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Convex Extension of Lower Semicontinuous Functions Defined on Normal Hausdorff Space

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We prove that any problem of minimization of proper lower semicontinuous function defined on a normal Hausdorff space is canonically equivalent to a problem of minimization of a proper weak-star lower semicontinuous convex function defined on a weak-star convex compact subset of some dual Banach space. We establish the existence of a bijective operator between the two classes of functions which preserves problems of minimization.

Keywords: Isomorphism, minimization problem, convex functions, normal Hausdorff space, the Stone-Čech compactification.

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1. Introduction.

Let (X, τ) be a completely regular and Hausdorff space. By $(C_b(X), \|.\|)$ we denote the Banach space of all real-valued, bounded and continuous functions, equipped with the sup-norm. The *continuous Dirac map* is defined by

$$\begin{array}{rcl}\Delta : (X, \tau) & \longrightarrow & (\Delta(X), w^*) \subset (B_{(C_b(X))^*}, w^*) \\ x & \mapsto & \Delta(x)\end{array}$$

where, $(B_{(C_b(X))^*}, w^*)$ is the closed unit ball of the dual space $(C_b(X))^*$ equipped with the weak-star topology and $\Delta(x)\colon C_b(X) \longrightarrow \mathbb{R}$ is the linear continuous map defined by $\Delta(x)(\varphi) = \varphi(x)$ for all $x \in X$ and all $\varphi \in C_b(X)$. It is well know that Δ is a homeomorphism from (X, τ) onto $(\Delta(X), w^*)$, it is also well know that the set $\overline{\Delta(X)}^{w^*}$ coincides (up to homeomorphism) with the Stone-Čech compactification βX. The Stone-Čech compactification βX, has the property that every continuous function φ from X into a compact Hausdorff space K, has a unique extension to a continuous function $\beta\varphi$ from βX into K. For more informations about the Stone-Čech compactification, we refer to [9].

We are interested in this paper in a canonical extension of a bounded from below lower semicontinuous function f from X into $\mathbb{R}$ to a bounded from below extended convex weak-star lower semicontinuous function $\mathcal{T}(f)$ defined on the convex weak-star compact set $\overline{\text{co}}^{w^*}(\Delta(X))$ (the weak-star closed convex hull of

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$\Delta(X)$, which can be seen as the convexification of the Stone-Čech compact βX). To prove the existence of a canonical extension of any bounded from below lower semicontinuous function f from X into $\mathbb{R}$, we need to assume that X is a normal Hausdorff space. Recall that a topological space X is a normal space if, given any disjoint closed sets E and F, there are neighbourhoods U of E and V of F that are also disjoint. The tools that will be used for this purpose, are based on a non convex analogue to Fenchel duality introduced in [1].

If A is a subset of $C_b(X)$, we denote by σ_A the *support function*, defined on the dual space by

$$\forall Q \in (C_b(X))^*; \ \ \sigma_A(Q) := \sup_{\varphi \in A} \langle Q, \varphi \rangle.$$

Definition 1.1. A nonempty subset A of $C_b(X)$ is said to be a Δ-*set* (or an X-*convex subset* of $C_b(X)$), if and only if, for all $x \in X$ there exist a real number $\lambda_x \in \mathbb{R}$, such that

$$A = \cap_{x \in X} \{\varphi \in C_b(X) / \langle \Delta(x), \varphi \rangle \leq \lambda_x\} := \cap_{x \in X} \{\varphi \in C_b(X) / \varphi(x) \leq \lambda_x\}. \quad \square$$

A Δ-set of $C_b(X)$ is necessarily a closed convex subset of $C_b(X)$. We denote by $SCI(X)$ the set of all real-valued functions $f \colon X \longrightarrow \mathbb{R}$, bounded from below and lower semicontinuous and by $\Sigma(\overline{\text{co}}^{w^*}(\Delta(X))$ we denote the set of all convex weak-star lower semicontinuous functions that are the restrictions to $\overline{\text{co}}^{w^*}(\Delta(X))$ of the functions σ_A, where A is a Δ-set of $C_b(X)$:

$$\Sigma(\overline{\text{co}}^{w^*}(\Delta(X))) := \{(\sigma_A)_{|\overline{\text{co}}^{w^*}(\Delta(X))} / A \text{ is a } \Delta\text{-set of } C_b(X)\}.$$

It is easy to see that $SCI(X)$ is a convex cone. The aim of this paper is to prove the following result.

Theorem 1.2. *Let X be a normal Hausdorff space.*
Then the set $\Sigma(\overline{\text{co}}^{w^*}(\Delta(X)))$ is a convex cone and there exists a convex cone isomorphism $\mathcal{T} \colon SCI(X) \longrightarrow \Sigma(\overline{\text{co}}^{w^*}(\Delta(X)))$, that is, $\mathcal{T}$ is bijective and for all $f, g \in SCI(X)$ and all $\alpha, \beta \in \mathbb{R}^+$, we have

$$\mathcal{T}(\alpha f + \beta g) = \alpha \mathcal{T}(f) + \beta \mathcal{T}(g).$$

This isomorphism satisfies also the following properties.

(1) *For all $f \in SCI(X)$, $\mathcal{T}(f) \circ \Delta = f$, this means that $\mathcal{T}(f)$ is a convex weak-star lower semicontinuous extension of f to $\overline{\text{co}}^{w^*}(\Delta(X))$, up to the identification between X and $\Delta(X)$.*

(2) *The convex cone isomorphime $\mathcal{T}$ is isotone i.e. for all $f, g \in SCI(X)$, we have that*
$$f \leq g \Longleftrightarrow \mathcal{T}(f) \leq \mathcal{T}(g).$$

(3) *For all bounded continuous function $\varphi \in C_b(X)$ and all $Q \in \overline{\text{co}}^{w^*}(\Delta(X))$, we have $\mathcal{T}(\varphi)(Q) = \langle Q, \varphi \rangle =: \hat{\varphi}(Q)$. This means that for all $\varphi \in C_b(X)$, the function $\mathcal{T}(\varphi)$ is affine and weak-star continuous on $\overline{\text{co}}^{w^*}(\Delta(X))$.*

(4) *For all $f \in SCI(X)$, we have $\inf_X f = \min_{\overline{\Delta(X)}^{w^*}} \mathcal{T}(f) = \min_{\overline{\text{co}}^{w^*}(\Delta(X))} \mathcal{T}(f)$.*

In the last part of the above theorem, the weak-star convex lower semicontinuous function $\mathcal{T}(f)$ always attains its minimum on $\overline{\Delta(X)}^{w^*}$ since it is a compact set, this is not the case in general for f. Note that if X is a compact Hausdorff space then $\overline{\Delta(X)}^{w^*} = \Delta(X)$ (see Corollary 2.8). From parts (1) and (4) of the above theorem, we get that a point $x \in X$ is a minimum for a lower semicontinuous function f iff $\Delta(x)$ is a minimum for the convex weak-star lower semicontinuous function $\mathcal{T}(f)$. More generally, if $Y \subset C_b(X)$ is a class of perturbations, then from the fact that $\mathcal{T}$ is a cone isomorphism we have that $\mathcal{T}(f+\varphi) = \mathcal{T}(f) + \mathcal{T}(\varphi)$. Using part (3), we get that $\mathcal{T}(f+\varphi) = \mathcal{T}(f) + \hat{\varphi}$. Thus, from parts (1) and (4), we have that for each $\varphi \in Y$, $f+\varphi$ has a minimum at some point $x \in X$ iff $\mathcal{T}(f) + \hat{\varphi}$ has a minimum at $\Delta(x)$. This shows that a non linear variational principle for lower semicontinuous functions f defined on normal Hausdorff space is equivalent to a linear variational principle for convex weak-star lower semicontinuous functions $\mathcal{T}(f)$ defined on a convex weak-star compact subset of some dual Banach space. The just mentioned remark can be seen as a linearization of the Deville-Godefroy-Zizler variational principle [2], [3], [4].

This main result will be proved in Section 4 when preliminary results are established in the next sections. These preliminary results are of interest in themselves. In the last section (Section 5) we investigate the case of Lipschitz-Free Spaces.

2. Duality and linearization results

Let X be a topological space and $C(X)$ the space of all real-valued continuous functions on X. Let Y be a non empty subset of $C(X)$. Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function such that $dom(f) := \{x \in X / f(x) < +\infty\} \neq \emptyset$. We define

$$A_Y(f) := \{\varphi \in Y / \varphi \leq f\}.$$

We introduced in [1] a non convex analogue to Fenchel duality, where relations between well-posedness and differentiability was established. We recall that the conjugate of f depending on the class of functions Y is defined as follows:

$$\text{for all } \varphi \in Y, \qquad f^{\times}(\varphi) := \sup_{x \in X}\{\varphi(x) - f(x)\}.$$

Note that $dom(f^{\times}) \neq \emptyset$ if and only if, there exists a real number $c \in \mathbb{R}$ such that $A_Y(f + c) \neq \emptyset$, this condition is satisfed in particular if f is bounded from below and Y contains a bounded continuous function. The second conjugate of f is defined on X as follows: for all $x \in X$

$$f^{\times\times}(x) := \sup_{\varphi \in Y}\{\varphi(x) - f^{\times}(\varphi)\}.$$

Note that we always have that $f^{\times\times} \leq f$.

2.1. Conjugacy of lower semicontinuous function

If the class Y is a vector subspace of $C(X)$, then $f^{\times}$ is a convex function on Y as a supremum of affine maps on Y. If moreover, Y is a vector subspace of $C_b(X)$ (the space of all real-valued bounded and continuous functions), then it is easy to see that $f^{\times}$ is a convex and 1-Lipschitz map for the norm $\|.\|_{\infty}$, for every bounded from below function f. But in general $f^{\times\times}$ is not convex on X even if X is a vector space. Actually, we get in Theorem 2.3 that under general conditions on the pair (X, Y), we have that f is lower semicontinuous function if and only if, $f^{\times\times} = f$. This result was initially obtained in [1, Theorem 1], when X is a metric space and Y is a subspace of $C_b(X)$ containing a bump function (a bounded function on X with a nonempty support). In fact, this result is true in a more general setting with essentially the same proof. Since we need to use this result in a general topological space, we give its proof in its general setting.

Definition 2.1. Let X be a topological space and Y be a nonempty subset of $C(X)$. We say that the pair (X, Y) satisfies the *property* (H) if and only if, for each $x \in X$ and each open neighborhood U of x, there exists $\sigma \colon X \longrightarrow [0, 1]$ such that $\sigma \in Y$, $\sigma(x) = 1$ and $\sigma(y) = 0$ for all $y \in X \setminus U$. □

Example 2.2. We have the following examples.

(1) If X is a normal Hausdorff space, then $(X, C_b(X))$ satisfies (H) tanks to the Urysohn's lemma.

(2) If (X, d) is a metric space, then it is easy to see that $(X, Lip_b(X))$ satisfies (H), where $Lip_b(X)$ denotes the space of all real-valued bounded and Lipschitz map on X.

(3) If X is a Banach space having a k-times ($k \in \mathbb{N} \cup \{+\infty\}$) continuously differentiable and uniformly bounded bump function (see [3] for examples of Banach spaces having this property), then of course $(X, C^k_b(X))$ satisfies (H), where $C^k_b(X)$ denotes the space of all k-continuously differentiable and uniformly bounded functions on X.

Theorem 2.3. *Let X be a topological space and Y a convex cone included in $C(X)$. Suppose that the pair (X, Y) satisfies hypothesis (H). Let $f \colon X \to \mathbb{R} \cup \{+\infty\}$ be a function with $dom(f) \neq \emptyset$ and $dom(f^{\times}) \neq \emptyset$. Then, f is lower semi-contiuous if and only if $f^{\times\times} = f$.*

Proof. The "only if" part follows from the definition of $f^{\times\times}$ and the fact that, the supremum of continuous functions is a lower semicontinuous function. Let us prove the "if" part. Since $dom(f^{\times}) \neq \emptyset$, there exists $\varphi_0 \in Y$ and $c \in \mathbb{R}$ such that $\varphi_0 \leq f + c$. Noting that $(f + c)^{\times\times} = f^{\times\times} + c$, we may assume without loss of generality that $c = 0$. Set $g := f - \varphi_0 \geq 0$ and let us proof that $g^{\times\times} = g$. Let $x \in X$ and take any real number a such that $a < g(x)$. We prove that $a \leq g^{\times\times}(x)$. Indeed, since g is lower semicontinuous, there exists an open neighborhood U of x such that $a < g(y)$ for all $y \in U$. From the hypothesis (H), there exists a continuous function $\sigma \colon X \longrightarrow [0, 1]$ such that $\sigma \in Y$, $\sigma(x) = 1$ and $\sigma(y) = 0$ for all $y \in X \setminus U$. We define $\varphi_x(y) := (g(x) - \inf_X g)\sigma(y)$ for all $y \in X$. Clearly, $\varphi_x \in Y$ for all $x \in X$, since Y is a cone and $\sigma \in Y$. By separately examining the case where $y \in U$ and $y \in X \setminus U$, we can easily verify that

$$\forall y \in X, \quad \varphi_x(y) - g(y) < \varphi_x(x) - a.$$

Taking the supremum over $y \in X$, we obtain that $g^{\times}(\varphi_x) \leq \varphi_x(x) - a$. Thus,we obtain $a \leq \varphi_x(x) - g^{\times}(\varphi_x) \leq g^{\times\times}(x)$. This proves that $a \leq g^{\times\times}(x)$ for all $a < g(x)$. Hence $g(x) \leq g^{\times\times}(x)$ for all $x \in X$. On the other hand, it follows from the definition of the second conjugacy that $g^{\times\times}(x) \leq g(x)$ for all $x \in X$. Thus, $g^{\times\times}(x) = g(x)$ for all $x \in X$. Now, replacing g by $f - \varphi_0$ and using the fact that Y is a convex cone we get that $g^{\times\times} \leq f^{\times\times} - \varphi_0$. Since it has already been proved that $g = g^{\times\times}$, then

$$f = g + \varphi_0 = g^{\times\times} + \varphi_0 \leq f^{\times\times},$$

and since the inequality $f^{\times\times} \leq f$ is always true, the conclusion follows. □

Corollary 2.4. *Let X be a topological space and Y be a convex cone included in $C(X)$ and containing the constants. Suppose that the pair (X, Y) satisfies (H). Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function with $dom(f) \neq \emptyset$ and such that $A_Y(f) \neq \emptyset$. Then, f is lower semicontinuous if and only if, $f(x) = \sup_{\varphi \in A_Y(f)} \varphi(x)$ for all $x \in X$. In other words, f is the supremum of functions from Y which remains smaller than f .*

Proof. Suppose that f is lower semicontinuous. We have $\sup_{\varphi \in A_Y(f)} \varphi(x) \leq f(x)$ for all $x \in X$ on one hand. On the other hand we know from Theorem 2.3 that

$$f(x) = f^{\times\times}(x) = \sup_{\psi \in Y}\{\psi(x) - f^{\times}(\psi)\} \leq \sup_{\varphi \in A_Y(f)} \varphi(x),$$

since $\psi(.) - f^{\times}(\psi) \leq f$ and $\psi(.) - f^{\times}(\psi) \in Y$ for each $\psi \in Y$. Now, if $f(x) = \sup_{\varphi \in A_Y(f)} \varphi(x)$ for all $x \in X$, then f is lower semicontinuous as supremum of continuous functions. □

Example 2.5. From Corollary 2.4 and Example 2.2, we get that

(1) if X is a normal Hausdorff space, then each lower semicontinuous function f with a nonempty domain and bounded from below by a continuous function, is the supremum of the continuous functions which remains smaller than f .

(2) if X is a metric space, then each lower semicontinuous function f with a nonempty domain and bounded from below by a Lipschitz continuous function, is the supremum of the Lipschitz continuous functions which remains smaller than f .

(3) if X is a Banach space having a k-times ($k \in \mathbb{N} \cup \{+\infty\}$) continuously differentiable bump function, then each lower semicontinuous function f with a nonempty domain and bounded from below by a k-times continuously differentiable function, is the supremum of the k-times continuously differentiable function which remains smaller than f .

2.2. A convex extension of lower semicontinuous functions

Let X be a normal Hausdorff space and $(Y, \|.\|)$ is a Banach space included in $C_b(X)$ such that $\|.\| \geq \|.\|_\infty$ and (X, Y) satisfies the hypothesis (H). Clearly, this conditions implies the following properties:

(1) Y separates the points of X

(2) for each $x \in X$, there exists $\varphi \in Y$ such that $\varphi(x) = 1$.

For each $x \in X$, we denote by δ_x the Dirac evaluation defined by $\delta_x(\varphi) = \varphi(x)$ for all $\varphi \in Y$. The continuity of the linear map δ_x is guaranteed by the condition $\|.\| \geq \|.\|_\infty$. Clearly, $\delta_x \in B_{Y^*}$ (the unit ball of the topological dual space of Y) for all $x \in X$. We define $\Delta_Y : (X, \tau) \longrightarrow (Y^*, w^*)$ by $\Delta_Y(x) = \delta_x$ for all $x \in X$. We need the following proposition.

Proposition 2.6. *Let (X, τ) be a normal Hausdorff space and $(Y, \|.\|)$ be a Banach space included in $C_b(X)$ such that $\|.\| \geq \|.\|_\infty$ and (X, Y) satisfies the hypothesis (H). Then, the map Δ_Y is an homeomorphism from (X, τ) onto $(\Delta_Y(X), w^*)$.*

Proof. Let $x, y \in X$ such that $x \neq y$. Since Y separates the points of X, there exists $\sigma \in Y$ such that $\sigma(x) \neq \sigma(y)$ and so Δ_Y is one-to-one. Clearly, Δ_Y is τ-w^*-continuous, since $Y \subset C_b(X)$. We prove that Δ is open: Let $a \in X$ and U an open set such that $a \in U$. We prove that there exists an open set V of $\Delta_Y(X)$ such that $\Delta_Y(a) \in V$ and $V \subset \Delta_Y(U)$. Indeed, by the hypothesis (H), there exists $\sigma : X \longrightarrow [0, 1]$ such that $\sigma \in Y$, $\sigma(a) = 1$ and $\sigma(y) = 0$ for all $y \in X \setminus U$. Set $W := \{y^* \in Y^*/y^*(\sigma) := \hat{\sigma}(y^*) > 0\}$. We have $W = \hat{\sigma}^{-1}(]0, +\infty[)$ and so it is a weak-star open subset of Y^*, moreover $\Delta_Y(a) \in W$. Set $V := W \cap \Delta_Y(X)$. $\square$

The set $\overline{\Delta_Y(X)}^{w^*} \subset B_{Y^*}$ is a weak-star compact subset of Y^* by the Banach-Alaoglu theorem. Note that when we take $Y = C_b(X)$, then the set $\overline{\Delta_Y(X)}^{w^*}$ coincides (up to a homeomorphism) with the Stone-Čech compactification βX.

For all $\varphi \in Y$ and all $Q \in Y^*$, we will use, according to the situations, the following equivalent notations

$$\langle Q, \varphi \rangle := Q(\varphi) =: \hat{\varphi}(Q).$$

Now, given a bounded from below function f defined on X, we denote by $\mathcal{F}(f)$ the Fenchel transform of the conjugacy $f^\times$ defined on the dual space Y^* by :

$$\mathcal{F}(f)(Q) := (f^\times)^*(Q) := \sup_{\varphi \in Y} \{\langle Q, \varphi \rangle - f^\times(\varphi)\}, \quad \forall Q \in Y^*.$$

We know that $\mathcal{F}(f)$ is convex and weak-star lower semicontinuous as Fenchel transform of the convex 1-Lipschitz function $f^\times$ on Y (see [1, Proposition 1]).

In the following lemma, we study some properties of the operator $\mathcal{F}$ which will be used in the next sections.

Lemma 2.7. *Let X be a normal Hausdorff space and $(Y, \|.\|)$ is a Banach space included in $C_b(X)$ such that $\|.\| \geq \|.\|_\infty$ and that the pair (X, Y) satisfies (H). Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a bounded from below and lower semicontinuous fonction with $dom(f) \neq \emptyset$. Then, the following assertions hold.*

(1) *The function $\mathcal{F}(f)$ is convex weak-star lower semicontinuous and bounded from below. We have that $dom(\mathcal{F}(f)) \subset \overline{\text{co}}^{w^*}(\Delta_Y(X))$ and $f = \mathcal{F}(f) \circ \Delta_Y$. In other words, the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "\Delta_Y"] \arrow[dr, "f"'] & Y^* \arrow[d, "\mathcal{F}(f)"] \\ & \mathbb{R} \cup \{+\infty\} \end{tikzcd}$$

(2) *When $f \equiv c$, where $c \in \mathbb{R}$, we have $\mathcal{F}(c) = c + i_{\overline{\mathrm{co}}^{w^*}(\Delta_Y(X))}$, where $i_{\overline{\mathrm{co}}^{w^*}(\Delta_Y(X))}$ is the indicator function of $\overline{\mathrm{co}}^{w^*}(\Delta_Y(X))$ which is equal to 0 on $\overline{\mathrm{co}}^{w^*}(\Delta_Y(X))$ and $+\infty$ otherwise.*

(3) *For all $\varphi \in Y$ we have $\mathcal{F}(f - \varphi) = \mathcal{F}(f) - \hat{\varphi}$, and, in particular, we have $\mathcal{F}(\varphi) = \hat{\varphi} + i_{\overline{\mathrm{co}}^{w^*}(\Delta_Y(X))}$ for all $\varphi \in Y$.*

(4) *We have the conservation of the infinima :*

$$\inf_{x \in X} f(x) = \min_{Q \in \overline{\Delta_Y(X)}^{w^*}} \mathcal{F}(f)(Q) = \min_{Q \in \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))} \mathcal{F}(f)(Q).$$

(5) *If $(x_n)_n$ is a sequence that minimize the function f on X, then $(\delta_{x_n})_n$ is a sequence that minimize $\mathcal{F}(f)$ on Y^*.*

Proof. (1) From the definition of $f^\times$ we have

$$f^\times(\varphi) \leq \sup_{x \in X} \varphi(x) - \inf_{x \in X} f(x) = \sup_{x \in X} \varphi(x) + f^\times(0)$$

for all $\varphi \in Y$. Thus, for all $\varphi \in Y$ and all $Q \in Y^*$ we have

$$Q(\varphi) - f^\times(\varphi) \geq Q(\varphi) - \sup_{x \in X} \varphi(x) - f^\times(0).$$

Let $Q \notin \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))$, by the Hahn-Banach theorem, there exist $\varphi_0 \in Y$ and a real number $t \in \mathbb{R}$ such that $Q(\varphi_0) = \hat{\varphi}_0(Q) > t \geq \sup_{S \in \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))} \hat{\varphi}_0(S)$. On the other hand, since $\Delta_Y(X) \subset \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))$, we have

$$\sup_{S \in \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))} \hat{\varphi}_0(S) \geq \sup_{x \in X} \varphi_0(x).$$

Thus

$$Q(\varphi_0) - \sup_{x \in X} \varphi_0(x) \geq Q(\varphi_0) - \sup_{S \in \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))} \hat{\varphi}_0(S) > 0.$$

Hence, for all $\lambda \in \mathbb{R}_+$, we have

$$Q(\lambda\varphi_0) - \sup_{x \in X} \lambda\varphi_0(x) \geq \lambda \left(Q(\varphi_0) - \sup_{\overline{\mathrm{co}}^{w^*}(\Delta_Y(X))} (\varphi_0) \right) \overset{\lambda \to +\infty}{\longrightarrow} +\infty.$$

This implies $\mathcal{F}(f)(Q) = +\infty$ whenever $Q \notin \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))$. Thus $dom(\mathcal{F}(f)) \subset \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))$. Now, the fact that $f = \mathcal{F}(f) \circ \Delta_Y$ follows from Theorem 2.3. To see that $\mathcal{F}(f)$ is bounded from below it suffices to see that for all $Q \in Y^*$

$$\mathcal{F}(f)(Q) = \sup_{\varphi \in Y} \{\langle Q, \varphi \rangle - f^\times(\varphi)\} \geq -f^\times(0) = \inf_X f > -\infty.$$

(2) By definition we have :

$$\mathcal{F}(c)(Q) = \sup_{\varphi \in Y} \left\{ Q(\varphi) - c^{\times}(\varphi) \right\} = \sup_{\varphi \in Y} \left\{ Q(\varphi) - \sup_{x \in X} \varphi(x) \right\} + c$$

We deduce from the above equality that $\mathcal{F}(c)(Q) \geq c$ for all $Q \in Y^*$. On the other hand, let $Q \in \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))$, there exists a net $(Q_\alpha)_\alpha \subset \mathrm{conv}(\Delta_Y(X))$ such that $Q_\alpha \xrightarrow{w^*} Q$. For each α, there exists $(\lambda_i^\alpha)_{i=1}^n \subset \mathbb{R}_+$ and $(x_i^\alpha)_{i=1}^n \subset X$ such that $\sum_{i=1}^n \lambda_i^\alpha = 1$ and $Q_\alpha = \sum_{i=1}^n \lambda_i^\alpha \delta_{x_i^\alpha}$. We then have

$$Q_\alpha(\varphi) = \sum_{i=1}^n \lambda_i^\alpha \varphi(x_i^\alpha) \leq \sup_{x \in X} \varphi(x)$$

for all $\varphi \in Y$. By taking the weak-star limit, we obtain that $Q(\varphi) \leq \sup_{x \in X} \varphi(x)$ for all $\varphi \in Y$. Thus, using the above formula, we get $\mathcal{F}(c)(Q) \leq c$ for all $Q \in \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))$ and so we have that $\mathcal{F}(c)(Q) = c$ for $Q \in \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))$. From part (1), since $dom(\mathcal{F}(c)) \subset \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))$, we conclude that $\mathcal{F}(c) = c + i_{\overline{\mathrm{co}}^{w^*}(\Delta_Y(X))}$.

(3) Let $\varphi \in Y$. By definition, we have for all $Q \in Y^*$ that

$$\mathcal{F}(f - \varphi)(Q) := \sup_{\psi \in Y} \left\{ Q(\psi) - (f - \varphi)^{\times}(\psi) \right\}.$$

Also by definition we have $(f - \varphi)^{\times}(\psi) = (f)^{\times}(\varphi + \psi)$ for all $\varphi, \psi \in Y$. By a change of variable we obtain for all $Q \in Y^*$:

$$\begin{aligned} \mathcal{F}(f - \varphi)(Q) &:= \sup_{\psi \in Y} \left\{ Q(\psi - \varphi) - (f)^{\times}(\psi) \right\} \\ &= \sup_{\psi \in Y} \left\{ Q(\psi) - (f)^{\times}(\psi) \right\} - Q(\varphi) = \mathcal{F}(f)(Q) - \hat{\varphi}(Q). \end{aligned}$$

Thus $\mathcal{F}(f - \varphi) = \mathcal{F}(f) - \hat{\varphi}$. In particular, when we take $f \equiv 0$ and by using part (2), we obtain that $\mathcal{F}(\varphi) = \hat{\varphi} + i_{\overline{\mathrm{co}}^{w^*}(\Delta_Y(X))}$, for all $\varphi \in Y$.

(4) Let us prove that

$$\inf_{x \in X} f(x) = \min_{Q \in \overline{\Delta_Y(X)}^{w^*}} \mathcal{F}(f)(Q) = \min_{Q \in \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))} \mathcal{F}(f)(Q).$$

Note that $\mathcal{F}(f)$ is weak-star lower semicontinuous and that the sets $\overline{\Delta_Y(X)}^{w^*}$ and $\overline{\mathrm{co}}^{w^*}(\Delta_Y(X))$ are weak-star compact. Hence $\mathcal{F}(f)$ attains its minimum on these sets. Now, since $\Delta(X) \subset \overline{\Delta_Y(X)}^{w^*} \subset \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))$, using part (1) we have

$$\min_{Q \in \overline{\mathrm{co}}^{w^*}(\Delta_Y(X))} \mathcal{F}(f)(Q) \leq \min_{Q \in \overline{\Delta_Y(X)}^{w^*}} \mathcal{F}(f)(Q) \leq \inf_{Q \in \Delta_Y(X)} \mathcal{F}(f)(Q) = \inf_{x \in X} f(x).$$

On the other hand, it follows from the definition that

$$\mathcal{F}(f)(Q) \geq -f^{\times}(0) := \inf_{x \in X} f(x)$$

for all $Q \in (Y)^*$. Thus, $\inf_{Q\in\overline{\text{co}}^{w^*}(\Delta_Y(X))} \mathcal{F}(f)(Q) \geq \inf_{x\in X} f(x)$. Hence,

$$\inf_{x\in X} f(x) = \min_{Q\in\overline{\Delta_Y(X)}^{w^*}} \mathcal{F}(f)(Q) = \min_{Q\in\overline{\text{co}}^{w^*}(\Delta_Y(X))} \mathcal{F}(f)(Q).$$

(5) Let $(x_n)_n$ be a sequence that minimize f on X. Since $f(x) = \mathcal{F}(f)(\delta_x)$ for all $x \in X$ and $\inf_{x\in X} f(x) = \min_{Q\in\overline{\text{co}}^{w^*}(\Delta_Y(X))} \mathcal{F}(f)(Q)$ then the seqence $(\delta_{x_n})_n$ minimize $\mathcal{F}(f)$ on $\overline{\text{co}}^{w^*}(\Delta_Y(X))$. □

Now, we prove the following particular case in the compact framework.

Corollary 2.8. *Let (X,τ) be a compact Hausdorff space. Then, there exists a weak-star compact subset K of some dual Banach space Y^* and an homeomorphism $H\colon (X,\tau) \longrightarrow (K,w^*)$, satisfying the following property: for every proper lower semicontinuous function $f\colon (X,\tau) \longrightarrow \mathbb{R}\cup\{+\infty\}$, there exists a proper convex weak-star lower semicontinuous function $F\colon (\overline{\text{co}}^{w^*}(K),w^*) \longrightarrow \mathbb{R}\cup\{+\infty\}$, such that the following diagramm commutes*

$$\begin{tikzcd}
(X,\tau) \arrow[r, "H"] \arrow[dr, "f"'] & (K,w^*) \arrow[d, "F_{|K}"] \\
& \mathbb{R}\cup\{+\infty\}
\end{tikzcd}$$

and $\quad \text{argmin}(f) = H^{-1}(\text{argmin}(F_{|K})) = H^{-1}(\text{argmin}(F)\cap K).$

Proof. By applying Proposition 2.6 with $Y = C(X)$, we get that the compact Hausdorff space (X,τ) is homeomorphic to the weak-star compact subset $(\Delta_Y(X),w^*)$ of $(C(X))^*$ $(\Delta_Y(X) = \overline{\Delta_Y(X)}^{w^*}$, since (X,τ) is a compact Hausdorff space). Thus, the conclusion follows from Lemma 2.7 by setting $F := \mathcal{F}(f)$ and $K := \Delta_Y(X)$. □

3. Duality and inf-convolution

In this section, we give the proof of Theorem 3.1 below. This theorem has a know analogous in the classical Fenchel duality. In all this section, we assume that X is a normal Hausdorff space and Y is a subset of $C(X)$. Recall that

$$A_Y(f) := \{\varphi \in Y/\varphi \leq f\}.$$

Recall that the inf-convolution of two functions k and l defined on vector space Z is defined for all $z \in Z$ as follows:

$$k \diamond l(z) \quad := \quad \inf_{y\in Z}\{k(y) + l(z-y)\}$$

Recall also that the hypothesis (H) is satisfied for the pair $(X, C_b(X))$ by the Urysohn's lemma (see Exemple 2.2).

Theorem 3.1. *Let X be a normal Hausdorff space, $Y = C_b(X)$, $f, g\colon X \longrightarrow \mathbb{R}$ lower semicontinuous functions bounded from below (here $dom(f) = dom(g) = X$). Then $(f+g)^\times = f^\times \diamond g^\times$ on $C_b(X)$.*

The proof of this theorem will be given after some preliminary results.

Lemma 3.2. *Let X be a normal Hausdorff space and $Y = C_b(X)$. Then, we have that $A_Y(f+g) = A_Y(f) + A_Y(g)$ for all $f, g\colon X \longrightarrow \mathbb{R}$ bounded from below and lower semicontinuous (here $dom(f) = dom(g) = X$).*

Proof. Clearly we have that $A_Y(f) + A_Y(g) \subseteq A_Y(f+g)$. Let us prove the converse. Indeed, let $\varphi \in A_Y(f+g)$. Then we have $\varphi \leq f + g$. In other words, $\varphi - g \leq f$, with $\varphi - g$ uper semicontinuous and f lower semicontinuous. Since X is a normal space, using the insertion theorem [7, Theorem 1], there exists a continuous function ψ on X such that $\varphi - g \leq \psi \leq f$. Since ψ is not necessarily bounded, we define $\tilde{\psi}(x) = \max(\min(M, \psi(x)), m)$ for all $x \in X$, where $M = \sup_X(\varphi - g)$ and $m = \inf_X f$. Then, $\tilde{\psi}$ is bounded, continuous and satisfies $\varphi - g \leq \tilde{\psi} \leq f$. Thus, we have $\tilde{\psi} \leq f$ i.e. $\tilde{\psi} \in A_Y(f)$ and $\varphi - \tilde{\psi} \leq g$ i.e. $\varphi - \tilde{\psi} \in A_Y(g)$ with $\varphi = \tilde{\psi} + (\varphi - \tilde{\psi})$. Hence $\varphi \in A_Y(f) + A_Y(g)$. $\square$

Lemma 3.3. *Let X be a normal Hausdorff space and $(Y, \|.\|)$ is a Banach space included in $C_b(X)$ such that $\|.\| \geq \|.\|_\infty$ and that the pair (X, Y) satisfies (H). Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a bounded from below lower semicontinuous function with $dom(f) \neq \emptyset$. Then, for all $\xi \in Y$, we have that*

$$f^\times(\xi) = \inf_{\varphi \in A_Y(f)} \varphi^\times(\xi).$$

Proof. It suffices to show $f^\times(0) = \inf_{\varphi \in A_Y(f)} \varphi^\times(0)$, since $f^\times(\xi) = (f - \xi)^\times(0)$ and $A_Y(f - \xi) = A_Y(f) - \xi$, for all $\xi \in Y$. Also since f is bounded from below, by replacing f by $f - \inf_X f$ we can assume without loss of generality that $\inf_X f = 0$. From part (4) in Lemma 2.7, we have for all $\varphi \in Y$ that

$$\varphi^\times(0) := -\inf_{x \in X} \varphi(x) = -\min_{Q \in \overline{\text{co}}^{w^*}(\Delta_Y(X))} \mathcal{F}(\varphi)(Q),$$

Thanks to part (3) of Lemma 2.7, we get that $\varphi^\times(0) = -\min_{Q \in \overline{\text{co}}^{w^*}(\Delta_Y(X))} \hat{\varphi}(Q)$, for all $\varphi \in Y$. Thus, we have

$$\inf_{\varphi \in A_Y(f)} \varphi^\times(0) = \inf_{\varphi \in A_Y(f)} (-\min_{Q \in \overline{\text{co}}^{w^*}(\Delta_Y(X))} \hat{\varphi}(Q)) = -\sup_{\varphi \in A_Y(f)} \min_{Q \in \overline{\text{co}}^{w^*}(\Delta_Y(X))} \hat{\varphi}(Q). \quad (1)$$

Applying the minimax theorem [8, Cor. 2] to $\langle \cdot, \cdot \rangle : \overline{\text{co}}^{w^*}(\Delta_Y(X)) \times A_Y(f) \to \mathbb{R}$, defined by $\langle Q, \varphi \rangle =: \hat{\varphi}(Q) = Q(\varphi)$, we get

$$\sup_{\varphi \in A_Y(f)} \min_{Q \in \overline{\text{co}}^{w^*}(\Delta_Y(X))} \hat{\varphi}(Q) = \min_{Q \in \overline{\text{co}}^{w^*}(\Delta_Y(X))} \sup_{\varphi \in A_Y(f)} \hat{\varphi}(Q).$$

Hence, using (1) we obtain

$$\inf_{\varphi \in A_Y(f)} \varphi^\times(0) = -\min_{Q \in \overline{\text{co}}^{w^*}(\Delta_Y(X))} \sup_{\varphi \in A_Y(f)} \hat{\varphi}(Q). \quad (2)$$

Since $f \geq 0 = \inf_X f$ then $\sup_{\varphi\in A_Y(f)} \hat{\varphi}(Q) \geq 0 = \inf_X f$ for all $Q \in \overline{\text{co}}^{w^*}(\Delta_Y(X))$. This implies that $\min_{Q\in\overline{\text{co}}^{w^*}(\Delta_Y(X))} \sup_{\varphi\in A_Y(f)} \hat{\varphi}(Q) \geq 0 = \inf_X f$. Thus, from (2) we obtain

$$\inf_{\varphi\in A_Y(f)} \varphi^\times(0) \leq 0 = -\inf_{x\in X} f := f^\times(0).$$

Now, to see the converse, since $\Delta(X) \subset \overline{\text{co}}^{w^*}(\Delta_Y(X))$, we have that

$$\min_{Q\in\overline{\text{co}}^{w^*}(\Delta_Y(X))} \sup_{\varphi\in A_Y(f)} \hat{\varphi}(Q) \leq \inf_{x\in X} \sup_{\varphi\in A_Y(f)} \varphi(x) \leq \inf_{x\in X} f(x).$$

Since $\inf_{x\in X} f(x) := -f^\times(0)$, using (2), we get $f^\times(0) \leq \inf_{\varphi\in A_Y(f)} \varphi^\times(0)$. □

Proof of Theorem 3.1. It is easy to see that for all $\varphi, \psi \in C_b(X)$, we have

$$f^\times(\psi) + g^\times(\varphi - \psi) \geq (f+g)^\times(\varphi).$$

Taking the infinimum over $\psi \in C_b(X)$, we get that $(f+g)^\times \leq f^\times \diamond g^\times$.

Claim. If $\varphi, \psi \in C_b(X)$, then $(\varphi+\psi)^\times = \varphi^\times \diamond \psi^\times$ on $C_b(X)$.

Proof of the claim. It suffices to show that $\varphi^\times \diamond \psi^\times \leq (\varphi+\psi)^\times$. Indeed, for all $\xi \in C_b(X)$,

$$\varphi^\times \diamond \psi^\times(\xi) = \inf_{\mu\in C_b(X)} \{\varphi^\times(\mu) + \psi^\times(\xi - \mu)\}$$
$$\leq \varphi^\times(\xi - \psi) + \psi^\times(\xi - (\xi - \psi)) = (\varphi+\psi)^\times(\xi) + \psi^\times(\psi) = (\varphi+\psi)^\times(\xi).$$ □

Now, let $\theta, \xi \in C_b(X)$. From Lemma 3.3 we have

$$\begin{aligned} f^\times(\xi) + g^\times(\theta - \xi) &= \inf_{\varphi\in A_Y(f)} \varphi^\times(\xi) + \inf_{\psi\in A_Y(g)} \psi^\times(\theta - \xi) \\ &= \inf_{(\varphi,\psi)\in A_Y(f)\times A_Y(g)} \{\varphi^\times(\xi) + \psi^\times(\theta - \xi)\} \end{aligned}$$

By taking the infinimum over ξ in the above formula, we obtain

$$f^\times \diamond g^\times(\theta) = \inf_{(\varphi,\psi)\in A_Y(f)\times A_Y(g)} \{\varphi^\times \diamond \psi^\times(\theta).\}$$

Using the claim we get

$$f^\times \diamond g^\times(\theta) = \inf_{(\varphi,\psi)\in A_Y(f)\times A_Y(g)} \{(\varphi+\psi)^\times(\theta)\}.$$

From Lemma 3.2 we have $f^\times \diamond g^\times(\theta) = \inf_{\mu\in A_Y(f+g)} \mu^\times(\theta)$.

Using again Lemma 3.3, we get $f^\times \diamond g^\times(\theta) = (f+g)^\times(\theta)$ for all $\theta \in C_b(X)$. □

4. Proof of Theorem 1.2.

For the proof of Theorem 1.2, we also need the following propositions.

Proposition 4.1. *Let X be a topological space, $Y = C_b(X)$ and $\emptyset \neq A \subset C_b(X)$. Then, A is Δ-set if and only if, there exists a bounded from below lower semicontinuous function $f\colon X \longrightarrow \mathbb{R}$ such that $A = A_Y(f) := \{\psi \in C_b(X)/\psi \leq f\}$.*

Proof. Let us prove the "only if part". Since A is Δ-set, there exists real numbers $\lambda_x \in \mathbb{R}$, for all $x \in X$, such that $A = \cap_{x\in X}\{\varphi \in C_b(X) : \varphi(x) \leq \lambda_x\}$. Let us set $f(x) := \sup_{\psi\in A} \psi(x)$, for all $x \in X$. Thus, we have $f(x) \leq \lambda_x < +\infty$ for all $x \in X$. It follows that $A_Y(f) \subset A$, that $dom(f) = \mathbb{R}$ and that the function f is lower semicontinuous as supremum of continuous function. It follows also that f is bounded from below, since there exists a bounded continuous function $\varphi \in A$ such that $-\infty < \inf_X \varphi \leq \varphi \leq f$. On the other hand, if $\varphi \in A$, then for all $x \in X$ we have $\varphi(x) \leq \sup_{\psi\in A} \psi(x) := f(x)$. This shows that $\varphi \in A_Y(f)$ and so that $A \subset A_Y(f)$. Hence $A = A_Y(f)$. The "if part" is clear. □

Proposition 4.2. *Under the hypothesis of Lemma 3.3, let $f\colon X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a bounded from below lower semicontinuous function with $dom(f) \neq \emptyset$. Then, for all $Q \in \overline{\text{co}}^{w^*}(\Delta_Y(X))$, we have that*

$$\mathcal{F}(f)(Q) := (f^{\times})^*(Q) = \sup_{\varphi\in A_Y(f)} \langle Q, \varphi\rangle := \sigma_{A_Y(f)}(Q).$$

Proof. From Lemma 3.3, we have that $f^{\times}(\xi) = \inf_{\varphi\in A_Y(f)} \varphi^{\times}(\xi)$, for all $\xi \in Y$. So, by applying the Fenchel conjugacy to $f^{\times}$, we get

$$(f^{\times})^*(Q) = \sup_{\varphi\in A_Y(f)} (\varphi^{\times})^*(Q)$$

for all $Q \in Y^*$. Hence, $\mathcal{F}(f)(Q) = \sup_{\varphi\in A_Y(f)} \mathcal{F}(\varphi)(Q)$ for all $Q \in Y^*$. Using part (3) of Lemma 2.7, we obtain that $\mathcal{F}(f)(Q) = \sup_{\varphi\in A_Y(f)}\langle Q, \varphi\rangle$, for all $Q \in \overline{\text{co}}^{w^*}(\Delta_Y(X))$. □

Proof of Theorem 1.2. First, we define $\mathcal{T}$ as follows: for all $f \in SCI(X)$,

$$\mathcal{T}(f) := \mathcal{F}(f)_{|\overline{\text{co}}^{w^*}(\Delta_Y(X))},$$

the restriction of $\mathcal{F}(f)$ to $\overline{\text{co}}^{w^*}(\Delta_Y(X))$. From Proposition 4.2, we know that $\mathcal{T}(f) \in \Sigma(\overline{\text{co}}^{w^*}(\Delta(X)))$ for all $f \in SCI(X)$. We now proceed to prove that $\mathcal{T}\colon SCI(X) \longrightarrow \Sigma(\overline{\text{co}}^{w^*}(\Delta(X)))$ is a bijective map. Indeed, using part (1) of Lemma 2.7, we get that $\mathcal{T}$ is one to one. To see that $\mathcal{T}$ is onto, take some $g \in \Sigma(\overline{\text{co}}^{w^*}(\Delta(X)))$. Then there exists a Δ-set A such that $g = (\sigma_A)_{|\overline{\text{co}}^{w^*}(\Delta(X))}$. Using Proposition 4.1, there exist a bounded from below lower semicontinuous function $f\colon X \longrightarrow \mathbb{R}$ such that $A = A_Y(f) := \{\psi \in C_b(X)/\psi \leq f\}$. Thus, by using Proposition 4.2, we get that $g = \mathcal{T}(f)$ i.e. $\mathcal{T}$ is onto. Hence, $\mathcal{T}$ is a bijective map. Now, we prove that for all $f, g \in SCI(X)$ and all $\alpha, \beta \in \mathbb{R}^+$, we have

$$\mathcal{T}(\alpha f + \beta g) = \alpha\mathcal{T}(f) + \beta\mathcal{T}(g).$$

Indeed, let $\alpha \in \mathbb{R}^+$. If $\alpha = 0$, then from part (2) of Lemma 2.7, we have that $\mathcal{T}(0) = \mathcal{F}(0) = 0$ on $\overline{\text{co}}^{w^*}(\Delta_Y(X))$. If $\alpha \neq 0$, it is easy to see that $(\alpha f)^\times(\varphi) = \alpha f^\times(\frac{\varphi}{\alpha})$ for all $f \in SCI(X)$. Thus, $((\alpha f)^\times)^* = \alpha(f^\times)^*$ which implies that $\mathcal{T}(\alpha f) = \alpha\mathcal{T}(f)$ for all $f \in SCI(X)$.

On the other hand, if $f, g \in SCI(X)$, then by applying Theorem 3.1, we get that $(f+g)^\times = f^\times \diamond g^\times$. Hence by the properties of the Fenchel conjugacy, $((f+g)^\times)^* = (f^\times \diamond g^\times)^* = (f^\times)^* + (g^\times)^*$. In other words, we have that $\mathcal{T}(f+g) = \mathcal{T}(f) + \mathcal{T}(g)$. Thus, $\mathcal{T}(\alpha f + \beta g) = \alpha\mathcal{T}(f) + \beta\mathcal{T}(g)$, for all $f, g \in SCI(X)$ and all $\alpha, \beta \in \mathbb{R}^+$. It follows from this formula that the set $\Sigma(\overline{\text{co}}^{w^*}(\Delta(X)))$ is a convex cone and that $\mathcal{T}$ is a convex cone isomorphism.

Parts (1), (3) and (4) of the theorem, follow repectively from the parts (1), (3) and (4) of Lemma 2.7. Now, we prove part (2) of the theorem. Let $f, g \in SCI(X)$. If $f \leq g$, then we see easily from the definition that $g^\times \leq f^\times$. Also from the definition of the Fenchel conjugacy we get $(f^\times)^* \leq (g^\times)^*$ which implies $\mathcal{T}(f) \leq \mathcal{T}(g)$. Now, suppose that $\mathcal{T}(f) \leq \mathcal{T}(g)$. Since $dom(\mathcal{F}(f)) \subset \overline{\text{co}}^{w^*}(\Delta_Y(X))$ (see part (1) of Lemma 2.7), we also have $\mathcal{F}(f) \leq \mathcal{F}(g)$, i.e. $(f^\times)^* \leq (g^\times)^*$. This implies that $(g^\times)^{**} \leq (f^\times)^{**}$. Since $f^\times$ and $g^\times$ are convex and 1-Lipschitz continuous functions, using the classical Fenchel-Moreau theorem, we obtain that $g^\times \leq f^\times$. This implies that $f^{\times\times} \leq g^{\times\times}$ and so by applying Theorem 2.3, we get $f \leq g$. $\square$

5. The case of Lipschitz-free spaces

Let K be a pointed metric space, that is, a metric space equipped with a distinguished point denoted 0. The space $Lip_0(K)$ is the space of real-valued Lipschitz functions on K which vanish at 0. When equipped with the Lipschitz norm

$$\|f\|_L := \sup\{\frac{f(x) - f(y)}{d(x,y)} : x \neq y \in K\}.$$

the space $Lip_0(K)$ becomes a Banach space whose dual contains in particular the Dirac measure $\Delta(x) = \delta_x$ at any point $x \in K$. The Dirac map $\Delta\colon K \longrightarrow Lip_0(K)^*$ defined by $\Delta(x)(\varphi) = \varphi(x)$ for all $\varphi \in Lip_0(K)$ is an isometric embedding from K to a subset of the dual of $Lip_0(K)$. We denote by $\mathcal{F}(K)$ the norm-closed linear span of $\Delta(K)$ in the dual space $Lip_0(K)^*$. This space is called *Lipschitz-free space over* K by Godefroy and Kalton in [6]. The Lipschitz-free space $\mathcal{F}(K)$ is an isometric predual of the space $Lip_0(K)$ whose w^*-topology coincides on the unit ball of $Lip_0(K)$ with the pointwise convergence on K. For more details about Lipschitz-free space we refer to [6] and [5].

A subset A of $Lip_0(K)$ will be said to be a *Lipschitz-Δ-set* (or *X-convex subset of* $Lip_0(K)$) if and only if for all $x \in K$ there exist a real number $\lambda_x \in \mathbb{R}$, such that

$$A \;:=\; \cap_{x\in X}\{\varphi \in Lip_0(K) : \varphi(x) \leq \lambda_x\}.$$

By σ_A we denote the *support function*, defined on the dual space by

$$\forall Q \in Lip_0(K)^*;\;\; \sigma_A(Q) := \sup_{\varphi\in A}\langle Q, \varphi\rangle.$$

By $\Sigma(\overline{\text{co}}(\Delta(K)))$ we denote the set of all functions of the form $\sigma_{A|\overline{\text{co}}(\Delta(K))}$ where A is a Lipschitz-Δ-set of $Lip_0(K)$ and $\overline{\text{co}}(\Delta(K))$ denotes the norm closed convex hull of $\Delta(K)$ in the Lipschitz-free space $\mathcal{F}(K)$:

$$\Sigma(\overline{\text{co}}(\Delta(K))) := \{\sigma_{A|\overline{\text{co}}(\Delta(K))} : A \text{ is Lipschitz-}\Delta\text{-set of } Lip_0(K)\}.$$

Finally, we denote by $SCI_0(K)$ the convex cone of all lower semicontinuous functions from K into $\mathbb{R}$ such that $A_Y(f) \neq \emptyset$ where $Y = Lip_0(K)$.

The aim of this section is to obtain an equivalent of Theorem 1.2 by replacing the dual space $C_b(K)^*$ (resp. the weak-star topology on $C_b(K)^*$) by the Lipschitz-free space $\mathcal{F}(K)$ (resp. the norm topology on $\mathcal{F}(K)$) and by working with $Y = Lip_0(K)$. To do this, we need to give somme adaptations of the results of this note since, for example, the pair $(K, Lip_0(K))$ does not satisfy the property (H).

All the results in this section work for bounded metric space K, except in Proposition 5.2 and so also in Theorem 5.1 , where we need to assume that K is compact metric space. In what follows, the conjugacy $f^\times$ is related to the the class $Y = Lip_0(K)$.

We obtain in the following theorem an equivalent to Theorem 1.2 for the Lipschitz-free space in the compact metric framework.

Theorem 5.1. *Let K be a compact metric space. Then $\Sigma(\overline{\text{co}}(\Delta(K)))$ is a convex cone and there exists a convex cone isomorphism $\mathcal{T}: SCI_0(K) \to \Sigma(\overline{\text{co}}(\Delta(K)))$. This isomorphism satisfies the following properties:*

(1) *For all $f \in SCI_0(K)$, $\mathcal{T}(f) \circ \Delta = f$, this means that $\mathcal{T}(f)$ is a convex lower semicontinuous extension of f to $\overline{\text{co}}(\Delta(K))$, up to the isometric identification between K and $\Delta(K)$.*

(2) *The convex cone isomorphime $\mathcal{T}$ is isotone, i.e., for all $f, g \in SCI_0(K)$:*
$$f \leq g \Longleftrightarrow \mathcal{T}(f) \leq \mathcal{T}(g).$$

(3) *For all $\varphi \in Lip_0(K)$ and all $Q \in \overline{\text{co}}(\Delta(K))$: $\mathcal{T}(\varphi)(Q) = \langle Q, \varphi\rangle$.*

(4) *For all $f \in SCI_0(K)$, we have $\min\limits_K f = \min\limits_{\overline{\text{co}}(\Delta(K))} \mathcal{T}(f)$.*

For the proof of the above theorem, we define $\mathcal{T}(f) := (f^\times)^*_{|\overline{\text{co}}(\Delta(K))}$ for all $f \in SCI_0(K)$ and we follow the proof of Theorem 1.2 using the following results. Note that $\overline{\text{co}}(\Delta(K))$ is a compact subset of the Lipschitz-free space $\mathcal{F}(K)$, since K is assumed to be compact metric space.

We prove the equivalent of Theorem 3.1 in the compact framework.

Proposition 5.2. *Let K be a compact metric space and $f, g: K \longrightarrow \mathbb{R}$ be two positive lower semicontinuous functions such that $A_Y(f) \neq \emptyset$ and $A_Y(g) \neq \emptyset$ where $Y = Lip_0(K)$. Then, $(f+g)^\times = f^\times \diamond g^\times$ on $Lip_0(K)$.*

Proof. We use Theorem 3.1, the density of $Lip_0(K)$ in $(C_0(K), \|.\|_\infty)$ (the space of all real-valued continuous functions that vanish at 0) for the sup-norm $\|.\|_\infty$

and the fact that the $\|.\|_\infty$-Lipschitz functions $f^\times, g^\times \colon Lip_0(K) \longrightarrow \mathbb{R}$ extend to $C(K)$. Indeed, for all $\xi \in Lip_0(K)$, by the density of $Lip_0(K)$ in $C_0(K)$ we have

$$f^\times \diamond g^\times(\xi) := \inf_{\varphi\in Lip_0(K)}\{f^\times(\varphi) + g^\times(\xi-\varphi)\} = \inf_{\varphi\in C_0(K)}\{f^\times(\varphi) + g^\times(\xi-\varphi)\}.$$

On the other hand, by considering the operator $(.)^\times$ on $C(K)$ and using Theorem 3.1 we have

$$(f+g)^\times(\xi) = f^\times \diamond g^\times(\xi) = \inf_{\varphi\in C(K)}\{f^\times(\varphi) + g^\times(\xi-\varphi)\}$$
$$= \inf_{\varphi\in C(K)}\{f^\times(\varphi-\varphi(0)) + g^\times(\xi-(\varphi-\varphi(0)))\} = \inf_{\varphi\in C_0(K)}\{f^\times(\varphi) + g^\times(\xi-\varphi)\}.$$

Hence, $(f+g)^\times = f^\times \diamond g^\times$ where the operator $(.)^\times$ is considered on $Lip_0(K)$. $\square$

Remark 5.3. Let K be a bounded metric space. The diameter $\alpha_K := diam(K)$ of K is finite and so we have that $\|.\|_L \geq \alpha_K\|.\|_\infty$. Note also that $\varphi \in Lip(K)$ if and only if $\varphi - \varphi(0) \in Lip_0(K)$. In fact we have $Lip(K) = Lip_0(K) \oplus \mathbb{R}$ and we can equip $Lip(K)$ with the norm

$$\|\varphi\| := \frac{1}{\alpha_K}\|\varphi-\varphi(0)\|_L + |\varphi(0)| \geq \|\varphi\|_\infty$$

for all $\varphi \in Lip(K)$. The pair $(K, Lip_0(K))$ does not satisfy property (H), but the pair $(K, Lip(K))$ satisfies it. It follows that except for the delicate Proposition 5.2 and so Theorem 5.1, all the other results of this section concerning $Lip_0(K)$ (where K is bounded metric not necessarily compact) can be deduced easily from the space $Lip(K)$ following the previous results established in this paper.

Corollary 5.4. *Let K be a bounded metric space and assume $Y = Lip_0(K)$. Let $f \colon X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a function with $dom(f) \neq \emptyset$ such that $A_Y(f) \neq \emptyset$. Then, f is lower semi-continuous if and only if $f^{\times\times} = f$.*

Proof. The "if" part is clear. For the "only if" part, it is clear that $(M, Lip(K))$ has the property (H) where $Lip(K)$ is the space of all Lipschitz maps on M. Thus from Theorem 2.3 applied with the pair $(M, Lip(K))$ we have that for all $x \in M$

$$f(x) \quad = \quad \sup_{\varphi\in Lip(K)}\{\varphi(x) - \sup_{y\in M}\{\varphi(y) - f(y)\}\}$$

It follows that

$$f(x) = \sup_{\varphi\in Lip(K)}\{(\varphi-\varphi(0))(x) - \sup_{y\in M}\{(\varphi-\varphi(0))(y) - f(y)\}\}$$
$$= \sup_{\psi\in Lip_0(K)}\{\psi(x) - \sup_{y\in M}\{\psi(y) - f(y)\}\} = f^{\times\times}(x).$$

A simple adaptation of Lemma 2.7 will permit to obtain the following proposition. By $\overline{\text{co}}^{w^*}(\Delta(K))$ we denote the weak-star closed convex hull of $\Delta(K)$ in $Lip_0(K)^*$.

Proposition 5.5. *Let K be a bounded metric space (not necessarily compact). Let $f\colon K \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous fonction with $dom(f) \neq \emptyset$ and $A_Y(f) \neq \emptyset$ where $Y = Lip_0(K)$. Then, the following assertions hold:*

(1) *The function $\mathcal{F}(f) = (f^{\times})^*\colon Lip_0(K)^* \longrightarrow \mathbb{R}\cup\{+\infty\}$ is convex weak-star lower semicontinuous and bounded from below, in particular the restriction $\mathcal{F}(f)_{|\mathcal{F}(K)}\colon \mathcal{F}(K) \longrightarrow \mathbb{R}\cup\{+\infty\}$ is convex (norm) lower semicontinuous and bounded from below. We have that $dom(\mathcal{F}(f)) \subset \overline{\text{co}}^{w^*}(\Delta(K))$ and $f = \mathcal{F}(f)\circ\Delta$. In other words, the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[dr, "f"'] & \mathcal{F}(K) \arrow[d, "\mathcal{F}(f)_{|\mathcal{F}(K)}"] \\ & \mathbb{R}\cup\{+\infty\} \end{tikzcd}$$

(2) *When $f \equiv c$, where $c \in \mathbb{R}$, we have $\mathcal{F}(c) = c + i_{\overline{\text{co}}^{w^*}(\Delta(K))}$, where $i_{\overline{\text{co}}^{w^*}(\Delta(K))}$ is the indicator function of $\overline{\text{co}}^{w^*}(\Delta(K))$ which is equal to 0 on $\overline{\text{co}}^{w^*}(\Delta(K))$ and $+\infty$ otherwise.*

(3) *For all $\varphi \in Lip_0(K)$ we have $\mathcal{F}(f-\varphi) = \mathcal{F}(f) - \hat{\varphi}$. In particular we have $\mathcal{F}(\varphi) = \hat{\varphi} + i_{\overline{\text{co}}^{w^*}(\Delta(K))}$ for all $\varphi \in Lip_0(K)$.*

(4) *If f is bounded from below, we have the conservation of the infinimums :*

$$\inf_K f = \inf_{\mathcal{F}(K)} \mathcal{F}(f) = \inf_{\Delta(K)} \mathcal{F}(f) = \min_{\overline{\Delta(K)}^{w^*}} \mathcal{F}(f) = \min_{\overline{\text{co}}^{w^*}(\Delta(K))} \mathcal{F}(f).$$

(5) *If f is bounded from below and $(x_n)_n$ is a sequence that minimize the function f on K, then $(\delta_{x_n})_n$ is a sequence that minimize $\mathcal{F}(f)$ on $\mathcal{F}(K)$.*

Similarly to Proposition 4.1 it is easy to obtain the following proposition.

Proposition 5.6. *Let K be a bounded metric space. A nonempty subset A of $Lip_0(K)$ is Lipschitz-Δ-set if and only if, there exists a lower semicontinuous function $f\colon K \longrightarrow \mathbb{R}$ such that $A = A_Y(f) \neq \emptyset$.*

Replacing Lemma 2.7 by Proposition 5.5 and by following the proof of Lemma 3.3, we obtain the following proposition.

Proposition 5.7. *Let K be a bounded metric space and $Y = Lip_0(K)$. Let $f\colon K \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function with $dom(f) \neq \emptyset$ and $A_Y(f) \neq \emptyset$ where $Y = Lip_0(K)$. Then, for all $\xi \in Y$, we have that*

$$f^{\times}(\xi) = \inf_{\varphi\in A_Y(f)} \varphi^{\times}(\xi).$$

Replacing Lemma 2.7 by Proposition 5.5 and by following the proof of Proposition 4.2, we obtain the following proposition.

Proposition 5.8. *Under the hypothesis of Proposition* 5.5*, let* $f : K \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous function with* $dom(f) \neq \emptyset$ *and* $A_Y(f) \neq \emptyset$ *where* $Y = Lip_0(K)$*. Then, for all* $Q \in \overline{\text{co}}^{w^*}(\Delta(K))$*, we have that*

$$\mathcal{F}(f)(Q) := (f^\times)^*(Q) = \sup_{\varphi \in A_Y(f)} \langle Q, \varphi \rangle := \sigma_{A_Y(f)}(Q).$$

The proof of Theorem 5.1 follows exactly the path of the proof of Theorem 1.2 using the adapted propositions mentioned above. We mention here that all the results of this section work in the bounded metric framework that is not necessarily compact, with the exception of Proposition 5.2 which requires compactness. Since in our approach Theorem 5.1 uses this proposition, we need the hypothesis of compactness also in this theorem. We do not know at present if the hypothesis of compactness can be omitted in Theorem 5.1. This question will be studied in future work.

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