1$ we have $\lim_{\alpha\to\infty} y_{\infty,\alpha} = 0$. This together with (52) shows that $\tilde{z}_\alpha(1)$ and $z_{\infty,\alpha}$ do not coincide in the limit when $\alpha \to \infty$, hence the solutions $\xi^\alpha_{\mathcal{R}}$ and $\xi^\alpha_{\mathcal{V}}$ must differ also for some particular choice of the parameter α.

The fact that the limit of the A-viscous trajectory as $\alpha \to \infty$ lies on the intersection of the x-axis with ∂K^* independently of the input direction is indeed very strange from the physical point of view and can be hardly justified within the concept of rate independence. It is therefore appropriate to handle the vanishing viscosity technique with maximal care.

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