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Opial and Saitoh Type Inequalities on Time Scales

Donal O'Regan
School of Mathematics, National University of Ireland, Galway, Ireland

Samir H. Saker
Department of Mathematics, Faculty of Science, University of Mansoura, Egypt

S. S. Rabie
Department of Mathematics, Faculty of Science, University of Mansoura, Egypt

Ravi P. Agarwal
Department of Mathematics, Texas A & M University, Kingsville, TX 78363, U.S.A.
Ravi.Agarwal@tamuk.edu

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Using Hölder's inequality, the chain rule on time scales and the properties of geometrically convex and concave functions we prove some new dynamic inequalities and their converses on time scales. As a special case, we derive the classical Saitoh integral inequality.

Keywords: Opial's inequality, Saitoh's inequality, time scales, Hölder's inequality.

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1. Introduction

In 1960 Opial [14] proved the inequality

$$\int_0^h \left| f(t)f^{'}(t) \right| dt \leq \frac{h}{4} \int_0^h \left| f^{'}(t) \right|^2 dt, \tag{1}$$

where $f \in C^1[0,h]$, $f(0) = f(h) = 0$, and $f > 0$ on $(0,h)$ (the constant $h/4$ is the best possible) and in the same year Olech [13] proved that if $f(0) = 0$, then

$$\int_0^h \left| f(t)f^{'}(t) \right| dt \leq \frac{h}{2} \int_0^h \left| f^{'}(t) \right|^2 dt. \tag{2}$$

The discrete analogy of the (1) was proved in [11] and the discrete analogy of (2) was proved in [3, Theorem 5.2.2].

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In 1995, S. Saitoh [15] proved for a real-valued function $f \in C^1[0,1]$ with $f(0)=0$ and $\int_0^1 f'(x)^2 dx < 1$, we have

$$\int_0^1 \left(\left(\frac{f(x)}{1-f(x)}\right)'\right)^2 \left(1-\int_0^1 f'(x)^2 dx\right)(1-x)^2 dx \leq \int_0^1 f'(x)^2 dx. \quad (3)$$

Inequality (3) gives another relation between the function and its derivative and to see the connection between (2) and (3) we refer the reader to Theorem 3.1 (see also Remark 3.5, Theorem 3.9 and Corollary 3.15) in this paper. Dynamic inequalities of Opial type on time scales $\mathbb{T}$ was considered in the literature; see [2, Chapter 3] and the references cited therein. The cases when the time scale is equal to the reals or to the integers represent the classical theories of integral and of discrete inequalities.

In this paper, we prove some new dynamic inequalities and their converses on time scales using properties of geometrically convex and concave functions. In particular, we prove some inequalities of the form

$$\int_a^b \frac{\left|(G' \circ f)(x)\right|^p \left|f^\Delta(x)\right|^p}{((G \circ F)^\Delta(x))^{p-1}} \Delta x \leq G\left(\int_a^b \frac{\left|f^\Delta(x)\right|^p}{(F^\Delta(x))^{p-1}} \Delta x\right),$$

and their converse on time scales. As a special case, we obtain some inequalities of Saitoh type and of Opial type. The paper is organized as follows. In Section 2, we present some concepts related to time scales and some basic definitions and lemmas. In Section 3, we prove the main results.

2. Preliminaries on Time Scales

In this section, for completeness, we recall the following concepts related to the notion of time scales and the basic lemmas that will be used in the proof of the main results; for more details of time scale analysis we refer the reader to the books by Bohner and Peterson [6], [7]. A *time scale* $\mathbb{T}$ is an arbitrary non-empty closed subset of the real numbers $\mathbb{R}$.

Without loss of generality, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$. The *forward jump operator* is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. The *graininess function* μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be *right-dense, right-scattered*, if $\sigma(t) = t$, $\sigma(t) > t$, respectively. The *backward jump operator* $\rho\colon \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. A point $t \in \mathbb{T}$ is said to be *left-dense, left-scattered*, if $\rho(t) = t$, $\rho(t) < t$, respectively. A function $f\colon \mathbb{T} \to \mathbb{R}$ is said to be *right-dense continuous* (*rd-continuous*) provided f is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$.

For any function $f\colon \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. We will refer to the (delta) integral which we define as follows: If $F^\Delta(t) = f(t)$, then the *Cauchy* *(delta) integral* of f is defined by $\int_a^t f(s)\Delta s := F(t) - F(a)$. It was shown (see [6]) that if $f \in C_{rd}(\mathbb{T})$, then the Cauchy integral $F(t) := \int_{t_0}^t f(s)\Delta s$, $t_0 \in \mathbb{T}$, exists and satisfies $F^{\Delta}(t) = f(t)$, $t \in \mathbb{T}$. An infinite integral is defined as

$$\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t.$$

The integration by parts formula is given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t, \tag{4}$$

and Hölder's inequality, see [6, Theorem 6.13], on time scales is given by

$$\int_a^b |f(t)g(t)|\Delta t \leq \left[\int_a^b |f(t)|^{\gamma}\Delta t\right]^{\frac{1}{\gamma}} \left[\int_a^b |g(t)|^{\nu}\Delta t\right]^{\frac{1}{\nu}}, \tag{5}$$

where $a, b \in \mathbb{T}$ and f, $g \in \mathrm{C}_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$, $\gamma > 1$ and $1/\gamma + 1/\nu = 1$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g\colon \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g\colon \mathbb{T} \to \mathbb{R}$ is delta differentiable and the chain rule

$$(f\circ g)^{\Delta}(t) = \left\{\int_0^1 f'\left(g(t) + h\mu(t)\, g^{\Delta}(t)\right) dh\right\} g^{\Delta}(t), \tag{6}$$

holds. A special case of (6) is

$$\left[u^{\lambda}(t)\right]^{\Delta} = \lambda \int_0^1 \left[hu^{\sigma} + (1-h)\, u\right]^{\lambda-1} u^{\Delta}(t)\, dh.$$

Throughout the paper, we assume (sometimes without mentioning) that the functions are rd-continuous (and differentiable where appropriate) and the integrals are assumed to exist (and are finite).

For a given $0 < L \leq \infty$, let Γ_L^0 be the class of all functions $G \in C^1(-L, L)$ satisfying the following assumptions:

(a_0) $G(0) = 0$,

(b_0) $|G'(x)| \leq G'(|x|)$ for all $x \in (-L, L)$ (so in particular $G'(|x|) \geq 0$), where $G'(u) = dG/du$, and $G'(y) > 0$ for $y \in (0, L)$,

(c_0) $G'(x) \leq (G'(y))^{\alpha}\,(G'(z))^{1-\alpha}$ if $x \leq y^{\alpha}z^{1-\alpha}$, for $\alpha \in [0,1]$, and $0 \leq x$, y, $z < L$.

For a given $0 < L \leq \infty$, let Ω_L^0 be the class of all functions $G \in C^1(-L, L)$ satisfying the following assumptions:

(α_0) $G(0) = 0$,

(β_0) $|G'(x)| \geq G'(|x|)$ for all $x \in (-L, L)$ and $G'(y) > 0$ for $y \in (0, L)$,

(γ_0) $G'(x) \geq (G'(y))^{\alpha}\,(G'(z))^{1-\alpha}$ if $x \leq y^{\alpha}z^{1-\alpha}$, for $\alpha \in [0,1]$, and $0 \leq x$, y, $z < L$.

Remark 2.1. The function G in the class Γ_L^0 that satisfies condition (c_0) is called a geometrically convex function and the function G in the class Ω_L^0 that satisfies condition (γ_0) is called a geometrically concave function; for more details we refer the reader to [12, 17].

For a given $0 < L \leq \infty$, let Γ_L^1 be the class of all functions $G \in C^1[0, L)$ satisfying the following assumption:

$$G'(x) \leq G'(y) \quad \text{if} \quad x \leq y \quad \text{for} \quad 0 \leq x,\, y < L. \tag{7}$$

For a given $0 < L \leq \infty$, let Ω_L^1 be the class of all functions $G \in C^1[0, L)$ satisfying the following assumption:

$$G'(x) \geq G'(y) \quad \text{if} \quad x \leq y \quad \text{for} \quad 0 \leq x,\, y < L. \tag{8}$$

For a given $0 < L \leq \infty$, let Γ_L^2 be the class of all functions $G \in C^1(0, L)$ satisfying the following assumptions:

(a_2) $G(0) = 0$,

(b_2) $|G'(x)| \leq G'(|x|)$ for all $x \in (0, L)$, and $G'(y) > 0$ for $y \in (0, L)$,

(c_2) $G'(x) \leq (G'(y))^{\alpha} (G'(z))^{1-\alpha}$ if $x \leq y^{\alpha} z^{1-\alpha}$, for $\alpha \in [0, 1]$, and $0 < x,\, y,\, z < L$.

For a given $0 < L \leq \infty$, let Γ_L^3 be the class of all functions $G \in C^1[0, L)$ satisfying the following assumptions:

(a_3) $G(0) = 0$,

(b_3) $|G'(x)| \leq G'(|x|)$ for all $x \in [0, L)$, and $G'(y) > 0$ for $y \in (0, L)$,

(c_3) $G'(x) \leq (G'(y))^{\alpha} (G'(z))^{1-\alpha}$ if $x \leq y^{\alpha} z^{1-\alpha}$, for $\alpha \in [0, 1]$, and $0 \leq x,\, y,\, z < L$.

For a given $0 < L \leq \infty$, let Γ_L^4 be the class of all functions $G \in C^1(-L, L)$ satisfying the following assumptions:

(a_4) $G(0) = 0$,

(b_4) $|G'(x)| \leq G'(|x|)$ for all $x \in (-L, 0)$, and $G'(y) > 0$ for $y \in (0, L)$,

(c_4) $G'(x) \leq (G'(y))^{\alpha} (G'(z))^{1-\alpha}$ if $x \leq y^{\alpha} z^{1-\alpha}$, for $\alpha \in [0, 1]$, and $0 < x,\, y,\, z < L$.

For a given $0 < L \leq \infty$, let Γ_L^5 be the class of all functions $G \in C^1(-L, L)$ satisfying the following assumptions:

(a_5) $G(0) = 0$,

(b_5) $|G'(x)| \leq G'(|x|)$ for all $x \in (-L, 0]$, and $G'(y) > 0$ for $y \in (0, L)$,

(c_5) $G'(x) \leq (G'(y))^{\alpha} (G'(z))^{1-\alpha}$ if $x \leq y^{\alpha} z^{1-\alpha}$, for $\alpha \in [0, 1]$, and $0 \leq x,\, y,\, z < L$.

Remark 2.2. Similarly we could create Ω_L^2, Ω_L^3, Ω_L^4 and Ω_L^5.

In the following, we prove two basic lemmas which will be used in the proofs of the main results.

Lemma 2.3. *Let $G \in \Gamma_L^1$ (for a given L) and assume that F is a nonnegative rd-continuous and $\Delta-$differentiable function. Assume F^{Δ} is either always nonpositive or nonnegative with $F^{\sigma}(x) < L$ for $x \in \mathbb{T}$ if $F^{\Delta}(x) \geq 0$ for $x \in \mathbb{T}$ and $F(x) < L$ for $x \in \mathbb{T}$ if $F^{\Delta}(x) \leq 0$ for $x \in \mathbb{T}$. Then*

$$F^{\Delta}(x)(G' \circ F)(x) \leq (G \circ F)^{\Delta}(x) \leq F^{\Delta}(x)(G' \circ F^{\sigma})(x), \ x \in \mathbb{T}. \tag{9}$$

Proof. From the chain rule (6) for $h \in (0,1)$ and $x \in \mathbb{T}$, we see that

$$(G \circ F)^{\Delta}(x) = \left\{ \int_0^1 G'\left(hF^{\sigma}\left(x\right) + (1-h)F\left(x\right)\right) dh \right\} F^{\Delta}\left(x\right).$$

First assume that $F^{\Delta}(x) \geq 0$ for $x \in \mathbb{T}$. Then, we see that $F^{\sigma}(x) \geq F(x)$ so for $h \in (0,1)$ we have

$$hF\left(x\right) + (1-h)F\left(x\right) \leq hF^{\sigma}\left(x\right) + (1-h)F\left(x\right) \leq hF^{\sigma}\left(x\right) + (1-h)F^{\sigma}\left(x\right).$$

Now apply the condition (7) of the class Γ_L^1 twice and we see that

$$\begin{aligned} F^{\Delta}(x)(G' \circ F)(x) &= \left\{ \int_0^1 G'\left(hF\left(x\right) + (1-h)F\left(x\right)\right) dh \right\} F^{\Delta}\left(x\right) \\ &\leq \left\{ \int_0^1 G'\left(hF^{\sigma}\left(x\right) + (1-h)F\left(x\right)\right) dh \right\} F^{\Delta}\left(x\right) \\ &= (G \circ F)^{\Delta}(x) \leq \left\{ \int_0^1 G'\left(hF^{\sigma}\left(x\right) + (1-h)F^{\sigma}\left(x\right)\right) dh \right\} F^{\Delta}\left(x\right) \\ &= F^{\Delta}(x)(G' \circ F^{\sigma})(x). \end{aligned}$$

which is (9). Next, assume that $F^{\Delta}(x) \leq 0$ for $x \in \mathbb{T}$. In this case we see that $F^{\sigma}(x) \leq F(x)$ and then using the condition (7) of the class Γ_L^1 twice we have

$$\begin{aligned} \left\{ \int_0^1 G'\left(hF\left(x\right) + (1-h)F\left(x\right)\right) dh \right\} &\geq \left\{ \int_0^1 G'\left(hF^{\sigma}\left(x\right) + (1-h)F\left(x\right)\right) dh \right\} \\ &\geq \left\{ \int_0^1 G'\left(hF^{\sigma}\left(x\right) + (1-h)F^{\sigma}\left(x\right)\right) dh \right\}. \end{aligned}$$

Multiply by $F^{\Delta}(x)$ (note that $F^{\Delta}(x) \leq 0$) and we get

$$\begin{aligned} &\left\{ \int_0^1 G'\left(hF\left(x\right) + (1-h)F\left(x\right)\right) dh \right\} F^{\Delta}(x) \\ \leq\ & F^{\Delta}(x) \left\{ \int_0^1 G'\left(hF^{\sigma}\left(x\right) + (1-h)F\left(x\right)\right) dh \right\} \\ \leq\ & F^{\Delta}(x) \left\{ \int_0^1 G'\left(hF^{\sigma}\left(x\right) + (1-h)F^{\sigma}\left(x\right)\right) dh \right\}. \end{aligned}$$

This gives us that

$$F^{\Delta}(x)(G' \circ F)(x) \leq (G \circ F)^{\Delta}(x) \leq F^{\Delta}(x)(G' \circ F^{\sigma})(x),$$

which is (9). The proof is complete. $\square$

Lemma 2.4. *Let $G \in \Omega_L^1$ (for a given L) and assume that F is a nonnegative rd-continuous and $\Delta-$differentiable function. Assume F^Δ is either always nonpositive or nonnegative with $F^\sigma(x) < L$ for $x \in \mathbb{T}$ if $F^\Delta(x) \geq 0$ for $x \in \mathbb{T}$ and $F(x) < L$ for $x \in \mathbb{T}$ if $F^\Delta(x) \leq 0$ for $x \in \mathbb{T}$. Then*

$$F^\Delta(x)(G' \circ F)(x) \geq (G \circ F)^\Delta(x) \geq F^\Delta(x)(G' \circ F^\sigma)(x), \; x \in \mathbb{T}. \tag{10}$$

Proof. From the chain rule (6) for $h \in (0,1)$ and $x \in \mathbb{T}$, we see that

$$(G \circ F)^\Delta(x) = \left\{ \int_0^1 G'\left(hF^\sigma(x) + (1-h)F(x)\right) dh \right\} F^\Delta(x).$$

First assume that $F^\Delta(x) \geq 0$ for $x \in \mathbb{T}$. Then, we see that $F^\sigma(x) \geq F(x)$ and then using (8) twice we see that

$$\begin{aligned} F^\Delta(x)(G' \circ F)(x) &= \left\{ \int_0^1 G'\left(hF(x) + (1-h)F(x)\right) dh \right\} F^\Delta(x) \\ \geq & \left\{ \int_0^1 G'\left(hF^\sigma(x) + (1-h)F(x)\right) dh \right\} F^\Delta(x) \\ = & (G \circ F)^\Delta(x) \geq \left\{ \int_0^1 G'\left(hF^\sigma(x) + (1-h)F^\sigma(x)\right) dh \right\} F^\Delta(x) \\ = & F^\Delta(x)(G' \circ F^\sigma)(x), \end{aligned}$$

which is (10). Next, assume that $F^\Delta(x) \leq 0$ for $x \in \mathbb{T}$. In this case, we see that $F^\sigma(x) \leq F(x)$ and then using (8) twice we have

$$\begin{aligned} \left\{ \int_0^1 G'\left(hF(x) + (1-h)F(x)\right) dh \right\} &\leq \left\{ \int_0^1 G'\left(hF^\sigma(x) + (1-h)F(x)\right) dh \right\} \\ &\leq \left\{ \int_0^1 G'\left(hF^\sigma(x) + (1-h)F^\sigma(x)\right) dh \right\}. \end{aligned}$$

Multiply by $F^\Delta(x)$ (note that $F^\Delta(x) \leq 0$) and we get

$$\begin{aligned} & \left\{ \int_0^1 G'\left(hF(x) + (1-h)F(x)\right) dh \right\} F^\Delta(x) \\ \geq \; & F^\Delta(x) \left\{ \int_0^1 G'\left(hF^\sigma(x) + (1-h)F(x)\right) dh \right\} \\ \geq \; & F^\Delta(x) \left\{ \int_0^1 G'\left(hF^\sigma(x) + (1-h)F^\sigma(x)\right) dh \right\}. \end{aligned}$$

This gives us that

$$F^\Delta(x)(G' \circ F)(x) \geq (G \circ F)^\Delta(x) \geq F^\Delta(x)(G' \circ F^\sigma)(x),$$

which again is (9). The proof is complete. $\square$

3. Main results

In this section, we present the main results. We will assume that there exists a function $\Phi \in C_{rd}([a,\sigma(b)]_{\mathbb{T}},\mathbb{R})$ with

$$\Phi(a)=0,\ \Phi^{\Delta}(t)>0 \ \text{ for } \ t\in(a,\sigma(b))_{\mathbb{T}}\ \text{, and } \ \Phi^{\sigma}(x)<L \ \text{ for } \ x\in(a,b)_{\mathbb{T}},$$

for a given $0<L\leq\infty$.

Theorem 3.1. *Assume that $G\in\Gamma_L^0$ (for a given L) and $f\in C_{rd}([a,\sigma(b)]_{\mathbb{T}},\mathbb{R})$. If $f(a)=0$, $p>1$ and*

$$F^{\sigma}(x)<L \ \text{ for } \ x\in(a,b)_{\mathbb{T}} \ \text{ where } \ F(x)=\int_a^x \frac{\left|f^{\Delta}(t)\right|^p}{(\Phi^{\Delta}(t))^{p-1}}\Delta t, \tag{11}$$

then
$$\int_a^b \frac{|(G'\circ f)(x)|^p\left|f^{\Delta}(x)\right|^p}{[(G\circ\Phi)^{\Delta}(x)]^{p-1}}\Delta x\leq G\left(\int_a^b\frac{\left|f^{\Delta}(x)\right|^p}{(\Phi^{\Delta}(x))^{p-1}}\Delta x\right). \tag{12}$$

Proof. Let $x\in(a,b)_{\mathbb{T}}$. Since $f(a)=0$, we have

$$|f(x)|=\left|\int_a^x f^{\Delta}(y)\Delta y\right|\leq\int_a^x\left|f^{\Delta}(y)\right|\Delta y=\int_a^x\frac{\left|f^{\Delta}(y)\right|}{(\Phi^{\Delta}(y))^{1/q}}\left(\Phi^{\Delta}(y)\right)^{1/q}\Delta y. \tag{13}$$

Apply Hölder's inequality on the term

$$\int_a^x\frac{\left|f^{\Delta}(y)\right|}{(\Phi^{\Delta}(y))^{1/q}}\left(\Phi^{\Delta}(y)\right)^{1/q}\Delta y,$$

with indices p and q such that $1/p+1/q=1$, and we have

$$|f(x)|\leq\left(\int_a^x\frac{\left|f^{\Delta}(y)\right|^p}{(\Phi^{\Delta}(y))^{p/q}}\Delta y\right)^{1/p}\left(\int_a^x\Phi^{\Delta}(y)\Delta y\right)^{1/q} \tag{14}$$
$$=\left(\int_a^x\frac{\left|f^{\Delta}(y)\right|^p}{(\Phi^{\Delta}(y))^{p/q}}\Delta y\right)^{1/p}(\Phi(x)-\Phi(a))^{1/q}=\Phi^{1/q}(x)\left(\int_a^x\frac{\left|f^{\Delta}(y)\right|^p}{(\Phi^{\Delta}(y))^{p-1}}\Delta y\right)^{1/p}.$$

From the condition (c) of the class Γ_L^0, note for $x\in(a,b)_{\mathbb{T}}$ that $0<\Phi(x)<\Phi^{\sigma}(x)<L$,

$$0\leq\int_a^x\frac{\left|f^{\Delta}(y)\right|^p}{(\Phi^{\Delta}(y))^{p-1}}\Delta y\leq\int_a^{\sigma(x)}\frac{\left|f^{\Delta}(y)\right|^p}{(\Phi^{\Delta}(y))^{p-1}}\Delta y<L,$$

and so $0\leq|f(x)|<L$), we have

$$G'(|f(x)|)\leq(G'(\Phi(x)))^{1/q}\left(G'\left(\int_a^x\frac{\left|f^{\Delta}(t)\right|^p}{(\Phi^{\Delta}(t))^{p-1}}\Delta t\right)\right)^{1/p}.$$

$$\text{Thus} \qquad (G'(|f(x)|))^p \le (G'(\Phi(x)))^{p/q}\, G'\left(\int_a^x \frac{\left|f^\Delta(t)\right|^p}{(\Phi^\Delta(t))^{p-1}}\Delta t\right)$$
$$= (G'(\Phi(x)))^{p-1}\, G'\left(\int_a^x \frac{\left|f^\Delta(t)\right|^p}{(\Phi^\Delta(t))^{p-1}}\Delta t\right). \tag{15}$$

Also from Lemma 2.3 (note $\Phi^\sigma(x) < L$ for $x \in (a,b)_{\mathbb{T}}$ and $G'(z) > 0$ for $z \in (0,L)$) we have $(G\circ\Phi)^\Delta(x) \ge G'(\Phi(x))\Phi^\Delta(x) > 0$. Multiply both sides of (15) by $\left|f^\Delta(x)\right|^p / \left((G\circ\Phi)^\Delta(x)\right)^{p-1}$, and using $|G'(y)| \le G'(|y|)$ (here $y = f(x)$ so that $y \in (-L, L)$), and we have

$$\frac{|G'(f(x))|^p \left|f^\Delta(x)\right|^p}{((G\circ\Phi)^\Delta(x))^{p-1}} \le \frac{(G'(|f(x)|))^p \left|f^\Delta(x)\right|^p}{((G\circ\Phi)^\Delta(x))^{p-1}}$$
$$\le \frac{(G'(\Phi(x)))^{p-1} \left|f^\Delta(x)\right|^p}{((G\circ\Phi)^\Delta(x))^{p-1}} G'\left(\int_a^x \frac{\left|f^\Delta(y)\right|^p}{(\Phi^\Delta(y))^{p-1}}\Delta y\right), \quad \text{so} \tag{16}$$

$$\frac{|G'(f(x))|^p \left|f^\Delta(x)\right|^p}{((G\circ\Phi)^\Delta(x))^{p-1}} \le \frac{(G'(\Phi(x)))^{p-1} \left|f^\Delta(x)\right|^p}{((G\circ\Phi)^\Delta(x))^{p-1}} G'\left(\int_a^x \frac{\left|f^\Delta(y)\right|^p}{(\Phi^\Delta(y))^{p-1}}\Delta y\right). \tag{17}$$

As above from Lemma 2.3, $(G\circ\Phi)^\Delta(x) \ge G'(\Phi(x))\Phi^\Delta(x) > 0$, so from (17), we obtain

$$\frac{|G'(f(x))|^p \left|f^\Delta(x)\right|^p}{((G\circ\Phi)^\Delta(x))^{p-1}} \le \frac{(G'(\Phi(x)))^{p-1} \left|f^\Delta(x)\right|^p}{(G'(\Phi(x)))^{p-1}(\Phi^\Delta(x))^{p-1}} G'\left(\int_a^x \frac{\left|f^\Delta(y)\right|^p}{(\Phi^\Delta(y))^{p-1}}\Delta y\right)$$
$$= \frac{\left|f^\Delta(x)\right|^p}{(\Phi^\Delta(x))^{p-1}} G'\left(\int_a^x \frac{\left|f^\Delta(y)\right|^p}{(\Phi^\Delta(y))^{p-1}}\Delta y\right).$$

Now applying Lemma 2.3 (note $F^\sigma(x) < L$ for $x \in (a,b)_{\mathbb{T}}$) on the right hand side of the last inequality with

$$F(x) = \int_a^x \frac{\left|f^\Delta(y)\right|^p}{(\Phi^\Delta(y))^{p-1}}\Delta y,$$

since $F^\Delta(x) = \left|f^\Delta(x)\right|^p / (\Phi^\Delta(x))^{p-1} \ge 0$, we obtain that

$$\frac{|G'(f(x))|^p \left|f^\Delta(x)\right|^p}{((G\circ\Phi)^\Delta(x))^{p-1}} \le \left[G\left(\int_a^x \frac{\left|f^\Delta(y)\right|^p}{(\Phi^\Delta(y))^{p-1}}\Delta y\right)\right]^\Delta. \tag{18}$$

Integrate both sides of (18) from a to b and we have

$$\int_a^b \frac{|G'(f(x))|^p \left|f^\Delta(x)\right|^p}{((G\circ\Phi)^\Delta(x))^{p-1}}\Delta x \le \int_a^b \left[G\left(\int_a^x \frac{\left|f^\Delta(y)\right|^p}{(\Phi^\Delta(y))^{p-1}}\Delta y\right)\right]^\Delta \Delta x$$
$$= \left[G\left(\int_a^b \frac{\left|f^\Delta(x)\right|^p}{(\Phi^\Delta(x))^{p-1}}\Delta x\right) - G(0)\right] = G\left(\int_a^b \frac{\left|f^\Delta(x)\right|^p}{(\Phi^\Delta(x))^{p-1}}\Delta x\right),$$

which is (12). The proof is complete. □

From the proof above we see that if f is monotonic, we obtain the following results.

Theorem 3.2. (a) *Assume that $G \in \Gamma_L^2$ (for a given L), $f \in C_{rd}([a,\sigma(b)]_{\mathbb{T}},\mathbb{R})$, $f(a)=0$, $p>1$ and* (11) *holds. Suppose $f^{\Delta}(x)>0$ for $x\in(a,b)_{\mathbb{T}}$. Then*

$$\int_a^b \frac{[(G'\circ f)(x)]^p\left[f^{\Delta}(x)\right]^p}{[(G\circ\Phi)^{\Delta}(x)]^{p-1}}\Delta x \le G\left(\int_a^b \frac{\left[f^{\Delta}(x)\right]^p}{(\Phi^{\Delta}(x))^{p-1}}\Delta x\right). \tag{19}$$

(b) *Assume that $G \in \Gamma_L^3$ (for a given L), $f \in C_{rd}([a,\sigma(b)]_{\mathbb{T}},\mathbb{R})$, $f(a)=0$, $p>1$ and* (11) *holds. Suppose $f^{\Delta}(x)\ge 0$ for $x\in(a,b)_{\mathbb{T}}$. Then*

$$\int_a^b \frac{[(G'\circ f)(x)]^p\left[f^{\Delta}(x)\right]^p}{[(G\circ\Phi)^{\Delta}(x)]^{p-1}}\Delta x \le G\left(\int_a^b \frac{\left[f^{\Delta}(x)\right]^p}{(\Phi^{\Delta}(x))^{p-1}}\Delta x\right). \tag{20}$$

(c) *Assume that $G \in \Gamma_L^4$ (for a given L), $f \in C_{rd}([a,\sigma(b)]_{\mathbb{T}},\mathbb{R})$, $f(a)=0$, $p>1$ and* (11) *holds. Suppose $f^{\Delta}(x)<0$ for $x\in(a,b)_{\mathbb{T}}$. Then*

$$\int_a^b \frac{|(G'\circ f)(x)|^p\left|f^{\Delta}(x)\right|^p}{[(G\circ\Phi)^{\Delta}(x)]^{p-1}}\Delta x \le G\left(\int_a^b \frac{\left|f^{\Delta}(x)\right|^p}{(\Phi^{\Delta}(x))^{p-1}}\Delta x\right). \tag{21}$$

(d) *Assume that $G \in \Gamma_L^5$ (for a given L), $f \in C_{rd}([a,\sigma(b)]_{\mathbb{T}},\mathbb{R})$, $f(a)=0$, $p>1$ and* (11) *holds. Suppose $f^{\Delta}(x)\le 0$ for $x\in(a,b)_{\mathbb{T}}$. Then*

$$\int_a^b \frac{|(G'\circ f)(x)|^p\left|f^{\Delta}(x)\right|^p}{[(G\circ\Phi)^{\Delta}(x)]^{p-1}}\Delta x \le G\left(\int_a^b \frac{\left|f^{\Delta}(x)\right|^p}{(\Phi^{\Delta}(x))^{p-1}}\Delta x\right). \tag{22}$$

Remark 3.3. Suppose the assumptions in Theorem 3.2 are satisfied and in addition assume there is a constant $M>0$ with $(G'\circ f^{\sigma})(x)\le M\,(G'\circ f)(x)$ for $x\in(a,b)_{\mathbb{T}}$. Then from Lemma 2.3 (note an argument similar to (14) guarantees that $f^{\sigma}(x)<L$ for $x\in(a,b)_{\mathbb{T}}$) we have

$$(G\circ f)^{\Delta}(x)\le f^{\Delta}(x)(G'\circ f^{\sigma})(x)\le M\,f^{\Delta}(x)(G'\circ f)(x)$$

for $x\in(a,b)_{\mathbb{T}}$; so (19) gives

$$\int_a^b \frac{\left[(G\circ f)^{\Delta}(x)\right]^p}{[(G\circ\Phi)^{\Delta}(x)]^{p-1}}\Delta x \le M^p\,G\left(\int_a^b \frac{\left[f^{\Delta}(x)\right]^p}{(\Phi^{\Delta}(x))^{p-1}}\Delta x\right). \tag{23}$$

Note that $|(G'\circ f)(x)|^p\,|f'(x)|^p=\left|(G\circ f)'(x)\right|^p$. Hence, if $\mathbb{T}=\mathbb{R}$ then Theorem 3.1 reduces to the following result. For our result let $\Phi\in C([a,b]_{\mathbb{R}},\mathbb{R})$ with

$$\Phi(a)=0,\ \Phi'(t)>0\ \text{ for }\ t\in(a,b)_{\mathbb{R}},\ \text{ and }\ \Phi(x)<L\ \text{ for }\ x\in(a,b)_{\mathbb{R}},$$

for a given $0<L\le\infty$.

Theorem 3.4. *Assume that $G \in \Gamma_L^0$ (for a given L) and $f \in C^1([a,b]_{\mathbb{R}}, \mathbb{R})$. If $f(a) = 0$, $p > 1$ and*

$$F(x) < L, \text{ for } x \in (a,b)_{\mathbb{R}} \text{ where } F(x) = \int_a^x \frac{|f'(t)|^p}{(\Phi'(t))^{p-1}}\, dt, \tag{24}$$

then
$$\int_a^b \frac{\left|(G \circ f)'(x)\right|^p}{[(G \circ \Phi)'(x)]^{p-1}}\, dx \le G\left(\int_a^b \frac{|f'(x)|^p}{(\Phi'(x))^{p-1}}\, dx\right). \tag{25}$$

Remark 3.5. Let $G(x) = x/(1-x)$, $\Phi(x) = x$, $p = 2$, $a = 0$ and $L = 1 < \infty$ in Theorem 3.4. Note $G'(x) = 1/(1-x)^2$, $G''(x) = 2/(1-x)^3$ and it is easy to see (consider the cases $w \in [0,1)$ and $w \in (-1,0)$ separately) that $|G'(w)| \le G'(|w|)$ for $w \in (-1,1)$ and $G'(x) \le (G'(y))^\alpha (G'(z))^{1-\alpha}$ if $x \le y^\alpha z^{1-\alpha}$, for $\alpha \in [0,1]$, and $0 \le x,\, y,\, z < 1$ (note using Young's inequality twice that

$$\begin{aligned}(1-y)^\alpha (1-z)^{1-\alpha} &\le \alpha(1-y) + (1-\alpha)(1-z)\\ &= 1 - [\alpha\, y + (1-\alpha) z] \le 1 - y^\alpha z^{1-\alpha} \le 1 - x\end{aligned}$$

so $(1-x)^2 \ge (1-y)^{2\alpha}(1-z)^{2(1-\alpha)}$). Using this in Theorem 3.4 and assuming $f(0) = 0$ and $\int_0^1 |f'(t)|^2\, dt < 1$, we obtain Saitoh's inequality:

$$\int_0^1 (1-x)^2 \left|\left(\frac{f(x)}{1-f(x)}\right)'\right|^2 dx \le \frac{\left(\int_0^1 |f'(x)|^2\, dx\right)}{\left(1 - \left(\int_0^1 |f'(x)|^2\, dx\right)\right)}.$$

Remark 3.6. Let $\mathbb{T} = \mathbb{N}$, $\Phi(x) = x$, $p = 2$, $a = 0$ and $b = N+1$ (here $N \in \mathbb{N}$) in Theorem 3.1. Let $L = N + 1 + \epsilon$ (here $\epsilon > 0$) and $G(x) = x/(L-x)$. Note $G'(x) = L/(L-x)^2$ and a similar argument to the above shows that $G \in \Gamma_{N+1+\epsilon}$.

Note $\displaystyle\int_a^b \left|f^\Delta(t)\right|^2 \Delta t = \sum_{n=0}^{N} |\Delta f(n)|^2$ and $\displaystyle\int_a^{\sigma(b)} \left|f^\Delta(t)\right|^2 \Delta t = \sum_{n=0}^{N+1} |\Delta f(n)|^2$.

Also note that $\Phi^\Delta(n) = 1$, and for $x \in (a,b)_{\mathbb{T}} = (0, N+1)_{\mathbb{T}} = \{1, ..., N\}$ we have $\Phi^\sigma(x) = x + 1 < L = N + 1 + \epsilon$. Finally note

$$\begin{aligned}(G \circ \Phi)^\Delta(n) &= \Delta G(n) = G(n+1) - G(n)\\ &= \frac{n+1}{L-(n+1)} - \frac{n}{L-n} = \frac{L}{[L-(n+1)][L-n]}.\end{aligned}$$

From Theorem 3.1 we have if $f(0) = 0$ and $\sum_{n=0}^{N+1} |\Delta f(n)|^2 < N + 1 + \epsilon$, then

$$\begin{aligned}&\sum_{n=0}^{N} \frac{(N+1+\epsilon-n)(N+\epsilon-n)}{N+1+\epsilon} \left|\frac{N+1+\epsilon}{(N+1+\epsilon) - f(n)}\right|^2 |\Delta f(n)|^2\\ &\qquad \le \frac{\sum_{n=0}^{N} |\Delta f(n)|^2}{(N+1+\epsilon) - \sum_{n=0}^{N} |\Delta f(n)|^2},\end{aligned}$$

i.e.
$$\sum_{n=0}^{N}(N+1+\epsilon-n)(N+\epsilon-n)\,\frac{N+1+\epsilon}{[(N+1+\epsilon)-f(n)]^2}\,(\Delta f(n))^2$$
$$\leq \frac{\sum_{n=0}^{N}|\Delta f(n)|^2}{(N+1+\epsilon)-\sum_{n=0}^{N}|\Delta f(n)|^2},$$
which is a new discrete inequality. Note that we can let $\epsilon\to 0^+$ if we wish.

In the following theorem, we will prove a new inequality using properties of geometrically concave functions. Again we assume $\Phi\in C_{rd}([a,\sigma(b)]_{\mathbb{T}},\mathbb{R})$ with
$$\Phi(a)=0,\ \Phi^{\Delta}(t)>0 \text{ for } t\in(a,\sigma(b))_{\mathbb{T}}, \text{ and } \Phi^{\sigma}(x)<L \text{ for } x\in(a,b)_{\mathbb{T}},$$
for a given $0<L\leq\infty$.

Theorem 3.7. *Assume that $G\in\Omega_L^0$ (for a given L), $f\in C_{rd}([a,\sigma(b)]_{\mathbb{T}},\mathbb{R})$, $f(a)=0$, $p>1$ and (11) holds. Then*
$$\int_a^b \frac{|(G'\circ f)(x)|^p\,\left|f^{\Delta}(x)\right|^p}{[(G\circ\Phi)^{\Delta}(x)]^{p-1}}\Delta x \geq G\left(\int_a^b \frac{\left|f^{\Delta}(x)\right|^p}{(\Phi^{\Delta}(x))^{p-1}}\Delta x\right). \tag{26}$$

Proof. Let $x\in(a,b)_{\mathbb{T}}$. Since $f(a)=0$, we have
$$|f(x)|=\left|\int_a^x f^{\Delta}(t)\Delta t\right|\leq\int_a^x\left|f^{\Delta}(t)\right|\Delta t=\int_a^x\frac{\left|f^{\Delta}(t)\right|}{(\Phi^{\Delta}(t))^{1/q}}\left(\Phi^{\Delta}(t)\right)^{1/q}\Delta t. \tag{27}$$
Apply Hölder's inequality on the right hand side, with indices p and q such that $1/p+1/q=1$, and we have
$$\begin{aligned}|f(x)| &\leq \left(\int_a^x\frac{\left|f^{\Delta}(t)\right|^p}{(\Phi^{\Delta}(t))^{p/q}}\Delta t\right)^{1/p}\left(\int_a^x\Phi^{\Delta}(t)\Delta t\right)^{1/q}\\ &= \Phi^{1/q}(x)\left(\int_a^x\frac{\left|f^{\Delta}(t)\right|^p}{(\Phi^{\Delta}(t))^{p-1}}\Delta t\right)^{1/p}.\end{aligned} \tag{28}$$
From the condition (γ_0) of the class Ω_L^0 we have that
$$G'(|f(x)|)\geq(G'(\Phi(x)))^{1/q}\left(G'\left(\int_a^x\frac{\left|f^{\Delta}(t)\right|^p}{(\Phi^{\Delta}(t))^{p-1}}\Delta t\right)\right)^{1/p},$$
and so
$$\begin{aligned}(G'(|f(x)|))^p &\geq (G'(\Phi(x)))^{p/q}\,G'\left(\int_a^x\frac{\left|f^{\Delta}(t)\right|^p}{(\Phi^{\Delta}(t))^{p-1}}\Delta t\right)\\ &= (G'(\Phi(x)))^{p-1}\,G'\left(\int_a^x\frac{\left|f^{\Delta}(t)\right|^p}{(\Phi^{\Delta}(t))^{p-1}}\Delta t\right).\end{aligned} \tag{29}$$
From Lemma 2.4 (note $\Phi^{\sigma}(x)<L$ for $x\in(a,b)_{\mathbb{T}}$ and $G'(z)>0$ for $z\in(0,L)$) we have $(G\circ\Phi)^{\Delta}(x)\geq G'(\Phi^{\sigma}(x))\Phi^{\Delta}(x)>0$.

Multiply both sides of (29) by $\left|f^{\Delta}(x)\right|^{p} / \left((G\circ\Phi)^{\Delta}(x)\right)^{p-1}$, and we have

$$\frac{(G'(|f(x)|))^{p}\left|f^{\Delta}(x)\right|^{p}}{((G\circ\Phi)^{\Delta}(x))^{p-1}} \geq \frac{(G'(\Phi(x)))^{p-1}\left|f^{\Delta}(x)\right|^{p}}{((G\circ\Phi)^{\Delta}(x))^{p-1}} G'\left(\int_{a}^{x}\frac{\left|f^{\Delta}(t)\right|^{p}}{(\Phi^{\Delta}(t))^{p-1}}\Delta t\right).$$

Now using the condition (γ_0) of the class Ω_L^0 (with $y = f(x)$ so $y \in (-L, L)$) we see that

$$\frac{|G'(f(x))|^{p}\left|f^{\Delta}(x)\right|^{p}}{((G\circ\Phi)^{\Delta}(x))^{p-1}} \geq \frac{(G'(|f(x)|))^{p}\left|f^{\Delta}(x)\right|^{p}}{((G\circ\Phi)^{\Delta}(x))^{p-1}},$$

and thus

$$\frac{|G'(f(x))|^{p}\left|f^{\Delta}(x)\right|^{p}}{((G\circ\Phi)^{\Delta}(x))^{p-1}} \geq \frac{(G'(\Phi(x)))^{p-1}\left|f^{\Delta}(x)\right|^{p}}{((G\circ\Phi)^{\Delta}(x))^{p-1}} G'\left(\int_{a}^{x}\frac{\left|f^{\Delta}(t)\right|^{p}}{(\Phi^{\Delta}(t))^{p-1}}\Delta t\right). \quad (30)$$

Also from Lemma 2.4, $(G\circ\Phi)^{\Delta}(x) \leq G'(\Phi(x))\Phi^{\Delta}(x)$, so from (30) we have

$$\frac{|G'(f(x))|^{p}\left|f^{\Delta}(x)\right|^{p}}{((G\circ\Phi)^{\Delta}(x))^{p-1}} \geq \frac{\left|f^{\Delta}(x)\right|^{p}}{(\Phi^{\Delta}(x))^{p-1}} G'\left(\int_{a}^{x}\frac{\left|f^{\Delta}(t)\right|^{p}}{(\Phi^{\Delta}(t))^{p-1}}\Delta t\right).$$

Now applying Lemma 2.4 (note $F^{\sigma}(x) < L$ for $x \in (a,b)_{\mathbb{T}}$) with

$$F(x) = \int_{a}^{x}\frac{\left|f^{\Delta}(t)\right|^{p}}{(\Phi^{\Delta}(t))^{p-1}}\Delta t,$$

since $F^{\Delta}(x) = \left|f^{\Delta}(x)\right|^{p} / \left(\Phi^{\Delta}(x)\right)^{p-1} \geq 0$, we see that

$$\frac{|G'(f(x))|^{p}\left|f^{\Delta}(x)\right|^{p}}{((G\circ\Phi)^{\Delta}(x))^{p-1}} \geq G^{\Delta}\left(\int_{a}^{x}\frac{\left|f^{\Delta}(t)\right|^{p}}{(\Phi^{\Delta}(t))^{p-1}}\Delta t\right). \quad (31)$$

Integrate both sides of (31) from a to b, and we have

$$\begin{aligned}\int_{a}^{b}\frac{|G'(f(x))|^{p}\left|f^{\Delta}(x)\right|^{p}}{((G\circ\Phi)^{\Delta}(x))^{p-1}}\Delta x &\geq \int_{a}^{b}\left[G^{\Delta}\left(\int_{a}^{x}\frac{\left|f^{\Delta}(t)\right|^{p}}{(\Phi^{\Delta}(t))^{p-1}}\Delta t\right)\right]\Delta x \\ &= G\left(\int_{a}^{b}\frac{\left|f^{\Delta}(x)\right|^{p}}{(\Phi^{\Delta}(x))^{p-1}}\Delta x\right),\end{aligned}$$

which is (26). The proof is complete. □

Remark 3.8. If F is monotonic then we can obtain results using the obvious spaces Ω_L^2, Ω_L^3, Ω_L^4 and Ω_L^5.

In the following, we derive some Opial type inequalities. For simplicity, we assume that the constant L in the above theorems is infinity (i.e. $G \in \Gamma_\infty$ or $G \in \Omega_\infty$).

Theorem 3.9. *Assume that s is a nonnegative function on $(a,b)_{\mathbb{T}}$ and r is a positive function on $(a,\sigma(b))_{\mathbb{T}}$, $p>1$ and $\int_a^{\sigma(b)} r^{-1/(p-1)}(x)\Delta x<\infty$.*
Let $\Phi(x)=\int_a^x r^{-1/(p-1)}(t)\Delta t$, $G\in\Gamma_\infty$ and suppose that f satisfies the conditions in Theorem 3.1 (with $L=\infty$). Then

$$\int_a^b |(G'\circ f)(x)|^q \left|f^\Delta(x)\right|^q s(x)\Delta x \le K^q \left(G\left[\int_a^b \left|f^\Delta(x)\right|^p r(x)\Delta x\right]\right)^{q/p}, \tag{32}$$

where $1/p+1/\gamma=1/q$, $\gamma>0$ and

$$K:=\left[\int_a^b \left[(G\circ\Phi)^\Delta(x)\right]^{\gamma(1-1/p)} s^{\gamma/q}(x)\Delta x\right]^{1/\gamma}<\infty.$$

Proof. The integral on the left-hand side of (32) can be written as

$$\begin{aligned}
&\int_a^b |(G'\circ f)(x)|^q \left|f^\Delta(x)\right|^q s(x)\Delta x\\
=\ &\int_a^b \frac{|(G'\circ f)(x)|^q \left|f^\Delta(x)\right|^q}{((G\circ\Phi)^\Delta(x))^\alpha}\left[(G\circ\Phi)^\Delta(x)\right]^\alpha s(x)\Delta x,
\end{aligned}$$

where $\alpha=q(p-1)/p$. Apply Hölder's inequality with p/q (note $p/q>1$ since $1/p=1/q-1/\gamma<1/q$) and γ/q, and we have

$$\begin{aligned}
\int_a^b |(G'\circ f)(x)|^q \left|f^\Delta(x)\right|^q s(x)\Delta x &\le \left(\int_a^b \frac{|(G'\circ f)(x)|^p \left|f^\Delta(x)\right|^p}{((G\circ\Phi)^\Delta(x))^{p-1}}\Delta x\right)^{q/p}\\
\times\left(\int_a^b \left[(G\circ\Phi)^\Delta(x)\right]^{\gamma(p-1)/p}(s(x))^{\gamma/q}\Delta x\right)^{q/\gamma}.
\end{aligned}$$

Let
$$K=\left[\int_a^b \left[(G\circ F)^\Delta(x)\right]^{\gamma(1-1/p)} s^{\gamma/q}(x)\Delta x\right]^{1/\gamma}$$

Note that $\Phi^\Delta(t)>0$ for $t\in(a,\sigma(b))_{\mathbb{T}}$ since r is a positive function on $(a,\sigma(b))_{\mathbb{T}}$ and $\Phi^\sigma(x)<\infty$ for $x\in(a,b)_{\mathbb{T}}$ since $\int_a^{\sigma(b)} r^{-1/(p-1)}(x)\Delta x<\infty$. Therefore, applying Theorem 3.1 we have

$$\begin{aligned}
&\int_a^b |(G'\circ f)(x)|^q \left|f^\Delta(x)\right|^q s(x)\Delta x \le K^q\left(\int_a^b \frac{|(G'\circ f)(x)|^p \left|f^\Delta(x)\right|^p}{((G\circ\Phi)^\Delta(x))^{p-1}}\Delta x\right)^{q/p}\\
&\le K^q\left(G\left(\int_a^b \frac{\left|f^\Delta(x)\right|^p}{(\Phi^\Delta(x))^{p-1}}\Delta x\right)\right)^{q/p} = K^q\left(G\left(\int_a^b \left|f^\Delta(x)\right|^p r(x)\Delta x\right)\right)^{q/p},
\end{aligned}$$

which is (32). The proof is complete. □

The proof of the following theorem is similar to the proof of Theorem 3.9 (apply Theorem 3.7).

Theorem 3.10. *Assume that s is a nonnegative function on $(a,b)_{\mathbb{T}}$ and r is a positive function on $(a,\sigma(b))_{\mathbb{T}}$, $p>1$ and $\int_a^{\sigma(b)} r^{-1/(p-1)}(x)\Delta x<\infty$.*
Let $\Phi(x)=\int_a^x r^{-1/(p-1)}(t)\Delta t$, $G\in\Omega_\infty$ and suppose that the function f satisfies the conditions in Theorem 3.7 (with $L=\infty$). Then

$$\left[\int_a^b |(G'\circ f)(x)|^q \left|f^\Delta(x)\right|^q s(x)\Delta x\right]^{1/q} \geq M\left(G\left[\int_a^b \left|f^\Delta(x)\right|^p r(x)\Delta x\right]\right)^{1/p}, \tag{33}$$

where $1/p+1/\gamma=1/q$, $\gamma>0$ and

$$M:=\left[\int_a^b \left[(G\circ\Phi)^\Delta(x)\right]^{\gamma(1-1/p)} s^{\gamma/q}(x)\Delta x\right]^{1/\gamma}<\infty.$$

As a special case of Theorem 3.9 when $\mathbb{T}=\mathbb{R}$, we have the following result.

Corollary 3.11. *Assume that s is a nonnegative function on $(a,b)_{\mathbb{R}}$ and r is a positive function on $(a,b)_{\mathbb{R}}$, $p>1$ and $\int_a^b r^{-1/(p-1)}(x)dx<\infty$.*
Let $\Phi(x)=\int_a^x r^{-1/(p-1)}(t)dt$, $G\in\Gamma_\infty$, and suppose that the function f satisfies the conditions in Theorem 3.4 (with $L=\infty$). Then

$$\left[\int_a^b |(G\circ f)'(x)|^q s(x)dx\right]^{1/q} \leq K\left(G\left[\int_a^b |f'(x)|^p r(x)dx\right]\right)^{1/p}, \tag{34}$$

where $1/p+1/\gamma=1/q$, $\gamma>0$ and

$$K:=\left[\int_a^b \left[(G\circ\Phi)'(x)\right]^{\gamma(1-1/p)} s^{\gamma/q}(x)dx\right]^{1/\gamma}<\infty.$$

Corollary 3.12. *Assume that s is a nonnegative function on $(a,b)_{\mathbb{R}}$ and r is a positive function on $(a,b)_{\mathbb{R}}$, $p>1$ and $\int_a^b r^{-1/(p-1)}(x)dx<\infty$.*
Let $\Phi(x)=\int_a^x r^{-1/(p-1)}(t)dt$, and suppose that the function f satisfies the conditions in Theorem 3.4 (with $L=\infty$). Then

$$\int_a^b |f(x)|^p |f'(x)|^q s(x)dx \leq M\left(\int_a^b r(x)|f'(x)|^p dx\right)^{\frac{p+q}{p}}, \tag{35}$$

where

$$M=\left(\frac{q}{q+p}\right)^q\left(\frac{p}{q}\right)^{q\left(\frac{p-1}{p}\right)}\left[\int_a^b\left[\int_a^x \frac{1}{r^{1/(p-1)}}(t)dt\right]^{p-1} r^{-\gamma/p}(x)s^{\gamma/q}(x)dx\right]^{1/\gamma}, \tag{36}$$

where $1/p+1/\gamma=1/q$ and $\gamma>0$.

Proof. Since $1/p + 1/\gamma = 1/q$ we have $p > q$, and therefore $(p+q)/q > 2$. Let $G(u) = |u|^{(p+q)/q}$. For $u \in (0,\infty)$, $G'(u) = \left(\frac{p+q}{q}\right) u^{\frac{p}{q}}$, for $u \in (-\infty, 0)$,

$$G'(u) = -\left(\frac{p+q}{q}\right)(-u)^{\frac{p}{q}}, \quad G'(0) = 0$$

and
$$|G'(u)| = G'(|u|) = \left(\frac{p+q}{q}\right)|u|^{\frac{p}{q}} \quad \text{for} \quad u \in (-\infty,\infty).$$

Also if $x \le y^{\alpha} z^{1-\alpha}$, for $\alpha \in [0,1]$, and $0 \le x,\, y,\, z < 1$ then

$$\begin{aligned} G'(x) &= \left(\frac{p+q}{q}\right) x^{\frac{p}{q}} \le \left(\frac{p+q}{q}\right)\left(y^{\alpha} z^{1-\alpha}\right)^{\frac{p}{q}} \\ &= \left(\frac{p+q}{q}\right)^{\alpha}\left(y^{\frac{p}{q}}\right)^{\alpha}\left(\frac{p+q}{q}\right)^{1-\alpha}\left(z^{\frac{p}{q}}\right)^{1-\alpha} = \left(G'(y)\right)^{\alpha}\left(G'(z)\right)^{1-\alpha}. \end{aligned}$$

Thus $G \in \Gamma_{\infty}$. Apply Corollary 3.11, and note the left side of (34) is

$$\begin{aligned} \left[\int_a^b |(G \circ f)'(x)|^q\, s(x)dx\right]^{1/q} &= \left[\int_a^b |(G' \circ f)(x)|^q\, |f'(x)|^q\, s(x)dx\right]^{1/q} \\ &= \frac{p+q}{q}\left[\int_a^b |f(x)|^p\, |f'(x)|^q\, s(x)dx\right]^{1/q}, \end{aligned}$$

and the right hand side of (34) is

$$K_1\left(\left[\int_a^b |f'(x)|^p\, r(x)dx\right]^{\frac{p+q}{q}}\right)^{1/p} = K_1\left(\int_a^b |f'(x)|^p\, r(x)dx\right)^{(p+q)/pq},$$

where

$$\begin{aligned} K_1 &= \left[\int_a^b \left[\left(\left[\int_a^x r^{-1/(p-1)}(t)dt\right]^{p/q}\right)'\right]^{\gamma(1-1/p)} s(x)^{\gamma/q}dx\right]^{1/\gamma} \\ &= \left[\int_a^b \left[\frac{p}{q}\left(\int_a^x r^{-1/(p-1)}(t)dt\right)^{p/q-1} r^{-1/(p-1)}(x)\right]^{\gamma(1-1/p)} s^{\gamma/q}(x)dx\right]^{1/\gamma} \\ &= \left(\frac{p}{q}\right)^{1-1/p}\left[\int_a^b \left[\int_a^x r^{-1/(p-1)}(t)dt\right]^{p-1} r^{-\gamma/p}(x) s^{\gamma/q}(x)dx\right]^{1/\gamma}. \end{aligned}$$

Thus
$$\int_a^b |f(x)|^p\, |f'(x)|^q\, s(x)dx \le M\left(\int_a^b |f'(x)|^p\, r(x)dx\right)^{(p+q)/p},$$

where M is given in (36). The proof is complete. □

Remark 3.13. In (35), if we applied Holder's inequality on the right hand side with indices $(p+q)/p$ and $(p+q)/q$, we see that

$$\begin{aligned}\left(\int_a^b r(x)\left|f'(x)\right|^p dx\right)^{\frac{p+q}{p}} &= \left(\int_a^b r^{1-\frac{p}{p+q}}(x) r^{\frac{p}{p+q}}(x)\left|f'(x)\right|^p dx\right)^{\frac{p+q}{p}} \\ &\leq \left(\int_a^b r(x)\left|f'(x)\right|^{p+q} dx\right)\left(\int_a^b r^{\left(1-\frac{p}{p+q}\right)\frac{p+q}{q}}(x)dx\right)^{\left(\frac{p+q}{p}\right)\left(\frac{q}{p+q}\right)} \\ &= \left(\int_a^b r(x)\left|f'(x)\right|^{p+q} dx\right)\left(\int_a^b r(x)dx\right)^{\frac{q}{p}}.\end{aligned}$$

Thus we have the following result.

Corollary 3.14. *Assume that s is a nonnegative function on $(a,b)_{\mathbb{R}}$ and r is a positive function on $(a,b)_{\mathbb{R}}$, $p>1$ and $\int_a^b r^{-1/(p-1)}(x)dx<\infty$.*
Let $\Phi(x)=\int_a^x r^{-1/(p-1)}(t)dt$, and suppose that the function f satisfies the conditions in Theorem 3.4 (with $L=\infty$). Then

$$\int_a^b |f(x)^p|\left|f'(x)\right|^q s(x)dx \leq M_2\left(\int_a^b r(x)\left|f'(x)\right|^{p+q} dx\right), \tag{37}$$

where

$$M_2=\left(\frac{q}{q+p}\right)^q\left(\frac{p}{q}\right)^{q\left(\frac{p-1}{p}\right)}\left(\int_a^b r(x)dx\right)^{\frac{q}{p}}\left[\int_a^b\left[\int_a^x \frac{1}{r^{\frac{1}{p-1}}(t)}dt\right]^{p-1} r^{-\frac{\gamma}{p}}(x)s^{\gamma/q}(x)dx\right]^{1/\gamma},$$

where $1/p+1/\gamma=1/q$ and $\gamma>0$.

Corollary 3.15. *Assume that s is a nonnegative function on $(a,b)_{\mathbb{R}}$ and r is a positive function on $(a,b)_{\mathbb{R}}$, $p>1$ and $\int_a^b r^{-1/(p-1)}(x)dx<\infty$.*
Let $\Phi(x)=\int_a^x r^{-1/(p-1)}(t)dt$, and suppose that the function f satisfies the conditions in Theorem 3.4 (with $L=\infty$). Then

$$\int_a^b s(x)\left|f(x)\right|^{p-q}\left|f'(x)\right|^q dx \leq M_1\int_a^b r(x)\left|f'(x)\right|^p dx, \tag{38}$$

where

$$M_1=\left(\frac{q}{p}\right)^{\frac{q}{p}}\left[\int_a^b\left[\int_a^x r^{-1/(p-1)}(t)dt\right]^{p-1} r^{-\gamma/p}(x)s^{\gamma/q}(x)dx\right]^{q/\gamma}, \tag{39}$$

$1/p+1/\gamma=1/q$, and $\gamma>0$.

Proof. Note since $1/p+1/\gamma=1/q$ then $p>q$ (i.e. $p/q>1$). Let $G(u)=|u|^{p/q}$. Note that $G\in\Gamma_\infty$ and $|G'(u)|=\frac{p}{q}|u|^{\frac{p}{q}-1}$ for $u\in(-\infty,\infty)$.

Apply Corollary 3.11, and note that the left side of (34) is

$$\left[\int_a^b \left|\frac{p}{q} f^{p/q-1}(x) f'(x)\right|^q s(x)dx\right]^{1/q} = \frac{p}{q}\left[\int_a^b |f(x)|^{p-q} |f'(x)|^q s(x)dx\right]^{1/q},$$

and the right hand side of (34) is

$$K\left(\int_a^b |f'(x)|^p r(x)dx\right)^{1/q},$$

where

$$\begin{aligned} K &= \left[\int_a^b \left[\left(\left[\int_a^x r^{-1/(p-1)}(t)dt\right]^{p/q}\right)'\right]^{\gamma(1-1/p)} s(x)^{\gamma/q}dx\right]^{1/\gamma} \\ &= \left[\int_a^b \left[\frac{p}{q}\left(\int_a^x r^{-1/(p-1)}(t)dt\right)^{p/q-1} r^{-1/(p-1)}(x)\right]^{\gamma(1-1/p)} s^{\gamma/q}(x)dx\right]^{1/\gamma} \\ &= \left(\frac{p}{q}\right)^{1-1/p}\left[\int_a^b \left[\int_a^x r^{-1/(p-1)}(t)dt\right]^{p-1} r^{-\gamma/p}(x)s^{\gamma/q}(x)dx\right]^{1/\gamma}. \end{aligned}$$

Thus
$$\int_a^b |f(x)|^{p-q} |f'(x)|^q s(x)dx \leq M_1 \int_a^b |f'(x)|^p r(x)dx,$$

where M_1 is defined as in (39). The proof is complete. □

In Corollary 3.15, let $p = 2$, $q = 1$ and $r(x) = s(x) = 1$. Then $\Phi(x) = x$ and it is easy see that $\gamma = 2$ and

$$M_1 = \frac{1}{\sqrt{2}}\left(\int_a^b (x-a)\,dx\right)^{\frac{1}{2}} = \frac{(b-a)}{2}.$$

Then we have an Opial type inequality.

Corollary 3.16. *If $f \in C^1([a,b]_{\mathbb{R}}, \mathbb{R})$ and $f(a) = 0$, then*

$$\int_a^b |f(x)|\,|f'(x)|\,dx \leq \frac{(b-a)}{2}\int_a^b |f'(x)|^2\,dx.$$

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