Journal of Convex Analysis
Volume 27 (2020), No. 4, 1091–1098

Barrier Functions in Subdifferential Theory

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Received: March 29, 2019
Accepted: October 4, 2019

We present a new method for proving Correa-Jofré-Thibault's theorem that monotonicity of subdifferential implies convexity of the function. This new method is based on barrier functions. Barrier functions help overcome some of the main technical difficulties when working with lower semicontinuous functions.

Keywords: Subdifferential, monotonicity, convex function, barrier function.

2010 Mathematics Subject Classification: 49J52, 47H05, 52A41.

1. Introduction

One of the basic theorems of Calculus says that a differentiable function is convex if and only if its derivative is monotone. So, one would expect that generalizations of derivative should satisfy this property as well.

Clarke in his classical book [1] demonstrated that a convex locally Lipschitz function can be characterized by monotonicity of its Clarke subdifferential. This result has been extended to lower semicontinuous function.

A first extension is due to Poliquin [12] who established the result in finite dimensional space.

Correa, Jofré and Thibault proved in a series of papers that a convex lower semicontinuous function can be characterized by a monotonicity property of its subdifferential – in a reflexive Banach space for the Clarke subdifferential in [3] and in any Banach space for an axiomatically introduced subdifferential in [4], and for a more general axiomatic presubdifferential in [5].

The main tool for proving this characterization is the Mean Value Theorem of Zagrodny [15], which holds for a lower semicontinuous function in a Banach space for any presubdifferential, see [13].

In setting up the axiomatic framework we follow [14], but we pick the apparently minimal set of axioms under which proofs can work. In this way our results are

*Supported by Bulgarian National Scientific Fund under grant KP-06-H22/4.
†Supported by Scientific Fund of Sofia University under grant 80-10-20/09.04.2019.

ISSN 0944-6532 / $ 2.50

slightly more general. As mentioned below, adding another natural axiom can significantly simplify the presentation.

We work in a real Banach space X with dual space X^*.

Definition 1.1. (Axioms for subdifferentials) A multi-valued operator ∂ which associates to any function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ and any $x \in X$ a (possibly empty) subset $\partial f(x) \subset X^*$ is a *feasible subdifferential* if

(P1) $\partial f(x) = \partial^c f(x)$ whenever f is a convex and continuous function in a neighbourhood U of x, where ∂^c stands for the Fenchel subdifferential, i.e.

$$\partial^c f(x) := \{x^* \in X^* : \langle x^*, y - x\rangle \le f(y) - f(x),\ \forall y \in U\};$$

(P2) For f lower semicontinuous and g convex and continuous in a neighbourhood of $x \in \operatorname{dom} f$,

$$0 \in \partial g(x) + \limsup_{y \to x,\ f(y) \to f(x)} \partial f(y) \tag{1}$$

whenever x is a local minimum point of $f + g$.

In more details (1) means that there are elements $y_n^* \in \partial f(y_n)$ such that $y_n \to x$, $f(y_n) \to f(x)$ and the sequence y_n^* converges in the w^* topology to some y^* such that $-y^* \in \partial g(x)$.

Note that all presubdifferentials considered by Correa, Jofré and Thibault in [3, 4, 5], as well as subdifferentials considered by Jules and Lassonde in [8] are feasible subdifferentials in the sense of Definition 1.1.

In terms of Ioffe's extensive classification, see [6], (P1) is called *contiguity*, while (P2) is a weak-star form of *trustworthness*.

We prove the above mentioned result for the feasible subdifferential in a different way by using barrier functions instead of (a variant of) Zagrodny's Theorem.

For convenience of notation we often identify the map $\partial f\colon X \rightrightarrows X^*$ with its graph, that is, $(x, x^*) \in \partial f$ is a shorthand for $x^* \in \partial f(x)$.

The main contribution of this work is a new method, based on [7], see also [11, p.569], for proving the following result of Correa, Jofré and Thibault.

Theorem 1.2. (Correa-Jofré-Thibault) *Let X be a Banach space and let ∂ be a feasible subdifferential. Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. If ∂f is monotone, then f is convex.*

Recall that monotonicity of ∂f means

$$(x_i, x_i^*) \in \partial f, \quad i = 1, 2 \Rightarrow \langle x_2^* - x_1^*, x_2 - x_1\rangle \ge 0.$$

The routine way of demonstrating the above result can be sketched like this: examining the proof in [13] it is clear that Zagrodny's Mean Value Theorem holds for any feasible subdifferential. Using it, one can prove a Minty test for optimality [9] of the following form

$$\langle x^*, x - x_0\rangle \ge 0,\ \forall (x, x^*) \in \partial f \Rightarrow 0 \in \partial^c f(x_0),$$

where f is proper and lower semicontinuous and ∂ is feasible subdifferential.

If, additional to (P1) and (P2), ∂ also satisfies the natural axiom

(P3) $\partial(f + x^*) = \partial f + x^*, \quad \forall x^* \in X^*$

(called *stability property* in [8], which is a rather limited form of the *calculability* axiom in [6]), then from Minty's test it immediately follows that $\partial \cup \partial^c$ has the somewhat surprising property (first noted by Jules and Lasonde) to be *maximal with respect to the monotonicity relation.* That is, if (x_0, x_0^*) is monotonely related to ∂f:

$$\langle x^* - x_0^*, x - x_0 \rangle \geq 0, \quad \forall (x, x^*) \in \partial f,$$

then $(x_0, x_0^*) \in \partial^c f$. From the latter Moreau-Rockafellar Theorem about maximal monotonicity of $\partial^c f$ for convex, proper and lower semicontinuous f follows immediately, but the surprising fact is that the above is true even if ∂f is not itself monotone. (For a precise statement see Theorem 4.1.)

So, in particular if ∂ is feasible and satisfies in addition (P3), then $\partial f \subset \partial^c f$, whenever ∂f is monotone. Further the proof can be completed as we do in the presented proof of Theorem 1.2.

Note that the additional axiom (P3) is not really necessary.

Our approach is based on a different technique involving barrier functions instead of Zagrodny's Theorem. We will also make the effort to obtain Theorem 1.2 for general feasible subdifferental.

The paper is organized as follows: In Section 2 we construct and consider the class of barrier functions we use. In Section 3 we show some additional properties of the feasible subdifferential linked to (P2) axiom. Finally, in Section 4 we present our proof of Correa-Jofré-Thibault theorem.

2. Barrier functions

Let $U \subset X$ be an open, convex and bounded neighbourhood of 0. Let μ be the *Minkowski functional* of U, that is,

$$\mu(x) = \inf\{\lambda : \lambda > 0,\ x \in \lambda U\}.$$

It is clear that $\mu(x) < 1$ if $x \in U$, and $\mu(x) \geq 1$ if $x \notin U$.

Let us first list a few properties of Minkowski functionals, see e.g. [2]:

(i) μ has values in $[0, +\infty)$;

(ii) μ is positively homogeneous, i.e. $\mu(tx) = t\mu(x)$ for all $x \in X$ and $t \geq 0$;

(iii) $\mu(x + y) \leq \mu(x) + \mu(y)$ for all $x, y \in X$ and hence μ is convex;

(iv) $\{x : \mu(x) \leq 1\} = \operatorname{cl} U \Rightarrow \{x : \mu(x) = 1\} = \partial U := \operatorname{cl} U \setminus U$, where $\operatorname{cl} U$ denotes the topological closure of U and ∂U denotes its boundary.

Moreover,

(v) There exist $c > b > 0$ such that $c\|x\| \geq \mu(x) \geq b\|x\|$ for all $x \in X$;

(vi) μ is Lipschtz continuous.

Indeed, let $\delta > 0$ be such that $\delta B_X \subset U$, where $B_X := \{x \in X : \|x\| \leq 1\}$.

Then $\dfrac{\delta x}{\|x\|} \in \delta B_X \subset U$, so $\mu\Big(\dfrac{\delta x}{\|x\|}\Big) \leq 1$, and $\mu(x) \leq \delta^{-1}\|x\|$.

Further, let $U \subset sB_X$, then $\dfrac{sx}{\|x\|} \notin U$, so $\mu\Big(\dfrac{sx}{\|x\|}\Big) \geq 1$, and $\mu(x) \geq \dfrac{1}{s}\|x\|$.

Hence, (v) holds with $c := \delta^{-1}$ and $b := s^{-1}$.

Finally, $\mu(x) = \mu((x-y)+y) \leq \mu(x-y) + \mu(y)$ by (ii).

Hence, $\mu(x) - \mu(y) \leq \mu(x-y)$ and using (v) we get $\mu(x) - \mu(y) \leq b\|x-y\|$, which yields $|\mu(x) - \mu(y)| \leq b\|x-y\|$ for all $x, y \in X$ and (vi) holds.

For $x \in U$, define the function $k(x) := \dfrac{\mu(x)}{1-\mu(x)}$. (2)

The next lemma shows that k is a *barrier function* for U, i.e. a continuous function whose values tend to infinity while the arguments tend to ∂U (see e.g. [10]).

Lemma 2.1. *The function k defined by* (2) *has the following properties:*

(a) $k(0) = 0$ *and for some* $b > 0$, $k(x) \geq \mu(x) \geq b\|x\|$.

(b) $\lim\limits_{x \to \partial U} k(x) = +\infty$*;*

(c) k *is convex and Lipschitz on each level set* $\{x : k(x) \leq \lambda\}$.

Proof. The properties (a) and (b) are clear. Since

$$k(x) = \varphi(\mu(x)), \quad \text{where} \quad \varphi(t) := \frac{t}{1-t}\,,$$

φ is increasing and convex on $[0,1[$, and μ is convex, the function k is also convex. The Lipschitz continuity of k on each level set is inherited by the Lipschitz continuity of μ. □

3. Properties of feasible subdifferential

For the sake of clarity, we will take two technical parts out of the proof of the main result. Namely, we will show some additional properties of feasible subdifferential linked to (P2) axiom.

Lemma 3.1. *Let ∂ be a feasible subdifferential. Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and bounded below on the open, convex and bounded set $U \subset X$. Let, moreover,* $\operatorname{dom} f \cap U \neq \emptyset$. *Let g be a convex continuous and bounded below barrier function for U. Let $\bar{x} \in X$ be fixed.*

Then there exist sequences $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subset U$ and $x_n^* \in \partial f(x_n)$, $y_n^* \in \partial^c g(y_n)$ such that

$$\|x_n - y_n\| \to 0, \tag{3}$$

$$f(x_n) + g(y_n) \to \inf_U (f+g), \tag{4}$$

$$\langle x_n^*, x_n - \bar{x}\rangle + \langle y_n^*, y_n - \bar{x}\rangle \to 0. \tag{5}$$

Proof. Define $$M := \sup\{\|y - \bar{x}\| : \ y \in U\}. \tag{6}$$

Since U is bounded, $M \in \mathbb{R}$.

Let
$$\overline{g}(x) := \begin{cases} g(x), & x \in U, \\ +\infty, & x \in X \setminus U. \end{cases}$$

It is clear that $\overline{g}$ is lower semicontinuous and convex. It is also clear that

$$\inf_X (f + \overline{g}) = \inf_U (f + g) \in \mathbb{R}.$$

Fix a sequence $(\varepsilon_n)_{n=1}^\infty$ such that $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$.

Pick $z_n \in X$ such that $f(z_n) + \overline{g}(z_n) < \inf_X (f + \overline{g}) + \varepsilon_n$.

By the Ekeland Variational Principle there are $y_n \in X$ such that

$$f(y_n) + \overline{g}(y_n) < \inf_X (f + \overline{g}) + \varepsilon_n,$$

and the function $x \to f(x) + \overline{g}(x) + \varepsilon_n \|x - y_n\|$ attains its minimum at y_n.

Since $\overline{g}(y_n) < +\infty$, in fact $y_n \in U$. So, we actually have that

$$f(y_n) + g(y_n) < \inf_U (f + g) + \varepsilon_n, \tag{7}$$

and the function $x \to f(x) + g(x) + \varepsilon_n \|x - y_n\|$ attains a local minimum at $y_n \in U$. Fix any $n \in \mathbb{N}$. By (P2) there are $u_k \to y_n$, as $k \to \infty$, and $u_k^* \in \partial f(u_k)$ such that $f(u_k) \to f(y_n)$ and

$$w^*\!-\lim u_k^* = u^* \in -\partial(g + \varepsilon_n \| \cdot - y_n \|)(y_n).$$

By the Sum Theorem for the Fenchel subdifferential and the obvious fact that $\partial \varepsilon_n \| \cdot - y_n \| = \varepsilon_n \partial \| \cdot - y_n \|$ it follows that there are $y_n^* \in \partial g(y_n)$ such that

$$\|u^* + y_n^*\| \le \varepsilon_n. \tag{8}$$

Since $(u_k, f(u_k)) \to (y_n, f(y_n))$ as $k \to \infty$, there is $k_1(n) \in \mathbb{N}$ such that

$$\|u_k - y_n\| < \varepsilon_n, \ |f(u_k) - f(y_n)| < \varepsilon_n, \quad \forall k > k_1(n). \tag{9}$$

By (7) we have

$$|f(u_k) + g(y_n) - \inf_U (f + g)| \le 2\varepsilon_n, \quad \forall k > k_1(n). \tag{10}$$

Note that

$$\langle u_k^*, u_k - \bar{x} \rangle + \langle y_n^*, y_n - \bar{x} \rangle = \langle u^* + y_n^*, y_n - \bar{x} \rangle + \langle u_k^* - u^*, y_n - \bar{x} \rangle + \langle u_k^*, u_k - y_n \rangle.$$

Now, $|\langle u^* + y_n^*, y_n - \bar{x} \rangle| \le \|u^* + y_n^*\| \|y_n - \bar{x}\| \le M\varepsilon_n$ by (6) and (8). Since u_k^* weak-star converges to u^*, we have $\langle u_k^* - u^*, y_n - \bar{x} \rangle \to 0$ as $k \to \infty$. Also, since by the Banach-Steinhaus Theorem the sequence $(u_k^*)_{k=1}^\infty$ is bounded, and $u_k \to y_n$ as $k \to \infty$, we have $|\langle u_k^*, u_k - y_n \rangle| \le \|u_k^*\| \|u_k - y_n\| \to 0$ as $k \to \infty$. Therefore, there is $k_2(n) \in \mathbb{N}$ such that

$$|\langle u_k^*, u_k - \bar{x} \rangle + \langle y_n^*, y_n - \bar{x} \rangle| \le (M + 1)\varepsilon_n, \quad \forall k > k_2(n). \tag{11}$$

Set $\quad k(n) = \max\{k_1(n), k_2(n)\} + 1$ and $(x_n, x_n^*) := (u_{k(n)}, u_{k(n)}^*)$.

From (9), (10) and (11) it follows that so constructed sequences $(x_n, x_n^*)_{n=1}^\infty$ and $(y_n, y_n^*)_{n=1}^\infty$ satisfy (3), (4) and (5). □

Proposition 3.2. *Let ∂ be a feasible subdifferential. Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semicontinuous function. Then* $\operatorname{dom} \partial f$ *is nonempty and*

$$f(\overline{x}) \le \sup\{f(x) + \langle x^*, \overline{x} - x\rangle : \ (x, x^*) \in \partial f\}, \ \textit{for all } \overline{x} \in X. \tag{12}$$

Proof. Fix arbitrary $\bar{x} \in X$ and $r < f(\bar{x})$. Fix some $y \in \operatorname{dom} f$.

Since f is lower semicontinuous and the segment $[\bar{x}, y]$ is compact, there is $\delta > 0$ such that f is bounded below on the set

$$C := [\bar{x}, y] + \delta B_X^\circ,$$

where $B_X^\circ := \{x \in X : \|x\| < 1\}$. Let k be the barrier function defined by (2) for the set $U := C - \{\bar{x}\}$.

Let $\xi > 0$ be such that $f > r$ on $\bar{x} + \xi B_X$. Fix $a > 0$ and such that

$$a > (r - \inf_U f)/(\xi b),$$

where $b > 0$ and $k(x) \ge \mu(x) \ge b\|x\|$ (see Lemma 2.1 (a)).

Then it is immediate that

$$f(x) + ak(x - \bar{x}) > r, \quad \forall x \in U. \tag{13}$$

Apply Lemma 3.1 to f and the barrier function $g = ak(\cdot - \overline{x})$ to get $x_n^* \in \partial f(x_n)$ and $y_n^* \in \partial^c k(\cdot - \bar{x})(y_n)$ such that (3) and (4) are fulfilled and, moreover,

$$\langle x_n^*, x_n - \bar{x}\rangle + a\langle y_n^*, y_n - \bar{x}\rangle = \alpha_n, \text{ where } \lim_{n\to\infty} \alpha_n = 0. \tag{14}$$

Since $y_n^* \in \partial^c k(y_n - \bar{x})$, we have

$$k(0) \ge k(y_n - \bar{x}) + \langle y_n^*, \bar{x} - y_n\rangle \iff \langle y_n^*, y_n - \bar{x}\rangle \ge k(y_n - \bar{x}),$$

From the boundedness below of f and (4) it follows that $(k(y_n - \bar{x}))_{n=1}^\infty$ is a bounded sequence. Since k is Lipschitz on its level sets (see Lemma 2.1 (c)), from (3) it follows that

$$k(y_n - \bar{x}) = k(x_n - \bar{x}) + \beta_n, \text{ where } \lim_{n\to\infty} \beta_n = 0.$$

This and (14) yield

$$\langle x_n^*, x_n - \bar{x}\rangle = \alpha_n - a\langle y_n^*, y_n - \bar{x}\rangle \le \alpha_n - ak(y_n - \bar{x}) = -ak(x_n - \bar{x}) + \alpha_n - a\beta_n.$$

So, $\langle x_n^*, x_n - \bar{x}\rangle \le -ak(x_n - \bar{x}) + \gamma_n$, where $\lim_{n\to\infty} \gamma_n = 0$.

From (13) it follows that $\langle x_n^*, x_n - \bar{x}\rangle < f(x_n) - r + \gamma_n$, or, equivalently,

$$r < f(x_n) + \langle x_n^*, \bar{x} - x_n\rangle + \gamma_n.$$

Therefore,

$$r \le \sup\{f(x_n) + \langle x_n^*, \bar{x} - x_n\rangle : \ n \in \mathbb{N}\} \le \sup\{f(x) + \langle x^*, \bar{x} - x\rangle : \ (x, x^*) \in \partial f\}.$$

Since $r < f(\bar{x})$ were arbitrary, we are done. □

4. Monotonicity and convexity

We start with the following extension to the case of feasible subdifferential of a result of Jules and Lassonde [8].

Theorem 4.1. *Let ∂ be a feasible subdifferential. Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $(x_0, x_0^*) \in X \times X^*$ be in monotone relation to ∂f, that is,*

$$\langle x^* - x_0^*, x - x_0 \rangle \geq 0, \quad \forall (x, x^*) \in \partial f. \tag{15}$$

Then $(x_0, x_0^*) \in \partial^c f$.

Proof. Let $r \in \mathbb{R}$ be such that $$r < f(x_0). \tag{16}$$

Let $y \in \operatorname{dom} f$ be arbitrary. Since f is lower semicontinuous and the segment $[x_0, y]$ is compact, there is $\delta > 0$ such that f is bounded below on

$$C := [x_0, y] + \delta B_X^\circ.$$

Let k be the barrier function defined by (2) for the set $U := C - \{x_0\}$.

Let $a > 0$ be arbitrary. Obviously, $g(x) := -\langle x_0^*, x - x_0 \rangle + ak(x - x_0)$ is a convex and continuous barrier for U. So, we can apply Lemma 3.1 to f and g. Since

$$\partial^c g = -x_0^* + a\partial^c k(\cdot - x_0),$$

there are $x_n^* \in \partial f(x_n)$ and $y_n^* \in \partial^c k(\cdot - x_0)(y_n)$ such that (3) and (4) are fulfilled and, moreover,

$$\langle x_n^*, x_n - x_0 \rangle + \langle -x_0^* + a y_n^*, y_n - x_0 \rangle \to 0. \tag{17}$$

Clearly, $\langle x_0^*, x_n - y_n \rangle \to 0$, so (17) is equivalent to

$$\langle x_n^* - x_0^*, x_n - x_0 \rangle + a \langle y_n^*, y_n - x_0 \rangle \to 0.$$

This and (15) give $$\limsup_{n\to\infty} \langle y_n^*, y_n - x_0 \rangle \leq 0. \tag{18}$$

Since $y_n^* \in \partial^c k(\cdot - x_0)(y_n)$, we have

$$k(0) \geq k(y_n - x_0) + \langle y_n^*, x_0 - y_n \rangle \iff \langle y_n^*, y_n - x_0 \rangle \geq k(y_n - x_0), \tag{19}$$

because $k(0) = 0$. But $k(y_n - x_0) \geq b\|y_n - x_0\|$ (cf. Lemma 2.1 (a)), so (18) and (19) imply that

$$\limsup_{n\to\infty} \|y_n - x_0\| \leq 0 \iff \lim_{n\to\infty} y_n = x_0.$$

From (3) it follows that $x_n \to x_0$ as well; and from the lower semicontinuity of f and (4) it follows that

$$f(x_0) + g(x_0) = \inf_U (f + g).$$

In particular, $f(y) - \langle x_0^*, y - x_0 \rangle + ak(y - x_0) \geq f(x_0)$.

Since $a > 0$ was arbitrary, $f(y) \geq f(x_0) + \langle x_0^*, y - x_0 \rangle$.

So, $x_0 \in \operatorname{dom} f$ and, since $y \in \operatorname{dom} f$ was arbitrary, $x_0^* \in \partial^c f(x_0)$. $\square$

After all the preparations above the proof of Correa-Jofré-Thibault is now almost immediate.

Theorem 4.2. (Correa-Jofré-Thibault)*Let X be a Banach space and ∂ be a feasible subdifferential. Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. If ∂f is monotone, then f is convex.*

Proof. Consider $g\colon X \to \mathbb{R} \cup \{+\infty\}$ defined as

$$g(\overline{x}) := \sup\{f(x) + \langle x^*, \overline{x} - x\rangle : \ (x, x^*) \in \partial f\}.$$

As a supremum of linear functions, g is convex and lower semicontinuous.

By (12) we have that $f \le g$. From Theorem 4.1 and the monotonicity of ∂f we have that $\partial f \subset \partial^c f$, which implies $f \ge g$. Therefore, $f = g$. $\square$

Acknowledgement. We thank Prof. Lionel Thibault for his helpful comments.

References

[1] F. Clarke: *Optimization and Nonsmooth Analysis*, Wiley, New York (1983).

[2] F. Clarke: *Functional Analysis, Calculus of Variations and Optimal Control*, Graduate Texts in Mathematics 264, Springer, London (2013).

[3] R. Correa, A. Jofré, L. Thibault: *Characterization of lower semicontinuous convex functions*, Proc. Amer. Math. Soc. 116 (1992) 67–72.

[4] R. Correa, A. Jofré, L. Thibault: *Subdifferential monotonicity as characterization of convex functions*, Numer. Funct. Anal. Optimization 15 (1994) 531–535.

[5] R. Correa, A. Jofré, L. Thibault: *Subdifferential characterization of convexity*, in: *Recent Advances in Non-Smooth Optimization*, D.-Z. Du, L. Qi and R. S. Womersly (eds.), World Scentific, Singapour (1995) 18–23.

[6] A. Ioffe: *On the theory of subdifferentials*, Adv. Nonlinear Analysis 1 (2012) 47–120.

[7] M. Ivanov, N. Zlateva: *Maximal monotonicity of the subdifferential of a convex function: a direct proof*, J. Convex Analysis 24 (2017) 1307–1311.

[8] F. Jules, M. Lassonde: *Subdifferential test for optimality*, J. Global Optimization 59 (2014) 101–106.

[9] G. J. Minty: *On the monotonicity of the gradient of a convex function*, Pacific J. Math. 14 (1964) 243–247.

[10] J. Nocedal, S. Wright: *Numerical Optimization*, Springer, New York (1999).

[11] J.-P. Penôt: *Analysis: From Concepts to Applications*, Springer, Cham (2016).

[12] R. A. Poliquin: *Subgradient monotonicity and convex functions*, Nonlinear Anal. Theory Methods Appl. 14 (1990) 305–317.

[13] L. Thibault: *A note on the Zagrodny mean value theorem*, Optimization 35 (1995) 127–130.

[14] L. Thibault, N. Zlateva: *Integrability of subdifferentials of directionally Lipschitz functions*, Proc. Amer. Math. Soc. 133(10) (2005) 2939–2948.

[15] D. Zagrodny: *Approximate mean value theorem for upper subderivatives*, Nonlinear Analysis 12 (1988) 1413–1428.