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On Moving Chords in Constant Curvature 2-Manifolds*

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An introduction to non-Euclidean geometry is given and a new approach to prove a generalization of Holditch's theorem in 2-dimensional constant curvature manifolds is presented. Moreover, the same procedure also makes possible to obtain generalized versions of Barbier's theorem for constant width curves and of Steiner's formulae for parallel curves. The main fact is that a general formula relating the geodesic curvatures of the involved curves is found, in such a way these results can be derived from there. Regularity of some curves generated geodesically from another is also studied.

Keywords: Holditch's theorem, Steiner's formula for parallel curves, Barbier's theorem, total geodesic curvature.

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1. Introduction

Holditch's theorem, [12], is an old geometrical result on planar areas generated by moving chords. Suppose that a constant length chord can move along with its extremities on a convex planar closed curve α a full revolution. If a point is chosen on such a chord at a distance p from one end and at a distance q from the other, then another curve will be traced out by that point. That curve is called the Holditch curve of α for the chord length $p+q$. Holditch's theorem states that the difference between the areas of both curves is equal to $\pi\, p\, q$ (see Figure 1.1). This area is called the Holditch area. At first, the most interesting point may be that the Holditch area only depends on the lengths in which the chosen point divides the chord and not on the shape or size of the initial curve.

Some extensions of this theorem in the plane have been given before. See, for instance, [3], [4], [20] and [24]. Moreover, there is a natural extension to space curves in $\mathbb{R}^3$ given recently by the authors in [18].

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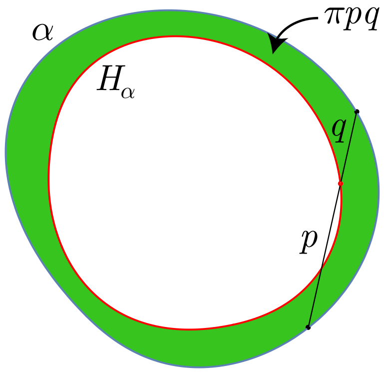

Figure 1.1: Representation of Holditch's theorem. The curve H_α is the Holditch curve of α for a chord length $p+q$.

There also exists a generalization of the theorem to constant curvature surfaces, replacing the straight chord of the plane by a geodesic chord in the surface and computing surface areas. Nevertheless, there the Holditch area depends on the area of the initial curve besides the curvature of the surface and the chosen lengths in the chord. The first extension of Holditch's theorem in that direction was done by Elliott in [8], but only for the sphere where it was presented together with another theorems of spherical kinematics. The first general setting for any constant curvature surface was given originally by Vidal Abascal in [28] and revisited by the same author and Rodeja in [31]. Later on, Santaló in [27] gave a different proof of the same theorem and extended it to n dimensions to continue the approach of Kurita, who in [14] gave the result in the Euclidean space of n dimensions for volumes.

The proof of Vidal Abascal comes from the theory of surfaces. In Santaló's own words: "Vidal Abascal and Rodeja use the methods from the differential geometry of constant curvature surfaces. Although the geometry on these surfaces is equivalent to the non-Euclidean geometry on the plane, we think that it could be of some interest to obtain the same result using the computation methods of this kind of geometry". Thus, in Santaló's paper a brief introduction to non-Euclidean geometry is given and methods involving differential forms are used.

Nevertheless, these three papers (among others) by Vidal Abascal and Santaló which presented Holditch's theorem and related topics in constant curvature surfaces were written in Spanish and some of them didn't have so much diffusion. In fact, a few decades later, some other authors studied spherical kinematics on the light of [19] with works as [11] or [13], where different Holditch-type theorems were found, but no citation to the Spanish papers was done.

Vidal Abascal in his paper computed the Holditch area as a swept out area by the moving chord, which has some disadvantages if the chosen p distance from one end is too long because overlapping area problems have to be managed. The fact of finding at the end a theorem on a difference of areas is a consequence of some assumptions on convexity.

Our aim in this paper is to revisit these results in modern language to make them more accessible and to give a new proof of the theorem without using any background on differential forms as Santaló. All needed prerequisites are presented in Section 2. Actually, instead of Holditch curves, a more general kind of curves are considered under the name of *generated curves* (Section 3). The study of regularity of these generated curves is done in Section 4. By this approach, an extension of Holditch's theorem in 2-dimensional constant curvature manifolds when two initial curves are considered instead of only one follows directly. In addition, the constant curvature 2-manifold version of Barbier's theorem for constant width curves and of Steiner's formulae for parallel curves are also deduced with the same procedure. The key point is Lemma 5.1 where we relate the geodesic curvatures of a generated curve and the initial one. These fundamental results are presented in Section 5.

2. Introduction to non-Euclidean geometry

Here we give a brief introduction to non-Euclidean geometry. The interested reader can see [27], [5] or [22]. It is also worth to take a look at the nice introduction in [23] to the hyperboloid model of hyperbolic geometry.

We will introduce the setting as a quadratic space and we will see that it can also be seen as a 2-dimensional Riemannian manifold of constant curvature. All needed prerequisites together with the expression of the naturally defined frame in such a manifold will be presented.

Thus, we will work with the non-Euclidean metric space defined by the quadratic space $\left(\mathbb{R}^3, \mathcal{Q}_K\right)$, where $\mathcal{Q}_K$ is the quadratic form

$$\mathcal{Q}_K(x_0, x_1, x_2) = \operatorname{sgn}(K)\, x_0^2 + x_1^2 + x_2^2,$$

with K being a non-zero real constant (namely, the space curvature).

The polarization of the quadratic form $\mathcal{Q}_K$ is the bilinear map defined by

$$\mathcal{B}(\mathbf{u}, \mathbf{v}) = \frac{\mathcal{Q}_K(\mathbf{u}+\mathbf{v}) - \mathcal{Q}_K(\mathbf{u}) - \mathcal{Q}_K(\mathbf{v})}{2}$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, which gives the *inner product* to consider in our space. The norm associated with $\mathcal{B}$ is the function $\|\cdot\|\colon \mathbb{R}^3 \to \mathbb{C}$ defined by $\|\mathbf{u}\| = \sqrt{\mathcal{B}(\mathbf{u}, \mathbf{u})}$. Notice that although it is standard to use the term *inner product* for $\mathcal{B}$, it is actually not an inner product in the usual sense because it is not positive-definite if $K < 0$.

Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, we define *generalized cross product of* $\mathbf{u}$ *and* $\mathbf{v}$ as

$$\mathbf{u} \mathbin{\hat{\wedge}} \mathbf{v} := \begin{pmatrix} \operatorname{sgn}(K) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\mathbf{u} \wedge \mathbf{v}),$$

where $\wedge$ denotes the usual cross product of $\mathbb{R}^3$. Note, in fact, that when $K > 0$, $\hat{\wedge}$ is the usual cross product of $\mathbb{R}^3$ and when $K < 0$, $\hat{\wedge}$ is the so-called Lorentzian cross product of $\mathbb{R}^{1,2}$. The cross product $\hat{\wedge}$ satisfies similar properties as the usual one. First, it is such that $\mathcal{B}(\mathbf{u}, \mathbf{u} \mathbin{\hat{\wedge}} \mathbf{v}) = \mathcal{B}(\mathbf{v}, \mathbf{u} \mathbin{\hat{\wedge}} \mathbf{v}) = 0$. More useful properties are stated in the following proposition (see [22] for more details).

Proposition 2.1. *Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and let $\alpha\colon I \to \mathbb{R}^3$ and $\beta\colon I \to \mathbb{R}^3$ be two functions defined on a real open interval I. Then*

(1) $\big(\alpha(t) \mathbin{\hat{\wedge}} \beta(t)\big)' = \alpha'(t) \mathbin{\hat{\wedge}} \beta(t) + \alpha(t) \mathbin{\hat{\wedge}} \beta'(t)$.

(2) $\mathcal{B}(\mathbf{u},\, \mathbf{v} \mathbin{\hat{\wedge}} \mathbf{w}) = \det(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

(3) $\mathbf{u} \mathbin{\hat{\wedge}} (\mathbf{v} \mathbin{\hat{\wedge}} \mathbf{w}) = \operatorname{sgn}(K)\big(\mathbf{v}\,\mathcal{B}(\mathbf{u}, \mathbf{w}) - \mathbf{w}\,\mathcal{B}(\mathbf{u}, \mathbf{v})\big)$.

Henceforth we will denote by $\wedge$ the generalized cross product $\hat{\wedge}$. This abuse of notation is justified by the fact that both operators have similar properties, taking care only on small differences involving the sign of K.

We will understand by an *orthonormal basis* a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^3$ such that $\mathcal{B}(\mathbf{u}_1, \mathbf{u}_1) = \operatorname{sgn}(K)$ and $\mathcal{B}(\mathbf{u}_i, \mathbf{u}_j) = \delta_{ij}$ otherwise. Note that if $K > 0$ it is the common definition of orthonormal basis of $\mathbb{R}^3$ and when $K < 0$ it corresponds to the definition of a Lorentz orthonormal basis of $\mathbb{R}^{1,2}$.

Now, given $K \neq 0$, a differentiable manifold can be defined as follows:

$$M^K = \Big\{(x_0, x_1, x_2) \in \mathbb{R}^3 : \mathcal{Q}_K(x_0, x_1, x_2) = \frac{1}{K},\ x_0 \in D_K\Big\},$$

where $D_K =]0, +\infty[$ if $K < 0$ or $D_K = \mathbb{R}$ otherwise.

Since M^K is a subset of $\mathbb{R}^3$, the tangent space to M^K at a point $p \in M^K$, T_pM^K, is a 2-dimensional vector subspace of $\mathbb{R}^3$. Indeed, it is easy to check that

$$T_pM^K = \big\{\mathbf{v} \in \mathbb{R}^3 : \mathcal{B}(p, \mathbf{v}) = 0\big\}.$$

Given $p \in M^K$, a Riemannian metric at p for M^K, $g_p\colon T_pM^K \times T_pM^K \to \mathbb{R}$, can be defined in terms of the bilinear function:

$$g_p(\mathbf{v}, \mathbf{w}) := g(\mathbf{v}, \mathbf{w})(p) = \mathcal{B}(\mathbf{v}, \mathbf{w}).$$

Notice that now the restriction of $\mathcal{B}$ to the tangent space is a positive-definite operator. Therefore, $\big(M^K, g\big)$ is a Riemannian manifold and it can be shown that K is, indeed, the Gauss curvature of M^K. We can outfit a metric space structure to M^K: given two points $p, q \in M^K$, the distance between them is defined by

$$\mathrm{d}(p, q) := \frac{1}{\sqrt{K}} \arccos\big(K\,\mathcal{B}(p, q)\big).$$

If $p \in M^K$ and $\mathbf{v} \in T_pM^K$ with $\|\mathbf{v}\| = 1$, the geodesic that goes through p in the direction $\mathbf{v}$ can be parameterized by arc length as

$$G(u) = \cos\big(\sqrt{K}\,u\big)\,p + \frac{\sin\big(\sqrt{K}\,u\big)}{\sqrt{K}}\,\mathbf{v}.$$

We define the *chord between p and q* on M^K as the shortest piece of the geodesic curve that passes through p and q and has these points as endpoints.

The (signed) geodesic curvature of an arc-length parameterized regular curve $\alpha\colon I \to M^K$, where I is some interval, can be defined as the real-valued function

$$\kappa_g^\alpha = \sqrt{|K|}\,\mathcal{B}\big(\alpha'',\, \alpha \wedge \alpha'\big).$$

Observe that we are working intrinsically with the frame

$$\mathcal{F}(s) = \big\{\sqrt{|K|}\,\alpha(s), \alpha'(s), \sqrt{|K|}\,(\alpha \wedge \alpha')(s)\big\},$$

which is orthonormal in $\mathbb{R}^3$, with

$$\big\{\alpha'(s), \sqrt{|K|}\,(\alpha \wedge \alpha')(s)\big\}$$

being a basis of $T_{\alpha(s)}M^K$. In general, if a curve γ is not parameterized by arc-length, we can use the expression

$$\kappa_g^\gamma(t) = \sqrt{|K|}\,\frac{\det\big(\gamma(t), \gamma'(t), \gamma''(t)\big)}{\|\gamma'(t)\|^3} \tag{1}$$

to compute the (signed) geodesic curvature of γ.

It is straightforward to show that the variation of $\mathcal{F}$ is given by

$$\begin{pmatrix} \sqrt{|K|}\,\alpha(s) \\ \alpha'(s) \\ \sqrt{|K|}\,(\alpha \wedge \alpha')(s) \end{pmatrix}' = \begin{pmatrix} 0 & \sqrt{|K|} & 0 \\ -\operatorname{sgn}(K)\sqrt{|K|} & 0 & \kappa_g^\alpha(s) \\ 0 & -\kappa_g^\alpha(s) & 0 \end{pmatrix} \begin{pmatrix} \sqrt{|K|}\,\alpha(s) \\ \alpha'(s) \\ \sqrt{|K|}\,(\alpha \wedge \alpha')(s) \end{pmatrix}.$$

Finally, given a compact region R of M^K (defined by the boundary of some simple closed curve ϕ in M^K) contained in some coordinate neighborhood $\mathbf{x}(U)$ consistent with the orientation of M^K, the area of R is defined, as it is usual in Riemannian manifolds (see [7]), by

$$\mathcal{A}(\phi) = \iint_{\mathbf{x}^{-1}(R)} \sqrt{(g_{11}\,g_{22} - g_{12}^2)(u,v)}\;\mathrm{d}u\;\mathrm{d}v,$$

where g_{ij} is the local representation of the Riemannian metric g in the coordinate system $\mathbf{x}$. Also, by using Proposition 2.1, it is easy to show that

$$\big\|(\mathbf{x}_u \wedge \mathbf{x}_v)(u,v)\big\|^2 = \operatorname{sgn}(K)\,(g_{11}\,g_{22} - g_{12}^2)(u,v).$$

Throughout the following sections we will work with the 2-dimensional Riemannian manifold $\big(M^K, g\big)$ of non-zero constant Gauss curvature K introduced above.

3. Generation of a new curve

Let $I = [0, L]$ and let $\alpha\colon I \to M^K$ be an arc-length parameterized curve (of length L) positively oriented in M^K. Assume α to be regular and simple. Let $\beta\colon I \to \cup_{p\in M} T_pM^K$ be the function such that for each $s \in I$, $\beta(s) \in T_{\alpha(s)}M^K$ is the unit direction of a geodesic chord from $\alpha(s)$ in M^K.

Given $s \in I$, the arc-length parameterized geodesic that goes from $\alpha(s)$ in the direction of $\beta(s)$ will be denoted by

$$\lambda_s(u) = \cos\big(\sqrt{K}\,u\big)\,\alpha(s) + \frac{\sin\big(\sqrt{K}\,u\big)}{\sqrt{K}}\,\beta(s).$$

We can write the function β in terms of the frame $\mathcal{F}$ as

$$\beta(s) = \cos\nu(s)\,\alpha'(s) + \sin\nu(s)\,\sqrt{|K|}\,(\alpha\wedge\alpha')(s),$$

where $\nu\colon I \to \mathbb{R}$ is the oriented angle function (according to the orientation of M^K) from $\alpha'(s)$ to $\beta(s)$.

We summarize in the following hypothesis the conditions on α and β we have assumed.

Hypothesis $\mathcal{S}(\alpha)$. The initial curve $\alpha\colon I \to M^K$ is arc-length parameterized, positively oriented in M^K, differentiable, simple and regular, and for each point $\alpha(s)$ there is only one direction $\beta(s) \in T_{\alpha(s)}M^K$. Suppose that α has a finite length L, so $I = [0, L]$.

Definition 3.1. Suppose $\mathcal{S}(\alpha)$ holds. Given a continuous function $p\colon I \to [0, +\infty[$ (called the *length function*), we generate from α the following curve:

$$\gamma(s) := \lambda_s\big(p(s)\big) = \cos\big(\sqrt{K}\,p(s)\big)\,\alpha(s) + \frac{\sin\big(\sqrt{K}\,p(s)\big)}{\sqrt{K}}\,\beta(s).$$

That is to say, for each $s \in I$, $\gamma(s)$ is generated by going from $\alpha(s)$ the length $p(s)$ geodesically in the direction forming an angle $\nu(s)$ with $\alpha'(s)$. We will refer to γ as the *generated curve*.

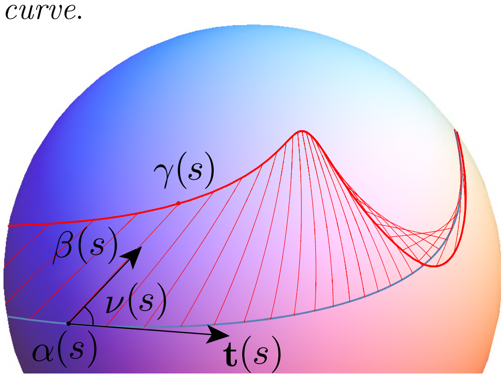

Figure 3.1: The generated curve $\gamma(s)$ is obtained from $\alpha(s)$ by going a distance $p(s)$ in the geodesic determined by the direction $\beta(s)$. This direction forms an angle $\nu(s)$ with the tangent vector $\mathbf{t}(s)$ of α at $\alpha(s)$.

See in Figure 3.1 an example of a generated curve. Particular choices of $p(s)$ and $\nu(s)$ will lead to different kinds of generated curves from α such as parallel curves or Holditch curves.

The objective is to find some relations between areas and lengths involving the initial curve α and the generated one γ. Lemma 5.1 is the key point of our procedure to compute areas in the constant curvature manifold. It relates the geodesic curvatures of α and γ. Before that, we will study the regularity of γ.

4. Regularity of generated curves

In this section, regularity of generated curves will be studied. First, a brief remark on Jacobi fields may be convenient to simplify the notation (but also to understand better the movement of the chord). An analysis from this point of view in a particular case can be found in [16], where the moving chord represents a bicycle and the generated curve its front wheel.

Remark 4.1. The movement of a geodesic moving chord is a variation of geodesics, so it induces naturally for each point $s = s_0$ a Jacobi field:

$$\mathcal{J}_{s_0}(u) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=s_0} \lambda_s(u).$$

This Jacobi field can be decomposed into a tangential and a normal component and it can be computed explicitly in a 2-dimensional constant curvature manifold. In particular, the normal Jacobi field along the geodesic λ_s can be written as $B_s(u) = b_s(u)\,\mathbf{v}(s)$, where

$$b_s(u) := \cos\big(\sqrt{K}\,u\big)\,\sin\nu(s) - \frac{\sin\big(\sqrt{K}\,u\big)}{\sqrt{K}}\big(\nu'(s) + \kappa_g^\alpha(s)\big)$$

and
$$\mathbf{v}(s) = \sin\nu(s)\,\alpha'(s) - \cos\nu(s)\,\sqrt{|K|}\,(\alpha\wedge\alpha')(s).$$

If $b_s(u) \geq 0$, then $b_s(u) = \big\|B_s(u)\big\|$. But in general, the sign of $b_s(u)$ may change from $u = 0$ to $u = p(s)$.

It is easy to show with a straightforward computation that

$$\big\|\gamma'(s)\big\|^2 = \big(p'(s) + \cos\nu(s)\big)^2 + \Big(b_s\big(p(s)\big)\Big)^2. \tag{2}$$

Let $\mathbf{t}_g(s)$ be the tangent of the geodesic λ_s at $\lambda_s\big(p(s)\big)$. If $\eta(s)$ is the oriented angle function from $\gamma'(s)$ to $\mathbf{t}_g(s)$ in $T_{\gamma(s)}M^K$, then it can be seen that

$$\big\|\gamma'(s)\big\|\,\cos\eta(s) = p'(s) + \cos\nu(s) \tag{3}$$

and
$$b_s\big(p(s)\big) = \big\|\gamma'(s)\big\|\,\sin\eta(s). \tag{4}$$

The angle η and the expression b_s will be used later in the proof of Lemma 5.1. Its geometric interpretation will be of interest.

Notice that from Equation (2), the regularity of γ can be studied. In particular, the following hypothesis is equivalent to γ being regular.

Hypothesis $\mathcal{R}(\gamma)$. The length function $p(s)$ must satisfy for each $s \in I$

$$p'(s) + \cos\nu(s) \neq 0, \quad \text{or}$$

$$b_s\big(p(s)\big) = \cos\big(\sqrt{K}\,p(s)\big)\,\sin\nu(s) - \frac{\sin\big(\sqrt{K}\,p(s)\big)}{\sqrt{K}}\,\big(\nu'(s) + \kappa_g^\alpha(s)\big) \neq 0$$

Now, we will try to give some sufficient condition for the hypothesis $\mathcal{R}(\gamma)$. First of all, to manage easily the function b_s we will assume the following hypothesis.

Hypothesis $\mathcal{V}(\alpha)$. The hypothesis $\mathcal{S}(\alpha)$ holds and in addition, for all $s \in I$,

$$\nu'(s) + \kappa_g^\alpha(s) > 0.$$

Remark 4.2. Both in spherical and hyperbolic geometry it is possible to interpret the geodesic curvature κ_g^α of a convex curve α by means of the derivative of some angle σ (as happens in the plane). Any geodesic with only one common point or with a complete arc in common with α is called a supporting geodesic of α. For each point of α, there is a tangent supporting geodesic. Given $s \in I$, the angle $\sigma(s)$ is computed according to the metric as the space-like angle from the parallel transport of this tangent supporting geodesic and the corresponding supporting geodesic of α at $\alpha(s)$. See [25] and [26] for more details.

With that, the quantity $\nu'(s) + \kappa_g^\alpha(s) = \big(\nu(s) + \sigma(s)\big)'$ can be understood as the derivative of the angle from such a parallel tangent vector field to the moving chord. Hypothesis $\mathcal{V}(\alpha)$ establishes this angle (for convex curves) to be an increasing function.

For convex curves in M^K (see [25] and [26]), in the particular case of constructing Holditch curves under the condition $\mathcal{S}(\alpha)$, we claim that $\mathcal{V}(\alpha)$ holds (as happens in the plane), but we leave it as an open problem.

Proposition 4.3. *Suppose that $\mathcal{V}(\alpha)$ holds and, if $K > 0$, that $p(s) < \frac{\pi}{2\sqrt{K}}$ for all $s \in I$. If $\nu(s) < 0$ for all $s \in I$, then $\mathcal{R}(\gamma)$ holds. Otherwise, if in addition*

$$p(s) \neq \frac{1}{\sqrt{K}}\,\arctan\left(\frac{\sqrt{K}\,\sin\nu(s)}{\nu'(s) + \kappa_g^\alpha(s)}\right),$$

for all $s \in I$, then $\mathcal{R}(\gamma)$ holds.

Proof. First, notice that if $K < 0$, then

$$\cos\big(\sqrt{K}\,p(s)\big) = \cosh\big(\sqrt{-K}\,p(s)\big) \geq 0$$

and
$$\frac{\sin\big(\sqrt{K}\,p(s)\big)}{\sqrt{K}} = \frac{\sinh\big(\sqrt{-K}\,p(s)\big)}{\sqrt{-K}} \geq 0.$$

In the case $K > 0$, since $p < \frac{\pi}{2\sqrt{K}}$, both terms are also non-negative.

If $\nu(s) < 0$ for all $s \in I$, then $b_s\big(p(s)\big) < 0$ and so $\mathcal{R}(\gamma)$ holds. Otherwise, the condition $b_s\big(p(s)\big) \neq 0$ can be written in the form of the statement as a sufficient condition for $\mathcal{R}(\gamma)$. $\square$

In the particular case of being $p(s) = p$ constant (such as in the Holditch case), the condition of Proposition 4.3 is relaxed a little.

Theorem 4.4. *Suppose $\mathcal{V}(\alpha)$ holds and $p(s) = p$ is constant. Suppose $p < \frac{\pi}{2\sqrt{K}}$ if $K > 0$. If $\nu(s) \neq \frac{\pi}{2}$ for all $s \in I$ for such a chord, then $\mathcal{R}(\gamma)$ holds. Otherwise, if $\nu(s)$ reaches the value $\frac{\pi}{2}$ at some point and*

$$p < \frac{1}{\sqrt{K}} \arctan\left(\frac{\sqrt{K}}{\nu'_{\sup} + \kappa_g^{\sup}}\right), \tag{5}$$

or

$$p > \frac{1}{\sqrt{K}} \arctan\left(\frac{\sqrt{K}}{\nu'_{\inf} + \kappa_g^{\inf}}\right), \tag{6}$$

(where the indices sup *and* inf *in a function denotes the supremum or infimum, respectively, of such a function) then $\mathcal{R}(\gamma)$ also holds.*

Proof. From Equation (2), since $p(s) = p$ is constant, we have

$$\big\|\gamma'(s)\big\|^2 = \cos^2\nu(s) + b_s^2(p).$$

If $\nu(s) \neq \pm\frac{\pi}{2}$, then $\cos\nu(s) \neq 0$ and $\mathcal{R}(\gamma)$ holds. So, to prove the first part of the statement, we must show that if ν reaches the value $-\frac{\pi}{2}$, the condition $\mathcal{R}(\gamma)$ also holds. Indeed, if $\nu(s_0) = -\frac{\pi}{2}$ for some $s_0 \in I$, then $b_{s_0}(p) < 0$ and, therefore, we have $\mathcal{R}(\gamma)$.

Suppose now that for some $s_1 \in I$, $\nu(s_1) = \frac{\pi}{2}$. The condition $b_{s_1}(p) \neq 0$ can be written as

$$p \neq \frac{1}{\sqrt{K}} \arctan\left(\frac{\sqrt{K}}{\nu'(s_1) + \kappa_g^{\alpha}(s_1)}\right).$$

From that follows the hypothesis on p of the statement just by using that

$$\sup_{s\in I}\big(\nu'(s) + \kappa_g^{\alpha}(s)\big) \leq \sup_{s\in I}\nu'(s) + \sup_{s\in I}\kappa_g^{\alpha}(s)$$

and

$$\inf_{s\in I}\big(\nu'(s) + \kappa_g^{\alpha}(s)\big) \geq \inf_{s\in I}\nu'(s) + \inf_{s\in I}\kappa_g^{\alpha}(s). \qquad \square$$

Remark 4.5. The bounds (5) and (6) on p of Theorem 4.4 both avoid singularities on the generated curves. Nevertheless, since a small p is usually considered, the bound of interest is (5). In fact, if $\nu(s) = \omega$ is constant, that bound is the condition of regularity for parallel curves in constant curvature surfaces:

$$p < \frac{1}{\sqrt{K}} \arctan\left(\frac{\sqrt{K}}{\kappa_g^{\sup}}\right).$$

Of course, the limit

$$\lim_{K\to 0} \frac{1}{\sqrt{K}} \arctan\left(\frac{\sqrt{K}}{\kappa_g^{\sup}}\right) = \frac{1}{\kappa_{\sup}}$$

gives us the bound for having regular parallel curves in the plane, where here $\kappa_{\sup}$ is the supremum curvature of the initial curve (see [9]). It is also the condition for regularity of pipe surfaces (see [6], p. 399).

Remark 4.6. In the case of Holditch curves, a sufficiently small chord length ℓ will make $\nu(s) \neq \frac{\pi}{2}$ for all $s \in I$. This means that all Holditch curves of a simple regular curve for a sufficiently small chord length are always regular. Otherwise, if the chord length is so big, a sufficiently small choice of p can avoid singularities.

5. Main results on the generated curves

We will use the following notation for the total geodesic curvature of a curve $\phi\colon I \to M^K$ with trace ϕ^*:

$$\int_{\phi^*} \kappa_g^\phi := \int_I \kappa_g^\phi(t) \left\|\phi'(t)\right\| \,\mathrm{d}t.$$

Recall that the total geodesic curvature is invariant by reparameterizations (up to a sign if the orientation of the curve or of the manifold is changed).

Now, we can state the relation between the geodesic curvatures of the initial curve, α, and its generated one, γ.

Lemma 5.1. *If $\mathcal{S}(\alpha)$ holds and the generated curve γ is regular, hypothesis $\mathcal{R}(\gamma)$, then*

$$\begin{aligned}\int_{\gamma^*} \kappa_g^\gamma = \sqrt{K} \int_I &\sin\big(\sqrt{K}\, p(s)\big) \sin\nu(s) \,\mathrm{d}s \\ &+ \int_I \cos\big(\sqrt{K}\, p(s)\big) \big(\nu'(s) + \kappa_g^\alpha(s)\big) \,\mathrm{d}s - \int_I \eta'(s) \,\mathrm{d}s,\end{aligned}$$

where κ_g^γ and κ_g^α are the geodesic curvatures of γ and α, respectively, and η is the oriented angle function defined in Remark 4.1.

Proof. Mainly, it is a matter of computation of the geodesic curvature of γ with the expression (1). Since we are going to integrate, it is sufficient to compute

$$\sqrt{|K|}\, \frac{\mathcal{B}(\gamma, \gamma' \wedge \gamma'')(s)}{\left\|\gamma'(s)\right\|^2}.$$

(Notice that γ may not be arc-length parameterized.) With the frame $\mathcal{F}$, the computation of the geodesic curvature leads to a wide expression. Hence, it is convenient to make the same computations but using an orthonormal frame adapted to γ, such as

$$\mathcal{F}^*(s) = \left\{\sqrt{|K|}\, \gamma(s),\, \mathbf{t}_g(s),\, \sqrt{|K|}\, (\gamma \wedge \mathbf{t}_g)(s)\right\},$$

where $\mathbf{t}_g$ is the tangent of the geodesic λ_s at $\gamma(s)$:

$$\mathbf{t}_g(s) = \lambda_s'\big(p(s)\big) = -\sqrt{K}\,\sin\big(\sqrt{K}\,p(s)\big)\,\alpha(s) + \cos\big(\sqrt{K}\,p(s)\big)\,\beta(s).$$

Also (see Remark 4.1),

$$\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s) = -\frac{B_s(u)}{b_s(u)} = -\mathbf{v}(s).$$

The variation of $\mathcal{F}^*$ is given by the matrix

$$\sqrt{|K|}\begin{pmatrix} 0 & p'(s)+\cos\nu(s) & -b_s\big(p(s)\big) \\ -\operatorname{sgn}(K)\,\big(p'(s)+\cos\nu(s)\big) & 0 & -\frac{b_s'\left(p(s)\right)}{\sqrt{|K|}} \\ \operatorname{sgn}(K)\,b_s\big(p(s)\big) & \frac{b_s'\left(p(s)\right)}{\sqrt{|K|}} & 0 \end{pmatrix}.$$

In the frame $\mathcal{F}^*$, we can write

$$\gamma'(s) = \big(p'(s)+\cos\nu(s)\big)\,\mathbf{t}_g(s) - b_s\big(p(s)\big)\,\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s).$$

Thus, from (3) and (4):

$$\frac{\gamma'(s)}{\|\gamma'(s)\|} = \cos\eta(s)\,\mathbf{t}_g(s) - \sin\eta(s)\,\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s). \tag{7}$$

Differentiating (7), we get

$$\begin{aligned} \frac{\gamma''(s)}{\|\gamma'(s)\|} &+ \left(\frac{1}{\|\gamma'(s)\|}\right)'\gamma'(s) \\ &= \eta'(s)\,\big(-\sin\eta(s)\,\mathbf{t}_g(s) - \cos\eta(s)\,\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)\big) \\ &\quad + \cos\eta(s)\,\mathbf{t}_g'(s) - \sin\eta(s)\,\big(\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)\big)' \\ &= \eta'(s)\,\big(-\sin\eta(s)\,\mathbf{t}_g(s) - \cos\eta(s)\,\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)\big) \\ &\quad + b_s'\big(p(s)\big)\,\big(\sin\eta(s)\,\mathbf{t}_g(s) + \cos\eta(s)\,\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)\big) \\ &\quad + \big(\text{term in } \sqrt{|K|}\,\gamma(s)\big). \end{aligned} \tag{8}$$

Hence, making the cross product of (7) and (8),

$$\begin{aligned} \frac{(\gamma'\wedge\gamma'')(s)}{\|\gamma'(s)\|^2} &= \operatorname{sgn}(K)\,\Big(b_s'\big(p(s)\big) - \eta'(s)\Big)\,\sqrt{|K|}\,\gamma(s) \\ &\qquad + \Big(\text{terms in } \mathbf{t}_g(s) \text{ and } \sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)\Big). \end{aligned}$$

Therefore,
$$\sqrt{|K|}\,\frac{\mathcal{B}\big(\gamma,\gamma'\wedge\gamma''\big)(s)}{\big\|\gamma'(s)\big\|^2}=b_s'\big(p(s)\big)-\eta'(s). \tag{9}$$

Since

$$b_s'\big(p(s)\big)=\sqrt{K}\sin\big(\sqrt{K}\,p(s)\big)\sin\nu(s)+\cos\big(\sqrt{K}\,p(s)\big)\,\big(\nu'(s)+\kappa_g^\alpha(s)\big),$$

the result is obtained by integrating (9). □

The next step is to translate the expression of Lemma 5.1 in terms of areas when α and γ are both simple and closed and the length $p(s)$ is constant (Lemma 5.4). For that, we will use Gauss–Bonnet theorem (Remark 5.3). To avoid problems when computing areas we will suppose that the generated curve is simple.

Hypothesis $\mathcal{CR}(\alpha,\gamma)$. The hypothesis $\mathcal{S}(\alpha)$ holds and in addition α is closed and the functions $\beta(s)$ and $p(s)$ can be extended continuously by periodicity (so γ is also closed). Moreover, assume the generated curve γ to be positively oriented, simple and regular – thus, $\mathcal{R}(\gamma)$ holds.

Since regularity and simpleness of generated curves is dependent on the chosen length of the moving chord, the following definition makes sense.

Definition 5.2. We will say that a constant length p is *admissible* if the generated curve γ is regular and simple.

Remark 5.3. Given a compact region R of M^K with boundary ∂R defined by the trace of a positively oriented regular simple closed curve $\phi\colon I\to\partial R$, recall that Gauss–Bonnet theorem states that

$$\iint_R K\,\mathrm{d}\sigma+\int_{\partial R}\kappa_g^\phi=2\pi\,\chi(R),$$

where K is the Gauss curvature of R, κ_g^ϕ the geodesic curvature of ϕ and $\chi(R)$ the Euler–Poincaré characteristic of R. In our case, we have $\chi(R)=1$ and K constant, so we can write

$$\int_{\phi^*}\kappa_g^\phi=2\,\pi-K\mathcal{A}(\phi). \tag{10}$$

Lemma 5.4. *If $\mathcal{CR}(\alpha,\gamma)$ holds and γ is the generated curve for a chord of constant length $p(s)=p$, then*

$$\begin{aligned}\mathcal{A}(\gamma)=-\frac{\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}}\int_I\sin\nu(s)\,\mathrm{d}s+\frac{2\,\pi\,n}{K}\Big(1-\cos\big(\sqrt{K}\,p\big)\Big)\\ +\cos\big(\sqrt{K}\,p\big)\mathcal{A}(\alpha),\end{aligned} \tag{11}$$

where n is the number of revolutions done by the geodesic chord in M^K.

Proof. Since $p(s) = p$, the expression of Lemma 5.1 takes the form

$$\int_{\gamma^*} \kappa_g^\gamma = \sqrt{K}\,\sin\big(\sqrt{K}\,p\big)\int_I \sin\nu(s)\,\mathrm{d}s \\ + \cos\big(\sqrt{K}\,p\big)\int_I \big(\nu'(s) + \kappa_g^\alpha(s)\big)\,\mathrm{d}s - \int_I \eta'(s)\,\mathrm{d}s.$$

Since α and γ are positively oriented, we have

$$\int_I \nu'(s)\,\mathrm{d}s = \int_I \eta'(s)\,\mathrm{d}s = 2\,\pi\,m,$$

where m is the number of revolutions of the moving chord with respect to the tangent vector of α (or the same for γ). Thus,

$$\int_{\gamma^*} \kappa_g^\gamma = \sqrt{K}\,\sin\big(\sqrt{K}\,p\big)\int_I \sin\nu(s)\,\mathrm{d}s + 2\,\pi\,m\,\Big(\cos\big(\sqrt{K}\,p\big) - 1\Big) \\ + \cos\big(\sqrt{K}\,p\big)\int_{\alpha^*} \kappa_g^\alpha.$$

Using Equation (10) to both α and γ and arranging the expression we get:

$$\mathcal{A}(\gamma) = -\frac{\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}}\int_I \sin\nu(s)\,\mathrm{d}s \\ + \frac{2\,\pi\,(m+1)}{K}\Big(1 - \cos\big(\sqrt{K}\,p\big)\Big) + \cos\big(\sqrt{K}\,p\big)\mathcal{A}(\alpha).$$

Finally, notice that the tangent vector of α (or γ) with respect to a parallel tangent vector field along such a curve makes just a positive complete revolution because both α and γ are simple and positively oriented curves. Therefore, the number n of chord revolutions is equal to $m + 1$. □

Now, from Equation (11) of Lemma 5.4 we are going to deduce some results taking particular situations for the generated curve γ, restricting ourselves to the case of being such a curve regular and simple (admissible length).

5.1. Parallel curves

The first case we will treat is the case of γ being a parallel curve of α at some constant distance $p(s) = p$ in a direction maintaining some constant angle $\nu(s) = \omega$ with α (see Figure 5.1).

Next, we state what is known as the generalization to constant curvature surfaces of Steiner's formula for the area of parallel curves according to a constant angle (see [29] and [30]).

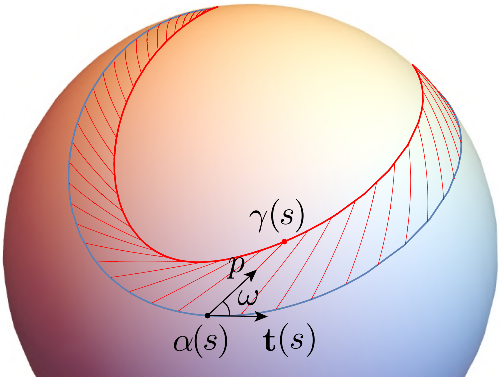

Figure 5.1: The generated curve $\gamma(s)$ is the parallel curve of α at a constant distance p in a direction maintaining a constant angle ω with α.

Theorem 5.5. (Steiner's formula for the area) *Let α be a positively oriented regular simple closed curve of finite length L in a 2-dimensional manifold of constant curvature K. Let γ be the parallel curve to α at a constant distance p maintaining a constant angle ω with α. If p is admissible, then*

$$\mathcal{A}(\gamma) = -L\,\frac{\sin\omega\,\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}} + \frac{2\pi}{K}\Big(1-\cos\big(\sqrt{K}\,p\big)\Big) + \cos\big(\sqrt{K}\,p\big)\mathcal{A}(\alpha). \quad (12)$$

Proof. The result follows just by taking the constant angle function $\nu(s) = \omega$ in Lemma 5.4. Notice that $n = 1$ in this case. $\square$

Remark 5.6. Making $K \to 0$ in Equation (12) and taking $\omega = \frac{\pi}{2}$, the classical planar Steiner formula for the area of inner offset curves is obtained:

$$\mathcal{A}(\gamma) = \mathcal{A}(\alpha) - L\,p + \pi\,p^2.$$

The analogous formula for outer parallel curves is obtained just by considering an angle $\omega = -\frac{\pi}{2}$ in Equation (12) and making $K \to 0$.

We also have the extension of Steiner's formula for the length of a parallel curve to constant curvature surfaces (see [29] and [30]).

Theorem 5.7. (Steiner's formula for the length) *Let α be a positively oriented regular simple closed curve of finite length L in a 2-dimensional manifold of constant curvature K. Let γ be the parallel curve to α found orthogonally at a constant distance p. If p is admissible and sufficiently small, then*

$$L(\gamma) = L\,\cos\big(\sqrt{K}\,p\big) \pm \frac{\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}}\big(2\pi - K\,\mathcal{A}(\alpha)\big). \quad (13)$$

The sign $\pm$ is $-$ for inner offsets and $+$ for outer offsets.

Proof. The length of the generated curve γ can be computed as

$$L(\gamma) = \int_I \big\|\gamma'(s)\big\|\,\mathrm{d}s.$$

We have that $p(s) = p$ and $\nu(s) = \omega = \pm\frac{\pi}{2}$ are both constant, so the first term of the right hand side of Equation (2) disappears and we have a perfect square. Since p is admissible, hypothesis $\mathcal{R}(\gamma)$ holds.

On the one hand, suppose $p < \frac{\pi}{2\sqrt{K}}$ if $K > 0$. On the other hand, consider p small enough to have constant sign in $b_s(p)$—positive for inner parallels (for which $\omega = \pi/2$ and notice that $b_s(0) = 1$) and negative for outer parallels (for which $\omega = -\pi/2$ and $b_s(0) = -1$). In that case,

$$L(\gamma) = \int_0^L \left|b_s(p)\right| \,\mathrm{d}s = \int_0^L \left(\cos\left(\sqrt{K}\,p\right) \pm \kappa_g(s)\,\frac{\sin\left(\sqrt{K}\,p\right)}{\sqrt{K}}\right) \mathrm{d}s.$$

Using (10), the result is obtained. □

Remark 5.8. Of course, the classical planar Steiner formula for the length of offset curves,

$$L(\gamma) = L \pm 2\,\pi\,p,$$

follows by making $K \to 0$ in Equation (13). In the case of outer offsets of a convex curve, the formula holds for any length p since $b_s(p) < 0$.

5.2. Constant width curves

Suppose that the angle function $\nu(s)$ is a constant right angle and that the other endpoint of the chord always lies in α at the constant distance $p(s) = \ell$. That setting is the definition of α being a curve of constant width ℓ (see, for instance, [25], [26] and [15]). The situation is shown in Figure 5.2.

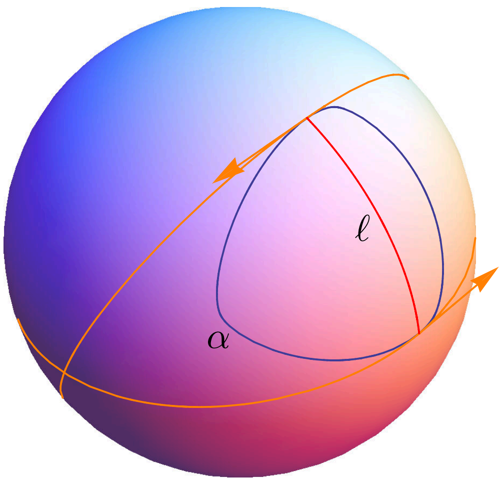

Figure 5.2: The generated curve from a curve α of constant width ℓ describes the other endpoint of the geodesic segment that measures such a constant width.

Now, Barbier's theorem for constant width curves in constant curvature surfaces (see [28]) can be deduced.

Theorem 5.9. (Barbier's theorem]) *Let α be a simple and regular curve of constant width ℓ in a 2-dimensional manifold of constant curvature K. Then*

$$L = \frac{\sqrt{K}}{\sin\big(\sqrt{K}\,\ell\big)} \left(\frac{2\pi}{K} - \mathcal{A}(\alpha)\right)\Big(1 - \cos\big(\sqrt{K}\,\ell\big)\Big). \tag{14}$$

Proof. Suppose α to be positively oriented. The result follows just by taking $\nu(s) = \frac{\pi}{2}$, $p(s) = \ell$ in Lemma 5.4 by noticing that, therefore, $\mathcal{A}(\alpha) = \mathcal{A}(\gamma)$ and $n = 1$. □

Remark 5.10. Making $K \to 0$ in Equation (14), Barbier's theorem in the plane, [1], for curves of constant width ℓ is found:

$$L = \pi\,\ell.$$

5.3. Holditch's extensions

In this subsection, the generated curve γ will be a Holditch curve. Notice that the hypothesis $\mathcal{S}(\alpha)$ demands, for each $\alpha(s)$, to be only one direction $\beta(s) \in T_{\alpha(s)}M^K$ to construct the generated curve γ. Implicitly this hypothesis avoids the so-called retrograde motion of the moving chord in a Holditch setting. Retrograde motion happens when the endpoints of the Holditch chord pass more than once through the same point. Sometimes that is not a desired phenomenon in some applications for industrial or mechanical engineering, so it must be controlled (see [10] for some applications of Holditch's theorem in the plane, some of which can be extended to constant curvature surfaces).

Restrictions on the chord length in order to avoid retrograde motions in the plane were commented in [2], studied in [20] and have been recently revisited in [17]. Results on convexity and singularities of planar Holditch curves given in [21] seem to be also related to the avoiding of retrograde movements, but a further detailed study is needed, both in the plane and in constant curvature 2-manifolds. Here, we will assume that no retrograde movements appear for the chosen length of the chord as a hypothesis.

Now we are going to use Lemma 5.4 to give a new proof of Holditch's theorem in 2-dimensional constant curvature manifolds for two initial curves α and $\bar{\alpha}$. See [27] for an equivalent expression of (15) as a determinant. See the setting in Figure 5.3.

Theorem 5.11. (Holditch's theorem for two initial curves) *Let α and $\bar{\alpha}$ be two positively oriented closed simple and regular curves in a 2-dimensional manifold of constant curvature K. Suppose that a chord of constant length ℓ moves forwards with each of its endpoints in each curve such that n chord revolutions (according to the orientation of M^K) are done. A point dividing the chord into two parts of lengths p and q will describe another closed curve γ. If p is admissible, then*

$$\mathcal{A}(\gamma) = \frac{\sin\big(\sqrt{K}\,q\big)\mathcal{A}(\alpha) + \sin\big(\sqrt{K}\,p\big)\mathcal{A}(\bar{\alpha})}{\sin\big(\sqrt{K}\,(p+q)\big)} - n\,C_{p,q}(K), \tag{15}$$

where n is the number of chord revolutions and

$$C_{p,q}(K) := 4\pi \frac{\sin\left(\sqrt{K}\,\frac{p}{2}\right)\sin\left(\sqrt{K}\,\frac{q}{2}\right)}{K\cos\left(\sqrt{K}\,\frac{p+q}{2}\right)}.$$

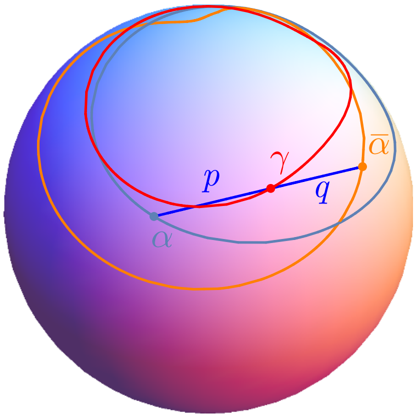

Figure 5.3: Two initial closed curves α and $\bar{\alpha}$ and the associated Holditch curve for a chord length $p+q$.

Proof. We will use Lemma 5.4. Suppose that at a constant distance p in the chord from α the curve γ is generated but also at a constant distance $\ell \geq p$ the curve $\bar{\alpha}$ is described. So, we have

$$\begin{aligned}\mathcal{A}(\gamma) = &-\frac{\sin\left(\sqrt{K}\,p\right)}{\sqrt{K}}\int_I \sin\nu(s)\,\mathrm{d}s + \frac{2\,\pi\,n}{K}\left(1-\cos\left(\sqrt{K}\,p\right)\right)\\ &+\cos\left(\sqrt{K}\,p\right)\mathcal{A}(\alpha)\end{aligned} \tag{16}$$

and

$$\begin{aligned}\mathcal{A}(\bar{\alpha}) = &-\frac{\sin\left(\sqrt{K}\,\ell\right)}{\sqrt{K}}\int_I \sin\nu(s)\,\mathrm{d}s + \frac{2\,\pi\,n}{K}\left(1-\cos\left(\sqrt{K}\,\ell\right)\right)\\ &+\cos\left(\sqrt{K}\,\ell\right)\mathcal{A}(\alpha).\end{aligned} \tag{17}$$

If we multiply Equation (16) by $\sin\left(\sqrt{K}\,\ell\right)$ and subtract Equation (17) multiplied by $\sin\left(\sqrt{K}\,p\right)$, we get

$$\begin{aligned}\sin\left(\sqrt{K}\,\ell\right)\mathcal{A}(\gamma) - \sin\left(\sqrt{K}\,p\right)\mathcal{A}(\bar{\alpha}) &= \mathcal{A}(\alpha)\,\sin\left(\sqrt{K}\,(\ell-p)\right)\\ &+\frac{2\,\pi\,n}{K}\left(\sin\left(\sqrt{K}\,\ell\right)-\sin\left(\sqrt{K}\,p\right)-\sin\left(\sqrt{K}\,(\ell-p)\right)\right).\end{aligned}$$

Arranging and noticing that

$$\frac{\sin\left(\sqrt{K}\,p\right)-\sin\left(\sqrt{K}\,\ell\right)+\sin\left(\sqrt{K}\,(\ell-p)\right)}{\sin\left(\sqrt{K}\,\ell\right)} = 2\,\frac{\sin\left(\sqrt{K}\,\frac{p}{2}\right)\sin\left(\sqrt{K}\,\frac{\ell-p}{2}\right)}{\cos\left(\sqrt{K}\,\frac{\ell}{2}\right)},$$

the expression of the statement is obtained. □

Remark 5.12. The constant $C_{p,q}(K)$ is called the Holditch constant because it generalizes the Holditch constant of the planar case:

$$\lim_{K\to 0} C_{p,q}(K) = \pi\, p\, q.$$

Moreover, making $K \to 0$ in (15), we obtain

$$\mathcal{A}(\gamma) = \frac{q\,\mathcal{A}(\alpha) + p\,\mathcal{A}(\bar{\alpha})}{p+q} - n\,\pi\, p\, q,$$

which is the extension of Holditch's theorem for two planar initial curves and any number n of chord revolutions. Sometimes it is called Broman's extension (see [3]), although the first proof of that result was due to Woolhouse among others more than 150 years ago (see [32], pp. 66–68, 96).

As a corollary of Theorem 5.11, we can deduce the statement of Holditch's theorem in constant curvature manifolds (see [28], [31] or [27]). See in Figure 5.4 this setting.

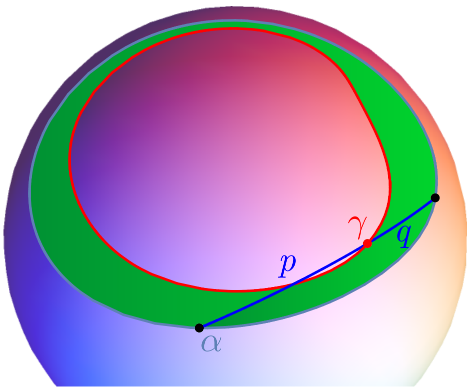

Figure 5.4: A closed curve, α, and its Holditch curve, γ, generated by a chord of constant length $p+q$. The Holditch area (shaded) is the difference between the areas of both closed curves.

Theorem 5.13. (Holditch's theorem) *Let α be a positively oriented closed simple and regular curve in a 2-dimensional manifold of constant curvature K. Suppose that a chord of constant length ℓ moves forwards with both endpoints lying in α such that a full chord revolution (according to the orientation of M^K) is done. A point dividing the chord into two parts of lengths p and q will describe another closed curve γ. If p is admissible, then*

$$\mathcal{A}(\alpha) - \mathcal{A}(\gamma) = C_{p,q}(K)\cdot\left(1 - \frac{K\,\mathcal{A}(\alpha)}{2\pi}\right), \tag{18}$$

where

$$C_{p,q}(K) := 4\pi\,\frac{\sin\!\left(\sqrt{K}\,\frac{p}{2}\right)\sin\!\left(\sqrt{K}\,\frac{q}{2}\right)}{K\cos\!\left(\sqrt{K}\,\frac{p+q}{2}\right)}.$$

Proof. We will use Theorem 5.11 when $\bar{\alpha}$ is another parameterization of the curve α. Since α is simple and both endpoints of the moving chord must be on α, we have $n = 1$. Thus, $\mathcal{A}(\alpha) = \mathcal{A}(\bar{\alpha})$ and the formula (15) become

$$\mathcal{A}(\gamma) = \frac{\sin\big(\sqrt{K}\,q\big) + \sin\big(\sqrt{K}\,p\big)}{\sin\big(\sqrt{K}\,(p+q)\big)}\,\mathcal{A}(\alpha) - C_{p,q}(K).$$

Therefore, we get

$$\mathcal{A}(\gamma) - \mathcal{A}(\alpha) = C_{p,q}(K) \cdot \frac{K}{2\pi}\,\mathcal{A}(\alpha) - C_{p,q}(K),$$

from which the expression of the statement is deduced. $\square$

Remark 5.14. Of course, making $K \to 0$ in the formula (18), the classical Holditch theorem in the plane is obtained: $\mathcal{A}(\alpha) - \mathcal{A}\big(\gamma\big) = \pi\, p\, q$.

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