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On Densely Complete Metric Spaces and Extensions of Uniformly Continuous Functions in ZF

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A metric space $\mathbf{X}$ is called densely complete if there exists a dense set D in $\mathbf{X}$ such that every Cauchy sequence of points of D converges in $\mathbf{X}$. One of the main aims of this work is to prove that the countable axiom of choice, **CAC** for abbreviation, is equivalent to the following statements:

(i) Every densely complete (connected) metric space $\mathbf{X}$ is complete.

(ii) For every pair of metric spaces $\mathbf{X}$ and $\mathbf{Y}$, if $\mathbf{Y}$ is complete and $\mathbf{S}$ is a dense subspace of $\mathbf{X}$, while $f\colon \mathbf{S} \to \mathbf{Y}$ is a uniformly continuous function, then there exists a uniformly continuous extension $F\colon \mathbf{X} \to \mathbf{Y}$ of f.

(iii) Complete subspaces of metric spaces have complete closures.

(iv) Complete subspaces of metric spaces are closed.

It is also shown that the restriction of (i) to subsets of the real line is equivalent to the restriction $\mathbf{CAC}(\mathbb{R})$ of **CAC** to subsets of $\mathbb{R}$. However, the restriction of (ii) to subsets of $\mathbb{R}$ is strictly weaker than $\mathbf{CAC}(\mathbb{R})$ because it is equivalent to the statement that $\mathbb{R}$ is sequential. Moreover, among other relevant results, it is proved that, for every positive integer n, the space $\mathbb{R}^n$ is sequential if and only if $\mathbb{R}$ is sequential. It is also shown that $\mathbb{R} \times \mathbb{Q}$ is not densely complete if and only if $\mathbf{CAC}(\mathbb{R})$ holds.

Keywords: Countable axiom of choice, complete metric spaces, connected metric spaces, sequential spaces.

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1. Notation and terminology

Let $\mathbf{X} = \langle X, d\rangle$ be a metric space. For $x \in X$ and $\varepsilon > 0$, let $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. The topology on X induced by d is denoted by $\tau(d)$. If $A \subseteq X$, the diameter of A in $\mathbf{X}$ is denoted by $\delta(A)$ or by $\delta_d(A)$ if it is better to point at the metric in use. In the sequel, boldface letters will denote metric spaces and lightface letters will denote their underlying sets. If $Y \subseteq X$, then $\mathbf{Y}$ denotes the metric subspace of $\mathbf{X}$, that is $\mathbf{Y} = \langle Y, d_{|Y}\rangle$ where $d_{|Y}(x, y) = d(x, y)$ for all $x, y \in Y$.

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If τ is a topology on a set Z and Y is a subset of Z, then, for $\tau_{|Y} = \{U \cap Y : U \in \tau\}$, the topological space $\langle Y, \tau_{|Y}\rangle$ can be denoted by Y and called the *(topological) subspace* of $\langle Z, \tau\rangle$. Topological subspaces of the metric space $\mathbf{X}$ are topological subspaces of the topological space $\langle X, \tau(d)\rangle$.

The set of all finite von Neumann ordinal numbers is denoted by ω, while $\mathbb{N} = \omega \setminus \{0\}$. We recall that if $n \in \omega$, then $n+1 = n \cup \{n\}$. We call a set Z *countable* if Z is equipotent with a subset of ω. A set equipotent with ω is called *countably infinite*. The power set of a set Z is denoted by $\mathcal{P}(Z)$.

In the Definitions 1.1–1.3, we recall several known notions.

Definition 1.1. Let $A \subseteq X$. We denote by A^* the set of all points $x \in X$ which have the property that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $A \setminus \{x\}$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. Then the set $\widetilde{A} = A \cup A^*$ is called the *sequential closure* of A in $\mathbf{X}$. The set A is called *sequentially closed* in $\mathbf{X}$ if $A^* \subseteq A$.

Definition 1.2. The space $\mathbf{X}$ is called:

(i) *sequential* if each sequentially closed subset of $\mathbf{X}$ is closed in $\mathbf{X}$;

(ii) *Fréchet-Urysohn* if, for every $F \in \mathcal{P}(X)$, all accumulation points of F belong to the sequential closure of F;

(iii) *complete* if each of its Cauchy sequences converges in $\mathbf{X}$;

(iv) *Cantor complete* if, for every descending family $\{G_n : n \in \mathbb{N}\}$ of non-empty closed subsets of $\mathbf{X}$ with $\lim_{n\to\infty} \delta(G_n) = 0$, the set $\bigcap_{n\in\mathbb{N}} G_n$ is a singleton.

Definition 1.3. Let f be a function from $\mathbf{X}$ into a metric space $\mathbf{Y}$. Then f is called:

(i) *sequentially continuous at* $x_0 \in X$ if, for every sequence $(x_n)_{n\in\mathbb{N}}$ of points of X such that $(x_n)_{n\in\mathbb{N}}$ converges in $\mathbf{X}$ to x_0, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges in $\mathbf{Y}$ to $f(x_0)$;

(ii) *sequentially continuous* if f is sequentially continuous at each point of $\mathbf{X}$.

To a great extent, this article is about new concepts introduced in the following definition:

Definition 1.4. (i) If ρ is a metric on a set Y and $\mathbf{Y} = \langle Y, \rho\rangle$, then:

(a) for $D \subseteq Y$, the metric space $\mathbf{Y} = \langle Y, \rho\rangle$ is called *densely complete with respect to* D if D is dense in $\mathbf{Y}$ and every Cauchy sequence of points of D converges in $\mathbf{Y}$;

(b) the metric space $\mathbf{Y}$ is called *densely complete* if there exists a subset D of Y such that $\mathbf{Y}$ is densely complete with respect to D;

(c) the metric ρ is called *densely complete* if the metric space $\mathbf{Y}$ is densely complete.

(ii) A topological space $\langle Y, \tau\rangle$ is called *densely completely metrizable* if there exists a densely complete metric ρ on Y such that $\tau = \tau(\rho)$.

The symbol $\mathbb{R}$ denotes the set of real numbers equipped with the metric induced by the standard absolute value. As usual, for $n \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let $\|x\| = \sqrt{\Sigma_{i\in n} x(i)^2}$ and, for $x, y \in \mathbb{R}^n$, let $d_{e(n)}(x, y) = \|x - y\|$. Then $d_{e(n)}$ is the

Euclidean metric on $\mathbb{R}^n$. As usual, the metric space $\langle \mathbb{R}^n, d_{e(n)}\rangle$, the set $\mathbb{R}^n$ and the topological space $\langle \mathbb{R}^n, \tau(d_{e(n)})\rangle$ are all denoted by $\mathbb{R}^n$ if this is not misleading.

We recall that if $\{\mathbf{X}_n : n \in \mathbb{N}\}$ is a family of metric spaces $\mathbf{X}_n = \langle X_n, d_n\rangle$, while $X = \prod_{n\in\mathbb{N}} X_n$ and, for all $n \in \mathbb{N}$ and $a, b \in X_n$, $\rho_n(a,b) = \min\{1, d_n(a,b)\}$, then the function $d : X \times X \to \mathbb{R}$ given by:

$$d(x,y) = \sum_{n\in\mathbb{N}} \frac{\rho_n(x(n), y(n))}{2^n} \tag{1}$$

is a metric such that the topology $\tau(d)$ coincides with the product topology of the family of topological spaces $\{\langle X_n, \tau(d_n)\rangle : n \in \mathbb{N}\}$. In the sequel, we shall always assume that whenever a family $\{\mathbf{X}_n : n \in \mathbb{N}\}$ of metric spaces $\mathbf{X}_n = \langle X_n, d_n\rangle$ is given, then $X = \prod\limits_{n\in\mathbb{N}} X_n$ and $\mathbf{X} = \langle X, d\rangle$ where d is the metric on X defined by (1).

A set which contains a countably infinite subset is called *Dedekind-infinite*. That X is Dedekind-infinite is denoted by $\mathbf{DI}(X)$. A set which is not Dedekind-infinite is called *Dedekind-finite*. By universal quantifying over X, $\mathbf{DI}(X)$ gives rise to the choice principle **IDI**: $\forall X(X \text{ infinite} \to \mathbf{DI}(X))$ that is, "every infinite set is Dedekind-infinite" (Form 9 of [8], Definition 2.13(1) in [6]). Below, we list some other weak forms of the axiom of choice we shall deal with in the sequel. For the known forms, we quote in their statements, the form number under which they are recorded in [8] or we refer to their definitions in [6]. The rest of the forms are new here.

We recall that if $\mathcal{A} = \{A_j : j \in J\}$ is an indexed collection of sets, then every element of the product $\prod_{j\in J} A_j$ is called a *choice function* of $\mathcal{A}$, while every choice function of a subcollection $\{A_j : j \in J^*\}$ of $\mathcal{A}$ for an infinite subset J^* of J is called a *partial choice function* of $\mathcal{A}$. Given a pairwise disjoint family $\mathcal{A} = \{A_j : j \in J\}$ (i.e., $A_{j_1} \cap A_{j_2} = \emptyset$ for each pair of distinct elements j_1, j_2 of J), a set C is called a *partial choice set* of $\mathcal{A}$ if the set $\{j \in J : C \cap A_j \text{ is a singleton}\}$ is infinite. If $\mathcal{A} \subseteq \mathcal{P}(X)$ for a set X, we can consider $\mathcal{A}$ as the indexed family $\{A : A \in \mathcal{A}\}$ to speak about (partial) choice functions of $\mathcal{A}$.

- $\mathbf{CAC}(X)$: Every family $\{A_n : n \in \omega\}$ of non-empty subsets of X has a choice function.
- **CAC** (Form 8 in [8], Definition 2.5 in [6]): For every infinite set X, $\mathbf{CAC}(X)$ holds. Equivalently, every family $\{A_n : n \in \omega\}$ of non-empty sets has a partial choice function (cf. Form [8 B] in [8]).
- **CMC** (Form 126 in [8], Definition 2.10 in [6]): For every collection $\{A_n : n \in \omega\}$ of non-empty sets, there exists a collection $\{F_n : n \in \omega\}$ of non-empty finite sets such that $F_n \subseteq A_n$ for each $n \in \omega$.
- $\mathbf{CAC}_{fin}$ (Form 10 in [8], Definition 2.9(3) in [6]): **CAC** restricted to countable families of non-empty finite sets.
- $\mathbf{CAC}(\mathbb{R})$ (Form 94 in [8], Definition 2.9(1) in [6]): **CAC** restricted to families of non-empty subsets of $\mathbb{R}$. Equivalently, every pairwise disjoint family $\{A_n : n \in \omega\}$ of non-empty subsets of the real line has a partial choice function (see, e.g., [7]).

- $\omega - \mathbf{CAC}(\mathbb{R})$ (Definition 4.56 in [6]): For every family $\mathcal{A} = \{A_n : n \in \omega\}$ of non-empty subsets of $\mathbb{R}$ there exists a family $\mathcal{B} = \{B_n : n \in \omega\}$ of countable non-empty subsets of $\mathbb{R}$ such that, for each $n \in \omega$, the inclusion $B_n \subseteq A_n$ holds.
- $\mathbf{IDI}(\mathbb{R})$ (Form 13 in [8], Definition 2.13(2) in [6]): **IDI** restricted to subsets of $\mathbb{R}$.
- **DC** (Form 43 in [8], Definition 2.11 (1) in [6]): For every pair $\langle X, \rho\rangle$, if X is a set, while $\rho \subseteq X \times X$ is such that, for each $x \in X$, there exists $y \in X$ with $\langle x, y\rangle \in \rho$, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of X such that $\langle x_n, x_{n+1}\rangle \in \rho$ for each $n \in \mathbb{N}$.
- **BPI** (Form 14 in [8], Definition 2.15 in [6]): Every Boolean algebra has a prime ideal.

It is known that **BPI** is equivalent to the statement: The Tychonoff product of compact T_2-spaces is compact (see, e.g., [16], Form [14 J] in [8] and Theorem 4.70 in [6]).

Our main aim is to investigate the following newly defined forms:

- $\mathbf{DCC}(\mathbf{X})$: Every densely complete subspace of $\mathbf{X}$ is complete.
- **DCC**: Every densely complete metric space is complete.
- $\mathbf{DCC}_c(\mathbf{X})$: Every densely complete connected subspace of $\mathbf{X}$ is complete.
- $\mathbf{DCC}_c$: Every densely complete connected metric space is complete.
- $\mathbf{UCE}(\mathbf{X})$: For every metric subspace $\mathbf{Z}$ of $\mathbf{X}$ and every complete metric subspace $\mathbf{Y}$ of $\mathbf{X}$, if S is a dense subset of $\mathbf{Z}$ and $f : \mathbf{S} \to \mathbf{Y}$ is uniformly continuous, then there exists a uniformly continuous extension $F : \mathbf{Z} \to \mathbf{Y}$ of f.
- **UCE**: For every metric space $\mathbf{X}$, $\mathbf{UCE}(\mathbf{X})$ holds.
- $\mathbf{UCE}(\mathbf{X}, \mathbb{R})$: For every metric subspace $\mathbf{Z}$ of $\mathbf{X}$ and every complete metric subspace $\mathbf{Y}$ of $\mathbb{R}$, if S is a dense subset of $\mathbf{Z}$ and $f : \mathbf{S} \to \mathbf{Y}$ is uniformly continuous, then there exists a uniformly continuous extension $F : \mathbf{Z} \to \mathbf{Y}$ of f.
- $\mathbf{UCE}_c$: For every compact metric space $\mathbf{X}$ and every complete metric space $\mathbf{Y}$, if S is a dense subset of $\mathbf{X}$ and $f : \mathbf{S} \to \mathbf{Y}$ is a uniformly continuous function, then there exists a uniformly continuous extension $F\colon \mathbf{X} \to \mathbf{Y}$ of f.
- $\mathbf{UCE}_{cc}(\mathbf{X})$: For every compact and connected metric subspace $\mathbf{Z}$ of $\mathbf{X}$ and every complete metric subspace $\mathbf{Y}$ of $\mathbf{X}$, if S is a dense subset of $\mathbf{Z}$ and $f : \mathbf{S} \to \mathbf{Y}$ is uniformly continuous, then there exists a uniformly continuous extension $F\colon \mathbf{Z} \to \mathbf{Y}$ of f.
- $\mathbf{UCE}_{cc}$: For every metric space $\mathbf{X}$, $\mathbf{UCE}_{cc}(\mathbf{X})$ holds.
- $\mathbf{UCE}_{cc}(\mathbf{X}, \mathbb{R})$: For every compact and connected metric subspace $\mathbf{Z}$ of $\mathbf{X}$ and every complete metric subspace $\mathbf{Y}$ of $\mathbb{R}$, if S is a dense subset of $\mathbf{Z}$ and $f : \mathbf{S} \to \mathbf{Y}$ is uniformly continuous, then there exists a uniformly continuous extension $F\colon \mathbf{Z} \to \mathbf{Y}$ of f.

- $\mathbf{UCE}_c(\mathbb{R})$: For every connected metric subspace $\mathbf{X}$ of $\mathbb{R}$ and every complete metric subspace $\mathbf{Y}$ of $\mathbb{R}$, if S is a dense subset of $\mathbf{X}$ and $f : \mathbf{S} \to \mathbf{Y}$ is a uniformly continuous function, then there exists a uniformly continuous extension $F : \mathbf{X} \to \mathbf{Y}$ of f.

Usually, a topological space which is simultaneously compact and connected is called a continuum. Since every non-empty continuum of $\mathbb{R}$ is either a singleton or an interval $[a, b]$ where $a, b \in \mathbb{R}$ and $a < b$, then the following proposition holds:

Proposition 1.5. $\mathbf{UCE}_{cc}(\mathbb{R}, \mathbb{R})$ *and* $\mathbf{UCE}_{cc}(\mathbb{R})$ *are equivalent.*

We recall that if $\mathbf{X}$ and $\mathbf{Y}$ are metric spaces, S is a dense set of $\mathbf{X}$, while $f : \mathbf{S} \to \mathbf{Y}$ is extendable to a uniformly continuous mapping from $\mathbf{X}$ to $\mathbf{Y}$, then there exists a unique uniformly continuous extension of f over X.

2. Introduction and some preliminary results

In this paper, the intended context for reasoning and statements of theorems will be the Zermelo-Fraenkel set theory $\mathbf{ZF}$ without the axiom of choice $\mathbf{AC}$. As usual, the system $\mathbf{ZF} + \mathbf{AC}$ is denoted by $\mathbf{ZFC}$. Our intention here is to study the set-theoretic strength of the new statements $\mathbf{DCC}$, $\mathbf{DCC}_c$, $\mathbf{DCC}(\mathbb{R})$, $\mathbf{DCC}_c(\mathbb{R})$, $\mathbf{UCE}$ and $\mathbf{UCE}(\mathbb{R})$, as well as to sort them in the hierarchy of choice principles. We also solve some non-trivial open problems posed in [15].

The paper is organized as follows. In Section 2, we give some preliminary results. Section 3 concerns $\mathbf{UCE}(\mathbb{R})$, $\mathbf{UCE}_c(\mathbb{R})$, $\mathbf{DCC}(\mathbb{R})$, $\mathbf{DCC}_c(\mathbb{R})$, as well as some modifications of $\mathbf{UCE}(\mathbb{R})$. In Section 3, among other results, it is proved that $\mathbf{UCE}(\mathbb{R})$ and $\mathbf{UCE}_c(\mathbb{R})$ are both equivalent to the sentence: $\mathbb{R}$ is sequential. It is also proved in Section 3 that $\mathbf{DCC}(\mathbb{R})$, $\mathbf{DCC}_c(\mathbb{R})$ and $\mathbf{CAC}(\mathbb{R})$ are all equivalent. Moreover, it is proved in Section 3 that, for each $n \in \mathbb{N}$, the space $\mathbb{R}^n$ is sequential if and only if $\mathbb{R}$ is sequential. This leads to immediate solutions of Problems 6.6 and 6.11 posed in [15].

In Section 4, it is proved that $\mathbf{DCC}$, $\mathbf{DCC}_c$ and $\mathbf{UCE}$ are all equivalent to $\mathbf{CAC}$; however, $\mathbf{UCE}_c$ does not imply $\mathbf{CAC}$ in $\mathbf{ZF}^0$. It has been established in [10] that $\mathbf{UCE}$ implies $\mathbf{CAC}_{fin}$. Therefore, in the forthcoming Theorem 4.12, we improve the latter result by showing that $\mathbf{CAC}$ and $\mathbf{UCE}$ are actually equivalent. Moreover, we investigate the problem on when countable products of densely complete metric spaces can be densely complete. We also show that $\mathbf{CAC}(\mathbb{R})$ is equivalent to the statement that $\mathbb{R} \times \mathbb{Q}$ is not densely complete.

It is well-known that, in the absence of the axiom of choice $\mathbf{AC}$, fundamental results of elementary analysis and topology may fail severely in some $\mathbf{ZF}$ models. As an example of this kind of models, we mention here the famous Cohen Basic Model $\mathcal{M}1$ in [8]. It is known that in $\mathcal{M}1$, the set A of all added Cohen reals is an infinite, Dedekind-finite subset of $\mathbb{R}$ disjoint from $\mathbb{Q}$. It is known that the following disasters may strike in $\mathcal{M}1$:

(i) A is a complete subset of $\mathbb{R}$ with $\overline{A} = \mathbb{R} \neq A$. Hence, complete metric subspaces of $\mathbb{R}$ need not be closed in $\mathcal{M}1$.

(ii) $A = \widetilde{A} \neq \overline{A} = \mathbb{R}$. So, $\mathbb{R}$ is not sequential in $\mathcal{M}1$.

(iii) $0 \in \overline{A} \backslash A$. This means that $\mathbb{R}$ is not Fréchet-Urysohn in $\mathcal{M}1$.

(iv) Any function $f : \mathbf{A} \to \mathbb{R}$ is sequentially continuous. Thus, sequential continuity of real-valued functions defined on complete metric subspaces of $\mathbb{R}$ is not equivalent to the usual $\varepsilon - \delta$ definition of continuity in $\mathcal{M}1$.

(v) $\mathbf{A}$ is not separable. Therefore, $\mathbb{R}$ is not hereditarily separable in $\mathcal{M}1$.

(vi) The open cover $\{(a-1, a+1) : a \in A\}$ of $\mathbb{R}$ has no countable subcover. So, $\mathbb{R}$ is not Lindelöf in $\mathcal{M}1$.

(vii) $(0,1) \cap A$ is a complete, non-closed metric subspace of the connected subspace $(0,1)$ of $\mathbb{R}$, but its closure in $(0,1)$ is not complete. So, closures of complete metric subspaces of connected metric spaces need not be complete in $\mathcal{M}1$!

For more disasters of this kind, we refer the reader to [6].

Proposition 2.1. $\mathbf{DCC}_c(\mathbb{R})$ *implies* $\mathbf{IDI}(\mathbb{R})$.

Proof. Assume the contrary and let D be an infinite Dedekind-finite subset of $\mathbb{R}$. N. Brunner showed in [1] that if there is a Dedekind-finite set D of $\mathbb{R}$, then there exists a dense Dedekind-finite subset of $\mathbb{R}$. This is why we may assume that D is dense in $\mathbb{R}$. Let $H = D \cap (0,1)$. Since H contains no countably infinite subsets, it follows that every Cauchy sequence of points of H is eventually constant, hence convergent in H, so in $(0,1)$. Hence $(0,1)$ is a densely complete connected, not complete metric subspace of $\mathbb{R}$. $\square$

The proof to Proposition 2.1 shows that all statements $\mathbf{DCC}_c(\mathbb{R})$, $\mathbf{DCC}(\mathbb{R})$, $\mathbf{DCC}_c$ and $\mathbf{DCC}$ fail in the model $\mathcal{M}1$. In consequence, we have the following corollary:

Corollary 2.2. *None of* $\mathbf{DCC}_c(\mathbb{R})$*,* $\mathbf{DCC}(\mathbb{R})$*,* $\mathbf{DCC}_c$*,* $\mathbf{DCC}$ *is a theorem of* $\mathbf{ZF}$.

Let us list in the following four theorems some known results we will need in the sequel.

Theorem 2.3. (Cf. [4], [9] and pp. 74–75 of [6].) *The following are equivalent:*

(a) $\mathbf{CAC}(\mathbb{R})$;

(b) *every subspace of* $\mathbb{R}$ *is sequential;*

(c) $\mathbb{R}$ *is Fréchet-Urysohn;*

(d) *for every* $x \in \mathbb{R}$ *and every function* $f : \mathbb{R} \to \mathbb{R}$*, if* f *is sequentially continuous at* x*, then* f *is continuous at* x.

Theorem 2.4. (Cf. [4].) *The following are equivalent:*

(a) $\mathbf{CAC}$;

(b) *every metric space is sequential;*

(c) *every metric space is Fréchet-Urysohn;*

(d) *every complete metric space is Cantor complete.*

Theorem 2.5. (i) (Cf. [6], p.75.) *$\mathbb{R}$ is sequential if and only if every complete metric subspace of $\mathbb{R}$ is closed.*

(ii) (Cf. [6], p.76.) *$\omega - \mathbf{CAC}(\mathbb{R})$ implies $\mathbb{R}$ is sequential.*

(iii) (Cf. [5].) *If there is a complete, non-closed metric subspace of $\mathbb{R}$, then there exists a dense, complete, non-closed metric subspace of $\mathbb{R}$.*

Theorem 2.6. (i) (Cf. [14].) *The family of all non-empty closed subsets of the real line has a choice function.*

(ii) (Cf. [13].) *If the family of all non-empty closed subsets of a compact metric space has a choice function, then the space is separable.*

Clearly, every complete metric space is trivially densely complete, and the proposition: *every densely complete metric space is complete* is a well-known criterion of completeness of metric spaces in $\mathbf{ZF} + \mathbf{CAC}$. It is also known that the Baire Category Theorem can fail for a complete metric space in a model of **ZF** and that every separable completely metrizable space is a Baire space (see, e.g., Section 4.10 of [6]). For the reader's convenience, we supply a proof to the following more general theorem here:

Theorem 2.7. *Let $\mathbf{X} = \langle X, d\rangle$ be a metric space, densely complete with respect to a set D. Then the following conditions are satisfied:*

(i) *If $\mathbf{CAC}(D)$ holds, then $\mathbf{X}$ is complete.*

(ii) *If D is well-orderable, then $\mathbf{X}$ is a Baire space.*

Proof. (i) Assume that $\mathbf{CAC}(D)$ holds. To show that $\mathbf{X}$ is complete, fix a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathbf{X}$ and, for each $n \in \mathbb{N}$, put $D_n = B_d(x_n, \frac{1}{n}) \cap D$. Since D is dense in $\mathbf{X}$, the sets D_n are non-empty. In view of $\mathbf{CAC}(D)$, there is a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in D_n$ for each $n \in \mathbb{N}$. We claim that $(y_n)_{n\in\mathbb{N}}$ is Cauchy. To see this, fix $\varepsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, it follows that there exists $n_0 \in \mathbb{N}$ such that $1/n_0 < \varepsilon/3$ and $d(x_n, x_m) < \varepsilon/3$ whenever $n, m \geq n_0$. For all natural numbers $n, m \geq n_0$, we have:

$$d(y_n, y_m) \leq d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) \leq 1/n + \varepsilon/3 + 1/m < \varepsilon.$$

Hence, $(y_n)_{n\in\mathbb{N}}$ is Cauchy as claimed. Therefore, $(y_n)_{n\in\mathbb{N}}$ converges to some point $x \in X$. Since for all $n \in \mathbb{N}$, $d(x_n, x) \leq d(x_n, y_n) + d(y_n, x)$, we get that

$$\lim_{n\to\infty} d(x_n, x) \leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, x) = 0.$$

Thus, $\lim_{n\to\infty} d(x_n, x) = 0$. This is why $\mathbf{X}$ is complete as required.

(ii) We notice that if D is well-orderable, then the base $\mathcal{B} = \{B_d(x, \frac{1}{n}) : x \in D$ and $n \in \mathbb{N}\}$ of $\mathbf{X}$ is well-orderable, so the usual proof in **ZFC** that complete metric spaces are Baire can be easily adopted to get a proof in **ZF** that $\mathbf{X}$ is Baire. $\square$

Corollary 2.8. (i) $\mathbf{CAC}$ *implies* $\mathbf{DCC}$.

(ii) $\mathbf{CAC}(\mathbb{R})$ *implies* $\mathbf{DCC}(\mathbb{R})$.

(iii) *For every metric space $\mathbf{X} = \langle X, d\rangle$, $\mathbf{CAC}(X)$ implies $\mathbf{DCC}(\mathbf{X})$.*

Theorem 2.9. **DC** *is equivalent to the statement: Every densely complete metric space is Baire.*

Proof. It follows from Theorem 4.106 of [6] that if every densely complete metric space is Baire, then **DC** holds. Now, assume **DC** and assume that $\mathbf{X} = \langle X, d\rangle$ is a densely complete metric space. Let $D \subseteq X$ be such that $\mathbf{X}$ is densely complete with respect to a set D. To show that $\mathbf{X}$ is Baire, we slightly modify the part that (1) $\rightarrow$ (2) of the proof to Theorem 4.106 in [6]. Let $(G_n)_{n\in\omega}$ be a sequence of open dense sets in $\mathbf{X}$ such that $G_{n+1} \subseteq G_n$ for each $n \in \omega$. Let $G \in \tau(d)$ and

$$Y = \{\langle n, x, r\rangle \in \omega \times D \times \mathbb{R} : 0 < r < \frac{1}{2^n} \text{ and } \overline{B_d(x,r)} \subseteq G \cap G_n\}.$$

Define a binary relation ρ on Y as follows: if $y = \langle n, x, r\rangle \in Y$ and $\overline{y} = \langle \overline{n}, \overline{x}, \overline{r}\rangle \in Y$,

then: $$\langle y, \overline{y}\rangle \in \rho \leftrightarrow (n < \overline{n} \text{ and } \overline{B_d(\overline{x}, \overline{r})} \subseteq B_d(x,r)).$$

It follows from **DC** that there exists a sequence $(y_n)_{n\in\omega}$ of points of Y such that $\langle y_n, y_{n+1}\rangle \in \rho$ for each $n \in \omega$. If $y_n = \langle m_n, x_n, r_n\rangle$ for each $n \in \omega$, then $(x_n)_{n\in\omega}$ is a Cauchy sequence of points of D. Since $\mathbf{X}$ is densely complete with respect to D, the sequence $(x_n)_{n\in\omega}$ converges in $\mathbf{X}$ to a point x. Of course, $x \in G \cap \bigcap_{n\in\omega} G_n$, so $\mathbf{X}$ is Baire. $\square$

Theorem 2.10. *Let* $\mathbf{X} = \langle X, d\rangle$ *be a metric space and* $Y \subseteq X$. *Then the following conditions are satisfied:*

(i) *If* $b \in \overline{Y} \setminus Y$, *while* $Z = Y \cup \{b\}$, *then the identity function* $f : \mathbf{Y} \to \mathbf{Y}$ *does not extend continuously over* Z.

(ii) *If* $\mathbf{X}$ *is complete, then* $\mathbf{UCE}(\mathbf{X})$ *implies* $\mathbf{X}$ *is sequential.*

Proof. (i) Assume the contrary and let $F : \mathbf{Z} \to \mathbf{Y}$ be a continuous extension of f over Z. Then $F(b) = a \in Y$. Let $\varepsilon = d(a,b) > 0$ ($a \neq b$), and fix, by the continuity of F at b, a number $\delta \in (0, \varepsilon/2)$ such that:

$$\text{for each } x \in Z \text{ with } d(x,b) < \delta, d(F(x), F(b)) < \varepsilon/2.$$

Since Y is dense in $\mathbf{Z}$, we can fix $x \in Y$ with $d(x,b) < \delta$. Since

$$d(x,a) = d(F(x), F(b)) < \varepsilon/2,$$

we get $$\varepsilon = d(a,b) \leq d(b,x) + d(x,a) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

The contradiction obtained implies that f does not extend continuously over Z. This finishes the proof of (i).

(ii) Suppose that $\mathbf{X} = \langle X, d\rangle$ is a complete metric space having a sequentially closed subset A such that $\overline{A} \setminus \widetilde{A} \neq \emptyset$. Then the metric subspace $\mathbf{A}$ of $\mathbf{X}$ is complete. It follows from (i) that, for every $a \in \overline{A} \setminus A$, the identity function $f : A \to A$ does not extend continuously over $A \cup \{a\}$. This contradicts $\mathbf{UCE}(\mathbf{X})$ and finishes the proof of (ii). $\square$

Another well-known result in elementary analysis which fails in the model $\mathcal{M}1$ is the proposition **UCE**($\mathbb{R}$). Indeed, the set A of all added Cohen reals is complete and dense in the metric subspace $\mathbf{X}$ of $\mathbb{R}$ where $X = A \cup \{0\}$. Hence, by Theorem 2.10(i), the uniformly continuous function $f : \mathbf{A} \to \mathbf{A}$ does not extend continuously over X. Therefore, **UCE**($\mathbb{R}$) and, in consequence, **UCE** fail in $\mathcal{M}1$.

Corollary 2.11. *Neither* **UCE** *nor* **UCE**($\mathbb{R}$) *is a theorem of* **ZF**.

3. On UCE($\mathbb{R}$) and DCC($\mathbb{R}$)

Our first result in this section shows that **UCE**($\mathbb{R}$) is equivalent to the strictly weaker consequence "$\mathbb{R}$ is sequential" of **CAC**($\mathbb{R}$). Before we proceed, we need to establish the following lemma:

Lemma 3.1. *Let $\mathbf{Y} = \langle Y, \rho\rangle$ be a Cantor complete metric space, let $\mathbf{X} = \langle X, d\rangle$ be a metric space and S a dense set in $\mathbf{X}$. Suppose that $f : \mathbf{S} \to \mathbf{Y}$ is uniformly continuous. Then there exists a uniformly continuous extension $F : \mathbf{X} \to \mathbf{Y}$ of f.*

Proof. For every $x \in X$, let $\mathcal{F}(x) = \{\overline{f[B_d(x, 1/n) \cap S]} : n \in \mathbb{N}\}$.
Since S is dense in $\mathbf{X}$, each set from $\mathcal{F}(x)$ is non-empty and closed in $\mathbf{Y}$. By the uniform continuity of f, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k > k$ and the inclusion $f[B_d(x, 1/n_k) \cap S] \subseteq B_\rho(f(x), 1/k)$ holds for each $x \in X$. Thus,

$$\lim_{n\to\infty} \delta_\rho(\overline{f[B_d(x, 1/n) \cap S]}) = 0 \text{ for each } x \in X.$$

Therefore, $\bigcap \mathcal{F}(x)$ cannot have two distinct points. Moreover, since $\mathbf{Y}$ is Cantor complete, for each $x \in X$, the set $\bigcap \mathcal{F}(x)$ is non-empty. Hence, we can define a mapping $F : \mathbf{X} \to \mathbf{Y}$ by putting $F(x)$ as the unique point of $\bigcap \mathcal{F}(x)$ for each $x \in X$. As in standard textbooks of elementary analysis or topology, it is a routine work to verify that F is the required extension of f (see, e.g., the proof to Lemma 4.3.16 in [3]). □

Corollary 3.2. *For a given $m \in \mathbb{N}$, suppose that $\mathbf{Y}$ is a complete, closed metric subspace of $\mathbb{R}^m$. Let $\mathbf{X} = \langle X, d\rangle$ be a metric space, S a dense set in $\mathbf{X}$ and $f : \mathbf{S} \to \mathbf{Y}$ a uniformly continuous function. Then there exists a uniformly continuous extension $F : \mathbf{X} \to \mathbf{Y}$ of f.*

Proof. It suffices to apply Lemma 3.1 and simple observation that, since every bounded closed subset of $\mathbb{R}^m$ is compact, every complete closed metric subspace of $\mathbb{R}^m$ is Cantor complete. □

Theorem 3.3. (i) *The following are equivalent:*

(a) **UCE**($\mathbb{R}$)*;*

(b) $\mathbf{UCE}_c(\mathbb{R})$*;*

(c) $\mathbf{UCE}_{cc}(\mathbb{R})$*;*

(d) $\mathbb{R}$ *is sequential.*

(ii) *The conjunction of* **UCE**($\mathbb{R}$) *and of the negation of* **CAC**($\mathbb{R}$) *is relatively consistent with* **ZF**. *Hence,* **UCE**($\mathbb{R}$) *is strictly weaker than* **CAC**($\mathbb{R}$).

Proof. (i) The implications (a) $\to$ (b) $\to$ (c) are obvious.

(c) $\to$ (d) We assume (c). In view of Theorem 2.5(i), to prove that $\mathbb{R}$ is sequential, it suffices to show that every complete metric subspace of $\mathbb{R}$ is closed. Assume the contrary and fix a complete, non-closed metric subspace $\mathbf{S}$ of $\mathbb{R}$. By Theorem 2.5(iii), we may assume that S is dense in $\mathbb{R}$. Fix $a \in \overline{S} \setminus S$. Then, for $Y = S \cap [a-1, a+1]$, the metric subspace $\mathbf{Y}$ of $\mathbb{R}$ is complete. The metric subspace $\mathbf{X}$ of $\mathbb{R}$, where $X = [a-1, a+1]$, is compact and connected. Clearly, the set Y is dense in $\mathbf{X}$. The identity function $f : \mathbf{Y} \to \mathbf{Y}$ is uniformly continuous. By Theorem 2.10(i), f does not extend continuously to a function from $Z = \{a\} \cup Y$ into Y. Therefore, f does not extend continuously over X. This, in view of Proposition 1.5, contradicts $\mathbf{UCE}_{cc}(\mathbb{R})$.

(d) $\to$ (a) Assuming that $\mathbb{R}$ is sequential, we fix a metric subspace $\mathbf{X}$ of $\mathbb{R}$, a complete metric subspace $\mathbf{Y}$ of $\mathbb{R}$, a dense subset S of $\mathbf{X}$ and a uniformly continuous function $f : \mathbf{S} \to \mathbf{Y}$. By our hypothesis and Theorem 2.5(i), $\mathbf{Y}$ is closed. So, by Corollary 3.2, f extends to a uniformly continuous function from $\mathbf{X}$ into $\mathbf{Y}$.

(ii) To prove (ii), we notice that it is known that $\omega - \mathbf{CAC}(\mathbb{R})$ holds true but $\mathbf{CAC}(\mathbb{R})$ fails in the Feferman-Levy Model $\mathcal{M}9$ of [8] (see, e.g., [5] and Remark 4.59 of [6]). Therefore, in view of Theorem 2.5(ii), $\mathbf{UCE}(\mathbb{R})$ also holds true in $\mathcal{M}9$. $\square$

Remark 3.4. We can regard the interval $[0,1]$ as the unique (up to an equivalence) two-point Hausdorff compactification of $\mathbb{R}$. The unit circle $S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$, denoted by $\mathbf{S}^1$ as a topological subspace of $\mathbb{R}^2$, can be regarded as the unique (up to an equivalence) one-point Hausdorff compactification of $\mathbb{R}$.

The question which pops up at this moment is whether one can replace in $\mathbf{UCE}_{cc}(\mathbb{R})$ the two-point compactification $[0,1]$ with the one-point compactification $\mathbf{S}^1$ of $\mathbb{R}$, i.e., if

- $\mathbf{UCE}_{1pc}(\mathbb{R})$: For every complete metric subspace $\mathbf{Y}$ of $\mathbb{R}^2$, if D is a dense subset of the unit circle $\mathbf{S}^1$ of $\mathbb{R}^2$ and $f : \mathbf{D} \to \mathbf{Y}$ is a uniformly continuous function, then there exists a unique uniformly continuous extension $F : \mathbf{S}^1 \to \mathbf{Y}$ of f.

is equivalent to $\mathbf{UCE}(\mathbb{R})$. We intend to show that $\mathbf{UCE}_{1pc}(\mathbb{R})$ can be added to the list of equivalents of Theorem 3.3 (i). We need the following new theorem which leads to solutions of Problems 6.6 and 6.11 of [15]:

Theorem 3.5. *For each $m \in \mathbb{N}$, the space $\mathbb{R}^m$ is sequential if and only if $\mathbb{R}$ is sequential.*

Proof. Of course, if $\mathbb{R}^m$ is sequential, so is $\mathbb{R}$. Let us assume that $\mathbb{R}$ is sequential. We shall prove by induction that $\mathbb{R}^m$ is sequential for each $m \in \mathbb{N}$. For $m \in \mathbb{N}$, let

$$\mathcal{B}(m) = \{B_{d_{e(m)}}(x, \tfrac{1}{n}) : x \in \mathbb{Q}^m, n \in \mathbb{N}\}.$$

For $G \subseteq \mathbb{R}^m$ and each $i \in m$, let $G_i = \{t \in \mathbb{R} : \exists x \in G, x(i) = t\}$.

We notice that, to prove that $\mathbb{R}^m$ is sequential, it suffices to show that each sequentially closed subset of $\mathbb{R}^m$ is separable.

First, suppose that we have already proved that each bounded sequentially closed subset of $\mathbb{R}^m$ is closed in $\mathbb{R}^m$, so compact. Consider a sequentially closed subset H of $\mathbb{R}^m$ and, for each $n \in \mathbb{N}$, put $K_n = \{x \in H : \|x\| \leq n\}$. The sets K_n are sequentially closed in $\mathbb{R}^m$ and bounded. By our hypothesis, the sets K_n are compact in $\mathbb{R}^m$. Using the base $\mathcal{B}(m)$, the compactness of every K_n and a straightforward induction, we can easily define a sequence $(\langle A_n, \psi_n\rangle)_{n\in\mathbb{N}}$ such that each A_n is a set dense in K_n, each ψ_n is an injection $\psi_n : A_n \to \mathbb{N}$. Then the set $A = \bigcup_{n\in\mathbb{N}} A_n$ is countable and dense in H. If $a \in \overline{H}$, we can inductively define a sequence $(a_n)_{n\in\mathbb{N}}$ of points of A such that $\lim_{n\to\infty} a_n = a$. Then $a \in H$, which shows that H is closed in $\mathbb{R}^m$.

We shall prove by induction that $\mathbb{R}^m$ is sequential. We have already assumed that $\mathbb{R}^1 = \mathbb{R}$ is sequential. Suppose that $m \in \mathbb{N}$ is such that $\mathbb{R}^m$ is sequential. Let G be a non-empty sequentially closed bounded subset of $\mathbb{R}^{m+1}$. Fix $i \in m+1$. We claim that G_i is sequentially closed in $\mathbb{R}$. For $t \in G_i$, let $l(t) = \{x \in \mathbb{R}^{m+1} : x(i) = t\}$ and $L(t) = G \cap l(t)$. Notice that the subspace $l(t)$ of $\mathbb{R}^{m+1}$ is closed, so the set $L(t)$ is sequentially closed in $\mathbb{R}^{m+1}$. Since $l(t)$ is homeomorphic with $\mathbb{R}^m$ and $\mathbb{R}^m$ is sequential, we infer that $l(t)$ is sequential. This implies that $L(t)$ is closed in $l(t)$, so $L(t)$ is also closed in $\mathbb{R}^{m+1}$. Since $L(t)$ is also bounded, it is compact in $\mathbb{R}^{m+1}$. Now, fix a sequence $(x_n)_{n\in\mathbb{N}}$ of points of G_i such that $(x_n)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ to x. For each $n \in \mathbb{N}$, the set $L(x_n)$ is a non-empty compact subset of $\mathbb{R}^{m+1}$. Therefore, using the base $\mathcal{B}(m+1)$, we can inductively define a sequence $(y_n)_{n\in\mathbb{N}}$ of points of $\mathbb{R}^{m+1}$ such that $y_n \in L(x_n)$ for each $n \in \mathbb{N}$. The sequence $(y_n)_{n\in\mathbb{N}}$ is bounded, so it has a subsequence which converges in $\mathbb{R}^{m+1}$ to a point y.

For simplicity, we assume that $(y_n)_{n\in\mathbb{N}}$ converges to y. Then $y \in G$ because G is sequentially closed. Since $y_n(i) = x_n$ for each $n \in \mathbb{N}$, we have $y(i) = x$. Hence $x \in G_i$. This proves that G_i is sequentially closed in $\mathbb{R}$. Since $\mathbb{R}$ is sequential, G_i is closed in $\mathbb{R}$. Hence, by Theorem 4.55 of [6], there exists a countable set $D \subseteq G_i$ such that D is dense in G_i. Let $D = \{t_n : n \in \mathbb{N}\}$. Using the base $\mathcal{B}(m+1)$ and the compactness of the non-empty sets $L(t_n)$ for $n \in \mathbb{N}$, we can inductively define a sequence $(\langle C_n, \phi_n\rangle)_{n\in\mathbb{N}}$ such that each C_n is a dense set in $L(t_n)$ and each ϕ_n is an injective function from C_n into $\mathbb{N}$. The set $C = \bigcup_{n\in\mathbb{N}} C_n$ is countable and dense in G. Hence, if a point y of $\mathbb{R}^{m+1}$ is an accumulation point of G, there exists a sequence of points of C which converges to y, so $y \in G$ because G is sequentially closed. This implies that G is closed in $\mathbb{R}^{m+1}$. By the mathematical induction, we have proved that if $\mathbb{R}$ is sequential, so is $\mathbb{R}^m$ for each $m \in \mathbb{N}$. □

Now, we are in a position to prove the following theorem which is an analogue for $\mathbb{R}^m$ of Theorem 4.55 of [6]:

Theorem 3.6. *For each $m \in \mathbb{N}$, the following conditions are all equivalent:*

(i) $\mathbb{R}^m$ *is sequential;*

(ii) *every complete metric subspace of* $\mathbb{R}^m$ *is closed in* $\mathbb{R}^m$*;*

(iii) *every complete and bounded metric subspace of* $\mathbb{R}^m$ *is compact;*

(iv) *every sequentially compact subspace of* $\mathbb{R}^m$ *is compact;*
(v) *every complete metric subspace of* $\mathbb{R}^m$ *is separable;*
(vi) *every complete unbounded metric subspace of* $\mathbb{R}^m$ *contains an unbounded sequence;*
(vii) *every non-empty countable family of non-empty complete metric subspaces of* $\mathbb{R}^m$ *has a choice function;*
(viii) *every non-empty family of non-empty complete metric subspaces of* $\mathbb{R}^m$ *has a choice function.*

Proof. First, we notice that a metric subspace $\mathbf{A}$ of $\mathbb{R}^m$ is complete if and only if A is sequentially closed in $\mathbb{R}^m$. It has been shown in the proof to Theorem 3.5 that it follows from (i) that every sequentially closed subset of $\mathbb{R}^m$ is separable and, moreover, every separable sequentially closed subspace of $\mathbb{R}^m$ is closed in $\mathbb{R}^m$. Hence (i) and (v) are equivalent. The implications (i)$\rightarrow$ (ii)$\rightarrow$ (iii)$\rightarrow$(iv) and (v)$\rightarrow$(vi), as well as (viii)$\rightarrow$(vii) are obvious.

Let $H \subseteq \mathbb{R}^m$ be sequentially closed in $\mathbb{R}^m$. If H is not closed in $\mathbb{R}^m$, then there exists $n_0 \in \mathbb{N}$ such that the set $H_{n_0} = \{x \in H : \|x\| \leq n_0\}$ is not closed in $\mathbb{R}^m$. On the other hand, H_{n_0} is sequentially compact, so closed if (iv) holds. Hence (iv) implies (i). Clearly, (vi) implies that every complete unbounded subspace of $\mathbb{R}$ contains an unbounded sequence. Hence, in view of Theorem 4.55 of [6], (vi) implies that $\mathbb{R}$ is sequential. This, taken together with Theorem 3.5, proves that (vi) implies (i). In consequence, conditions (i)–(vi) are all equivalent.

To show that (i) implies (viii), let us consider any non-empty family $\mathcal{F}$ of non-empty complete metric subspaces of $\mathbb{R}^m$. For each $F \in \mathcal{F}$, let $n(F)$ be the least $n \in \mathbb{N}$ such that $\{x \in F : \|x\| \leq n\} \neq \emptyset$. Assuming that (i) holds, we obtain that, for each $F \in \mathcal{F}$, the set $K(F) = \{x \in F : \|x\| \leq n(F)\}$ is compact. It is known that every non-empty family of non-empty compact subsets of $\mathbb{R}^m$ has a choice function. Therefore, the family $\{K(F) : F \in \mathcal{F}\}$ has a choice function. Hence (i) implies (viii). To complete the proof, let us deduce that (viii) implies (vi). Let E be an unbounded complete metric subspace of $\mathbb{R}^m$. Inductively, we can define a sequence $(n_k)_{k\in\omega}$ of natural numbers such that, for each $k \in \omega$, $n_k < n_{k+1}$ and $\emptyset \neq \{x \in E : n_k \leq \|x\| \leq n_{k+1}\}$. For each $k \in \omega$, the metric subspace $E_k = \{x \in E : n_k \leq \|x\| \leq n_{k+1}\}$ of $\mathbb{R}^m$ is complete. Assuming that (viii) holds, we can choose $y \in \prod_{k\in\omega} E_k$. Then $(y(k))_{k\in\omega}$ is an unbounded sequence of points of E. Hence (viii) implies (vi). $\square$

Remark 3.7. Let us notice that positive answers to the questions posed in Problems 6.6 and 6.11 of [15] follow immediately from Theorems 3.5 and 3.6.

Remark 3.8. It follows directly from Theorem 4.54 of [6] and from Exercise E.3 to Section 4.6 of [6] that, for each $n \in \mathbb{N}$, $\mathbf{CAC}(\mathbb{R})$ is equivalent to: $\mathbb{R}^n$ is Fréchet-Urysohn.

Let us leave for another research project a satisfactory answer to the following question:

Question 3.9. If $\mathbb{R}$ is sequential, must $\mathbb{R}^\omega$ be sequential?

Proposition 3.10. *Let $m \in \mathbb{N}$. If $\mathbb{R}$ is sequential, then every closed subspace of $\mathbb{R}^m$ is sequential.*

Proof. Assume that $\mathbb{R}$ is sequential. Let $\mathbf{X}$ be a closed subspace of $\mathbb{R}^m$ and let $A \subseteq X$ be sequentially closed in $\mathbf{X}$. Since the set X is closed in $\mathbb{R}^m$, the set A is sequentially closed in $\mathbb{R}^m$. In the light of Theorem 3.5, the space $\mathbb{R}^m$ is sequential, so A is closed in $\mathbb{R}^m$. This implies that A is closed in $\mathbf{X}$. $\square$

Proposition 3.11. *Let $m, n \in \mathbb{N}$ and let $\alpha\mathbb{R}^n$ be a compactification of $\mathbb{R}^n$ such that $\alpha\mathbb{R}^n \subseteq \mathbb{R}^m$. Then $\mathbb{R}$ is sequential if and only if $\alpha\mathbb{R}^n$ is sequential.*

Proof. If $\mathbb{R}$ is sequential, then $\alpha\mathbb{R}^n$ is sequential by Proposition 3.10. Now, assume that $\alpha\mathbb{R}^n$ is sequential. Let A be a sequentially closed subset of $\mathbb{R}^n$ and let $C = \alpha[A] \cup (\alpha\mathbb{R}^n \setminus \alpha[\mathbb{R}^n])$. It is easy to notice that C is sequentially closed in $\alpha\mathbb{R}^n$. Therefore, C is closed in $\alpha\mathbb{R}^n$. This implies that $A = \alpha^{-1}[C]$ is closed in $\mathbb{R}^n$. Hence $\mathbb{R}^n$ is sequential and, in consequence, so is $\mathbb{R}$. $\square$

Now, let us give a more general statement than $\mathbf{UCE}_{1pc}(\mathbb{R})$. Namely, suppose that, for $m, n \in \mathbb{N}$, $\alpha\mathbb{R}^n$ is a compactification of $\mathbb{R}^n$ such that $\alpha\mathbb{R}^n \subseteq \mathbb{R}^m$. Then we define $\mathbf{UCE}_\alpha(\mathbb{R}^n)$ as follows:

- $\mathbf{UCE}_\alpha(\mathbb{R}^n)$: For every complete metric subspace $\mathbf{Y}$ of $\mathbb{R}^m$, if D is a dense set in $\alpha\mathbb{R}^n$ and $f : \mathbf{D} \to \mathbf{Y}$ is a uniformly continuous function, then there exists a uniformly continuous extension $F : \alpha\mathbb{R}^n \to \mathbf{Y}$ of f.

Theorem 3.12. *Assume that $m, n \in \mathbb{N}$ and that $\alpha\mathbb{R}^n$ is a compactification of $\mathbb{R}^n$ such that $\alpha\mathbb{R}^n \subseteq \mathbb{R}^m$. Then $\mathbf{UCE}_\alpha(\mathbb{R}^n)$ and $\mathbf{UCE}(\mathbb{R})$ are equivalent.*

Proof. To show that $\mathbf{UCE}_\alpha(\mathbb{R}^n)$ implies $\mathbf{UCE}(\mathbb{R})$, assume that $\mathbf{UCE}_\alpha(\mathbb{R}^n)$ holds. In view of Theorem 3.3, it suffices to show that $\mathbb{R}$ is sequential. To this end, we assume the contrary and fix, by Theorem 2.5(iii), a dense, sequentially closed, non-closed subset A of $\mathbb{R}$. Then the set $B = \{x \in \mathbb{R}^n : x(0) \in A\}$ is a sequentially closed, non-closed dense subset of $\mathbb{R}^n$. The set $Y = \alpha[B] \cup (\alpha\mathbb{R}^n \setminus \alpha[\mathbb{R}^n])$ is dense and sequentially closed in $\alpha\mathbb{R}^n$. If Y were closed in $\alpha\mathbb{R}^n$, the set B would be closed in $\mathbb{R}^n$. Hence Y is not closed in $\alpha\mathbb{R}^n$. Let $a \in \overline{Y} \setminus Y$. The metric subspace $\mathbf{Y}$ of $\mathbb{R}^m$ is complete, while, in view of Theorem 2.10(i), the identity function $f : \mathbf{Y} \to \mathbf{Y}$ is not extendable to any continuous function $F : Y \cup \{a\} \to Y$. This contradicts $\mathbf{UCE}_\alpha(\mathbb{R}^n)$. Hence $\mathbb{R}$ is sequential.

To show that $\mathbf{UCE}(\mathbb{R})$ implies $\mathbf{UCE}_\alpha(\mathbb{R}^n)$, we assume that $\mathbb{R}$ is sequential. Then, by Theorems 3.5 and 3.6, every complete metric subspace of $\mathbb{R}^m$ is closed, so $\mathbf{UCE}_\alpha(\mathbb{R}^n)$ follows from Corollary 3.2. $\square$

Corollary 3.13. $\mathbf{UCE}_{1pc}(\mathbb{R})$ *and* $\mathbf{UCE}(\mathbb{R})$ *are equivalent.*

Let us have a look at the following statement:

- $\mathbf{CAC}_D(\mathbb{R})$: Every family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of dense subsets of $\mathbb{R}$ has a partial choice function.

In contrast to the non-equivalence of $\mathbf{UCE}(\mathbb{R})$ and $\mathbf{CAC}(\mathbb{R})$, we are going to show that $\mathbf{DCC}(\mathbb{R})$ is equivalent to $\mathbf{CAC}(\mathbb{R})$. The following theorem will be applied:

Theorem 3.14. $\mathbf{CAC}(\mathbb{R})$ *and* $\mathbf{CAC}_D(\mathbb{R})$ *are equivalent.*

Proof. Since it is obvious that $\mathbf{CAC}(\mathbb{R})$ implies $\mathbf{CAC}_D(\mathbb{R})$, we assume that $\mathbf{CAC}_D(\mathbb{R})$ holds and that $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint family of non-empty subsets of $\mathbb{R}$. We show that $\mathcal{A}$ has a partial choice function. Let $((p_n, q_n))_{n\in\mathbb{N}}$ be an enumeration of the family of all open intervals with rational endpoints. Let $f : \mathbb{R} \to (0,1)$ be a homeomorphism. For each $n \in \mathbb{N}$, let $g_n : \mathbb{R} \to \mathbb{R}$ be a linear function such that $g_n(0) = p_n$ and $g_n(1) = q_n$. For each $n \in \mathbb{N}$, we put $f_n = g_n \circ f$ and we consider the sets

$$D_n = \bigcup\{f_i[A_n] : i \in \mathbb{N}\}.$$

It is easy to see that, for each $n \in \mathbb{N}$, D_n is a dense subset of $\mathbb{R}$. By $\mathbf{CAC}_D(\mathbb{R})$, there exists an infinite subset N of $\mathbb{N}$ such that $\mathcal{D} = \{D_n : n \in N\}$ has a choice function. Hence, there exists $h \in \prod_{n\in N} D_n$. For each $n \in N$, let

$$k_n = \min\{i \in \mathbb{N} : h(n) \in f_i[A_n]\}.$$

Since, for each $i \in \mathbb{N}$, the function f_i is a bijection, it follows that $C = \{f_{k_n}^{-1}(h(n)) : n \in N\}$ is a partial choice set of $\mathcal{A}$. $\square$

Theorem 3.15. *For each* $n \in \mathbb{N}$*, the conditions* $\mathbf{CAC}(\mathbb{R})$*,* $\mathbf{DCC}(\mathbb{R}^n)$ *and* $\mathbf{DCC}_c(\mathbb{R}^n)$ *are all equivalent.*

Proof. Let $n_0 \in \mathbb{N}$ be given. We notice that $\mathbf{CAC}(\mathbb{R}^{n_0})$ and $\mathbf{CAC}(\mathbb{R})$ are equivalent because $\mathbb{R}$ and $\mathbb{R}^{n_0}$ are equipotent in $\mathbf{ZF}$. Therefore, that $\mathbf{CAC}(\mathbb{R})$ implies $\mathbf{DCC}(\mathbb{R}^n)$ follows from Corollary 2.8. It is straightforward that $\mathbf{DCC}_c(\mathbb{R})$ follows from $\mathbf{DCC}_c(\mathbb{R}^{n_0})$ and $\mathbf{DCC}(\mathbb{R}^{n_0})$ implies $\mathbf{DCC}_c(\mathbb{R}^{n_0})$. To complete the proof, it suffices to show that if $\mathbf{CAC}(\mathbb{R})$ is false, then so is $\mathbf{DCC}_c(\mathbb{R})$.

Assume that $\mathbf{CAC}(\mathbb{R})$ is false. By Theorem 3.14, we can fix a family

$$\mathcal{D} = \{D_{p,q} : p, q \in \mathbb{Q} \cap (0,1], p < q\}$$

of dense subsets of $\mathbb{R}$ having no partial choice function. For $p, q \in \mathbb{R} \cap (0,1]$ with $p < q$, we put $A_{p,q} = D_{p,q} \cap (p,q)$. The collection

$$\mathcal{A} = \{A_{p,q} : p, q \in \mathbb{Q} \cap (0,1], p < q\}$$

does not have a partial choice function, and the set $A = \bigcup \mathcal{A}$ is dense in the connected subspace $(0,1]$ of $\mathbb{R}$.

We claim that every Cauchy sequence of points of A converges to some point in $(0,1]$. To this end, it suffices to show that $0 \notin \widetilde{A}$. Assume the contrary and fix a strictly descending sequence $(x_n)_{n\in\omega}$ of points of A converging to 0. Via a straightforward induction, we construct a subsequence $(x_{k_n})_{n\in\omega}$ of $(x_n)_{n\in\omega}$ and a sequence of open intervals $((p_n, q_n))_{n\in\omega}$ with rational endpoints in $(0,1]$ such that, for each $n \in \mathbb{N}$, $x_{k_{n+1}} < p_n$ and $x_{k_n} \in A_{p_n,q_n}$.

First, we fix a well ordering on $\mathbb{Q} \times \mathbb{Q}$. For $n = 0$, we simply pick any $p_0, q_0 \in \mathbb{Q} \cap (0,1]$ such that $p_0 < q_0$ and $x_0 \in A_{p_0,q_0}$. We put $k_0 = 0$.

Assume that $n \in \omega$ is such that the numbers k_i and intervals (p_i, q_i) have been defined for each $i \in n+1$ in such a way that, for all $i, j \in n+1$ with $i < j$, $k_i < k_j$ and, moreover, the following condition is satisfied:

$$x_{k_i} \in A_{p_i,q_i} \text{ for each } i \in n+1, \text{ while } x_{k_{i+1}} < p_i \text{ for each } i \in n.$$

Since $0 < p_n$ and $(x_n)_{n\in\mathbb{N}}$ converges to 0, it follows that infinitely many terms of $(x_n)_{n\in\mathbb{N}}$ are strictly less than p_n. Let $\langle p, q\rangle$ be the least member of $\mathbb{Q} \times \mathbb{Q}$ such that $0 < p < q \leq 1$, $p < p_n$ and there exists $m \in \omega$ such that $k_n \in m$, while $x_m \in A_{p,q}$. Put

$$k_{n+1} = \min\{m \in \omega : k_n \in m \text{ and } x_m \in A_{p,q}\}, p_{n+1} = p, q_{n+1} = q$$

to terminate the induction.

We notice that if $m, n \in \omega$ and $m \neq n$, then $\langle p_m, q_m\rangle \neq \langle p_n, q_n\rangle$. Hence, if $h(\langle p_n, q_n\rangle) = x_{k_n}$ for each $n \in \omega$, then h is a partial choice function of $\mathcal{A}$. This contradics our assumption on $\mathcal{D}$. Thus, $(0, 1]$ is densely complete as required.

Since if $\mathbf{CAC}(\mathbb{R})$ fails, then $(0, 1]$ is a connected, densely complete, not complete metric subspace of $\mathbb{R}$, it follows that $\mathbf{DCC}_c(\mathbb{R})$ implies $\mathbf{CAC}(\mathbb{R})$. □

4. On DCC and UCE

Theorem 4.1. *The following are equivalent:*

(i) **DCC**;

(ii) **CAC**;

(iii) *for every metric space* $\mathbf{X}$ *and every complete metric subspace* $\mathbf{Y}$ *of* $\mathbf{X}$, $\overline{Y}$ *is complete;*

(iv) *for every metric space* $\mathbf{X}$, *every complete metric subspace of* $\mathbf{X}$ *is closed in* $\mathbf{X}$;

(v) *functions between metric spaces are continuous if and only if they are sequentially continuous;*

(vi) (Cf. [2].) *real-valued functions on metric spaces are continuous if and only if they are sequentially continuous;*

(vii) *every complete metric space is Fréchet-Urysohn;*

(viii) *every complete metric space is sequential.*

Proof. That (ii) implies (i) follows from Corollary 2.8.

(i)→(ii) Assuming that **CAC** is false, we can consider a pairwise disjoint family $\mathcal{A} = \{A_{n,m} : n, m \in \mathbb{N}\}$ of non-empty sets having no partial choice set. Put $A = \bigcup \mathcal{A}$ and let $\{\infty_n : n \in \mathbb{N}\}$ be a set of pairwise distinct elements, disjoint with A. For every $n \in \mathbb{N}$, let

$$X_n = \bigcup\{A_{n,m} : m \in \mathbb{N}\} \cup \{\infty_n\}, \tag{2}$$

and define a mapping $d_n : X_n \times X_n \to \mathbb{R}$ by requiring:

$$d_n(x,y) = d_n(y,x) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{nm} & \text{if } x \in A_{n,m},\ y \in A_{n,s} \text{ and } m \leq s\ . \\ \frac{1}{nm} & \text{if } x \in A_{n,m} \text{ and } y = \infty_n \end{cases} \tag{3}$$

It is a routine work to verify that, for every $n \in \mathbb{N}$, d_n is a metric on X_n. Put $X = A \cup \{\infty_n : n \in \mathbb{N}\}$ and define a function $d : X \times X \to \mathbb{R}$ by letting:

$$d(x,y) = d(y,x) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n} & \text{if } x \in X_n,\ y \in X_m \text{ and } n < m\ . \\ d_n(x,y) & \text{if } x, y \in X_n \end{cases}$$

One can easily check that d is a metric on X.

Claim $(\star)$: $\overline{A} = X$ and every Cauchy sequence of $\mathbf{A}$ converges in $\mathbf{X}$.

Proof of Claim $(\star)$. The first assertion of Claim $(\star)$ follows at once from the observation that, for every $n \in \mathbb{N}$, ∞_n is an accumulation point of

$$Y_n = \bigcup \{A_{n,m} : m \in \mathbb{N}\} \tag{4}$$

in the metric space $\mathbf{X}_n$. Hence, ∞_n is an accumulation point of A in $\mathbf{X}$.

To see the second assertion, fix a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathbf{A}$. We show that $(x_n)_{n\in\mathbb{N}}$ converges in $\mathbf{X}$. Assume the contrary and let $(x_n)_{n\in\mathbb{N}}$ have no accumulation point in $\mathbf{X}$. Then $\{x_n : n \in \mathbb{N}\}$ is a countably infinite subset of A. By passing to a subsequence, if necessary, we may assume that $(x_n)_{n\in\mathbb{N}}$ is injective. Since $\mathcal{A}$ has no partial choice set, it follows that there exist $k, m \in \mathbb{N}$ such that infinitely many terms of $(x_n)_{n\in\mathbb{N}}$ belong to $A_{k,m}$. For our convenience, let us assume that all the terms of $(x_n)_{n\in\mathbb{N}}$ belong to $A_{k,m}$. Then $d(x_t, x_s) = \frac{1}{km}$ for all $t, s \in \mathbb{N}$ with $t \neq s$, contradicting the fact that $(x_n)_{n\in\mathbb{N}}$ is Cauchy, and finishing the proof of Claim $(\star)$.

It follows from Claim $(\star)$ that $\mathbf{X}$ is densely complete. However, $(\infty_n)_{n\in\mathbb{N}}$ is easily seen to be a non-convergent Cauchy sequence in $\mathbf{X}$. Hence $\mathbf{X}$ is not complete, so **DCC** fails. This proves that (i) implies (ii).

The implications (ii)$\to$(iv)$\to$(iii), (ii)$\to$(v)$\to$(vi) and (ii)$\to$(vii)$\to$(viii) are straightforward. To prove that (iii)$\to$(ii), (vi)$\to$(ii) and (viii)$\to$(ii), let us assume that (ii) does not hold and let $\mathcal{A}$ be as in the proof of (i) $\to$ (ii). Let us use the notation established in the proof of (i) $\to$ (ii).

(iii)$\to$(ii) We claim that $\mathbf{A}$ is a complete and dense metric subspace of $\mathbf{X}$ but $\mathbf{X}$, as in the proof of (i) $\to$ (ii), is not complete. This will contradict the (iii) and terminate the proof of (iii) $\to$ (ii).

That $\overline{A} = X$ follows from Claim $(\star)$. To see that $\mathbf{A}$ is complete, we fix a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of points of $\mathbf{A}$. By Claim $(\star)$, $(x_n)_{n\in\mathbb{N}}$ converges to some point $x \in \mathbf{X}$. If $x = \infty_t$ for some $t \in \mathbb{N}$, then almost all terms of $(x_n)_{n\in\mathbb{N}}$ are included in Y_t, and for infinitely many $m \in \mathbb{N}$, $A_{t,m} \cap \{x_v : v \in \mathbb{N}\} \neq \emptyset$. Therefore, a partial choice function of $\mathcal{A}$ can be easily defined, contradicting our hypothesis for $\mathcal{A}$. Thus, $x \in A$ and $\mathbf{A}$ is complete.

(vi) $\rightarrow$ (ii) We consider the sets X_1 and Y_1, defined by (2) and (4), respectively. Let $f : \mathbf{X}_1 \rightarrow \mathbb{R}$ be the function given by:

$$f(x) = \begin{cases} 1 & \text{if } x \in Y_1 \\ 0 & \text{if } x = \infty_1 \end{cases}.$$

Clearly, f is a sequentially continuous (the only sequences of $\mathbf{X}_1$ converging to ∞_1 are those which are eventually equal to ∞_1), non-continuous (for $\varepsilon = 1$ and every $\delta > 0$, $|f(x) - f(\infty_1)| \geq 1$ for every $x \in X_1$) function. This contradicts (iii). Hence (iii) implies (ii).

(viii)$\rightarrow$(ii) Now, fix $n \in \mathbb{N}$ and consider the sets X_n, Y_n defined in the proof of (i)$\rightarrow$(ii). Working as in the proof to Claim ($\star$), we can show that $\mathbf{X}_n$ is complete, while Y_n is sequentially closed in $\mathbf{X}_n$. Since $\infty_n \in \overline{Y_n} \setminus \widetilde{Y}_n$, we infer that $\mathbf{X}_n$ is not sequential. Hence (viii) does not hold if (ii) is false. $\square$

Remark 4.2. Although Theorem 2.4 was proved in [4], it is worth noticing that we can give a new proof to Theorem 2.4 by applying Theorem 4.1. Namely, it is obvious that every Fréchet-Urysohn space is sequential in **ZF**. It is also obvious that **CAC** implies that all metric spaces are Fréchet-Urysohn. Hence, to get a proof to Theorem 2.4, it suffices to notice that, in view of Theorem 4.1, if every metric space is sequential, then **CAC** holds.

Theorem 4.3. *The following are equivalent:*

(i) **CAC**;

(ii) *closed metric subspaces of densely complete metric spaces are densely complete;*

(iii) *for every metric space* $\mathbf{X}$*, every sequential subspace* $\mathbf{Y}$ *of* $\mathbf{X}$ *and every* $a \in \overline{Y} \setminus \widetilde{Y}$*, if* $Z = Y \cup \{a\}$*, then the subspace* $\mathbf{Z}$ *of* $\mathbf{X}$ *is sequential.*

Proof. In view of Remark 4.2, it is obvious that (i) implies (iii). By Theorem 4.1, (i) implies densely complete metric spaces are complete. Since closed metric subspaces of complete metric spaces are complete, it is obvious that they are also densely complete. Hence (i) implies (ii)

Now, let us assume that **CAC** does not hold. Fix a pairwise disjoint family $\mathcal{A}$ as in the proof of (i)$\rightarrow$(ii) of Theorem 4.1. Let us also use the notation established in the proof of Theorem 4.1. Clearly, the set $K = \{\infty_n : n \in \mathbb{N}\}$ is closed in $\mathbf{X}$, while the metric subspace $\mathbf{K}$ of $\mathbf{X}$ fails to be densely complete. Moreover, the subspace $\mathbf{Y}_1$ of $\mathbf{X}_1$ is sequential, $\infty_1 \in \overline{Y}_1 \setminus \widetilde{Y}_1$, $X_1 = Y_1 \cup \{\infty_1\}$ but the space $\mathbf{X}_1$ is not sequential. $\square$

The proof to the following Theorem 4.4 is another proof that **DCC** and **CAC** are equivalent. However, since the densely complete, non-complete metric space $\mathbf{X}$ constructed in the proof to Theorem 4.1 has been shown useful for other problems, we cannot omit our first proof to the equivalence of (i) and (ii) of Theorem 4.1

Theorem 4.4. **DCC**$_c$ *and* **CAC** *are equivalent.*

Proof. Since **DCC** implies **DCC**$_c$ and, in view of Corollary 2.8(i), **CAC** implies **DCC**, we infer that **DCC**$_c$ follows from **CAC**.

Now, suppose that **CAC** fails and fix a pairwise disjoint family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of non-empty sets having no partial choice function. Fix a strictly increasing sequence $(a_n)_{n\in\omega}$ of points in $[0, 1)$ converging to 1. For each $n \in \omega$, let $r_n = \frac{a_{n+1}-a_n}{2}$, and $c_n = \frac{a_n+a_{n+1}}{2}$. In the complex plane $\mathbb{C}$, we consider the circles

$$C_n = \{z \in \mathbb{C} : |z - c_n| = r_n\}$$

where $n \in \omega$. If $n \in \omega$ and $x \in A_n$, let $C_{n,x} = (C_n \setminus \{a_n, a_{n+1}\}) \times \{x\}$. Let

$$Z = \{a_n : n \in \omega\} \cup \bigcup\{C_{n,x} : n \in \omega \text{ and } x \in A_n\}.$$

For every $n \in \omega$ and $a, b \in C_n$, let $\widehat{ab}$ denote the arc of the circle C_n starting at a and, moving counterclockwise, ending at b; moreover, let $l(ab)$ denote the minimum of the lengths of the arcs $\widehat{ab}, \widehat{ba}$, while $l_n = l(a_na_{n+1})$. Clearly, $l(ab) = l(ba)$. We define a metric ρ on Z by requiring:

$$\rho(\langle a, x\rangle, \langle b, x\rangle) = l(ab) \quad\text{and}\quad \rho(a_i, \langle a, x\rangle) = \rho(\langle a, x\rangle, a_i) = l(a_i, a)$$

if $a, b \in C_n \setminus \{a_n, a_{n+1}\}$ and $x \in A_n$, while $n \in \omega$ and $i \in \{n, n + 1\}$;

$$\rho(\langle a, x\rangle, \langle b, y\rangle) = \min\{l(aa_{n+1}) + l(ba_{n+1}), l(aa_n) + l(ba_n)\} \tag{5}$$

if $a, b \in C_n \setminus \{a_n, a_{n+1}\}$ and $x, y \in A_n$, while $x \neq y$ and $n \in \omega$;

$$\rho(\langle a, x\rangle, \langle b, y\rangle) = \rho(\langle b, y\rangle, \langle a, x\rangle) = l(a_{n+1}a) + l(bb_m) + \sum_{i=n+1}^{m-1} l_i$$

if $a\in C_n\setminus \{a_n, a_{n+1}\}$, $b\in C_m\setminus \{a_m, a_{m+1}\}$ for some $n\in m\in\omega$, while $x\in A_n$, $y\in A_m$;

$$\rho(a_n, a_n) = 0 \text{ and } \rho(a_n, a_m) = \rho(a_m, a_n) = \sum_{i=n}^{m-1} l_i \ \text{ if } n \in m \in \omega;$$

$$\rho(a_n, \langle a, x\rangle) = \rho(\langle a, x\rangle, a_n) = l(aa_m) + \sum_{i=n}^{m-1} l_i$$

when $n \in m \in \omega$, $a \in C_m \setminus \{a_m, a_{m+1}\}$ and $x \in A_m$;

$$\rho(a_n, \langle a, x\rangle) = \rho(\langle a, x\rangle, a_n) = l(aa_{m+1}) + \sum_{i=m+1}^{n-1} l_i$$

when $m + 1 \in n \in \omega$, $a \in C_m \setminus \{a_m, a_{m+1}\}$ and $x \in A_m$.

Clearly, for each $n \in \omega$ and $x \in A_n$, the subspace $\{a_n, a_{n+1}\}\cup C_{n,x}$ of $\mathbf{Z}$ is pathwise connected. This implies that the metric space $\mathbf{Z} = \langle Z, \rho\rangle$ is pathwise connected, so $\mathbf{Z}$ is connected. It is easy to see that the set $D = Z\setminus\{a_n : n\in\omega\}$ is dense in $\mathbf{Z}$.

Let us show that $\mathbf{Z}$ is densely complete with respect to D. To this end, fix a Cauchy sequence $(d_n)_{n\in\omega}$ of points of D. Without loss of generality we may assume that $(d_n)_{n\in\omega}$ is injective. Let

$$B = \{d_n : n \in \omega\} \text{ and } \mathcal{C} = \{(C_m \setminus \{a_m, a_{m+1}\}) \times A_m : m \in \omega\}.$$

If the set $\{C \in \mathcal{C} : C \cap B \neq \emptyset\}$ were infinite, using the countability of B, one could easily define a partial choice function of $\mathcal{A}$. Therefore, since $\mathcal{A}$ has no partial choice function, B meets only finitely many members of $\mathcal{C}$. This implies that there exists $t \in \omega$ such that $B \subseteq \bigcup\{(C_m \setminus \{a_m, a_{m+1}\}) \times A_m : m \in t+1\}$. There exists $k \in t+1$ such that the set $B \cap [(C_k \setminus \{a_k, a_{k+1}\}) \times A_k]$ is infinite. For our convenience, let us assume that $B \subseteq [(C_k \setminus \{a_k, a_{k+1}\}) \times A_k]$. If there exists $x \in A_k$ such that $B \cap C_{k,x}$ is infinite, then, by the completeness of the circle $\mathbf{C}_k$, the sequence $(d_n)_{n\in\mathbb{N}}$ converges to some point of $C_{k,x} \cup \{a_k, a_{k+1}\}$. Assume that, for each $x \in A_k$, the set $B \cap C_{k,x}$ is finite. We can replace, if necessary, the sequence $(d_n)_{n\in\omega}$ by its subsequence $(d_{n_j})_{j\in\omega}$ such that, for each $x \in A_k$, the set $\{d_{n_j} : j \in \omega\} \cap C_{k,x}$ consists of at most one point. Therefore, without loss of generality, we can assume that, for each $x \in A_k$, the set $B \cap C_{k,x}$ consists of at most one point. For each $n \in \omega$, let $d_n = \langle s_n, x_n\rangle \in C_{k,x_n}$. We may assume that the sequence $(x_n)_{n\in\omega}$ is injective. Since $(d_n)_{n\in\omega}$ is Cauchy in $\mathbf{Z}$, it follows that there exists $n_0 \in \omega$ such that if $n, m \in \omega$ and $n_0 \subseteq n \cap m$, then

$$\rho(d_n, d_m) < \tfrac{\pi r_k}{3}. \tag{6}$$

Let $n \in \omega$ be such that $n_0 \in n$. From (5) and (6), it follows that

$$\rho(d_n, d_{n_0}) = \min\{l(s_n a_{k+1}) + l(s_{n_0} a_{k+1}), l(s_n a_k) + l(s_{n_0} a_k)\} < \frac{\pi r_k}{3}.$$

Clearly, just one of $l(s_{n_0} a_{k+1})$ and $l(s_{n_0} a_k)$ is less than $\frac{\pi r_k}{3}$. Assume that $l(s_{n_0} a_k) < \frac{\pi r_k}{3}$. Then $l(s_{n_0} a_{k+1}) > \frac{2\pi r_k}{3}$ and $\rho(d_n, d_{n_0}) = l(s_n a_k) + l(s_{n_0} a_k) < \frac{\pi r_k}{3}$. Hence $\lim_{n\to\infty} l(s_n a_k) = 0$ and, in consequence, the sequence $(d_n)_{n\in\omega}$ converges in $\mathbf{Z}$ to a_k. This proves that $\mathbf{Z}$ is densely complete with respect to D as required. However, $(a_n)_{n\in\omega}$ is easily seen to be a Cauchy sequence in $\mathbf{Z}$ converging to no point of Z. This shows that $\mathbf{Z}$ is not complete and finishes the proof that $\mathbf{DCC}_c$ implies $\mathbf{CAC}$. □

Definition 4.5. Let J be a countable non-empty set and let $\{\mathbf{X}_j : j \in J\}$ be a collection of metric spaces $\mathbf{X}_j = \langle X_j, d_j\rangle$ where $j \in J$. Let $X = \prod_{j\in J} X_j$.

(i) If J is finite, let d be the metric on X defined as follows:

$$d(x, y) = \max\{d_j(x(j), y(j)) : j \in J\} \text{ for all } x, y \in X.$$

Then $\mathbf{X} = \prod_{j\in J} \mathbf{X}_j$ denotes the metric space $\langle X, d\rangle$ and $\mathbf{X}$ is called the *finite product* of the metric spaces $\mathbf{X}_j$.

(ii) If $J = \{j_1, j_2\}$ where $j_i \neq j_2$, then $\mathbf{X}_{j_1} \times \mathbf{X}_{j_2} = \langle X_{j_1} \times X_{j_2}, d\rangle$, where

$$d(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = \max\{d_{j_1}(x_1, x_2), d_{j_2}(y_1, y_2)\}$$

for all $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in X_{j_1} \times X_{j_2}$, is called the Cartesian product of the metric spaces $\mathbf{X}_{j_1}$ and $\mathbf{X}_{j_2}$.

(iii) If $J = \{j_n : n \in \mathbb{N}\}$, we put $X_n = X_{j_n}$, $d_n = d_{j_n}$ and $\mathbf{X}_n = \mathbf{X}_{j_n}$ for each $n \in \mathbb{N}$. Moreover, we put $X = \prod_{n\in\mathbb{N}} X_n$ and consider the metric d on X defined by:

$$d(x, y) = \sum_{n\in\mathbb{N}} \frac{\min\{d_n(x(n), y(n)), 1\}}{2^n}. \tag{7}$$

Then $\mathbf{X} = \langle X, d\rangle$ is the product $\prod_{n\in\mathbb{N}} \mathbf{X}_n$ of the collection $\{\mathbf{X}_j : j \in J\}$. For each $n \in \mathbb{N}$, let π_n denote the projection $\pi_n : X \to X_n$ defined by: $\pi_n(x) = x(n)$ for each $x \in X$.

Proposition 4.6. *Dense completeness is finitely productive, i.e., all finite products of densely complete metric spaces are densely complete metric spaces.*

Proof. Let J be a non-empty finite set and let $\{\mathbf{X}_j : j \in J\}$ be a collection of metric spaces $\mathbf{X}_j = \langle X_j, d_j\rangle$. Suppose that, for each $j \in J$, the space $\mathbf{X}_j$ is densely complete with respect to a set D_j. Then $\mathbf{X} = \prod_{j\in J} \mathbf{X}_j$ is densely complete with respect to the set $D = \prod_{j\in J} D_j$. □

In view of Proposition 4.6 and the countable productivity of completeness for metric spaces in **ZF**, see, e.g., [12], one may ask the following question:

Question 4.7. Is dense completeness in the class of metric spaces countably productive in **ZF**?

Our next two theorems give a partial answer to this question:

Theorem 4.8. *Each of the following statements implies the one beneath it:*

(i) **CMC**.

(ii) *For every family $\{\mathbf{X}_n : n \in \mathbb{N}\}$ of densely complete metric spaces, there exists a family $\{D_n : n \in \mathbb{N}\}$ such that, for every $n \in \mathbb{N}$, the space $\mathbf{X}_n$ is densely complete with respect to D_n.*

(iii) *Every countable product of densely complete metric spaces is densely complete.*

Proof. Let $\{\mathbf{X}_n : n \in \mathbb{N}\}$ be a collection of densely complete metric spaces $\mathbf{X}_n = \langle X_n, d_n\rangle$ and let $\mathbf{X} = \prod_{n\in\mathbb{N}} \mathbf{X}_n$. If $\prod_{n\in\mathbb{N}} X_n = \emptyset$, then $\mathbf{X}$ is trivially densely complete, so we may assume that $\prod_{n\in\mathbb{N}} X_n \neq \emptyset$.

(i)$\to$(ii) For each $n \in \mathbb{N}$, let $\mathcal{G}_n$ be the collection of all sets G such that $\mathbf{X}_n$ is densely complete with respect to G. Assume that **CMC** holds. Then, by **CMC**, there exists a collection $\{\mathcal{D}_n : n \in \mathbb{N}\}$ such that, for each $n \in \mathbb{N}$, $\mathcal{D}_n$ is a non-empty finite subcollection of $\mathcal{G}_n$. For each $n \in \mathbb{N}$, we put $D_n = \bigcup \mathcal{D}_n$. It is straightforward that, for each $n \in \mathbb{N}$, the space $\mathbf{X}_n$ is densely complete with respect to D_n.

(ii)$\to$(iii) Now, let us assume (ii). Fix a collection $\{D_n : n \in \mathbb{N}\}$ such that, for each $n \in \mathbb{N}$, the space $\mathbf{X}_n$ is densely complete with respect to D_n. It is possible that $\prod_{n\in\mathbb{N}} D_n = \emptyset$. However, we have assumed that $\prod_{n\in\mathbb{N}} X_n \neq \emptyset$, so we can fix $a \in \prod_{n\in\mathbb{N}} X_n$. For $n \in \mathbb{N}$, let $H_n = D_n \cup \{a(n)\}$. Of course, for each $n \in \mathbb{N}$, the space $\mathbf{X}_n$ is densely complete with respect to H_n, while the set $H = \prod_{n\in\mathbb{N}} H_n$ is dense in $\mathbf{X}$. It can be easily deduced from the classical theory of metric spaces that $\mathbf{X}$ is densely complete with respect to H. □

In [11], it has been shown that "sequentiality" of metric spaces is not countably productive in **ZF**. Next, we elaborate a little bit more on this matter in connection with the notion of dense completeness.

Theorem 4.9. *The following are equivalent:*

(i) **CAC**;

(ii) *all countable products of densely complete metric spaces are densely complete and sequential;*

(iii) *all countable products of discrete metric spaces are sequential;*

(iv) *all countable products of sequential metric spaces are sequential.*

Proof. That (i) implies (ii) follows directly from Theorems 4.1 and 4.8. It is known that (iv) follows from (i). It is obvious that (ii) implies (iii) and (iv) implies (iii). To complete the proof, it suffices to show that the negation of **CAC** implies the negation of (iii).

Assume that **CAC** does not hold and fix a pairwise disjoint family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of non-empty sets without a partial choice function. Choose any set ∞ such that $\infty \neq \bigcup \mathcal{A}$. For every $n \in \mathbb{N}$, let $X_n = A_n \cup \{\infty\}$ and let d_n be the discrete metric on X_n. Put $\mathbf{X}_n = \langle X_n, d_n \rangle$ and $\mathbf{X} = \prod_{n\in\mathbb{N}} \mathbf{X}_n$. Of course, all discrete spaces $\mathbf{X}_n$ are complete. We claim that $G = \bigcup_{n\in\mathbb{N}} \pi_n^{-1}[A_n]$ is sequentially closed in $\mathbf{X}$. To see this, fix a sequence $(x_n)_{n\in\mathbb{N}}$ of points of G which converges in $\mathbf{X}$ to a point x. Since $\mathcal{A}$ has no partial choice function, it follows that there exists $n_0 \in \mathbb{N}$ such that $(x_n)_{n\in\mathbb{N}} \subseteq K$, where $K = \bigcup_{n\in n_0} \pi_n^{-1}[A_n]$. Since K is a closed subspace of $\mathbf{X}$, we see that $x \in K \subseteq G$. Hence G is sequentially closed in $\mathbf{X}$. Let $a \in X$ be defined as follows: $a(n) = \infty$ for each $n \in \mathbb{N}$. Of course, $a \notin G$. Since, for every neighborhood V of a in $\mathbf{X}$, the set $V \cap G$ is non-empty, $a \in \overline{G}$. So, G is sequentially closed but not closed in $\mathbf{X}$. This contradicts (iii). □

Theorem 4.10. *The following conditions are all equivalent:*

(i) $\mathbf{CAC}(\mathbb{R})$;

(ii) *for all non-empty metric subspaces $\mathbf{X}_1, \mathbf{X}_2$ of $\mathbb{R}$, if the product $\mathbf{X}_1 \times \mathbf{X}_2$ is densely complete, then both $\mathbf{X}_1, \mathbf{X}_2$ are densely complete;*

(iii) *for all non-empty topological subspaces $\mathbf{X}_1, \mathbf{X}_2$ of $\mathbb{R}$, if the product $\mathbf{X}_1 \times \mathbf{X}_2$ is densely completely metrizable, then both $\mathbf{X}_1, \mathbf{X}_2$ are densely completely metrizable;*

(iv) *for each countably infinite, not complete metric subspace $\mathbf{C}$ of $\mathbb{R}$, the metric space $\mathbb{R} \times \mathbf{C}$ is not densely complete;*

(v) *the metric subspace $\mathbb{R} \times \{\frac{1}{n} : n \in \mathbb{N}\}$ of $\mathbb{R}^2$ is not densely complete;*

(vi) the topological subspace $\mathbb{R} \times \mathbb{Q}$ of $\mathbb{R}^2$ is not densely completely metrizable.

Proof. Since $\mathbf{CAC}(\mathbb{R})$ and $\mathbf{CAC}(\mathbb{R}^2)$ are equivalent, in the light of Theorem 2.7(i), conditions (ii)–(vi) follow from (i). Now, assume that (i) is not satisfied. In view of Theorem 3.14, we can fix a family $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$ of dense subsets of $\mathbb{R}$ such that $\mathcal{D}$ does not have a partial choice function. Let $C = \{c_n : n \in \mathbb{N}\}$ be a countably infinite subset of $\mathbb{R}$ and let $X = \mathbb{R} \times C$ be endowed with the Euclidean metric ρ. We may assume that $c_i \neq c_j$ for each pair i, j of distinct numbers from $\mathbb{N}$. Clearly,

$$D = \bigcup \{D_n \times \{c_n\} : n \in \mathbb{N}\}$$

is a dense subset of $\mathbf{X} = \langle X, \rho \rangle$ such that $\{D_n \times \{c_n\} : n \in \mathbb{N}\}$ has no partial choice function. To show that $\mathbf{X}$ is densely complete with respect to D, consider any Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ of points of D. Let $z_n = \langle x_n, y_n \rangle$ for each $n \in \mathbb{N}$. Then the set $\{y_n : n \in \mathbb{N}\}$ is finite because $\mathcal{D}$ does not have a partial choice function. Therefore, there exists $k \in \mathbb{N}$ such that $z_n \in \bigcup_{i=1}^{k}(D(i) \times \{c_i\})$ for each $n \in \mathbb{N}$. Hence $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of the complete metric subspace $\mathbb{R} \times \{c_1, \dots c_k\}$ of $\mathbb{R}^2$, so $(z_n)_{n\in\mathbb{N}}$ converges in $\mathbf{X}$. In consequence, if (i) does not hold, then all conditions (ii)-(vi) are false. $\square$

Remark 4.11. The equivalence of (i) and (v) of Theorem 4.10 is especially remarkable because $\mathbb{Q}$ is not densely completely metrizable in **ZF**, while $\mathbb{R} \times \mathbb{Q}$ is densely completely metrizable in every model of **ZF** in which **CAC**($\mathbb{R}$) fails.

Theorem 4.12. **UCE** *and* **CAC** *are equivalent.*

Proof. If **CAC** holds, then every complete metric space is Cantor complete; hence, by Lemma 3.1, **UCE** follows from **CAC**.

In view of Theorem 4.1, to prove that **CAC** follows from **UCE**, it suffices to show that **UCE** implies "for every metric space $\mathbf{X}$, every complete subspace $\mathbf{Y}$ of $\mathbf{X}$ is closed". Assume the contrary and fix a metric space $\mathbf{X}$ having a complete, non-closed subset Y. Fix, $x \in \overline{Y} \setminus Y$. Clearly, the identity function $f : \mathbf{Y} \to \mathbf{Y}$ is uniformly continuous, and $\mathbf{Y}$ is dense in $\mathbf{M}$ where $M = \{x\} \cup Y$. By Theorem 2.10(i), f cannot be extended to a continuous function on $\mathbf{M}$. This contradicts **UCE** and finishes the proof. $\square$

Remark 4.13. Clearly, **DCC** restricted to the class of compact metric spaces is a theorem of **ZF**. A compact metric space is always complete in **ZF**. However, as Theorem 4.4 indicates, the restriction $\mathbf{DCC}_c$ of **DCC** to the class of all connected metric spaces is not provable in **ZF**. Similarly, Theorem 3.3 shows that $\mathbf{UCE}_{cc}$ is unprovable in **ZF**. Of course, $\mathbf{UCE}_c$ implies $\mathbf{UCE}_{cc}$.

The following lemma will be applied in our proof that $\mathbf{UCE}_{cc}$ does not imply **CAC** in $\mathbf{ZF}^0$.

Lemma 4.14. *Let S be a subset of a metric space $\mathbf{X} = \langle X, d \rangle$, let $\mathbf{Y} = \langle Y, \rho \rangle$ be a complete metric space and let $f : \mathbf{S} \to \mathbf{Y}$ be a uniformly continuous mapping. Then, for $Z = \widetilde{S}$, there exists a uniformly continuous mapping $F : \mathbf{Z} \to \mathbf{Y}$ such that $F(x) = f(x)$ for each $x \in S$.*

Proof. Consider any $x \in \widetilde{S}$. There exists a sequence $(x_n)_{n\in\omega}$ of points of S which converges to x. Since f is uniformly continuous, the sequence $(f(x_n))_{n\in\omega}$ is a Cauchy sequence in $\mathbf{Y}$, so it converges in $\mathbf{Y}$. Let $(x_n)_{n\in\omega}$ and $(x_n^\star)_{n\in\omega}$ be sequences of points of S, both converging to x. Let y be the limit in $\mathbf{Y}$ of $(f(x_n))_{n\in\omega}$ and let $y^\star$ be the limit in $\mathbf{Y}$ of $(f(x_n^\star))_{n\in\omega}$.

Suppose that $y \neq y^\star$. Then $\varepsilon = \rho(y, y^\star) > 0$ and, by the uniform continuity of f, there exists $\delta > 0$ such that $\rho(f(t), f(t^\star)) < \frac{\varepsilon}{3}$ for all $t, t^\star \in S$ with $d(t, t^\star) < \delta$. There exists $k_1 \in \omega$ such that $\rho(y, f(x_n)) < \frac{\varepsilon}{3}$ and $\rho(y^\star, f(x_n^\star)) < \frac{\varepsilon}{3}$ for each $n \in \omega \setminus k_1$. There exists $k_2 \in \omega$ such that if $n \in \omega \setminus k_2$, then $d(x_n, x) < \frac{\delta}{2}$ and $d(x_n^\star, x) < \frac{\delta}{2}$. We notice that if $n \in \omega \setminus (k_1 \cup k_2)$, then $d(x_n, x_n^\star) \leq d(x_n, x) + d(x, x_n^\star) < \delta$, so $\rho(f(x_n), f(x_n^\star)) < \frac{\varepsilon}{3}$ and, therefore, $\rho(y, y^\star) \leq \rho(y, f(x_n)) + \rho(f(x_n), f(x_n^\star)) + \rho(f(x_n)^\star, y^\star) < \varepsilon$. This contradicts our assumption on ε. Hence $y = y^\star$. Let y_x be the unique point of Y such that, for every sequence $(x_n)_{n\in\omega}$ of points of S converging to x, the sequence $(f(x_n))_{n\in\omega}$ is convergent in $\mathbf{Y}$ to y_x. We define a mapping $F : \mathbf{Z} \to \mathbf{Y}$ by putting $F(x) = y_x$ for each $x \in \widetilde{S}$. Of course, $F(x) = f(x)$ for each $x \in S$. One can easily check in a standard way that F is uniformly continuous. □

Theorem 4.15. *The conjunction of* **BPI** *and* $\mathbf{CAC}(\mathbb{R})$ *implies* $\mathbf{UCE}_c$.
In particular $\mathbf{UCE}_c$*, and consequently* $\mathbf{UCE}_{cc}$*, does not imply* **CAC**.

Proof. Let us assume **BPI** and $\mathbf{CAC}(\mathbb{R})$. Let $\mathbf{X} = \langle X, d\rangle$ be a compact metric space. First, we show that **BPI** implies that the family $\mathcal{K}$ of all non-empty closed subsets of $\mathbf{X}$ has a choice function. Indeed, let ∞ be a set which does not belong to X and, for every $K \in \mathcal{K}$, let $\mathbf{X}_K$, where $X_K = K \cup \{\infty\}$, be the topological sum of $\mathbf{K}$ and the trivial space $\{\infty\}$. Clearly, K is a closed subset of the compact space $\mathbf{X}_K$. Since

$$\mathcal{G} = \{\pi_K^{-1}(K) : K \in \mathcal{K}\}$$

is a family of closed subsets of the Tychonoff product $\mathbf{Y} = \prod_{K\in\mathcal{K}} \mathbf{X}_K$, and $\mathbf{Y}$ is compact by **BPI** (see Theorem 4.70 in [6]), it follows that $\bigcap \mathcal{G} \neq \emptyset$. Clearly, any element $f \in \bigcap \mathcal{G}$ is a choice function of $\mathcal{K}$. By Theorem 2.6(ii), $\mathbf{X}$ is separable. Hence, in view of Exercise E.3 to Section 4.6 of [6], it follows from $\mathbf{CAC}(\mathbb{R})$ that every subspace of $\mathbf{X}$ is separable (see also [11]).

Now, suppose that $\mathbf{Y} = \langle Y, \rho\rangle$ is a complete metric space, S is a dense set in $\mathbf{X}$, and $f : \mathbf{S} \to \mathbf{Y}$ is uniformly continuous. Let D be a countable dense set in $\mathbf{S}$. Since $X = \widetilde{D}$, it follows from Lemma 4.14 that there exists a uniformly continuous extension $F : \mathbf{X} \to \mathbf{Y}$ of f. Hence $\mathbf{UCE}_c$ holds in every model of $\mathbf{ZF} + [\mathbf{BPI} + \mathbf{CAC}(\mathbb{R})]$.

The second assertion follows from the fact that in Mostowski's Linearly Ordered Model $\mathcal{N}3$ in [8], **BPI** and $\mathbf{CAC}(\mathbb{R})$ hold true, but **CAC** fails. □

Remark 4.16. As in the case of hedgehog metric spaces, it follows from the proof to Theorem 4.4 that, in **ZFC**, for every set X such that $|X| \geq |\mathbb{R}|$, there exists a connected metric space $\mathbf{Z}$ with $|X| = |Z|$.

We leave the following question unanswered:

Question 4.17. Does, for every set X having a subset equipotent with $\mathbb{R}$, there exist in **ZF** a connected metric space equipotent with X?

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