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Nonlinear Images of Sets.
II: Applications to Differential Games

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This paper is a continuation of our previous study on weak convexity of nonlinear images of sets [see G. E. Ivanov, *Nonlinear images of sets. I: Strong and weak convexity*, J. Convex Analysis 27(1) (2020) 363–382]. We consider smoothness of a set in a special sense called "bodily smoothness" and split this smoothness property of a set S into two properties: weak convexity of S and weak convexity of the closure of the complementary to S. We obtain sufficient conditions for bodily smoothness of the image of the Cartesian product of two sets $S \times U$, wherein S is bodily smooth and U is strongly convex. We apply this result for nonlinear differential games and obtain sufficient conditions for the game reachable sets to be smooth.

Keywords: $C^{1,1}$ mapping, covering mapping, bi-Lipschitz mapping, strong convexity, weak convexity, Minkowski sum, Minkowski operator, differential game.

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1. Introduction

The foundations of the theory of zero sum differential games were laid by Isaacs [7], Pontryagin [16, 17] and others. Typically algorithms to construct optimal (or suboptimal) control strategies in pursuit-evasion games provide some methods to calculate the game reachable sets and then use these sets for construction of the strategies. In this context convexity or smoothness of the game reachable sets are very important properties, which make possible to apply less or more complicated methods to approximate the game reachable sets and to construct the strategies. If the dynamics of the differential game is linear and the game target set is convex and compact, then the game reachable sets are convex too and may be determined as Pontryagin's alternating sums using operations of Minkowski sum and difference.

It is known that strong and weak convexity are very useful properties in the theory and numerical algorithms for optimal control and differential games. In particular in [14] it was shown that convergence of the alternating sums to the alternating integral has the second order rate provided that the differential game is linear and the admissible control set of the pursuer is strongly convex. In [13] the

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alternative theorem for a linear differential game with strongly convex admissible control sets and a smooth target set is obtained. The alternative theorem states that the program (open-loop) strategies are optimal for such a game. Theory and applications of strongly and weakly convex sets and akin classes of sets are presented in [1, 2, 3, 5, 6, 10, 15, 19].

As shown in [4] the game reachable sets and the control strategies for nonlinear differential games can be calculated based on the Minkowski operators which generalize the Minkowski operations to nonlinear case. So, the questions about convexity, weak convexity and smoothness of the images of the Minkowski operators arise. We consider smoothness of a set in a special sense called "bodily smoothness", which means that the set is the closure of its interior and the unit normal vector to the set is Lipschitz continuous. This smoothness property of a set S can be splitted into two properties: weak convexity of S and weak convexity of the closure of the complementary to S. Using sufficient conditions for weak convexity of the nonlinear image [11] we obtain sufficient conditions for smoothness of the image $f(S, U)$ of the Cartesian product $S \times U$ of two sets S and U, where S is a bodily smooth and U is strongly convex. Applying this result for the Minkowski operators, which can be expressed in terms of the image $f(S, U)$, we obtain sufficient conditions for the images of the Minkowski operators to be smooth. Using this result we obtain sufficient conditions for the game reachable sets to be smooth that allows the effective approximation algorithms to be used in such problems.

2. Auxiliary notions and results

We use $B_r(c)$ to denote the *closed ball* in a Banach space X with center $c \in X$ and radius $r \geq 0$:

$$B_r(c) = \{x \in X : \ \|x - c\| \leq r\}.$$

Let X and Y be Banach spaces. For a mapping $f\colon X \to Y$ and a set $S \subset X$ we use $\operatorname{lip}(f, S)$ to denote the *Lipschitz constant* of f on S, i.e. the infimum of $L > 0$ such that

$$\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\| \qquad \forall x_1, x_2 \in S.$$

As usual, the infimum of the empty set is defined as $+\infty$.

A mapping $f\colon X \to Y$ is called *co-Lipschitz* on $S \subset X$ if there exists $\beta > 0$ such that

$$\|f(x_1) - f(x_2)\| \geq \beta\|x_1 - x_2\| \qquad \forall x_1, x_2 \in S.$$

Denote by $\operatorname{colip}(f, S)$ the supremum of such $\beta > 0$. If no such $\beta > 0$ exists, set $\operatorname{colip}(f, S) = 0$.

If a mapping $f\colon X \to Y$ is Lipschitz continuous and co-Lipschitz on $S \subset X$, it is called *bi-Lipschitz* on S.

A mapping $f\colon X \to Y$ is called *covering (open at a linear rate)* on $S \subset X$ if there exists $\sigma > 0$ such that for every $x \in S$ and every $t > 0$, with $B_t(x) \subset S$, we have

$$B_{\sigma t}(f(x)) \subset f(B_t(x)).$$

Denote by $\operatorname{sur}(f, S)$ the supremum of such $\sigma > 0$. If no such $\sigma > 0$ exists, set $\operatorname{sur}(f, S) = 0$.

Denote by $\mathcal{L}(X,Y)$ the Banach space of all linear continuous operators from X to Y equipped with the operator norm. Given an open set $V \subset X$, a function $f\colon V \to Y$ and a point $x_0 \in V$, we denote by $\mathcal{D}f(x_0) \in \mathcal{L}(X,Y)$ the Fréchet derivative of f at x_0. Let $S \subset X$. A mapping $f\colon S \to Y$ is called *Fréchet differentiable* if it can be extended on an open subset $V \subset X$ with $S \subset V$ such that the extension $\tilde{f}\colon V \to Y$ is Fréchet differentiable on S. We use $\mathcal{D}f(x_0)$ to define the Fréchet derivative $\mathcal{D}\tilde{f}(x_0)$ of the extension $\tilde{f}\colon V \to Y$ at the point $x_0 \in S$. Denote $C^{1,1}(S,Y)$ the class of Fréchet differentiable functions $f\colon S \to Y$ such that the Fréchet derivative $\mathcal{D}f\colon S \to \mathcal{L}(X,Y)$ satisfies the Lipschitz condition

$$\|\mathcal{D}f(x_1) - \mathcal{D}f(x_2)\| \le \gamma\|x_1 - x_2\| \quad \forall x_1, x_2 \in S.$$

Denote by $\operatorname{lip}(\mathcal{D}f, S)$ the infimum of such γ.

Recall that the *Fréchet normal cone* of a set $S \subset X$ at a point $x_0 \in S$ is defined as

$$N(x_0,S) = \{x^* \in X^*\colon\ \forall\varepsilon>0\ \exists\delta>0\colon \forall x \in S\cap B_\delta(x_0)\ \ \langle x^*, x-x_0\rangle \le \varepsilon\|x-x_0\|\},$$

where $\langle x^*, x-x_0\rangle$ is the value of the functional $x^* \in X^*$ at $x - x_0$.

We shall use the following proposition which follows immediately from the definitions.

Proposition 2.1. *Let $x_0 \in S \subset X$ and $f\colon S \to Y$ be a diffeomorphism between S and $f(S)$. Then $N(x_0,S) = (\mathcal{D}f(x_0))^* N(f(x_0), f(S))$, where $(\mathcal{D}f(x_0))^*\colon Y^* \to X^*$ is the adjoint operator to $\mathcal{D}f(x_0)$.*

Let H be a Hilbert space. We identify H with its dual and use $\langle x,y\rangle$ to denote the inner product of $x, y \in H$. As usual, we consider H as a Banach space with the norm induced by the inner product. For a set $S \subset H$ by $\operatorname{int} S$, $\overline{S}$ and ∂S we denote the interior, the closure and the boundary of S respectively.

Let $r > 0$. A set $S \subset H$ is called *strongly convex* with radius r if it can be represented as an intersection of closed balls with radius r, i.e. there exists $C \subset H$ such that

$$S = \bigcap_{c\in C} B_r(c).$$

Denote by $\mathcal{R}^+(S)$ the infimum of such $r > 0$.

A closed set $S \subset H$ is called *weakly convex* with radius $r > 0$ if for any $x_0 \in S$ and $p \in N(x_0,S)$ with $\|p\| = 1$ one has $S \cap \operatorname{int} B_r(x_0 + rp) = \emptyset$. Denote by $\mathcal{R}^-(S)$ the supremum of such $r > 0$. If no such $r > 0$ exists, set $\mathcal{R}^-(S) = 0$.

Remark 2.2. For any $c \in H$ and $r > 0$ the set $S = H \setminus \operatorname{int} B_r(c)$ is weakly convex with radius r.

A set $S \subset H$ is called *weakly convex in the sense of Efimov–Stechkin (ES-weakly convex)* with radius $r > 0$ if for any $x \in H \setminus S$ there exists $c \in H$ such that $x \in \operatorname{int} B_r(c) \subset H \setminus S$. Denote by $\mathcal{R}^-_{ES}(S)$ the supremum of such $r > 0$. If no such $r > 0$ exists, set $\mathcal{R}^-_{ES}(S) = 0$.

Proposition 2.3. ([8, Theorem 7.1]). *If $S \subset H$ is closed, then $\mathcal{R}^-(S) \le \mathcal{R}^-_{ES}(S)$.*

Example 2.4. Fix some $\varepsilon > 0$. Consider $H = \mathbb{R}^2$ and $S_\varepsilon = S_\varepsilon^+ \cap S_\varepsilon^-$, where

$$S_\varepsilon^+ = \{(x,y) \in \mathbb{R}^2 : (x+1)^2 + y^2 \geq 1+\varepsilon^2\}, \quad S_\varepsilon^- = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \geq 1+\varepsilon^2\}.$$

One can see that $\mathcal{R}^-(S) = \varepsilon < \sqrt{1+\varepsilon^2} = \mathcal{R}^-_{ES}(S)$. This shows that $\mathcal{R}^-(S)$ and $\mathcal{R}^-_{ES}(S)$ may be different.

Proposition 2.5. ([8, Lemma 3.5]). *Let $S_\alpha \subset H$ for all $\alpha \in A$ and $S = \bigcap_{\alpha \in A} S_\alpha$. Then*

$$\mathcal{R}^-_{ES}\Big(\bigcap_{\alpha \in A} S_\alpha\Big) \geq \inf_{\alpha \in A} \mathcal{R}^-_{ES}(S_\alpha).$$

Remark 2.6. In general the inequality $\mathcal{R}^-\left(\bigcap_{\alpha \in A} S_\alpha\right) \geq \inf\limits_{\alpha \in A} \mathcal{R}^-(S_\alpha)$ is invalid. This can be seen from Example 2.4.

We say that $S \subset H$ is *weakly convex-concave* with positive constants R and $\widehat{R}$ and write $S \in WCC(R, \widehat{R}; H)$ if S is a proper subset of H, $S = \overline{\operatorname{int} S}$, S is weakly convex with radius R and $H \setminus \operatorname{int} S$ is weakly convex with radius $\widehat{R}$.

Remark 2.7. If $S \in WCC(R, \widehat{R}; H)$, then $(H \setminus \operatorname{int} S) \in WCC(\widehat{R}, R; H)$.

We say that a closed proper set $S \subset H$ is *bodily smooth with constant $L > 0$* if $S = \overline{\operatorname{int} S}$ and

$$\|p_1 - p_2\| \leq L\|a_1 - a_2\| \quad \forall a_1, a_2 \in S, \quad \forall p_i \in N(a_i, S) : \ \|p_i\| = 1, \ i = 1, 2.$$

Proposition 2.8. ([8, Lemma 7.2]). *If $S \in WCC(R, \widehat{R}; H)$, then for any $x \in \partial S$ there exists $p \in H$, $\|p\| = 1$ such that*

$$B_{\widehat{R}}(x - \widehat{R}p) \subset S, \quad S \cap \operatorname{int} B_R(x + Rp) = \emptyset.$$

Proposition 2.9. ([8, Theorem 7.2]). *Let $S \subset H$ be a proper closed set, R and $\widehat{R}$ be positive constants. Then the following assertions are equivalent:*

(i) *$S \in WCC(R, \widehat{R}; H)$;*

(ii) *$S = \overline{\operatorname{int} S}$, $\mathcal{R}^-_{ES}(S) \geq R$ and $\mathcal{R}^-_{ES}(H \setminus \operatorname{int} S) \geq \widehat{R}$.*

Proposition 2.10. ([9, Theorem 3], [10, Section 1.12]). *Let $S \subset H$ be a proper closed set and $R > 0$. Then the following assertions are equivalent:*

(i) *S is bodily smooth with constant $1/R$;*

(ii) *$S \in WCC(R, R; H)$.*

Proposition 2.11. ([11, Theorem 5.1] *Let X and Y be Hilbert spaces and $S \subset X$ be a closed and weakly convex set. Assume that mapping $f \in C^{1,1}(\operatorname{conv} S, Y)$ is bi-Lipschitz on* $\operatorname{conv} S$. *Then $f(S)$ is closed and weakly convex, wherein*

$$\mathcal{R}^-(f(S)) \geq \frac{\mathcal{R}^-(S) \cdot \operatorname{colip}^2(f, S)}{\operatorname{lip}(f, \operatorname{conv} S) + \mathcal{R}^-(S) \cdot \operatorname{lip}(\mathcal{D}f, \operatorname{conv} S)}.$$

Proposition 2.12. ([11, Theorem 5.2] *Let X, Y and Z be Hilbert spaces; let $S \subset X$ be a weakly convex set and $U \subset Y$ be a strongly convex set. Assume that $f \in C^{1,1}(\operatorname{conv} S \times U, Z)$ is such that for all $u \in U$ mapping $f(\cdot, u)$ is bi-Lipschitz on $\operatorname{conv} S$ and for all $x \in S$ mapping $f(x, \cdot)$ is covering on U. Let*

$$\mathcal{R}^-(S) > 0, \quad \mathcal{R}^+(U) > 0, \quad \inf_{x\in S} \operatorname{sur}(f(x,\cdot), U) > \mathcal{R}^+(U) \cdot \operatorname{lip}(\mathcal{D}f, \operatorname{conv} S \times U),$$

$$\varrho_x = \frac{\mathcal{R}^-(S) \cdot \inf\limits_{u\in U} \operatorname{colip}^2(f(\cdot,u), S)}{\sup\limits_{u\in U} \operatorname{lip}(f(\cdot,u), \operatorname{conv} S) + \mathcal{R}^-(S) \cdot \operatorname{lip}(\mathcal{D}f, \operatorname{conv} S \times U)},$$

$$\varrho_u = \frac{\mathcal{R}^+(U) \cdot \sup\limits_{x\in S} \operatorname{lip}^2(f(x,\cdot), U)}{\inf\limits_{x\in\operatorname{conv} S} \operatorname{sur}(f(x,\cdot), U) - \mathcal{R}^+(U) \cdot \operatorname{lip}(\mathcal{D}f, \operatorname{conv} S \times U)}.$$

$$\varrho = \varrho_x - \varrho_u > 0.$$

Then $f(S,U)$ is closed and weakly convex with $\mathcal{R}^-(f(S,U)) \geq \varrho$.

3. Smoothness of the image

Lemma 3.1. *Let X and Y be Hilbert spaces. Let $S \subset X$ be ES-weakly convex. Let $f \in C^{1,1}(X,Y)$ be a bi-Lipschitz bijection. Then $f(S)$ is ES-weakly convex with*

$$\mathcal{R}^-_{ES}(f(S)) \geq \varrho := \frac{\mathcal{R}^-_{ES}(S) \cdot \operatorname{colip}^2(f,X)}{\operatorname{lip}(f,X) + \mathcal{R}^-_{ES}(S) \cdot \operatorname{lip}(\mathcal{D}f,X)}.$$

Proof. Fix any positive $r < \mathcal{R}^-_{ES}(S)$. Fix any $y \in Y \setminus f(S)$. Since f is a bijection, one can find $x \in X \setminus S$ such that $y = f(x)$. Since S is ES-weakly convex with radius r, there exists $c \in X$ such that $x \in \operatorname{int} B_r(c) \subset X \setminus S$. According to Remark 2.2 the set $X \setminus \operatorname{int} B_r(c)$ is weakly convex with radius r. Hence, by Proposition 2.11 the set $A := f(X \setminus \operatorname{int} B_r(c))$ is weakly convex with

$$\mathcal{R}^-(A) \geq \varrho' := \frac{r \cdot \operatorname{colip}^2(f,X)}{\operatorname{lip}(f,X) + r \cdot \operatorname{lip}(\mathcal{D}f,X)}.$$

Proposition 2.3 implies that $\mathcal{R}^-_{ES}(A) \geq \mathcal{R}^-(A) \geq \varrho'$. Hence, A is ES-weakly convex with any radius $\varrho'' < \varrho'$. Since $y = f(x) \notin A$ there exists $z \in Y$ such that $y \in \operatorname{int} B_{\varrho''}(z) \subset Y \setminus A$. In view of the inclusion $\operatorname{int} B_r(c) \subset X \setminus S$ we have $S \subset X \setminus \operatorname{int} B_r(c)$. Consequently, $f(S) \subset f(X \setminus \operatorname{int} B_r(c)) = A$ and hence $y \in \operatorname{int} B_{\varrho''}(z) \subset Y \setminus A \subset Y \setminus f(S)$. It means that $f(S)$ is ES-weakly convex with any radius $\varrho'' < \varrho'$. So, $\mathcal{R}^-_{ES}(f(S)) \geq \varrho'$. Passing to the limit as $r \to \mathcal{R}^-_{ES}(S)$, we obtain $\mathcal{R}^-_{ES}(f(S)) \geq \varrho$. $\square$

Lemma 3.2. *Let X and Z be Hilbert spaces. Let $S \subset X$ be such that $X \setminus \operatorname{int} S$ be ES-weakly convex and U be an arbitrary set. Assume that $f : X \times U \to Z$ is such that for any $u \in U$ the mapping $f(\cdot, u)$ is of class $C^{1,1}(X,Z)$ and is a bi-Lipschitz bijection from X to Z. Then $Z \setminus f(\operatorname{int} S, U)$ is ES-weakly convex with*

$$\mathcal{R}^-_{ES}(Z \setminus f(\operatorname{int} S, U)) \geq \varrho := \frac{\mathcal{R}^-_{ES}(X \setminus \operatorname{int} S) \cdot \inf\limits_{u \in U} \operatorname{colip}^2(f(\cdot, u), X)}{\sup\limits_{u \in U} \operatorname{lip}(f(\cdot, u), X) + \mathcal{R}^-_{ES}(X \setminus \operatorname{int} S) \cdot \operatorname{lip}(\mathcal{D}f, X \times U)}.$$

Proof. Lemma 3.1 implies that the set $F_u := Z \setminus f(\operatorname{int} S, u) = f(X \setminus \operatorname{int} S, u)$ is ES-weakly convex with $\mathcal{R}^-_{ES}(F_u) \geq \varrho$ for any $u \in U$. Since

$$Z \setminus f(\operatorname{int} S, U) = \bigcap_{u \in U} Z \setminus f(\operatorname{int} S, u) = \bigcap_{u \in U} F_u,$$

then by Proposition 2.5 we have

$$\mathcal{R}^-_{ES}(Z \setminus f(\operatorname{int} S, U)) \geq \inf_{u \in U} \mathcal{R}^-_{ES}(F_u) \geq \varrho. \qquad \square$$

Lemma 3.3. *Suppose in addition to the assumptions of Proposition* 2.12 *that* $S = \overline{\operatorname{int} S}$ *and for all* $u \in U$ *the mapping* $f(\cdot, u)$ *is a bi-Lipschitz bijection from* X *to* Z*. Then* $F = f(S, U)$ *satisfies the equation* $F = \overline{\operatorname{int} F}$*.*

Proof. Fix any $f_0 \in F$. One can find $u_0 \in U$, $x_0 \in S$ such that $f_0 = f(x_0, u_0)$. Since $x_0 \in S = \overline{\operatorname{int} S}$ there exists a sequence $\{x_k\}$ in $\operatorname{int} S$ such that $x_k \to x_0$. By continuity of f we have $f(x_k, u_0) \to f(x_0, u_0) = f_0$. Since $x_k \in \operatorname{int} S$ and the mapping $f(\cdot, u_0)$ is a bi-Lipschitz bijection from X to Z, it follows that $f(x_k, u_0) \in \operatorname{int} f(S, u_0) \subset \operatorname{int} F$ for all $k \in \mathbb{N}$. Consequently $f_0 \in \overline{\operatorname{int} F}$. So, $F \subset \overline{\operatorname{int} F}$. On the other hand, according to Proposition 2.12 F is closed. Consequently, $\overline{\operatorname{int} F} \subset \overline{F} = F$. $\square$

Theorem 3.4. *Let* X, Y *and* Z *be Hilbert spaces; let* $S \in WCC(r_S, \widehat{r}_S; X)$ *and* $U \subset Y$ *be strongly convex. Assume that* $f \in C^{1,1}(X \times U, Z)$ *is such that for all* $u \in U$ *the mapping* $f(\cdot, u)$ *is a bi-Lipschitz bijection from* X *to* Z *and for all* $x \in X$ *the mapping* $f(x, \cdot)$ *is a covering bijection from* U *to* $f(x, U)$*. Let*

$$\inf_{x \in X} \operatorname{sur}(f(x, \cdot), U) > \mathcal{R}^+(U) \cdot \operatorname{lip}(\mathcal{D}f, X \times U),$$

$$\varrho_u = \frac{\mathcal{R}^+(U) \cdot \sup\limits_{x \in X} \operatorname{lip}^2(f(x, \cdot), U)}{\inf\limits_{x \in X} \operatorname{sur}(f(x, \cdot), U) - \mathcal{R}^+(U) \cdot \operatorname{lip}(\mathcal{D}f, X \times U)},$$

$$\varrho_x = \frac{r_S \cdot \inf\limits_{u \in U} \operatorname{colip}^2(f(\cdot, u), X)}{\sup\limits_{u \in U} \operatorname{lip}(f(\cdot, u), X) + r_S \cdot \operatorname{lip}(\mathcal{D}f, X \times U)},$$

$$\widehat{\varrho}_x = \frac{\widehat{r}_S \cdot \inf\limits_{u \in U} \operatorname{colip}^2(f(\cdot, u), X)}{\sup\limits_{u \in U} \operatorname{lip}(f(\cdot, u), X) + \widehat{r}_S \cdot \operatorname{lip}(\mathcal{D}f, X \times U)}.$$

Assume that

$$\varrho_u < \varrho_x.$$

Then $\operatorname{int} f(S, U) = f(\operatorname{int} S, U)$ *and* $f(S, U) \in WCC(\varrho_x - \varrho_u, \widehat{\varrho}_x; Z)$*.*

Proof. We first define $L_x = \sup_{u \in U} \operatorname{lip}(f(\cdot, u), X)$, $L_u = \sup_{x \in X} \operatorname{lip}(f(x, \cdot), U)$, $\sigma_x = \inf_{u \in U} \operatorname{colip}(f(\cdot, u), X)$, $\sigma_u = \inf_{x \in X} \operatorname{sur}(f(x, \cdot), U)$, $\gamma = \operatorname{lip}(\mathcal{D}f, X \times U)$, $r_U = \mathcal{R}^+(U)$, $F = f(S, U)$, and prove that

$$\operatorname{int} F = f(\operatorname{int} S, U). \tag{1}$$

Since $f(\cdot,u)$ is a homeomorphism for any $u\in U$, it follows that $f(\operatorname{int} S,u)$ is open for all $u\in U$ and, hence, $f(\operatorname{int} S,U)=\bigcup_{u\in U} f(\operatorname{int} S,u)$ is open. Consequently, $f(\operatorname{int} S,U)\subset \operatorname{int} F$. Let us prove the converse inclusion.

Fix any $f_0\in \operatorname{int} F$. One can find $x_0\in S$, $u_0\in U$ such that $f_0=f(x_0,u_0)$. To conclude the proof of (1) it suffices to show that

$$f_0\in f(\operatorname{int} S,U). \tag{2}$$

If $x_0\in \operatorname{int} S$, then (2) is proved. Assume that $x_0\in\partial S$. According to Proposition 2.8 there exists $\ell\in X$, $\|\ell\|=1$ such that

$$B_{\widehat{r}_S}(x_0-\widehat{r}_S\ell)\subset S. \tag{3}$$

and

$$S\cap \operatorname{int} B_{r_S}(x_0+r_S\ell)=\emptyset. \tag{4}$$

The last relation implies that $\ell\in N(x_0,S)$. Let $\mathcal{D}_x f(x_0,u_0)$ be the Fréchet derivative of the mapping $f(\cdot,u_0)$ at x_0 and $\mathcal{D}_u f(x_0,u_0)$ be the Fréchet derivative of the mapping $f(x_0,\cdot)$ at u_0. By Proposition 2.1 there exists $q\in N(f_0,f(S,u_0))$ such that $\ell=(\mathcal{D}_x f(x_0,u_0))^*q$. Denote $p=(\mathcal{D}_u f(x_0,u_0))^*q$, $p_1=\frac{p}{\|p\|}$, $q_1=\frac{q}{\|q\|}$. By the definitions of σ_u and L_x we have $\|p\|\ge\sigma_u\|q\|$, $1=\|\ell\|\le L_x\|q\|$.

Assume that $q\in N(f_0,f(x_0,U))$. Proposition 2.1 implies that $p\in N(u_0,U)$. According to [11, Proposition 3.1] and by (4) we obtain

$$\|u_0-r_Up_1-u\|\le r_U\quad\forall u\in U,\quad\text{and}\quad \|x_0+r_S\ell-x\|\ge r_S\quad\forall x\in S.$$

This can equivalently be rewritten as

$$2r_U\langle p,u-u_0\rangle\le-\|p\|\cdot\|u-u_0\|^2\qquad\forall u\in U, \tag{5}$$

$$2r_S\langle \ell,x-x_0\rangle\le\|x-x_0\|^2\qquad\forall x\in S. \tag{6}$$

Using [18, Lemma 2.7] we get for any $x\in S$, $u\in U$ and $f=f(x,u)$

$$\begin{aligned}\langle q_1,f-f_0\rangle&\le\frac{1}{\|q\|}\langle q,\mathcal{D}_u f(x_0,u_0)[u-u_0]+\mathcal{D}_x f(x_0,u_0)[x-x_0]\rangle\\&\qquad\qquad+\frac{\gamma}{2}(\|x-x_0\|^2+\|u-u_0\|^2)\\&=\frac{\langle p,u-u_0\rangle}{\|q\|}+\frac{\langle \ell,x-x_0\rangle}{\|q\|}+\frac{\gamma}{2}(\|x-x_0\|^2+\|u-u_0\|^2).\end{aligned}$$

Hence by (5), (6) we obtain

$$\begin{aligned}\langle q_1,f-f_0\rangle&\le-\Big(\frac{\sigma_u}{2r_U}-\frac{\gamma}{2}\Big)\|u-u_0\|^2+\Big(\frac{L_x}{2r_S}+\frac{\gamma}{2}\Big)\|x-x_0\|^2\\&=-\frac{L_u^2}{2\varrho_u}\|u-u_0\|^2+\frac{\sigma_x^2}{2\varrho_x}\|x-x_0\|^2.\end{aligned}$$

According to the definitions of L_u and σ_x we have

$$\begin{aligned}\sigma_x\|x-x_0\|&\le\|f(x,u_0)-f(x_0,u_0)\|\le\|f-f_0\|+\|f(x,u)-f(x,u_0)\|\\&\le\|f-f_0\|+L_u\|u-u_0\|.\end{aligned}$$

Consequently,

$$\begin{aligned}\langle q_1, f-f_0\rangle &\le -\frac{L_u^2}{2\varrho_u}\|u-u_0\|^2+\frac{\sigma_x^2}{2\varrho_x}\left(\frac{\|f-f_0\|+L_u\|u-u_0\|}{\sigma_x}\right)^2\\ &= -\frac{L_u^2(\varrho_x-\varrho_u)}{2\varrho_u\varrho_x}\|u-u_0\|^2+\frac{L_u}{\varrho_x}\|u-u_0\|\cdot\|f-f_0\|+\frac{\|f-f_0\|^2}{2\varrho_x}\\ &= -\frac{L_u^2(\varrho_x-\varrho_u)}{2\varrho_u\varrho_x}\left(\|u-u_0\|-\frac{\varrho_u}{L_u(\varrho_x-\varrho_u)}\|f-f_0\|\right)^2\\ &\qquad +\left(\frac{1}{2\varrho_x}+\frac{\varrho_u}{2\varrho_x(\varrho_x-\varrho_u)}\right)\|f-f_0\|^2 \le \frac{\|f-f_0\|^2}{2(\varrho_x-\varrho_u)}.\end{aligned}$$

So,
$$\langle q_1, f-f_0\rangle \le \frac{\|f-f_0\|^2}{2\varrho} \qquad \forall f\in F. \tag{7}$$

with $\varrho=\varrho_x-\varrho_u>0$. Since $f_0\in \operatorname{int} F$ it follows that $f_0+tq_1\in F$ for sufficiently small $t>0$. This contradicts (7). Consequently, $q\notin N(f_0, f(x_0,U))$.

Proposition 2.1 implies that $p\notin N(u_0,U)$. Since U is convex there exists $u_1\in U$ such that
$$\langle p, u_1-u_0\rangle > 0. \tag{8}$$

For any $t\in(0,1)$ denote $u_t=u_0+t(u_1-u_0)$. In view of convexity of U we have $u_t\in U$ for all $t\in(0,1)$. Using Lipschitz continuity we have
$$\|f(x_0,u_t)-f(x_0,u_0)\|\le L_u\|u_t-u_0\| = L_u\|u_1-u_0\|t.$$

Denoting $\delta_t=\frac{L_u\|u_1-u_0\|}{\sigma_x}t$ we get from the definition of σ_x and by [11, Prop. 2.1]
$$f_0\in B_{\sigma_x\delta_t}(f(x_0,u_t))\subset f(B_{\delta_t}(x_0),u_t).$$

Hence there exists $x_t\in B_{\delta_t}(x_0)$ such that
$$f_0=f(x_t,u_t). \tag{9}$$

Since $\|u_t-u_0\|=O(t)$ and $\|x_t-x_0\|=O(t)$ as $t\to+0$ it follows that
$$f_0=f(x_t,u_t)=f_0+\mathcal{D}_xf(x_0,u_0)[x_t-x_0]+\mathcal{D}_uf(x_0,u_0)[u_t-u_0]+o(t),\ t\to+0.$$

Consequently, $\langle q, \mathcal{D}_xf(x_0,u_0)[x_t-x_0]+\mathcal{D}_uf(x_0,u_0)[u_t-u_0]\rangle=o(t)$ for $t\to+0$,

and hence
$$\langle \ell, x_t-x_0\rangle+\langle p, u_t-u_0\rangle = o(t), \quad t\to+0. \tag{10}$$

Denote $\varepsilon=\frac12\langle p,u_1-u_0\rangle$. In view of (8) we have $\varepsilon>0$. According to (10) there exists $t_0>0$ such that for all $t\in(0,t_0)$
$$\langle \ell, x_t-x_0\rangle < -\langle p,u_t-u_0\rangle+\varepsilon t = -2\varepsilon t+\varepsilon t=-\varepsilon t.$$

Hence there exists $t>0$ such that $2\widehat{r}_S\langle \ell, x_t-x_0\rangle+\|x_t-x_0\|^2<0$, i.e. $x_t\in \operatorname{int} B_{\widehat{r}_S}(x_0-\widehat{r}_S\ell)$. Using (3) we get $x_t\in\operatorname{int} S$. According to (9) we obtain $f_0=f(x_t,u_t)\in f(x_t,U)\subset f(\operatorname{int} S,U)$. So, (2) and (1) are proved.

Propositions 2.3 and 2.12 imply that $\mathcal{R}^-_{ES}(F)\ge\mathcal{R}^-(F)\ge\varrho_x-\varrho_u$. Lemma 3.3 says that $F=\overline{\operatorname{int} F}$. By Lemma 3.2 we see that $\mathcal{R}^-_{ES}(Z\setminus f(\operatorname{int} S,U))\ge\widehat{\varrho}_x$. In view of (1) it means that $\mathcal{R}^-_{ES}(Z\setminus \operatorname{int} F)\ge\widehat{\varrho}_x$. So, according to Proposition 2.9 we obtain that $F\in WCC(\varrho_x-\varrho_u,\widehat{\varrho}_x;Z)$. □

4. Minkowski operators

Let X be a linear space. The *Minkowski operations (sum and difference)* for $A, C \subset X$ are

$$A + C = \{a + c : \ a \in A,\ c \in C\}, \quad A \overset{*}{-} C = \{x \in X : \ x + C \subset A\}.$$

The *Minkowski operators* of the first and the second kinds for a set-valued mapping $G\colon X \rightrightarrows X$ and a subset $S \subset X$ are defined as (see [4, 12])

$$\mathcal{M}_{1,G}(S) = \bigcup_{x \in S} (x + G(x)), \quad \mathcal{M}_{2,G}(S) = X \setminus \mathcal{M}_{1,G}(X \setminus S).$$

Remark 4.1. If set-valued mapping $G\colon X \rightrightarrows X$ is defined as $G(x) = g(x, U)$ for all $x \in X$, where $g\colon X \times U \to X$, then for any $S \subset X$ we have $\mathcal{M}_{1,G}(S) = f(S, U)$ with $f(x, u) = g(x, u) + x$ for all $x \in X$, $u \in U$.

Remark 4.2. If set-valued mapping $G\colon X \rightrightarrows X$ is constant $G(x) = G_0$ for all $x \in X$, then for any $S \subset X$ the values of the Minkowski operators coincide with the Minkowski sum and difference, respectively:

$$\mathcal{M}_{1,G}(S) = S + G_0, \quad \mathcal{M}_{2,G}(S) = S \overset{*}{-} (-G_0).$$

Theorem 4.3. *Let X and Y be Hilbert spaces. Let $S_0 \in WCC(r_S, \widehat{r}_S; X)$, $U \subset Y$ be strongly convex and let $G(x) = g(x, U)$ for all $x \in X$. Suppose that $g \in C^{1,1}(X \times U, X)$ is such that for all $x \in X$ the mapping $g(x, \cdot)$ is a covering bijection from U to $f(x, U)$. Let*

$$\sup_{u \in U} \operatorname{lip}(g(\cdot, u), X) < 1, \quad \inf_{x \in X} \operatorname{sur}(g(x, \cdot), U) > \mathcal{R}^+(U) \cdot \operatorname{lip}(\mathcal{D}g, X \times U),$$

$$\varrho_u = \frac{\mathcal{R}^+(U) \cdot \sup\limits_{x \in X} \operatorname{lip}^2(g(x, \cdot), U)}{\inf\limits_{x \in X} \operatorname{sur}(g(x, \cdot), U) - \mathcal{R}^+(U) \cdot \operatorname{lip}(\mathcal{D}g, X \times U)},$$

$$\varrho_x = \frac{r_S \cdot \left(1 - \sup\limits_{u \in U} \operatorname{lip}(g(\cdot, u), X)\right)^2}{1 + \sup\limits_{u \in U} \operatorname{lip}(g(\cdot, u), X) + r_S \cdot \operatorname{lip}(\mathcal{D}g, X \times U)},$$

$$\widehat{\varrho}_x = \frac{\widehat{r}_S \cdot \left(1 - \sup\limits_{u \in U} \operatorname{lip}(g(\cdot, u), X)\right)^2}{1 + \sup\limits_{u \in U} \operatorname{lip}(g(\cdot, u), X) + \widehat{r}_S \cdot \operatorname{lip}(\mathcal{D}g, X \times U)}.$$

(i) *If $\varrho_u < \varrho_x$, then*
$\operatorname{int} \mathcal{M}_{1,G}(S_0) = \mathcal{M}_{1,G}(\operatorname{int} S_0)$ *and* $\mathcal{M}_{1,G}(S_0) \in WCC(\varrho_x - \varrho_u, \widehat{\varrho}_x; X)$.

(ii) *If $\varrho_u < \widehat{\varrho}_x$, then*
$\operatorname{int} \mathcal{M}_{2,G}(S_0) = \mathcal{M}_{2,G}(\operatorname{int} S_0)$ *and* $\mathcal{M}_{2,G}(S_0) \in WCC(\varrho_x, \widehat{\varrho}_x - \varrho_u; X)$.

Proof. Observe that the function $f(x,u) = g(x,u) + x$, $x \in X$, $u \in U$ is of class $C^{1,1}(X \times U, X)$, and for all $x \in X$, $u \in U$ we have $\operatorname{lip}(g(x,\cdot),U) = \operatorname{lip}(f(x,\cdot),U)$, $\operatorname{sur}(f(x,\cdot),U) = \operatorname{sur}(g(x,\cdot),U)$, $\operatorname{lip}(f(\cdot,u),X) \le 1 + \operatorname{lip}(g(\cdot,u),X)$, and

$$\operatorname{colip}(f(\cdot,u),X) \ge 1 - \operatorname{lip}(g(\cdot,u),X), \quad \operatorname{lip}(\mathcal{D}f, X \times U) = \operatorname{lip}(\mathcal{D}g, X \times U).$$

Moreover, for any $u \in U$ the mapping $f(\cdot,u)$ is a bijection from X to X.

Applying Theorem 3.4 for $S = S_0$, we obtain assertion (i).

We prove assertion (ii): Assume that $\varrho_u < \widehat{\varrho}_x$. We take $S = X \setminus \operatorname{int} S_0$, apply Theorem 3.4 and get $\operatorname{int}\mathcal{M}_{1,G}(S) = \mathcal{M}_{1,G}(\operatorname{int} S)$ and $\mathcal{M}_{1,G}(S) \in WCC(\widehat{\varrho}_x - \varrho_u, \varrho_x; X)$. Hence, $\mathcal{M}_{1,G}(S) = \overline{\operatorname{int}\mathcal{M}_{1,G}(S)} = \overline{\mathcal{M}_{1,G}(\operatorname{int} S)}$.

Substituting $X \setminus \operatorname{int} S_0$ for S, we get $\mathcal{M}_{1,G}(X \setminus \operatorname{int} S_0) = \overline{\mathcal{M}_{1,G}(X \setminus S_0)}$. Consequently, $\mathcal{M}_{2,G}(\operatorname{int} S_0) = \operatorname{int}\mathcal{M}_{2,G}(S_0)$. Since $\mathcal{M}_{1,G}(S) \in WCC(\widehat{\varrho}_x - \varrho_u, \varrho_x; X)$ it follows by Remark 2.7 that $(X \setminus \operatorname{int}\mathcal{M}_{1,G}(S)) \in WCC(\varrho_x, \widehat{\varrho}_x - \varrho_u; X)$, that is $\mathcal{M}_{2,G}(S_0) \in WCC(\varrho_x, \widehat{\varrho}_x - \varrho_u; X)$. So, assertion (ii) is proved as well. □

Theorem 4.3 improves the results of [12]. In view of Remark 4.2 Theorem 4.3 imply the following result, obtained previously in [9].

Corollary 4.4. *Let $S \in WCC(r_S, \widehat{r}_S; X)$ and $U \subset X$ be strongly convex with radius r_U.*

(i) *If $r_U < r_S$, then*
$S + U \in WCC(r_S - r_U, \widehat{r}_S; X)$ *and* $\operatorname{int}(S + U) = (\operatorname{int} S) + U$.

(ii) *If $r_U < \widehat{r}_S$, then*
$S \overset{*}{-} U \in WCC(r_S, \widehat{r}_S - r_U; X)$ *and* $\operatorname{int}(S \overset{*}{-} U) = (\operatorname{int} S) \overset{*}{-} U$.

5. Application in nonlinear differential games

Consider the following nonlinear differential game

$$\frac{dx}{dt} = a(t, x(t), u(t)) + b(t, x(t), v(t)) \tag{11}$$

on a fixed time segment $t \in [0,T]$ with initial position $x(0) = x_0$ and the target set $M \subset \mathbb{R}^n$, where $x(t) \in \mathbb{R}^n$ is the phase vector of the system, $u(t) \in \mathbb{R}^m$ is the control of the pursuer, and $v(t) \in \mathbb{R}^k$ is the control of the evader. The controls of the players obey the geometric constraints

$$u(t) \in U, \quad v(t) \in V \quad \forall t \in [0,T], \tag{12}$$

where $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^k$ are given sets, $a\colon [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $b\colon [0,T] \times \mathbb{R}^n \times V \to \mathbb{R}^n$ are given functions. The goal of the pursuer is to bring the phase vector into the target set at a final time instant: $x(T) \in M$. The goal of the evader is the opposite: $x(T) \notin M$. If the condition $x(T) \in M$ is satisfied in some realization of the game, then we say that a capture has occurred. Otherwise, we say that an evasion has occurred.

We assume that $M \in WCC(R_M, \widehat{R}_M; \mathbb{R}^n)$ with some positive R_M and $\widehat{R}_M$; U is r_U-strongly convex and V is r_V-strongly convex; functions $a\colon [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}^n$ and $b\colon [0,T]\times\mathbb{R}^n\times V \to \mathbb{R}^n$ are bounded and satisfy the Lipschitz conditions

$$\|a(t_1,x_1,u_1) - a(t_2,x_2,u_2)\| \le L_a^t|t_1-t_2| + L_a^x\|x_1-x_2\| + L_a^u\|u_1-u_2\|,$$

$$\|b(t_1,x_1,u_1) - b(t_2,x_2,u_2)\| \le L_b^t|t_1-t_2| + L_b^x\|x_1-x_2\| + L_b^v\|v_1-v_2\|$$

for all $t_1,t_2 \in [0,T]$, $x_1,x_2 \in \mathbb{R}^n$, $u_1,u_2 \in U$ and $v_1,v_2 \in V$. Suppose also

$$a(t,\cdot,\cdot) \in C^{1,1}(\mathbb{R}^n \times U, \mathbb{R}^n), \quad b(t,\cdot,\cdot) \in C^{1,1}(\mathbb{R}^n \times V, \mathbb{R}^n) \quad \forall t \in [0,T].$$

Denote

$$\gamma_a = \sup_{t\in[0,T]} \operatorname{lip}(\mathcal{D}a(t,\cdot,\cdot), \mathbb{R}^n\times U), \quad \gamma_b = \sup_{t\in[0,T]} \operatorname{lip}(\mathcal{D}b(t,\cdot,\cdot), \mathbb{R}^n\times V),$$

$$\sigma_a = \inf_{t\in[0,T]} \inf_{x\in\mathbb{R}^n} \operatorname{sur}(a(t,x,\cdot),U), \qquad \sigma_b = \inf_{t\in[0,T]} \inf_{x\in\mathbb{R}^n} \operatorname{sur}(b(t,x,\cdot),V).$$

We assume further that $\sigma_a > r_U\gamma_a$, and $\sigma_b > r_V\gamma_b$ and define

$$\alpha_u = \frac{r_U(L_a^u)^2}{\sigma_a - r_U\gamma_a}, \qquad \alpha_v = \frac{r_V(L_b^v)^2}{\sigma_b - r_V\gamma_b}\,.$$

Consider a uniform partition $\{\tau_i\}_{i=0}^I$ of $[0,T]$, where $\tau_i = \tau\cdot i$, $i\in\{0,...,I\}$, $\tau = \frac{T}{I}$.

Define $$C_a = \sup_{t\in[0,T],\ x\in\mathbb{R}^n,\ u\in U} \|a(t,x,u\|, \quad C_b = \sup_{t\in[0,T],\ x\in\mathbb{R}^n,\ v\in V} \|b(t,x,v\|,$$

$$C = C_a + C_b, \quad L = L_a^x + L_b^x, \quad \gamma = \gamma_a + \gamma_b,$$

$$\Delta_u = \left(23L_a^xC_a(1+L_b^x\tau) + \frac{1}{2}(LC + L_a^t + L_b^t) + L_b^xC\right)\tau^2, \tag{13}$$

$$\Delta_v = \left(23L_b^xC_b(1+L_a^x\tau) + \frac{1}{2}(LC + L_a^t + L_b^t) + L_a^xC\right)\tau^2.$$

Pontryagin [16, 17] proposed an algorithm to construct optimal control strategies for linear differential games based on alternating integral. As shown in [4] the suboptimal strategies of the players can be constructed based on the game's reachable sets M_i^u, M_i^v determined by the following backward procedures:

$$M_I^u = M + B_\varepsilon(0),\ M_i^u = \mathcal{M}_{1,G_i^a}\mathcal{M}_{2,G_i^b}(M_{i+1}^u \stackrel{*}{-} B_{\Delta_u}(0)),\ i\in\{I-1,...,0\}, \tag{14}$$

$$M_I^v = M, \quad M_i^v = \mathcal{M}_{2,G_i^b}\mathcal{M}_{1,G_i^a}(M_{i+1}^v + B_{\Delta_v}(0)), \quad i\in\{I-1,...,0\}, \tag{15}$$

where $$G_i^a(x) = -\tau a(\tau_i,x,U), \quad G_i^b(x) = -\tau b(\tau_i,x,V), \quad x\in\mathbb{R}^n. \tag{16}$$

In [4] the backward procedures are written considering the errors caused by the approximation of sets. For the sake of simplicity, we do not take these errors into account. The positive real number ε should be chosen such that $M_0^v \subset M_0^u$ (see [4]).

Lemma 5.1. *$\forall\,\varrho_1 > 0$, $\forall\,\varrho_2 > \varrho_1$ $\exists\,\beta > 0$ $\exists\,I_0 \in \mathbb{N}$ such that $\forall\,I \in \mathbb{N}$ with $I \geq I_0$, $\forall\,i \in \{0, ..., I-1\}$, $\forall\,R, \widehat{R} \in [\varrho_1, \varrho_2]$:*

$$\begin{aligned} &\textit{if} \quad M^u_{i+1} \in WCC(R, \widehat{R}; \mathbb{R}^n) \\ &\textit{then} \quad M^u_i \in WCC(R - \alpha(R)\tau - \beta\tau^2, \widehat{R} - \widehat{\alpha}(\widehat{R})\tau - \beta\tau^2; \mathbb{R}^n), \end{aligned} \tag{17}$$

where

$$\alpha(R) = \alpha_u + 3LR + \gamma R^2, \quad \widehat{\alpha}(\widehat{R}) = \alpha_v + 3L\widehat{R} + \gamma\widehat{R}^2. \tag{18}$$

Proof. In view of (13) we have $\Delta_u < \varrho_1$ for sufficiently large I (i.e. for sufficiently small τ). Hence $\widehat{R} - \Delta_u \geq \varrho_1 - \Delta_u > 0$. By Corollary 4.4(ii) we have $M^u_{i+1} \overset{*}{-} B_{\Delta_u}(0) \in WCC(R, \widehat{R} - \Delta_u; \mathbb{R}^n)$. According to Theorem 4.3(ii) we get $\mathcal{M}_{2,G^b_i}(M^u_{i+1} \overset{*}{-} B_{\Delta_u}(0)) \in WCC(R_1, \widehat{R}_1; \mathbb{R}^n)$ with

$$R_1 = \frac{R(1 - L^x_b\tau)^2}{1 + L^x_b\tau + R\gamma_b\tau}, \qquad \widehat{R}_1 = \frac{(\widehat{R} - \Delta_u)(1 - L^x_b\tau)^2}{1 + L^x_b\tau + (\widehat{R} - \Delta_u)\gamma_b\tau} - \alpha_v\tau,$$

where $\tau > 0$ should be sufficiently small such that $\widehat{R}_1 > 0$. In view of Theorem 4.3(i) and relation (14) we obtain $M^u_i \in WCC(R_2, \widehat{R}_2; \mathbb{R}^n)$ with

$$R_2 = \frac{R_1(1 - L^x_a\tau)^2}{1 + L^x_a\tau + R_1\gamma_a\tau} - \alpha_u\tau, \qquad \widehat{R}_2 = \frac{\widehat{R}_1(1 - L^x_a\tau)^2}{1 + L^x_a\tau + \widehat{R}_1\gamma_a\tau},$$

where $\tau > 0$ should be sufficiently small such that $R_2 > 0$. By Taylor's theorem there exists $\beta > 0$, which depends only on ϱ_1 and ϱ_2 and does not depend on τ, R and $\widehat{R}$ such that $R_2 \geq R - \alpha(R)\tau - \beta\tau^2 > 0$ and $\widehat{R}_2 \geq \widehat{R} - \widehat{\alpha}(\widehat{R})\tau - \beta\tau^2 > 0$ for sufficiently small $\tau > 0$. Therefore

$$M^u_i \in WCC(R_2, \widehat{R}_2; \mathbb{R}^n) \subset WCC(R - \alpha(R)\tau - \beta\tau^2, \widehat{R} - \widehat{\alpha}(\widehat{R})\tau - \beta\tau^2; \mathbb{R}^n). \qquad \square$$

Theorem 5.2. *Let the length of the time segment is sufficiently small, namely $T < \min\{T_0, \widehat{T}_0\}$, where*

$$T_0 = \int_0^{R_M} \frac{dR}{\alpha_u + 3LR + \gamma R^2}, \qquad \widehat{T}_0 = \int_0^{\widehat{R}_M} \frac{dR}{\alpha_v + 3LR + \gamma R^2}.$$

Then for sufficiently small mesh of the time segment partition $\{\tau_i\}_{i=0}^I$ and sufficiently small $\varepsilon > 0$ the game reachable sets M^u_i and M^v_i are bodily smooth for all $i \in \overline{0, I}$.

Proof. Let the functions $\alpha(\cdot)$ and $\widehat{\alpha}(\cdot)$ be defined as in (18). Since

$$T < \min\left\{\int_0^{R_M} \frac{dR}{\alpha(R)}, \int_0^{\widehat{R}_M} \frac{dR}{\widehat{\alpha}(R)}\right\}$$

there exists $\varrho_1 \in \left(0, \frac{1}{2}\min\{R_M, \widehat{R}_M\}\right)$ such that

$$T < \min\left\{\int_{2\delta_1}^{R_M} \frac{dR}{\alpha(R)}, \int_{2\delta_1}^{\widehat{R}_M} \frac{dR}{\widehat{\alpha}(R)}\right\}.$$

Let $r\colon [0,T] \to \mathbb{R}$ and $\widehat{r}\colon [0,T] \to \mathbb{R}$ be the solutions of the differential equations

$$\frac{dr}{dt} = \alpha(r(t)), \quad \frac{d\widehat{r}}{dt} = \widehat{\alpha}(\widehat{r}(t)), \quad t \in [0,T] \tag{19}$$

with the initial conditions $r(T) = R_M$, $\widehat{r}(T) = \widehat{R}_M$. Solving the differential equation, we have $\frac{dr}{\alpha(r(t))} = dt$. In consequence,

$$\int_{r(t)}^{R_M} \frac{dr}{\alpha(r)} = T - t \le T < \int_{2\delta_1}^{R_M} \frac{dR}{\alpha(R)}.$$

Hence, $r(t) \in [2\varrho_1, R_M]$ for all $t \in [0,T]$. Similarly, $\widehat{r}(t) \in [2\varrho_1, R_M]$ for all $t \in [0,T]$. According to Lemma 5.1 there exists $\beta > 0$ and $I_0 \in \mathbb{N}$ such that for all $I \ge I_0$, $i \in \{0, ..., I-1\}$ and $R, \widehat{R} \in [\varrho_1, R_M + 1]$ we have (17). We define real numbers r_i and $\widehat{r}_i$ for $i \in \{I, ..., 0\}$ by the following backward procedure:

$$r_I = R_M - \varepsilon, \qquad r_i = r_{i+1} - \alpha(r_{i+1})\tau - \beta\tau^2, \quad i \in \{I-1, ..., 0\},$$

$$\widehat{r}_I = \widehat{R}_M, \qquad \widehat{r}_i = \widehat{r}_{i+1} - \widehat{\alpha}(\widehat{r}_{i+1})\tau - \beta\tau^2, \quad i \in \{I-1, ..., 0\}.$$

It follows from (19) that

$$\max_{i \in \overline{0,I}} |r_i - r(\tau_i)| \to 0, \quad \max_{i \in \overline{0,I}} |\widehat{r}_i - \widehat{r}(\tau_i)| \to 0 \quad \text{as} \quad \tau \to +0, \ \varepsilon \to +0.$$

Consequently, for sufficiently small τ and ε one has $r_i, \widehat{r}_i \in [\varrho_1, R_M + 1]$ for all $i \in \{0, ..., I\}$. According to Lemma 5.1 for all $i \in \{0, ..., I-1\}$

$$M^u_{i+1} \in WCC(r_{i+1}, \widehat{r}_{i+1}, \mathbb{R}^n) \quad \Rightarrow \quad M^u_i \in WCC(r_i, \widehat{r}_i, \mathbb{R}^n).$$

In view of Corollary 4.4(i) we get $M^u_I \in WCC(R_M - \varepsilon, \widehat{R}_M, \mathbb{R}^n) = WCC(r_I, \widehat{r}_I, \mathbb{R}^n)$. By induction we obtain $M^u_i \in WCC(r_i, \widehat{r}_i, \mathbb{R}^n)$ for all $i \in \{0, ..., I\}$. Proposition 2.10 implies that the reachable sets of the game M^u_i (and similarly M^v_i) are bodily smooth for all $i \in \{0, ..., I\}$. □

Smoothness of the game reachable sets stated in Theorem 5.2 allows the effective numerical approximation algorithms to be used.

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