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Ky-Fan and Sion Theorems for the Lexicographic Order and Applications to Vectorial Games and Min-Max Control Problems

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In many situations it is necessary to consider problems in which decisions have to be taken in a given sequence. For this purpose the lexicographic order of $\mathbb{R}^n$ appears to be an appropriate concept to employ. Therefore, we will study several of its properties in this paper. In a first step we introduce some definitions and regularity concepts with respect to this type of order. Secondly, we prove several classical results redesigned for the vectorial setting as well as several new specific results which are interesting consequences of our approach. Finally, as an application, we study min-max control problems with order. We investigate the existence of the value and we provide results concerning the existence of saddle points for a vectorial cost.

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1. Introduction

We start with a brief presentation of the the lexicographic order of $\mathbb{R}^n$. For this purpose, let us consider two vectors of $\mathbb{R}^n$, $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$. By definition, we say that

$$x \lesssim y \ \text{ if } x = y \text{ or for some } j \leq n \text{ and all } i < j,\ x_i = y_i \text{ and } x_j < y_j.$$

Moreover, we say that $x \prec y$ if $x \lesssim y$ and $x \neq y$. Note that, the lexicographic order is total but the infimum of a set is not well-defined. The notion of infimum of a set that we use in this paper was introduced for instance in [13] (see Definition 2.1). In order to formulate our main results we extend several notions such as lower semicontinuity, convexity and quasi convexity to the vectorial case.

After the above mentioned preliminaries, we focus on the vectorial Ky-Fan's inequality and its applications in the first part of the paper. For the proof of the

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Ky-Fan type inequality in the vectorial case we employ the well known Fan's theorem (see e. g. [26, 30]) that we recall, for completeness, in the following.

Theorem 1.1. (Fan) *Let $\Omega\colon X \longrightarrow X$ be a relation from a convex subset of a topological vector space to itself such that the following hold:*

(i) *The set Ω_x is closed and $x \in \Omega_x$ for all $x \in X$,*

(ii) *The set $\Omega^\star y = X \backslash \Omega_y^{-1}$ is convex for all $y \in X$,*

(iii) *There is at least one point $x_0 \in X$ for which Ω_{x_0} is compact.*

Then we have: $\displaystyle\bigcap_{x\in X} \Omega_x \neq \emptyset$.

We continue with the presentation of some consequences of the Ky-Fan's inequality in the vectorial setting. More precisely, the existence of a Nash equilibrium for a game where each player i has a preference related to the lexicographic order on some $\mathbb{R}^{n_i}$, an extension of Von Neumann min-max theorem and two extensions of Sion's theorem are presented. For the convenience of the reader, we recall Sion's theorem in the following:

Theorem 1.2. (Sion) *Let U, V be two convex compact subsets of topological vector spaces, $\phi\colon U \times V \longrightarrow \overline{\mathbb{R}}$ a function which verifies the following assumptions:*

(i) *$\phi(\cdot, v)$ is lower semicontinuous, quasi-convex for all $v \in V$,*

(ii) *$\phi(u, \cdot)$ is upper semicontinuous, quasi-concave[1] for all $u \in U$.*

Then we have: $\displaystyle\inf_{u\in U} \sup_{v\in V} \phi(u, v) = \sup_{v\in V} \inf_{u\in U} \phi(u, v)$.

In the second part of the paper we apply the new results to a min-max vectorial control problem with order. This kind of problems appear in many situations whenever it is necessary to deal with decisions which have to be taken in a given sequence. For this purpose the lexicographic order and the related notions of infimum and supremum appear appropriate concepts to employ.

There are a lot of works addressing min-max control problems in the last years. Several first results are presented for instance in [9, 15, 18, 19, 27, 28, 33]. Many results on differential games are obtained using the notion of viscosity solution. Some relevant papers on this subject are for instance [2, 3, 4, 5, 7, 10, 12, 16, 17, 29] and the survey paper [14].

In our game, the first player acting on a control u tries to minimize a cost and the second player acting on a control v tries to maximize the same cost, which is a function depending on u and v. In this context, two value functions are defined (the upper and respectively the lower value) depending on which player chooses first its control. Note that in this paper we do not employ the notion of strategies (see e. g. [19, 15, 28]) for defining the value functions.

Whenever the two value functions are equal we say that the game has a value. For obtaining the existence of a value the so-called Isaacs condition has to be

[1]Recall that a function $f\colon X \longrightarrow \overline{\mathbb{R}}$ is quasi-convex (resp. quasi-concave) if for any real number $\lambda \in \mathbb{R}$ the set $\{x \in X \mid f(x) \leq \lambda\}$ (resp. $\{x \in X \mid f(x) \geq \lambda\}$) is convex.

satisfied. The existence of a value is a way to modelize the fact that the players act simultaneously. Results proving the existence of a value can be found for instance in [6, 8, 9, 10, 11, 14, 15, 16, 18, 29, 31, 34, 35]. Moreover, in all these papers the Hamilton-Jacobi-Bellman equations are crucial in the proofs.

As mentioned before, in the last section we apply our results for the study of a vectorial min-max control problem/game. We use the linear formulation from [25] for the study of the separated dynamics. Linear programming formulations are considered in various papers for classical deterministic systems (cf. [20], [21], [22]), reflected systems (cf. [32]), or L^∞ control problems (cf. [23]). The non-vectorial case was treated in [25]. Note that it is not a surprise that in the vectorial case additional difficulties appear due to the particularities of the lexicographic order and it's associate concepts. We investigate the regularity of the vectorial value functions and we provide results concerning the existence of saddle points for a vectorial cost. We would like to underline that these results are due to the second Sion's theorem for the vectorial setting.

The paper is organized as follows. In Subsection 2.1, we present the basic notions and properties on the lexicographic order. We introduce for example the notions of infimum, semi-continuity, convexity and quasi-convexity in the vectorial setting. Section 3 is devoted to the vectorial Ky Fan's theorem. Nash equilibrium results are presented in Section 4. Two versions of Sion's theorem are provided in Section 5. Finally we apply the abstract results to the study a min-max control problem in the vectorial framework in Section 6.1. The linear version of the problem is described in Subsection 6.3. The existence of the value and saddle points is presented in Subsection 6.5.

2. Preliminaries

In this section we present several basic notions and properties of the lexicographic order.

2.1. Infimum of a set and vector lower semicontinuity

We begin by introducing the definitions of the infimim and supremun of a set and vector lower/upper semi-continuity as well as some of their proprieties. In this paper, we use the notation $I = \{1, \cdots, n\}$.

Definition 2.1. Let C be a subset of $\mathbb{R}^n$. Consider the functions $\Pi_i \colon \mathbb{R}^n \to \mathbb{R}^i$ and $p_i \colon \mathbb{R}^n \to \mathbb{R}$ defined by $\Pi_i(x_1, \cdots, x_n) = (x_1, \cdots, x_i)$ and $p_i(x_1, \cdots, x_n) = x_i$ for every $i \in I$. We denote by $\overrightarrow{\inf}\, C = (c_i)_{i\in I}$ the vector constructed as follows: Set $C_0 = C$ and $c_1 = \inf\limits_{x\in C_0} x_1$ and denote by

$$C_i = \begin{cases} C_{i-1} \bigcap p_i^{-1}(c_i) & \text{if there exists } \overline{x} \in C_{i-1} \text{ such that } p_i(\overline{x}) = c_i \\ \\ C_{i-1} & \text{otherwise} \end{cases}$$

and by $c_{i+1} = \liminf\limits_{\substack{x\in C_i \\ \Pi_i(x)\longrightarrow(c_1,\cdots,c_i)}} p_{i+1}(x)$ for $1 \leq i \leq n-1$.

Similarly, $\overrightarrow{\sup} C = (c_i)_{i\in I}$ is defined by $c_{i+1} = \limsup_{\substack{x\in C_i \\ \Pi_i(x)\longrightarrow(c_1,\cdots,c_i)}} p_{i+1}(x)$ where C_i is as above and $1 \leq i \leq n-1$. By convention we will say that $\overrightarrow{\inf} C = -\infty$ if there exists $i \in I$ such that $c_i = -\infty$, $\overrightarrow{\inf}\, \emptyset = +\infty$, $\overrightarrow{\sup} C = +\infty$ if there exists $i \in I$ such that $c_i = +\infty$ and $\overrightarrow{\sup}\, \emptyset = -\infty$. $\square$

Note that the notion of infimum for sets reduces to the usual definition when $n = 1$. Moreover, the vectorial infimum inherits several properties.

Remark 2.2. Let C be a subset of $\mathbb{R}^n$. We have the following assertions:

(i) $\overrightarrow{\inf} C \lesssim x$, for all $x \in C$,

(ii) Consider $\lambda \in \mathbb{R}^n$ such that $\lambda \lesssim x$, for all $x \in C$. Then we have $\lambda \lesssim \overrightarrow{\inf} C$.

Proposition 2.3. *Let $A \subset B \subset \mathbb{R}^n$ and let A be a closed set. Then we have that $\overrightarrow{\inf} B \lesssim \overrightarrow{\inf} A$.*

Proof. Note that it is enough to consider the case $n=2$. We define $\overrightarrow{\inf} B = (b_1, b_2)$ and $\overrightarrow{\inf} A = (a_1, a_2)$. Then, by definition we have that $b_1 \leq a_1$. If $b_1 < a_1$ or if $a_1 = -\infty$ the conclusion follows. If $b_1 = a_1$, the set $\{(a_1, x) \in A\}$ is closed. It follows that $b_2 = \inf_{(a_1,x_2)\in B} x_2 \leq \inf_{(a_1,x_2)\in A} x_2$ and the proof is finished. $\square$

Note that, in the vectorial framework, the closeness of A is the key-point in order to obtain the previous inequality. Indeed, let us consider for instance $A = \{(\frac{1}{n}, -1),\ n \in \mathbb{N}^\star\}$ and $B = A \cup \{(0,0)\}$. A simple calculation implies that $\overrightarrow{\inf} A = (0,-1) \prec \overrightarrow{\inf} B = (0,0)$.

We continue with the presentation of the lower/upper limits and lower/upper semicontinuity in the vectorial setting.

Definition 2.4. Let X be a metrisable space and $\Phi = (\Phi_i)_{i\in I} \colon X \mapsto \mathbb{R}^n \cup \{+\infty\}$ be a vector valued extended function. We define $\overrightarrow{\liminf_{x\longrightarrow x_0}} \Phi(x) = (c_i)_{i\in I}$ as follows:

$$\begin{cases} c_1 &= \liminf\limits_{x\longrightarrow x_0} \Phi_1(x) \text{ and} \\ c_i &= \liminf\limits_{\substack{x\longrightarrow x_0,\ x\in C_{i-1} \\ \Pi_{i-1}(\Phi(x))\longrightarrow(c_1,\cdots,c_{i-1})}} \Phi_i(x) \text{ for } 1 < i \leq n \end{cases}$$

Here $C_0 = \{\Phi(x) \mid x \in X\}$ and C_i is defined as in Definition 2.1 for $1 \leq i \leq n$.

Definition 2.5. Let X be a set, $\Phi \colon X \mapsto \mathbb{R}^n \cup \{+\infty\}$ be a vector valued extended function. The infimum of Φ is defined by $\overrightarrow{\inf}\, \Phi = \overrightarrow{\inf}\, \{\Phi(x) \mid x \in X\}$.

Definition 2.6. Let X be a topological space and $\Phi \colon X \mapsto \mathbb{R}^n \cup \{+\infty\}$ be a vector valued extended function. We say that Φ is vector level lower-semicontinuous if the sets $\{x \in X \mid \Phi(x) \lesssim \lambda\}$ are closed, for all $\lambda \in \mathbb{R}^n$.

We continue by pointing out some features of this class of functions.

Remark 2.7. We have the following assertions:

(1) If $\Phi\colon X \mapsto \mathbb{R}^n$ is vector level lower-semicontinuous, then Φ_1 is lower semicontinuous but, Φ_i is not necessary lower-semicontinuous, for $i > 1$. Indeed, an example is given by $\Phi\colon \mathbb{R} \mapsto \mathbb{R}^2$ defined by

$$\Phi(x) = \begin{cases} (1,0) & \text{if } x \neq 0, \\ (0,1) & \text{if } x = 0. \end{cases}$$

(2) If $\Phi_i\colon X \mapsto \mathbb{R}$ is lower semicontinuous for all $i \in I$, then $(\Phi_i)_{i\in I}$ can fail to be vector level lower semicontinuous. Indeed, an example is given by $\Phi\colon \mathbb{R} \mapsto \mathbb{R}^2$ defined by $\Phi(x) = (-|x|, 1)$. We can easily notice that for $\lambda = (0,0)$, the set $\{x \in \mathbb{R} \mid \Phi(x) \lesssim (0,0)\} = \mathbb{R}^\star$ is not closed.

Definition 2.8. Let X be a metrisable space and $\Phi\colon X \mapsto \mathbb{R}^n \cup \{+\infty\}$ be a vector valued extended function. We say that Φ is *vector lower-semicontinuous* if

$$\Phi(x_0) \lesssim \overrightarrow{\liminf_{x\longrightarrow x_0}}\, \Phi(x) \text{ for all } x_0 \in X.$$

Proposition 2.9. *Let X be a metrisable space and $\Phi\colon X \mapsto \mathbb{R}^n \cup \{+\infty\}$ be a vector valued extended function. If Φ is vector level lower-semicontinuous, then Φ is vector lower-semicontinuous.*

Proof. Let $x_0 \in X$. Suppose $\Phi(x_0) \succ \overrightarrow{\liminf_{x\longrightarrow x_0}}\, \Phi(x)$ and consider a vector $\beta \in \mathbb{R}^n$ with $\Phi(x_0) \succ \beta \succ \overrightarrow{\liminf_{x\longrightarrow x_0}}\, \Phi(x)$. We observe that the set $C_\beta = \{x \in X \mid \Phi(x) \succ \beta\}$ is open because Φ is vector lower-semicontinuous. Moreover, by definition we have that $x_0 \in C_\beta$. Therefore, there exists V a neighborough of x_0 included in C_β. By the definition of $\overrightarrow{\liminf}$, we deduce that $\overrightarrow{\liminf_{x\longrightarrow x_0}}\, \Phi(x) \gtrsim \beta$; which is obviously a contradiction. $\square$

Remark 2.10. If $\Phi_i\colon X \mapsto \mathbb{R}$ is lower-semicontinuous for all $i \in I$, then $(\Phi_i)_{i\in I}$ is vector lower-semicontinuous.

We continue with the study of families of functions with vector values.

Proposition 2.11. *Let X be a topological space. We assume that the map $\Phi^j\colon X \mapsto \mathbb{R}^n \cup \{+\infty\}$ is a vector level lower-semicontinuous function for all $j \in J$. Then $\overrightarrow{\sup_{j\in J}}\Phi^j$ is vector level lower-semicontinuous.*

Proof. Consider $\lambda \in \mathbb{R}^n$. We claim that

$$\{x \in X \mid \overrightarrow{\sup_{j\in J}}\, \Phi^j(x) \lesssim \lambda\} = \bigcap_{j\in J} \{x \in X \mid \Phi^j(x) \lesssim \lambda\}.$$

Let $x \in X$ be such that $\overrightarrow{\sup_{j\in J}}\, \Phi^j(x) \lesssim \lambda$. By applying Remark 2.2(i), we have that $\Phi^j(x) \lesssim \overrightarrow{\sup_{j\in J}}\, \Phi^j(x) \lesssim \lambda$ for all $j \in J$.

Hence, we obtain that

$$\{x \in X \mid \overrightarrow{\sup_{j\in J}}\, \Phi^j(x) \lesssim \lambda\} \subset \bigcap_{j\in J} \{x \in X \mid \Phi^j(x) \lesssim \lambda\}.$$

Conversely, let $x \in X$ be such that $\Phi^j(x) \lesssim \lambda$ for all $j \in J$. By applying Remark 2.2(ii), we obtain that $\overrightarrow{\sup}_{j\in J} \Phi^j(x) \lesssim \lambda$. Therefore our claim holds and consequently, the level set $\{x \in X \mid \overrightarrow{\sup}_{j\in J} \Phi^j(x) \lesssim \lambda\}$ is closed. □

Proposition 2.12. *Let X be a metrisable space and $\Phi^j \colon X \mapsto \mathbb{R}^n$, $j \in J$ be a family of vector lower-semicontinuous functions. Assume that for every $x \in X$ and for every $i \in I$, there exists $M_i(x) \in \mathbb{R}$ such that $\Phi^j_i(x) \le M_i(x)$, for all $j \in J$. Then $\overrightarrow{\sup}_{j\in J}\Phi^j$ is vector lower-semicontinuous and has values in $\mathbb{R}^n$.*

Proof. We start by observing that $\overrightarrow{\sup}_{j\in J}\Phi^j(x) \lesssim (M_i(x))_{i\in I}$ for all $x \in X$ and therefore, $\overrightarrow{\sup}_{j\in J}\Phi^j$ has values $\mathbb{R}^n$. We continue with the study of the lower-semicontinuity. We note that $\Phi^j_1 \colon X \longrightarrow \mathbb{R}$ is lower-semicontinuous for all $j \in J$ and consequently $\sup_{j\in J} \Phi^j_1$ is lower-semicontinuous. Consider $x \in X$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence converging to x such that $\liminf_{n\to\infty} \sup_{j\in J} \Phi^j_1(y_n) = \sup_{j\in J} \Phi^j_1(x)$. Observe that $(y_n)_{n\in\mathbb{N}}$ can be restricted to the constant sequence $(x)_{n\in\mathbb{N}}$. We obtain that

$$\Phi^j_2(x) \le \liminf_{n\to\infty} \Phi^j_2(y_n) \le \liminf_{n\to\infty} \sup_{j\in J} \Phi^j_2(y_n)$$

because Φ^j is vector lower-semicontinuous for all $j \in J$. Hence we obtain the following inequality

$$\sup_{j\in J} \Phi^j_2(x) \le \liminf_{n\to\infty} \sup_{j\in J} \Phi^j_2(y_n),$$

for all $j \in J$. Finally, we deduce that $\overrightarrow{\sup}_{j\in J} \Phi^j$ is vector lower-semicontinuous by induction. □

Proposition 2.13. *Let X be a compact subset of a topological space and assume $\Phi \colon X \longrightarrow \mathbb{R}^n$ to be vector lower-semicontinuous. Then there exist $\bar{x} \in X$ such that*

$$\Phi(\bar{x}) = \overrightarrow{\inf_{x\in X}} \Phi(x).$$

Proof. Observe that Φ_1 is lower-semicontinuous because Φ is vector lower-semicontinuous. Then, Weierstrass' theorem implies that there exists $x_1 \in X$ such that $\Phi_1(x_1) = \inf_{x\in X} \Phi_1(x)$. Moreover, as Φ_1 is lower-semicontinuous, the set of minimizers X_1 of Φ_1 is closed. Consequently, X_1 is compact. From the vector lower-semicontinuity of Φ, it follows that $(\Phi_i)_{i\ge 2}$ is vector lower-semicontinuous on X_1. Hence, there exists $x_2 \in X_1$ such that $\Phi_2(x_2) = \inf_{x\in X_1} \Phi_2(x)$.

Finally, we have that $(\Phi_i)_{i=1,2}(x_2) = \overrightarrow{\inf}_{x\in X}\ (\Phi_i)_{i=1,2}(x)$ and we conclude the proof by induction. □

3. Ky-Fan's theorem

Definition 3.1. Let X be a vector space. We say that $\Phi\colon X \longrightarrow \mathbb{R}^n$ is *convex* if

$$\Phi(tx+(1-t)y) \lesssim t\Phi(x)+(1-t)\Phi(y),\ \forall (x,y)\in X^2,\ \forall t\in[0,1]\,,$$

and $\Phi\colon X \longrightarrow \mathbb{R}^n$ is *quasi convex* if

$$\Phi(tx+(1-t)y) \lesssim \overrightarrow{\max}(\Phi(x),\Phi(y)),\ \forall (x,y)\in X^2,\ \forall t\in[0,1]\,.$$

Φ is concave if $-\Phi$ is convex and quasi concave if $-\Phi$ is quasi convex. □

Note that whenever $\Phi=(\Phi_i)$ is convex, Φ_1 is convex. Moreover, if Φ_1 is strictly convex then there is no condition on Φ_i for $i\geq 2$. Conversely, if Φ_i is convex for all $i\in I$, then Φ is convex.
We also remark that Φ beeing convex implies that Φ is quasi convex.
The quasi convexity of Φ is equivalent to the convexity of level sets i.e. the set $\{x\in X \mid \Phi(x)\lesssim\lambda\}$ is convex for all $\lambda\in\mathbb{R}^n$.

In the following we extend the Ky-Fan inequality as formulated in [1] to the lexicographic order framework.

Theorem 3.2. *Let K be a convex compact subset of a topological vector space and $\Phi\colon K\times K\longrightarrow\mathbb{R}^n$ be a function which verifies the following assumptions:*

(i) $\Phi(\cdot,y)$ *is vector level lower-semicontinuous for all* $y\in K$,

(ii) $\Phi(x,\cdot)$ *is quasi concave for all* $x\in K$,

(iii) $\Phi(y,y)\lesssim 0$ *for all* $y\in K$.

Then there exists $\bar{x}\in K$ such that $\Phi(\bar{x},y)\lesssim 0$ for all $y\in K$.

Proof. We denote by Ω the correspondence from K to K defined by

$$\Omega_y=\{x\in K \mid \phi(x,y)\lesssim 0\},\quad \text{for all } y\in K.$$

Assumption (i) implies that Ω_y is closed for all $y\in K$. To every $x\in K$ we associate the subset

$$\Omega^\star_x = K\backslash\Omega^{-1}_x=\{y\in K\mid 0\prec\Phi(x,y)\}.$$

Assumption (ii) implies that $\Phi(ty+(1-t)z)\gtrsim\min(\Phi(y),\Phi(z))\succ 0$, for all $x\in K$, $y,z\in\Omega^\star_x$, and $t\in[0,1]$. Consequently, $\Omega^\star_x$ is convex. Moreover, we have that $y\in\Omega_y$, $\forall y\in K$ because of (iii). We conclude by applying Theorem 1.1. □

Corollary 3.3. *Let U and V be two convex compact subspaces of two topological vector spaces. Consider that $\Phi\colon U\times V\longrightarrow\mathbb{R}^n$ is a function satisfying the following assumptions:*

(i) $\Phi(\cdot,v)$ *is vector level lower-semicontinuous and convex for all* $v\in V$,

(ii) $\Phi(u,\cdot)$ *is vector level upper-semicontinuous and concave for all* $u\in U$.

Then the following equality holds:

$$\overrightarrow{\min_{u\in U}}\ \overrightarrow{\max_{v\in V}}\ \Phi(u,v)=\overrightarrow{\max_{v\in V}}\ \overrightarrow{\min_{u\in U}}\ \Phi(u,v).$$

Proof. For $x = (u, v)$ and $y = (u', v')$ we define $f\colon (U \times V)^2 \to \mathbb{R}^n$ by

$$f(x, y) = \phi(u, v') - \phi(u', v).$$

Note that the function $f(\cdot, y)$ is vector level lower-semicontinuous for all $y \in U \times V$ and the function $f(x, \cdot)$ is concave for all $x \in U \times V$. Moreover, we have that $f(y, y) = 0$, for all $y \in U \times V$. By applying the previous theorem we obtain that there exists $(\bar{u}, \bar{v}) \in U \times V$ such that

$$\phi(\bar{u}, v) - \phi(u, \bar{v}) \lesssim 0, \text{ for all } (u, v) \in U \times V.$$

Hence, the following inequality holds

$$\phi(\bar{u}, v) \lesssim \phi(\bar{u}, \bar{v}) \lesssim \phi(u, \bar{v}), \text{ for all } (u, v) \in U \times V.$$

Remark 2.2 implies that $\overrightarrow{\sup}_{v \in V} \Phi(\bar{u}, v) \lesssim \Phi(\bar{u}, \bar{v})$ and that

$$\overrightarrow{\inf}_{u \in U} \overrightarrow{\sup}_{v \in V} \Phi(u, v) \lesssim \Phi(\bar{u}, \bar{v}) \lesssim \overrightarrow{\sup}_{v \in V} \overrightarrow{\inf}_{u \in U} \Phi(u, v).$$

Using again Remark 2.2 we have $\overrightarrow{\inf}_{u \in U} \Phi(u, v) \lesssim \Phi(u, v) \lesssim \overrightarrow{\sup}_{v \in V} \Phi(u, v)$ for all $u \in U$ and all $v \in V$. Consequently $\overrightarrow{\inf}_{u \in U} \Phi(u, v) \lesssim \overrightarrow{\inf}_{u \in U} \overrightarrow{\sup}_{v \in V} \Phi(u, v)$. It follows that

$$\overrightarrow{\sup}_{v \in V} \overrightarrow{\inf}_{u \in U} \Phi(u, v) \lesssim \overrightarrow{\inf}_{u \in U} \overrightarrow{\sup}_{v \in V} \Phi(u, v).$$

Therefore, we have the following equality

$$\overrightarrow{\min}_{u \in U} \overrightarrow{\max}_{v \in V} \Phi(u, v) = \overrightarrow{\max}_{v \in V} \overrightarrow{\min}_{u \in U} \Phi(u, v) = \Phi(\bar{u}, \bar{v})$$

which concludes the proof. □

4. Nash equilibrium

This subsection is concerned with a compact game G with n players in a topological vector space, E. More precisely, every player $i \in I$ has a non-empty, compact set of strategies denoted by X_i with $X_i \subset E$ and a preference which is a function $u_i\colon X \to \mathbb{R}^{n_i}$ with $n_i \in \mathbb{N}^*$ and $X = \prod_{i=1}^n X_i$.

We continue by introducing several notations and notions that we use in the rest of the paper. We denote by $u\colon X \longrightarrow \mathbb{R}^{\prod n_i}$ the vector $u(x) = (u_1(x), ..., u_n(x))$, by $X_{-i} = \prod_{j \neq i,\, j=1}^n X_j$ and by $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in X_{-i}$ for all $i \in I$.

Whenever X_i is convexe for all $i \in I$ and $u_i(\cdot, x_{-i})$ is quasi concave on X_i for all $x_{-i} \in X_{-i}$ we say that the game $G = (X_i, u_i)_{i \in I}$ is quasi concave.

In the following we present a result stating the existence of a Nash equilibrium i. e. a point $x \in X$ which satisfies $u_i(x_{-i}, y_i) \lesssim u_i(x)$ for all $y \in X$ and $i \in I$.

Theorem 4.1. *Consider a compact game $G = (X_i, u_i)_{i \in I}$. Assume that the function $\Phi = (\Phi_i)_{i\in I} \colon X \times X \longrightarrow \mathbb{R}^{\prod n_i}$ defined by $\Phi_i(x, y) = u_i(x_{-i}, y_i) - u_i(x)$, for all $i \in I$, is vector level lower semi-continuous with respect to x. Moreover, assume that the function $u\colon \prod_{j\in I} X_j \times \prod_{j\in I} X_j \longrightarrow \mathbb{R}^{\prod n_i}$ defined by $u(x, y) = (u_i(x_{-i}, y_i))_{i\in I}$ is quasi concave with respect to y. Then the game G admits a Nash equilibrium.*

Proof. Note that Φ verifies the assumptions of Theorem 3.2. Consequently, there exists $\bar{x} \in \prod X_i$ such that $\Phi(\bar{x}, y) \lesssim 0$ for all $y \in \prod X_i$. The conclusion follows if we apply the previous inequality to $y = (\bar{x}_{-i}, y_i)$ i. e. $u_i(\bar{x}) \gtrsim_{\mathbb{R}^{n_i}} u_i(\bar{x}_{-i}, y_i)$ for all $i \in I$. □

Note that Theorem 4.1 holds whenever Φ is vector level lower-semicontinuous. In case where the continuity hypotheses are made on the preferences u_i, $i \in I$, the existence of a Nash equilibrium can be derived by using of the Kakutany-Glicksberg's fixed point theorem which we recall in the following.

Theorem 4.2. (Kakutany-Glicksberg) *Let C be a non empty convex compact subset of a locally convex separated vector space and F be a correspondence from C to C such that:*

(i) *$F(c)$ is non empty, convex and compact for all $c \in C$,*

(ii) *The graph of F is closed.*

Then there exists $c \in C$ such that $c \in F(c)$.

Theorem 4.3. *We assume that E is metrisable and consider the compact quasi concave game $G = (X_i, u_i)_{i\in I}$. We assume that u_i^j is upper-semicontinuous for all $j \in \{1, \cdots, n_i\}$ and the function $u_i^j(y_i, \cdot)$ is continuous for all $y_i \in X_i$ and $i \in I$. Then the game admits a Nash equilibrium.*

Proof. Consider the best response correspondence $MR_i\colon \prod_{j\in I} X_j \longrightarrow X_i$ which satisfies $MR_i(x) = \arg\max_{y_i\in X_i} u_i(y_i, x_{-i})$ for all $i \in I$.

Let $MR(x) = \prod_{i\in I} MR_i(x_{-i})$. Proposition 2.13 implies that $MR_i(x)$ is non empty and convex because $u_i(\cdot, x_{-i})$ is vector upper semi-continuous and quasi concave for all $x \in \prod_{i\in I} X_i$ and $i \in I$.

Let $\left(x^k, y^k\right)_{k\in\mathbb{N}}$ be a sequence belonging to $graph(MR)$[2] converging to some (x, y). We have that $u_i(x^k_{-i}, y^k_i) \gtrsim u_i(x^k_{-i}, y)$ for every $k \in \mathbb{N}$, $i \in I$ and for all $y \in X_i$ by the definition of MR_i. Recall that u_i is vector upper-semi-continuous and $u_i(\cdot, y)$ is vector lower semi-continuous for all $i \in I$. Consequently, $u_i(\bar{x}_{-i}, \bar{y}_i) \gtrsim u_i(\bar{x}_{-i}, y)$ for all $i \in I$. Hence, the $graph(MR)$ is closed. We conclude by applying Theorem 4.2. □

Note that whenever for each player i, $n_i = 1$ and u_i is continuous, we obtain the usual Nash equilibrium. Moreover, under the assumptions of Theorem 4.1, $MR(x)$ is convex for all $x \in \prod_{i\in I} X_i$. However, we would like to underline that this set can be empty and that the graph of the correspondence is not necessarily closed.

[2]Here we denote by $graph(MR) = \left\{(x, y) \in \prod_{j\in I} X_j \times \prod_{j\in I} X_j, y \in MR(x)\right\}$.

5. Sion's theorem

This section is dedicated to our main results. More precisely, we extend Corollary 3.3 to quasi convex and respectively, quasi concave functions. Note that the case $n = 1$ is nothing else that the well known Sion's theorem.

Theorem 5.1. *Let U be a convex compact subset and V be a convex subset of two topological vector spaces and $\Phi\colon U \times V \longrightarrow \mathbb{R}^n$ a function which satisfies the following assumptions:*

(i) $\Phi(\cdot, v)$ *is vector level lower-semicontinuous and quasi convex for all $v \in V$,*

(ii) $\Phi(u, \cdot)$ *is vector level upper-semicontinuous and quasi concave for all $u \in U$.*

Then the following equality holds: $\overrightarrow{\inf_{u\in U}}\ \overrightarrow{\sup_{v\in V}}\ \Phi(u, v) = \overrightarrow{\sup_{v\in V}}\ \overrightarrow{\inf_{u\in U}}\ \Phi(u, v)$.

Before proving the previous main result we need to state the following lemma:

Lemma 5.2. *Let U be a convex compact subset and V be a convex subset of two topological vector spaces and let the assumptions* (i) *and* (ii) *of Theorem* 5.1 *hold. Then, for any finite subset $(v_j)_{j\in J}$ of V and any $\lambda \in \mathbb{R}^n$ with the property $\lambda \prec \overrightarrow{\inf_{u\in U}}\ \max_{j\in J} \Phi(u, v_j)$, there exists $v \in V$ verifying $\lambda \prec \overrightarrow{\inf_{u\in U}}\ \Phi(u, v)$.*

Proof. Note that it is enough consider the case $J = \{1, 2\}$. Assume that for all $v \in V$ we have $\lambda \gtrsim \overrightarrow{\inf_{u\in U}}\ \Phi(u, v)$ and choose β such that $\lambda \prec \beta \prec \overrightarrow{\inf_{u\in U}}\ \max_{j\in J} \Phi(u, v_j)$. Denote by $[v_1, v_2]$ the segment joining v_1 and v_2. For each $y \in [v_1, v_2]$, we define two sets by

$$C_y = \{u \in U \mid \Phi(u, y) \lesssim \lambda\} \text{ and } C'_y = \{u \in U \mid \Phi(u, y) \lesssim \beta\}.$$

We denote by $A = C'_{y_1}$ and $B = C'_{y_2}$. We observe that C_y, C'_y, A and B are non empty because of Proposition 2.13. Moreover, they are closed because $\Phi(\cdot, y)$ is vector level lower-semicontinuous and $A \cap B = \emptyset$. The quasi concavity of $\Phi(u, \cdot)$ implies that

$$\Phi(u, y) \gtrsim \min\left(\Phi(u, v_1), \Phi(u, v_2)\right)$$

for all $u \in U$, $y \in [v_1, v_2]$. Therefore we have $C'_y \subset A \cup B$. As $\Phi(\cdot, y)$ is quasi convexe, C'_y is convex and consequently connected. Hence $C_y \subset C'_y \subset A$ or $C_y \subset C'_y \subset B$. We continue by defining the two following sets

$$I' = \{y \in [v_1, v_2] \mid C_y \subset A\} \text{ and } J' = \{y \in [v_1, v_2] \mid C_y \subset B\}.$$

We notice that I' and J' are non empty and satisfy $I' \cap J' = \emptyset$ and $I' \cup J' = [v_1, v_2]$. Consider $(y_k)_{k\in\mathbb{N}}$ a sequence in I' converging to some $y \in [v_1, v_2]$. Moreover, we have that $\Phi(u, y) \lesssim \lambda \prec \beta$ for any $u \in C_y$. Suppose that $\Phi(u, y_k) \gtrsim \beta$ for all k. Then we have $\Phi(u, y) \gtrsim \beta$ because $\Phi(u, \cdot)$ is vector level upper-semicontinuous. Hence there exists k such that $\Phi(u, y_k) \prec \beta$. Consequently, $u \in C'_{y_k}$. We deduce that $C'_{y_k} \subset A$ and $u \in A$, since $C_{y_k} \subset A$. Consequently, we have $y \in I'$. Hence I' is closed. Similarly, we can show that the set J' is closed. This contradicts the connectedness of $[v_1, v_2]$. □

Proof of Theorem 5.1. We start by observing that the following inequality is obvious:

$$\overrightarrow{\sup_{v\in V}}\ \overrightarrow{\inf_{u\in U}}\ \Phi(u,v) \lesssim \overrightarrow{\inf_{u\in U}}\ \overrightarrow{\sup_{v\in V}}\ \Phi(u,v).$$

We continue by proving the reverse inequality. Denote by X_v the compact set $\{u \in U \mid \Phi(u,v) \lesssim \lambda\}$ for any $\lambda \prec \overrightarrow{\inf_{u\in U}}\ \overrightarrow{\sup_{v\in V}}\ \Phi(u,v)$ and any $v \in V$ and note that $\bigcap_{v\in V} X_v = \emptyset$. Consequently, there exists a finite family $(v_j)_{j\in J}$ in V such that $\bigcap_{j\in J} X_{v_j} = \emptyset$. Hence $\lambda \prec \overrightarrow{\inf_{u\in U}}\ \overrightarrow{\max_{i\in I}}\Phi(u,v_i)$. Lemma 5.2 implies that there exists $v \in V$ such that $\lambda \prec \overrightarrow{\inf_{u\in U}}\ \Phi(u,v)$. Consequently, $\lambda \prec \overrightarrow{\sup_{v\in V}}\ \overrightarrow{\inf_{u\in U}}\ \Phi(u,v)$. Therefore the following inequality holds: $\overrightarrow{\inf_{u\in U}}\ \overrightarrow{\sup_{v\in V}}\ \Phi(u,v) \lesssim \overrightarrow{\sup_{v\in V}}\ \overrightarrow{\min_{u\in U}}\ \Phi(u,v)$. □

Note that the level upper and lower vector semi-continuity can be weakened.

Theorem 5.3. *Let U be a convex compact subset, V a convex subset of two metrisable vector spaces and $\Phi = (\Phi_i)_{i\in I} : U \times V \longrightarrow \mathbb{R}^n$ a function such that Φ_i is bounded for all $i \in I$ and which verifies the following assumptions:*

(i) *$\Phi(\cdot, v)$ is quasi convex for all $v \in V$ and $\Phi_i(\cdot, v)$ is lower-semicontinuous for all $i \in I$,*

(ii) *$\Phi(u, \cdot)$ is quasi concave for all $u \in U$ and $\Phi_i(u, \cdot)$ is upper-semicontinuous for all $i \in I$.*

Then the following equality holds: $\overrightarrow{\inf_{u\in U}}\ \overrightarrow{\sup_{v\in V}}\ \Phi(u,v) = \overrightarrow{\sup_{v\in V}}\ \overrightarrow{\inf_{u\in U}}\ \Phi(u,v)$.

Proof. We begin by noticing that the inequality

$$\overrightarrow{\sup_{v\in V}}\ \overrightarrow{\inf_{u\in U}}\ \Phi(u,v) \lesssim \overrightarrow{\inf_{u\in U}}\ \overrightarrow{\sup_{v\in V}}\ \Phi(u,v)$$

is obvious. We continue by observing that the equality

$$\inf_{u\in U}\ \sup_{v\in V}\ \Phi_1(u,v) = \sup_{v\in V}\ \inf_{u\in U}\ \Phi_1(u,v) = \lambda_1$$

holds by Theorem 5.1.

Next, we consider λ_2 such that $(\lambda_i)_{i=1,2} \prec \overrightarrow{\inf_{u\in U}}\ \overrightarrow{\sup_{v\in V}}\ (\Phi_i(u,v))_{i=1,2}$ and $v \in V$.

We denote by X_v the set $\{u \in U \mid (\Phi_i)_{i=1,2}(u,v) \lesssim (\lambda_i)_{i=1,2}\}$. Note that X_v is closed because $\Phi_i(\cdot, v)$ is lower semicontinuous for i=1, 2. As $\bigcap_{v\in V} X_v = \emptyset$, there exists a finite family $(v_j)_{j\in J}$ in V such that $\bigcap_{j\in J} X_{v_j} = \emptyset$. Hence we obtain that

$$\lambda \prec \overrightarrow{\inf_{u\in U}}\ \max_{j\in J}\,((\Phi_i(u,v_j))_{i=1,2}).$$

Observe that there exist some $v_{j_0} \in (v_j)_{j\in J}$ such that $\inf_{u\in U} \Phi_1(u, v_{j_0}) = \lambda_1$.

Consider that $\inf_{u\in U} \Phi_1(u, v_j) < \lambda_1$ for $j \in J$, so that we have

$$\overrightarrow{\inf_{u\in U}} \left(\Phi_i(u, v_j)\right)_{i=1,2} \lesssim \overrightarrow{\inf_{u\in U}} \left(\Phi_i(u, v_{j_0})\right)_{i=1,2}.$$

Similarly with the proof of Lemma 5.2, we will show that there exists $v \in V$ verifying $\lambda \prec \overrightarrow{\inf_{u\in U}} (\Phi_i(u, v))_{i=1,2}$. By iteration, it is enough to consider $J = \{1, 2\}$ and $\inf_{u\in U} \Phi_1(u, v_j) = \lambda_1$ for $j \in J$. We assume that $\lambda \gtrsim \overrightarrow{\inf_{u\in U}} (\Phi_i(u, v))_{i=1,2}$ for all $v \in V$ and we choose $\beta = (\lambda_1, \beta_2)$ such that $\lambda \prec \beta \prec \overrightarrow{\inf_{u\in U}} \max_{j\in J}((\Phi_i(u, v_j))_{i=1,2})$. For each $y \in [v_1, v_2]$, we define

$$C_y = \{u \in U \mid (\Phi_i(u, y))_{i=1,2} \lesssim \lambda\},\ C'_y = \{u \in U \mid (\Phi_i(u, y))_{i=1,2} \lesssim \beta\},$$

$A = C'_{v_1}$ and $B = C'_{v_2}$. Note that C_y, C'_y, A and B are non empty, closed because of the lower-semicontinuity of $\Phi_i(\cdot, y)$ for $i = 1, 2$ and $A \cap B = \emptyset$. Moreover, the quasi concavity of $\Phi(u, \cdot)$ implies that

$$(\Phi_i(u, y))_{i=1,2} \gtrsim \min\ \left((\Phi_i(u, v_1))_{i=1,2}, (\Phi_i(u, v_2))_{i=1,2}\right), \tag{1}$$

for all $u \in U$, $y \in [u_1, u_2]$. It follows that $C'_y \subset A \cup B$. Additionally, C'_y is convex and consequently connexe because $(\Phi_i(\cdot, y))_{i=1,2}$ is quasi convexe. Hence $C_y \subset C'_y \subset A$ or $C_y \subset C'_y \subset B$. We continue by defining by

$$I' = \{y \in [v_1, v_2] \mid C_y \subset A\} \text{ and } J' = \{y \in [v_1, v_2] \mid C_y \subset B\}.$$

We observe that I' and J' are non empty, $I' \cap J' = \emptyset$ and $I' \cup J' = [v_1, v_2]$. Consider $(y_k)_{k\in\mathbb{N}}$ a sequence in I' converging to some $y \in [v_1, v_2]$. We have that $(\Phi_i(u, y))_{i=1,2} \lesssim \lambda \prec \beta$ for any $u \in C_y$. Consequently $\Phi_1(u, y) \leq \lambda_1$. As $\Phi_1(u, v_1) \geq \lambda_1$ and $\Phi_1(u, v_2) \geq \lambda_1$, (1) implies that $\Phi_1(u, v_1) = \lambda_1$ and $\Phi_1(u, v_2) = \lambda_1$. Therefore, $\Phi_1(u, z) = \lambda_1$ for all $z \in [v_1, v_2]$ and $\Phi_1(u, y_k) = \lambda_1$ for any $k \in \mathbb{N}$ because $\lambda_1 = \inf_{u\in U} \sup_{v\in V} \Phi_1(u, v) = \sup_{v\in V} \inf_{u\in U} \Phi_1(u, v)$ and $\Phi_1(u, \cdot)$ is quasi concave. Assume that we have $(\Phi_i(u, y_k))_{i=1,2} \gtrsim \beta$ for all $k \in \mathbb{N}$. Then $(\Phi_i(u, y))_{i=1,2} \gtrsim \beta$ because $\Phi_2(u, \cdot)$ is upper-semicontinuous. Hence, there exists $k \in \mathbb{N}$ such that $\Phi(u, y_k) \prec \beta$. Consequently, $u \in C'_{y_k}$. Since $C_{y_k} \subset A$, we deduce that $C'_{y_k} \subset A$ and $u \in A$. Therefore, $y \in I$ and I' is closed. Similarly we can prove that the set J' is closed. This contradicts the connectedness of $[v_1, v_2]$. Therefore, we have that

$$\overrightarrow{\inf_{u\in U}}\, \overrightarrow{\sup_{v\in V}} \left(\phi_i(u, v)\right)_{i=1,2} \lesssim \overrightarrow{\sup_{v\in V}}\, \overrightarrow{\inf_{u\in U}} \left(\Phi_i(u, v)\right)_{i=1,2}.$$

Repeating the same arguments, we conclude that

$$\overrightarrow{\inf_{u\in U}}\, \overrightarrow{\sup_{v\in V}} \phi(u, v) \lesssim \overrightarrow{\sup_{v\in V}}\, \overrightarrow{\inf_{u\in U}} \Phi(u, v)$$

and we obtain the desired equality. $\square$

We finish this section by proving the existence of a saddle point which is very important for our application to min-max control problems.

Theorem 5.4. *Let U, V be two convex compact subsets of metrisable vector spaces, $\Phi = (\Phi_i)_{i\in I} : U \times V \longrightarrow \mathbb{R}^n$ a function such that Φ_i is bounded for all $i \in I$ and which verifies the following assumptions:*

(i) *$\Phi(\cdot, v)$ is quasi convex for all $v \in V$, $\Phi_i(\cdot, v)$ is lower-semicontinuous for all $i \in I$,*

(ii) *$\Phi(u, \cdot)$ is quasi concave for all $u \in U$, $\Phi_i(u, \cdot)$ is upper-semicontinuous for all $i \in I$.*

Then the following equality holds: $\overrightarrow{\inf}_{u\in U} \overrightarrow{\sup}_{v\in V} \Phi(u, v) = \overrightarrow{\sup}_{v\in V} \overrightarrow{\inf}_{u\in U} \Phi(u, v)$.

Moreover, there exist $\bar{u} \in U$, $\bar{v} \in V$ such that

$$\Phi(\bar{u}, \bar{v}) = \overrightarrow{\inf_{u\in U}} \overrightarrow{\sup_{v\in V}} \Phi(u, v) = \overrightarrow{\sup_{v\in V}} \overrightarrow{\inf_{u\in U}} \Phi(u, v).$$

Proof. We begin by noticing that the following inequality is obvious:

$$\overrightarrow{\sup_{v\in V}} \overrightarrow{\inf_{u\in U}} \Phi(u, v) \lesssim \overrightarrow{\inf_{u\in U}} \overrightarrow{\sup_{v\in V}} \Phi(u, v).$$

Moreover, we have that $\overrightarrow{\sup}_{v\in V} \Phi(\cdot, v)$ is vector lower semicontinuous by Proposition 2.11. Additionally, there exists $\tilde{u} \in U$ such that $\overrightarrow{\sup}_{v\in V} \Phi(\tilde{u}, v) = \overrightarrow{\inf}_{u\in U} \overrightarrow{\sup}_{v\in V} \Phi(u, v)$ by Proposition 2.13. Hence, using similar arguments as in the proofs of Theorem 5.3 and Lemma 5.2, we obtain the following equalities:

$$\overrightarrow{\inf_{u\in U}} \overrightarrow{\sup_{v\in V}} \Phi(u, v) = \min_{u\in U} \max_{v\in V} \Phi(u, v)$$

and

$$\overrightarrow{\sup_{v\in V}} \overrightarrow{\inf_{u\in U}} \Phi(u, v) = \max_{v\in V} \min_{u\in U} \Phi(u, v).$$

Consequently, there exist $\bar{u} \in U$, $\bar{v} \in V$ such that

$$\Phi(\bar{u}, \bar{v}) = \overrightarrow{\inf_{u\in U}} \overrightarrow{\sup_{v\in V}} \Phi(u, v) = \overrightarrow{\sup_{v\in V}} \overrightarrow{\inf_{u\in U}} \Phi(u, v). \qquad \square$$

6. Application to vectorial min-max control problems

We start by presenting the setting of min-max control problems.

6.1. The bi-controlled dynamics

In this section, we consider the following bi-controlled system

$$\begin{cases} dx_t^{t_0,x_0,u} = f\left(t, x_t^{t_0,x_0,u}, u_t\right) dt, \\ dy_t^{t_0,y_0,v} = g\left(t, y_t^{t_0,y_0,v}, v_t\right) dt, \ t \geq t_0, \\ x_{t_0}^{t_0,x_0,u} = x_0 \in \mathbb{R}^N, \ y_{t_0}^{t_0,y_0,v} = y_0 \in \mathbb{R}^M. \end{cases} \tag{2}$$

We denote by $T > 0$ a finite time horizon. Here $t_0 \in [0, T]$, the control u (respectively v) takes its values in a compact, metric space U (respectively V) and is associated to the first (respectively the second) player. We recall that a control $(u_t)_{t_0 \le t \le T}$ is Lebesgue-measurable on $[t_0, T]$ and we let $\mathcal{U}$ denote the family of all controls on $[0, T]$. Similarly, one defines the family of, V-valued control functions v and let $\mathcal{V}$ denote this family. We assume that the coefficient functions $f\colon \mathbb{R} \times \mathbb{R}^N \times U \longrightarrow \mathbb{R}^N$ and $g\colon \mathbb{R} \times \mathbb{R}^M \times V \longrightarrow \mathbb{R}^M$ satisfy

$$\begin{cases} f \text{ and } g \text{ are bounded and uniformly continuous and} \\ |f(s,x,u) - f(t,x',u)| \le c\,(|x - x'| + |s - t|)\,, \\ |g(s,y,v) - g(t,y',v)| \le c\,(|y - y'| + |s - t|)\,, \end{cases} \tag{3}$$

for all $(s,t) \in \mathbb{R}^2$, $(x,x') \in \mathbb{R}^{2N}$, $(y,y') \in \mathbb{R}^{2M}$, $(u,v) \in U \times V$, for some positive real constant $c > 0$. Under these assumptions, for every control couple $u \in \mathcal{U}, v \in \mathcal{V}$ and every initial data $(t_0, x_0, y_0) \in [0,T] \times \mathbb{R}^{N+M}$, the system (2) admits a unique solution denoted by $\left(x_\cdot^{t_0,x_0,u}, y_\cdot^{t_0,y_0,v}\right)$.

6.2. Occupation measures

We start by recalling some results obtained in [25]. We fix $t_0 > 0$ and $x_0 \in \mathbb{R}^N$. To every $t > t_0$ and $u \in \mathcal{U}$, one can associate a couple of occupation measures $\gamma^{t_0,t,x_0,u} = \left(\gamma_1^{t_0,t,x_0,u}, \gamma_2^{t_0,t,x_0,u}\right) \in \mathcal{P}\left([t_0,t] \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right)$ defined by

$$\begin{cases} \gamma_1^{t_0,t,x_0,u}(A \times B \times C) = \frac{1}{t-t_0}\int_{t_0}^t 1_{A\times B\times C}\left(s, x_s^{t_0,x_0,u}, u_s\right) ds, \\ \gamma_2^{t_0,t,x_0,u} = \delta_{x_t^{t_0,x_0,u}}, \end{cases}$$

for all Borel sets $A \subset [t_0,t]$, $B \subset \mathbb{R}^N$ and $C \subset U$. Here, $\delta_\cdot$ stands for the Dirac measure. One can also define $\left(\gamma_1^{t_0,t_0,x_0,u}, \gamma_2^{t_0,t_0,x_0,u}\right) \in \mathcal{P}\left(\{t_0\} \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right)$ by setting

$$\gamma_1^{t_0,t_0,x_0,u} = \delta_{t_0,x_0,u_{t_0}}, \gamma_2^{t_0,t_0,x_0,u} = \delta_{x_0}.$$

For every $t \ge t_0$, one can consider the family of occupational couples

$$\Gamma^1(t_0,t,x_0) = \left\{ \ \left(\gamma_1^{t_0,t,x_0,u}, \gamma_2^{t_0,t,x_0,u}\right),\ u \in \mathcal{U} \ \right\} \tag{4}$$

as an alternative to controls.

We denote as usual by $C_b^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right)$ the class of bounded, differentiable functions $\psi\colon \mathbb{R}_+ \times \mathbb{R}^N \longrightarrow \mathbb{R}$ for which the first order derivatives are continuous and bounded. Whenever $\psi \in C_b^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right)$,

$$\begin{aligned} &-\psi(t_0,x_0) + \int_{\mathbb{R}^N} \psi(t,x)\,\gamma_2^{t_0,t,x_0,u}(dx) = -\psi(t_0,x_0) + \psi\left(t, x_t^{t_0,x_0,u}\right) \\ &= \int_{t_0}^t \left(\partial_s\psi\left(s, x_s^{t_0,x_0,u}\right) + \left\langle f\left(s, x_s^{t_0,x_0,u}\right), D\psi\left(s, x_s^{t_0,x_0,u}\right)\right\rangle\right) ds \\ &= \int_{[t_0,t]\times\mathbb{R}^N\times U} \left(\partial_s\psi(s,x) + \langle D\psi(s,x), f(s,x,u)\rangle\right) \gamma_1^{t_0,t,x_0,u}(dsdydu)\,. \end{aligned}$$

It follows that $\Gamma^1(t_0, t, x_0)$ can be embedded into the larger set

$$\Theta^1(t_0, t, x_0) = \left\{ \begin{array}{l} \gamma \in \mathcal{P}\left([t_0, t] \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right) : \forall \psi \in C_b^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right) \\ \psi(t_0, x_0) + \int\limits_{[t_0,t]\times\mathbb{R}^N\times U} (\partial_s \psi(s,x) + \langle D\psi(s,x), f(s,x,u)\rangle)\gamma_1(dsdxdu) \\ = \int\limits_{\mathbb{R}^N} \psi(t,x)\gamma_2(dx) \end{array} \right\}. \quad (5)$$

In fact, one can give a more precise relationship between the two sets.
The following result is taken from [23] (see also [24]):

Corollary 6.1. ([23], Corollary 2.5.) *The set $\Theta^1(t_0, t, x_0)$ is the closed convex hull of the family of occupational couples $\Gamma^1(t_0, t, x_0)$*

$$\Theta^1(t_0, t, x_0) = \overline{co}\left(\Gamma^1(t_0, t, x_0)\right),$$

*for all $t \geq t_0 \geq 0$. The operator $\bar{\cdot}$ designates the closure with respect to the topology induced by the weak (weak *) convergence of probability measures.*

Moreover, from [25] we have that whenever $\gamma \in \Theta^1(t_0, t, x_0)$

$$Supp(\gamma_1) \subset [t_0, t] \times K_{x_0,c} \times U, \; Supp(\gamma_2) \subset K_{x_0,c}.$$

where $K_{x_0,c} = \left\{x \in \mathbb{R}^N : |x| \leq c(1 + |x_0|)\right\}$. The constant c can be chosen independently of t_0, x_0, y_0, u, v and $\gamma_1(ds, K_{x_0,c} \times U) = \frac{1}{t-t_0}ds$. In particular, $\Theta^1(t_0, t, x_0)$ is compact.

Similarly, for every $0 \leq t_0 \leq t \leq T$ and every $y_0 \in \mathbb{R}^M$, one can define the sets of constraints for the second player by setting

$$\Gamma^2(t_0, t, y_0) = \left\{\left(\eta_1^{t_0,t,y_0,v}, \eta_2^{t_0,t,y_0,v}\right), \; v \in \mathcal{V}\right\}, \;\text{ and} \quad (6)$$

$$\Theta^2(t_0, t, y_0) = \left\{ \begin{array}{l} \eta \in \mathcal{P}\left([t_0, t] \times \mathbb{R}^M \times V\right) \times \mathcal{P}\left(\mathbb{R}^M\right) : \forall \phi \in C_b^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^M\right) \\ \phi(t_0, y_0) + \int\limits_{[t_0,t]\times\mathbb{R}^M\times U} (\partial_s \phi(s,y) + \langle D\phi(s,y), g(s,y,v)\rangle)\eta_1(dsdydv) \\ = \int\limits_{\mathbb{R}^M} \phi(t,y)\eta_2(dy) \end{array} \right\}. \quad (7)$$

As before,

(1) The set $\Theta^2(t_0, t, y_0)$ is the closed convex hull of the family of occupation couples $\Gamma^2(t_0, t, y_0)$.

(2) Whenever $\eta = (\eta_1, \eta_2) \in \Theta^2(t_0, t, y_0)$,

$$Supp(\eta_1) \subset [t_0, t] \times K_{y_0,c} \times V, \; Supp(\eta_2) \subset K_{y_0,c},$$

where $K_{y_0,c} = \left\{y \in \mathbb{R}^M : |y| \leq c(1 + |y_0|)\right\}$.

(3) Whenever $\eta = (\eta_1, \eta_2) \in \Theta^2(t_0, t, y_0)$, and if $t > t_0$ the marginal measure satisfies
$$\eta_1 \left(ds, K_{y_0,c} \times V\right) = \frac{1}{t - t_0} ds.$$

Similarly, for a control couple $(u, v) \in \mathcal{U} \times \mathcal{V}$ one introduces an occupation measure

$$\zeta^{t_0,t,x_0,y_0,u,v} = \left(\zeta_1^{t_0,t,x_0,y_0,u,v}, \zeta_2^{t_0,t,x_0,y_0,u,v}\right) \in \mathcal{P}\left([t_0, t] \times \mathbb{R}^{N+M} \times U \times V\right) \times \mathcal{P}\left(\mathbb{R}^{N+M}\right)$$

defined by

$$\begin{cases} \zeta_1^{t_0,t,x_0,y_0,u,v} (A \times B_1 \times B_2 \times C_1 \times C_2) \\ \qquad = \dfrac{1}{t - t_0} \displaystyle\int_{t_0}^{t} 1_{A \times B_1 \times B_2 \times C_1 \times C_2} (s, x_s^{t_0,x_0,u}, y_s^{t_0,y_0,v}, u_s, v_s) ds, \\ \zeta_2^{t_0,t,x_0,y_0,u,v} = \delta_{x_T^{t_0,x_0,u}, y_T^{t_0,y_0,v}}, \end{cases}$$

for all Borel sets $A \subset [t_0, t]$, $B_1 \times B_2 \subset \mathbb{R}^{N+M}$ and $C_1 \times C_2 \subset U \times V$. The set of such occupation measures is denoted by $\Gamma^{1,2}(t_0, t, x_0, y_0)$. These measures have as marginals the elements in $\Gamma^1(t_0, t, x_0)$, respectively $\Gamma^2(t_0, t, y_0)$. One can embed $\Gamma^{1,2}(t_0, t, x_0, y_0)$ into a larger convex, compact set $\Theta^{1,2}(t_0, t, x_0, y_0)$ such that $\Theta^{1,2}(t_0, t, x_0, y_0) = \overline{co}\Gamma^{1,2}(t_0, t, x_0, y_0)$ with

$$\Theta^{1,2}(t_0, t, x_0, y_0) = \left\{ \begin{array}{l} \zeta \in \mathcal{P}\left([t_0, t] \times K_{x_0,c} \times K_{y_0,c} \times U \times V\right) \times \mathcal{P}\left(K_{x_0,c} \times K_{y_0,c}\right) : \\ \forall \phi \in C_b^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^{N+M}\right) \\ \displaystyle\int_{[t_0,t] \times \mathbb{R}^{N+M} \times U \times V} \left(\begin{array}{c} \partial_s \phi(s, x, y) + \langle D_x \phi(s, x, y), f(s, x, u) \rangle \\ + \langle D_y \phi(s, x, y), g(s, y, v) \rangle \end{array} \right) \zeta_1(dsdxdydudv) \\ = \displaystyle\int_{\mathbb{R}^N \times \mathbb{R}^M} \phi(t, x, y) \zeta_2(dxdy) - \phi(t_0, x_0, y_0) \end{array} \right\}, \quad (8)$$

and every such measure $\zeta \in \Theta^{1,2}(t_0, t, x_0, y_0)$ has the normal Lebesgue measure as marginal (whenever $t > t_0$).

6.3. Linearized formulation of the bi-controlled dynamics

Let us fix, for the time being, $0 \le t_0 < T$, $x_0 \in \mathbb{R}^N$, $y_0 \in \mathbb{R}^M$. Similarly to the previous subsection, one can imagine a natural way to define a probability measure whose marginals are elements of $\Theta^1(t_0, T, x_0)$, respectively $\Theta^2(t_0, T, y_0)$.

To this purpose, we consider the space $\mathcal{M}([t_0, T] \times K_{x_0,c} \times U) \times \mathcal{M}(K_{x_0,c})$ of signed Radon measures endowed with the usual norm

$$|\gamma|_{\mathcal{M}\left([t_0,T] \times K_{x_0,c} \times U\right) \times \mathcal{M}\left(K_{x_0,c}\right)} = \\ \gamma_1^+ ([t_0, T] \times K_{x_0,c} \times U) + \gamma_1^- ([t_0, T] \times K_{x_0,c} \times U) + \gamma_2^+ (K_{x_0,c}) + \gamma_2^- (K_{x_0,c}). \quad (9)$$

We recall that $K_{x_0,c} = \left\{x \in \mathbb{R}^N : |x| \leq c\left(1+|x_0|\right)\right\}$, for a fixed generic constant $c>0$. It is known that $\mathcal{M}\left([t_0,T]\times K_{x_0,c}\times U\right)\times\mathcal{M}\left(K_{x_0,c}\right)$ can be isometrically identified with de dual of $C\left([t_0,T]\times K_{x_0,c}\times U;\mathbb{R}\right)\times C\left(K_{x_0,c};\mathbb{R}\right)$. Also, this space is weakly complete and locally convex w.r.t. the weak topology. Whenever we have $\gamma\in\Theta^1\left(t_0,T,x_0\right)$, the disintegration theorem yields the existence of some

$$\gamma_{1,\cdot}\in\mathbb{L}^1\left([t_0,T];\mathcal{P}\left(\mathbb{R}^N\times U\right)\right)$$

such that
$$\gamma_1\left(Adxdu\right)=\frac{1}{T-t_0}\int_A\gamma_{1,t}\left(dxdu\right)dt,$$

for all Borel set $A\subset[t_0,T]$. Whenever $\eta\in\mathcal{P}\left([t_0,T]\times K_{y_0,c}\times V\right)\times\mathcal{P}\left(K_{y_0,c}\right)$, we define the measure $\gamma\otimes\eta\in\mathcal{P}\left([t_0,T]\times K_{x_0,c}\times U\times K_{y_0,c}\times V\right)\times\mathcal{P}\left(K_{x_0,c}\times K_{y_0,c}\right)$ by setting

$$\begin{cases}\left(\gamma\otimes\eta\right)_1\left(Adxdudydv\right)=\int_A\gamma_{1,t}\left(dxdu\right)\eta_1\left(dtdydv\right),\\ \left(\gamma\otimes\eta\right)_2\left(dxdy\right)=\gamma_2\left(dx\right)\eta_2\left(dy\right),\end{cases}\tag{10}$$

for all Borel set $A\subset[t_0,T]$. We introduce the sets

$$\gamma\otimes\Theta^2\left(t_0,T,y_0\right)=\left\{\gamma\otimes\eta:\eta\in\Theta^2\left(t_0,T,y_0\right)\right\},\tag{11}$$

$$\gamma\otimes\Gamma^2\left(t_0,T,y_0\right)=\left\{\gamma\otimes\eta:\eta\in\Gamma^2\left(t_0,T,y_0\right)\right\},\text{whenever }\gamma\in\Theta^1\left(t_0,T,x_0\right).$$

Proposition 6.2. (cf. [25]) (1) *We consider* $\gamma\in\Theta^1\left(t_0,T,x_0\right)$ *and a sequence* $\left(\eta^n\right)_n\subset\Theta^2\left(t_0,T,y_0\right)$ *that converges weakly to some* $\eta\in\Theta^2\left(t_0,T,y_0\right)$. *Then* $\gamma\otimes\eta^n$ *converges weakly to* $\gamma\otimes\eta$.

(2) *Whenever* $\gamma\in\Theta^1\left(t_0,T,x_0\right)$, *one has* $\gamma\otimes\Theta^2\left(t_0,T,y_0\right)=\overline{co}\left(\gamma\otimes\Gamma^2\left(t_0,T,y_0\right)\right)$.

(3) *Whenever* $\eta\in\Theta^2\left(t_0,T,y_0\right)$, *one has* $\Theta^1\left(t_0,T,x_0\right)\otimes\eta=\overline{co}\left(\Gamma^1\left(t_0,T,x_0\right)\otimes\eta\right)$.

6.4. Vector cost functional

We are going to consider two bounded, measurable cost functions

$$h:=(h_1,h_2)\colon\mathbb{R}\times\mathbb{R}^N\times U\times\mathbb{R}^M\times V\to\mathbb{R}^2\ \text{ and }\ l:=(l_1,l_2)\colon\mathbb{R}^N\times\mathbb{R}^M\to\mathbb{R}^2.$$

Let us now consider the vector cost functional

$$J\left(t_0,\gamma,\eta\right)=\left(T-t_0\right)\int\limits_{[t_0,T]\times\mathbb{R}^{N+M}}h(s,x,u,\mathrm{y},v)(\gamma\otimes\eta)_1(dsdxUdyV)+\int\limits_{\mathbb{R}^{N+M}}l(x,y)(\gamma\otimes\eta)_2(dxdy)$$

for every $t_0\in[0,T]$, $\gamma\in\Theta^1(t_0,T,x_0)$ and every $\eta\in\mathcal{P}\left([t_0,T]\times\mathbb{R}^M\times V\right)\times\mathcal{P}\left(\mathbb{R}^M\right)$. The upper and lower value functions are defined by

$$\Lambda^+\left(t_0,x_0,y_0\right)=\overrightarrow{\inf}_{\gamma\in\Theta^1(t_0,T,x_0)}\overrightarrow{\sup}_{\eta\in\Theta^2(t_0,T,y_0)}J\left(t_0,\gamma,\eta\right)\ \text{ and}$$

$$\Lambda^-\left(t_0,x_0,y_0\right)=\overrightarrow{\sup}_{\eta\in\Theta^2(t_0,T,y_0)}\overrightarrow{\inf}_{\gamma\in\Theta^1(t_0,T,x_0)}J\left(t_0,\gamma,\eta\right),$$

for all $0\leq t_0\leq T$, and all $(x_0,y_0)\in\mathbb{R}^{N+M}$.

Proposition 6.3. *Suppose that the functions h and l are bounded and continuous. Then the following inequalities are satisfied*

$$W^{-}(t_0,x_0,y_0) \lesssim \Lambda^{-}(t_0,x_0,y_0) \lesssim \Lambda^{+}(t_0,x_0,y_0) \lesssim W^{+}(t_0,x_0,y_0),$$

where W are the control-against-control value functions for the min-max control problem/game

$$W^{+}(t_0,x_0,y_0) = \overrightarrow{\inf}_{u\in\mathcal{U}}\overrightarrow{\sup}_{v\in\mathcal{V}} \int_{t_0}^{T} h\left(s, x_s^{t_0,x_0,u}, u_s, y_s^{t_0,y_0,v}, v_s\right) ds + l\left(x_T^{t_0,x_0,u}, y_T^{t_0,y_0,v}\right),$$

$$W^{-}(t_0,x_0,y_0) = \overrightarrow{\sup}_{v\in\mathcal{V}}\overrightarrow{\inf}_{u\in\mathcal{U}} \int_{t_0}^{T} h\left(s, x_s^{t_0,x_0,u}, u_s, y_s^{t_0,y_0,v}, v_s\right) ds + l\left(x_T^{t_0,x_0,u}, y_T^{t_0,y_0,v}\right).$$

Proof. One easily sees that $\Lambda^{-}(t_0,x_0,y_0) \lesssim \Lambda^{+}(t_0,x_0,y_0)$. We prove that

$$\Lambda^{+}(t_0,x_0,y_0) \lesssim W^{+}(t_0,x_0,y_0),$$

the remaining inequality $W^{-} \lesssim \Lambda^{-}$ being similar. Note that for every $\epsilon = (\epsilon_i)_{i=1,2}$ with $\epsilon_i > 0$ there exists $\gamma_\epsilon \in \Gamma^1(t_0,T,x_0)$ such that

$$\begin{aligned}
&W^{+}(t_0,x_0,y_0) + \epsilon = \overrightarrow{\inf}_{\gamma\in\Gamma^1(t_0,T,x_0)}\overrightarrow{\sup}_{\eta\in\Gamma^2(t_0,T,y_0)} J(t_0,\gamma,\eta) + \epsilon \\
&\succ \overrightarrow{\sup}_{\eta\in\Gamma^2(t_0,T,y_0)} J(t_0,\gamma_\epsilon,\eta) \\
&= \overrightarrow{\sup}_{\nu\in\gamma_\epsilon\otimes\Gamma^2(t_0,T,y_0)} \left[\begin{array}{c} (T-t_0) \displaystyle\int_{[t_0,T]\times\mathbb{R}^N\times U\times\mathbb{R}^M\times V} h(s,x,u,y,v)\,\nu_1(dsdxdudydv) \\ + \displaystyle\int_{\mathbb{R}^{N+M}} l(x,y)\,\nu_2(dxdy) \end{array}\right],
\end{aligned}$$

and consequently

$$\begin{aligned}
&W^{+}(t_0,x_0,y_0) + \epsilon = \overrightarrow{\inf}_{\gamma\in\Gamma^1(t_0,T,x_0)}\overrightarrow{\sup}_{\eta\in\Gamma^2(t_0,T,y_0)} J(t_0,\gamma,\eta) + \epsilon \\
&\succ \overrightarrow{\sup}_{\nu\in\overline{co}(\gamma_\epsilon\otimes\Gamma^2(t_0,T,y_0))} \left[\begin{array}{c} (T-t_0) \displaystyle\int_{[t_0,T]\times\mathbb{R}^N\times U\times\mathbb{R}^M\times V} h(s,x,u,y,v)\nu_1(dsdxdudydv) \\ + \displaystyle\int_{\mathbb{R}^{N+M}} l(x,y)\nu_2(dxdy) \end{array}\right].
\end{aligned}$$

The last inequality is due to:

$$\begin{aligned}
&W_1^{+}(t_0,x_0,y_0) + \epsilon = \inf_{\gamma\in\Gamma^1(t_0,T,x_0)}\sup_{\eta\in\Gamma^2(t_0,T,y_0)} J_1(t_0,\gamma,\eta) + \epsilon \\
&> \sup_{\eta\in\Gamma^2(t_0,T,y_0)} J_1(t_0,\gamma_\epsilon,\eta) \\
&= \sup_{\nu\in\gamma_\epsilon\otimes\Gamma^2(t_0,T,y_0)} \left[\begin{array}{c} (T-t_0) \displaystyle\int_{[t_0,T]\times\mathbb{R}^N\times U\times\mathbb{R}^M\times V} h_1(s,x,u,y,v)\,\nu_1(dsdxdudydv) \\ + \displaystyle\int_{\mathbb{R}^{N+M}} l_1(x,y)\,\nu_2(dxdy) \end{array}\right] \\
&= \sup_{\nu\in\overline{co}(\gamma_\epsilon\otimes\Gamma^2(t_0,T,y_0))} \left[\begin{array}{c} (T-t_0) \displaystyle\int_{[t_0,T]\times\mathbb{R}^N\times U\times\mathbb{R}^M\times V} h_1(s,x,u,y,v)\,\nu_1(dsdxdudydv) \\ + \displaystyle\int_{\mathbb{R}^{N+M}} l_1(x,y)\,\nu_2(dxdy) \end{array}\right].
\end{aligned}$$

The proof is completed by Proposition 6.2. □

We would like to point out that the inequality can be strict as shown by the following example.

Example 6.4. We consider the control spaces $U = V = \{-1, 1\}$ and the functions $f\colon \mathbb{R}^2 \times U \to \mathbb{R}^2$ and $g\colon \mathbb{R}^2 \times V \to \mathbb{R}^2$ given by

$$f(x_1, x_2, u) = (-ux_2, ux_1), \; g(y_1, y_2, v) = (-vy_2, vy_1), \tag{12}$$

To the previous dynamics we associate the following cost:

$$J\left(t_0, x^0, y^0, u, v\right) = \int_t^T l\left(s, x_s^{t_0,x^0,u}, y_s^{t_0,y^0,v}\right) ds + l\left(T, x_T^{t_0,x^0,u}, y_T^{t_0,y^0,v}\right), \tag{13}$$

for all $0 \le t_0 \le T$, and all $(x^0, y^0) \in \mathbb{R}^2 \times \mathbb{R}^2$. Here, $l_1\colon \mathbb{R}^5 \to \mathbb{R}$ is given by

$$l_1(t, x_1, x_2, y_1, y_2) = e^{-t} \arctan\left((x_1 - y_1)^2 + (x_2 - y_2)^2\right),$$

for all $(x_1, x_2, y_1, y_2) \in \mathbb{R}^4$, l_2 can be an arbitrary function and $h := l$

For $(x^0, y^0) = (1, 0, 1, 0)$, one easily proves that $W^+(0, x^0, y^0) > 0$. On the other hand, by choosing $u = v$,

$$0 \ge \sup_{v\in\mathcal{V}} \inf_{u\in\mathcal{U}} J\left(0, x^0, y^0, u, v\right) = W^-\left(0, x^0, y^0\right).$$

6.5. Existence of the value function and saddle points

In this section we generalize to the vector case the main result in [25] showing that playing constraint against constraint (γ against η) leads to a common value function and, moreover, it guarantees the existence of saddle points. Let us assume that U is a compact subset of some Euclidean space $\mathbb{R}^{N'}$ and V is a compact subset of $\mathbb{R}^{M'}$.

Theorem 6.5. *We assume that* $h := (h_1, h_2)\colon \mathbb{R} \times \mathbb{R}^N \times U \times \mathbb{R}^M \times V \longrightarrow \mathbb{R}^2$ *and* $l := (l_1, l_2)\colon \mathbb{R}^N \times \mathbb{R}^M \longrightarrow \mathbb{R}^2$ *are bounded and continuous in time uniformly w.r.t.* x, u, y, v *and lower-semicontinuous in* (x, u) *and upper-semicontinuous in* (y, v). *Then:*

(1) *The sets*

$$\begin{aligned}&\Theta^+(t_0, x_0, y_0)\\ &= \left\{\left(\gamma^+, \eta^+\right) \in \Theta^1(t_0, T, x_0) \times \Theta^2(t_0, T, y_0) : \Lambda^+(t_0, x_0, y_0) = J\left(t_0, \gamma^+, \eta^+\right)\right\},\\ &\Theta^-(t_0, x_0, y_0)\\ &= \left\{\left(\gamma^-, \eta^-\right) \in \Theta^1(t_0, T, x_0) \times \Theta^2(t_0, T, y_0) : \Lambda^-(t_0, x_0, y_0) = J\left(t_0, \gamma^-, \eta^-\right)\right\}\end{aligned}$$

are nonempty for every $(t_0, x_0, y_0) \in [0, T] \times \mathbb{R}^{N+M}$.

(2) *The linearized min-max control problems/game has a value i.e.* $\Lambda = \Lambda^+ = \Lambda^-$.

Proof. In [25] it is shown that the functions

$$\Theta^1(t_0, T, x_0) \ni \; \gamma \mapsto J_i(t_0, \gamma, \eta)$$

are convex and lower-semicontinuous for every $\eta \in \Theta^2(t_0, T, y_0)$ and $i \in 1, 2$. Similarly, the functions

$$\Theta^2(t_0, T, x_0) \ni \gamma \mapsto J_i(t_0, \gamma, \eta)$$

are concave and upper-semicontinuous for every $\gamma \in \Theta^1(t_0, T, x_0)$ and $i \in 1, 2$. Therefore, the function

$$\Theta^1(t_0, T, x_0) \ni \ \gamma \mapsto J(t_0, \gamma, \eta)$$

is convex and lower-semicontinuous for every $\eta \in \Theta^2(t_0, T, y_0)$. Similarly, the function

$$\Theta^2(t_0, T, x_0) \ni \gamma \mapsto J(t_0, \gamma, \eta)$$

is concave and upper-semicontinuous for every $\gamma \in \Theta^1(t_0, T, x_0)$.
We conclude by applying Theorem 5.4. $\square$

For further details on the value of the min-max control problem/game in which at least one of the players uses relaxed controls, the reader is referred to [2].

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