Journal of Convex Analysis
Volume 27 (2020), No. 4, 1157–1176

Multivalued Strong Laws of Large Numbers for Triangular Arrays with Gap Topology

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Received: March 4, 2019
Accepted: November 8, 2019

We state some strong laws of large numbers for triangular arrays of random sets in separable Banach spaces with the gap topology and with or without compactly uniformly integrable condition.

Keywords: Random set, gap topology, triangular array, compactly uniformly integrable.

2010 Mathematics Subject Classification: 60F15, 60B12, 28B20.

1. Introduction

In recent decades, the theory of random sets has been extensively studied and gave rise to applications in several fields, such as optimization and control, stochastic and integral geometry, mathematical economics, statistics and related fields. In 1975, Artstein and Vitale [2] proved the strong law of large numbers (SLLN) for sequences of independent and identically distributed random compact sets in $\mathbb{R}^d$, with convergence in the Hausdorff metric. It was extended to random closed sets in $\mathbb{R}^d$ by Artstein and Hart [1]. It is somewhat harder to deal with both non-compact values and an infinite dimensional carrier space at the same time. The Hausdorff metric is too strong to hold the law of large numbers and so need to be replaced by an appropriate weaker topology among a plethora of candidates from the literature of hyperspace topology.

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ISSN 0944-6532 / $ 2.50

Hiai [11] extended the SLLN of Artstein and Hart [1] to separable Banach space by using the Mosco topology. Also, Hess [9] independently proved similar results for random sets in an infinite dimensional Banach space. Moreover, Hess [10] obtained a convergence result on the Wijsman topology of SLLN for sequence of pairwise independent identically distributed random sets in a separable Banach space. In particular, Hess showed that the SLLN for the slice topology follows from the SLLN for the Wijsman topology, provided the space has a strongly separable dual ([10, Proposition 3.9]). The SLLN for a double array of independent (or pairwise independent) random sets was introduced by Castaing et al. [6, 7] with Mosco convergence, Wijsman convergence and slice convergence. In [15], some SLLNs for arrays of rowwise independent random sets with Mosco convergence were provided.

The slice topology on the hyperspace is the weak (or initial) topology determined by the family of gap functionals $\{D(C, .) : C$ is a nonempty slice of a ball$\}$. However, in the practical applications of probability theory, we need to consider sets C that are bounded and convex. In this context, the best candidate seems to be the gap topology given by the weak (or initial) topology corresponding to the gap functionals $\{D(C, .) : C$ is a nonempty bounded convex set$\}$.

Using the gap topology, Terán [17] established the SLLN for sequences of pairwise independent and identically distributed random sets. This result improves upon and unifies the laws of large numbers with convergence in the Wijsman, Mosco and slice topologies, without requiring extra assumptions on either the properties of the space or the kind of sets that can be taken on by the random set as values. Quang et al. [16] stated some SLLNs for double arrays of independent (or pairwise independent) random sets under various setting. To this direction, in this paper, we establish several SLLNs for triangular arrays of rowwise independent random sets with the gap topology and with or without compactly uniformly integrable condition. Let us note that, even in the case of single-valued random variables, the SLLNs for sequences and for double arrays cannot imply SLLNs for triangular arrays.

This paper is organized as follows. In Section 2, we introduce some basic notions: hyperspace of a Banach space, random set, Painlevé-Kuratowski convergence, Wijsman convergence, slice convergence, convergence in the gap topology. In Section 3, we establish some lemmas which will be used to prove the main results. The strong laws of large numbers for triangular arrays of rowwise independent random sets with the gap topology under various settings are given in Section 4.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{A}, P)$ be a complete probability space, $(\mathfrak{X}, \|.\|)$ be a separable Banach space. Let $\mathfrak{X}^*$ be the dual space of $\mathfrak{X}$ and S^* be its unit sphere. $\mathbb{R}$ (resp. $\mathbb{Q}$) (resp. $\mathbb{N}$) will denote the set of all real numbers (resp. rational numbers) (resp. positive integers).

Let $c(\mathfrak{X})$ (resp. $k(\mathfrak{X})$) be the family of all nonempty closed (resp. compact) subsets of $\mathfrak{X}$. For any $A, B \in c(\mathfrak{X})$, cl$A$, co$A$, $\overline{\text{co}}A$ denote the *norm-closure*, the *convex* *hull* and the *closed convex hull* of A respectively; the *distance function* $d(.,A)$ of A, the *support function* $s(.,A)$ of A, the *norm* $\|A\|$ of A and the *gap* $D(A,B)$ between two sets A and B are defined by

$$d(x,A) = \inf\{\|x-y\| : y \in A\},\ x \in \mathfrak{X}, \quad s(x^*,A) = \sup\{\langle x^*, y\rangle : y \in A\},\ x^* \in \mathfrak{X}^*,$$

$$\|A\| = \sup\{\|x\| : x \in A\}, \quad D(A,B) = \inf\{\|x-y\| : x \in A, y \in B\},$$

and also, for any $r > 0$, we put $A^r = \{x \in \mathfrak{X} : d(x,A) \le r\}$.

Let $U^- := \{C \in c(\mathfrak{X}) : C \cap U \neq \emptyset\}$, where $U \subset \mathfrak{X}$. Let $\mathcal{B}_{c(\mathfrak{X})}$ be the σ-field on $c(\mathfrak{X})$ generated by the sets U^-, taken for all open subsets U of $\mathfrak{X}$. We call $\mathcal{B}_{c(\mathfrak{X})}$ the *Effrös σ-field.* A mapping $F\colon \Omega \to c(\mathfrak{X})$ is said to be *a random set* if it is $(\mathcal{A}, \mathcal{B}_{c(\mathfrak{X})})$- measurable, namely, for every open subset U of $\mathfrak{X}$, we have $F^{-1}(U^-) \in \mathcal{A}$.

Given a random set F, we define a sub-σ-field of $\mathcal{A}$ by $\mathcal{A}_F = \{F^{-1}(\mathcal{U}) : \mathcal{U} \in \mathcal{B}_{c(\mathfrak{X})}\}$.

For each $\mathcal{F}$-measurable random set F, the family of all $\mathcal{F}$-measurable random elements f with $f(\omega) \in F(\omega)$ a.s., is denoted by $S^0_F(\mathcal{F})$, where $\mathcal{F}$ is a sub-σ-field of $\mathcal{A}$. Shortly, $S^0_F(\mathcal{A})$ is denoted by S^0_F.

For $1 \le p < \infty$, $\mathbf{L}^p(\Omega, \mathcal{A}, P, \mathfrak{X}) = \mathbf{L}^p(\mathfrak{X})$ denotes the space of $\mathcal{A}$-measurable functions $f\colon \Omega \to \mathfrak{X}$ such that the norm $\|f\|_p = (E\|f\|^p)^{\frac{1}{p}}$ is finite.

Set $S^p_F(\mathcal{F}) = \{f \in \mathbf{L}^p(\Omega, \mathcal{F}, P, \mathfrak{X}) : f(\omega) \in F(\omega),\ \text{a.s.}\}$. If $\mathcal{F} = \mathcal{A}$, then $S^p_F(\mathcal{F})$ is denoted for short by S^p_F.

For any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$ and any $\mathcal{F}$-measurable random set F, the *expectation* of F over Ω, with respect to $\mathcal{F}$, is defined by $E(F,\mathcal{F}) = \{Ef : f \in S^1_F(\mathcal{F})\}$, where $Ef = \int_\Omega f dP$ is the usual Bochner integral of f. We denote $E(F,\mathcal{A})$ for short by EF. We note that EF is not always closed.

Let $\{r_j, j \ge 1\}$ be a symmetric Bernoulli sequence; that is, $\{r_j, j \ge 1\}$ is a sequence of independent identically distributed random variables with $P(r_1 = 1) = P(r_1 = -1) = 1/2$. The Banach space $\mathfrak{X}$ is said to be *of type* p $(1 \le p \le 2)$ if there exists a positive constant C such that

$$\left(E\Big\|\sum_{j=1}^{i} r_j v_j\Big\|^p\right)^{1/p} \le C\left(\sum_{j=1}^{i} \|v_j\|^p\right)^{1/p}$$

for every $i \ge 1$ and $v_j \in \mathfrak{X}$ $(1 \le j \le i)$.

Hoffmann-Jørgensen and Pisier [12, Theorem 2.1] proved that a real separable Banach space is of type p $(1 \le p \le 2)$ if and only if there exists a positive constant C such that $E\big\|\sum\limits_{j=1}^{n} f_j\big\|^p \le C \sum\limits_{j=1}^{n} E\|f_j\|^p$ for every finite collection $\{f_1, f_2, \dots, f_n\}$ of independent mean 0 random elements. If a real separable Banach space is of type p for some $1 < p \le 2$, then it is of type q for all $1 \le q < p$. Every real separable Hilbert space and real separable finite-dimensional Banach space is of type 2. In particular, the real line $\mathbb{R}$ is of type 2.

Let us denote by s the strong topology of $\mathfrak{X}$. Let $\{A_n : n \geq 1\}$ be a sequence in $c(\mathfrak{X})$. We put

$$s\text{-}\liminf_{n\to\infty} A_n = \{x \in \mathfrak{X} : x = s\text{-}\lim_{n\to\infty} x_n, x_n \in A_n, \forall n \geq 1\},$$

$$s\text{-}\limsup_{n\to\infty} A_n = \{x \in \mathfrak{X} : x = s\text{-}\lim_{k\to\infty} x_k, x_k \in A_{n_k}, k \geq 1\},$$

where $\{A_{n_k} : k \geq 1\}$ is a subsequence of $\{A_n : n \geq 1\}$.

A sequence $\{A_n : n \geq 1\}$ converges to A, in the *sense of Painlevé-Kuratowski*, relative to the topology s, as $n \to \infty$, if $s\text{-}\liminf_{n\to\infty} A_n = s\text{-}\limsup_{n\to\infty} A_n = A$. Clearly, this is true if and only if the following inclusions hold:

$$s\text{-}\limsup_{n\to\infty} A_n \subset A \subset s\text{-}\liminf_{n\to\infty} A_n.$$

The *Wijsman convergence* on $c(\mathfrak{X})$ is the pointwise convergence of distance functions. This means that a sequence $\{A_n : n \geq 1\} \subset c(\mathfrak{X})$ converges to A as $n \to \infty$ with respect to Wijsman convergence, if for every $x \in \mathfrak{X}$, one has $d(x, A) = \lim_{n\to\infty} d(x, A_n)$.

A sequence $\{A_n : n \geq 1\} \subset c(\mathfrak{X})$ *converges in the slice topology* to A as $n \to \infty$ if $D(A_n, C) \to D(A, C)$ as $n \to \infty$ for all sets C being a nonempty slice of a ball, and it *converges in the gap topology* to A as $n \to \infty$ if $D(A_n, C) \to D(A, C)$ as $n \to \infty$ for all nonempty bounded closed convex subsets C of $\mathfrak{X}$.

A triangular array of random elements $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ is said to be *compactly uniformly integrable* (CUI for short) if for every $\varepsilon > 0$, there exists a compact subset K of $\mathfrak{X}$ such that

$$\sup_{n,i} E\Big(\|f_{ni}\| \mathbf{1}_{(f_{ni}(.)\notin K)}\Big) < \varepsilon.$$

For a subset A of Ω the complement of A in Ω is denoted by A^c.

A triangular array of random sets $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ is said to be *compactly uniformly integrable* (CUI for short) if for every $\varepsilon > 0$, there exists a compact subset K of $\mathfrak{X}$ such that

$$\sup_{n,i} E\Big(\|F_{ni}\| \mathbf{1}_{(F_{ni}(.)\subset K)^c}\Big) < \varepsilon.$$

The random elements $\{f_{ni}\}$ are uniformly bounded by a random variable ξ if for all $n \geq 1$, $1 \leq i \leq n$ and for every real number $t > 0$,

$$P(\|f_{ni}\| > t) \leq P(|\xi| > t).$$

3. Lemmas

In this section, we give some lemmas which will be used later.

Lemma 3.1. *Assume that $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of CUI random sets. Then $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of CUI random elements, where $f_{ni} \in S^0_{F_{ni}}$ for all $n \geq 1, 1 \leq i \leq n$.*

Proof. Since $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of CUI random sets, for every $\varepsilon > 0$, there exists a compact subset K of $\mathfrak{X}$ such that

$$\sup_{n,i} E\Big(\|F_{ni}\|\mathbf{1}_{(F_{ni}(.)\subset K)^c}\Big) < \varepsilon.$$

Since $f_{ni} \in S^0_{F_{ni}}$ $(n \geq 1,\ 1 \leq i \leq n)$,

$$(f_{ni}(.) \notin K) \subset (F_{ni}(.) \subset K)^c \quad \text{for all } n \geq 1,\ 1 \leq i \leq n.$$

This implies

$$\|f_{ni}\|\mathbf{1}_{(f_{ni}(.)\notin K)} \leq \|F_{ni}\|\mathbf{1}_{(F_{ni}(.)\subset K)^c} \quad \text{for all } n \geq 1,\ 1 \leq i \leq n,$$

and so

$$E\Big(\|f_{ni}\|\mathbf{1}_{(f_{ni}(.)\notin K)}\Big) \leq E\Big(\|F_{ni}\|\mathbf{1}_{(F_{ni}(.)\subset K)^c}\Big) \quad \text{for all } n \geq 1,\ 1 \leq i \leq n.$$

Thus, $$\sup_{n,i} E\Big(\|f_{ni}\|\mathbf{1}_{(f_{ni}(.)\notin K)}\Big) \leq \sup_{n,i} E\Big(\|F_{ni}\|\mathbf{1}_{(F_{ni}(.)\subset K)^c}\Big) < \varepsilon.$$ □

The following lemma establishes the Chung's SLLN for triangular arrays of rowwise independent random variables in separable Banach spaces and it will be used to prove the main results.

Lemma 3.2. *Let $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ be a triangular array of rowwise independent and CUI random elements in a separable Banach space. Let $\psi(t)$ be a positive, even, convex, continuous function such that*

$$\frac{\psi(|t|)}{|t|^r} \uparrow \ \text{and}\ \frac{\psi(|t|)}{|t|^{r+1}} \downarrow \ \text{as } |t| \uparrow \infty, \tag{1}$$

for some nonnegative integer r and there exists a positive constant C_1 with

$$\psi(a+b) \leq C_1(\psi(a)+\psi(b)) \text{ for all real numbers } a,\ b. \tag{2}$$

Then the following conditions

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E(\psi(\|f_{ni}\|))}{\psi(n)} < \infty; \tag{3}$$

$$\sum_{n=1}^{\infty}\Big(\sum_{i=1}^{n} E\Big\|\frac{f_{ni}}{n}\Big\|^2\Big)^{2k} < \infty, \tag{4}$$

for some positive integer k, imply that

$$\Big\|\frac{1}{n}\sum_{i=1}^{n}(f_{ni} - Ef_{ni})\Big\| \to 0 \ a.s. \ \text{as } n \to \infty.$$

Proof. First, we prove this lemma for the case $r \geq 2$.

Since $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of CUI random elements, for all $\varepsilon > 0$, there exists a compact subset K of $\mathfrak{X}$ such that

$$\sup_{n,i} E\{\|f_{ni}\|\mathbf{1}_{(f_{ni}(.)\notin K)}\} < \varepsilon. \tag{5}$$

Let $g_{ni} = f_{ni}\mathbf{1}_{(f_{ni}(.)\in K)}$ and $h_{ni} = f_{ni}\mathbf{1}_{(f_{ni}(.)\notin K)}$.

By the compactness of K, there exists $x_1, x_2, \ldots, x_m \in K$ such that

$$K \subset \bigcup_{j=1}^{m} B(x_j, \varepsilon), \text{ where } B(x_j, \varepsilon) = \{x \in \mathfrak{X} : \|x - x_j\| \leq \varepsilon\}.$$

For $n \geq 1$, $1 \leq i \leq n$, we define $W_{ni} = U_{ni}\mathbf{1}_{(f_{ni}(.)\in K)}$, where

$$U_{ni} = x_1.\mathbf{1}_{(f_{ni}(.)\in B(x_1,\varepsilon))} + \sum_{j=2}^{m} x_j.\mathbf{1}_{[(f_{ni}(.)\in B(x_j,\varepsilon))\cap[\cup_{t=1}^{j-1}(f_{ni}(.)\in B(x_t,\varepsilon))]^c]}.$$

It is easy to check that $W_{ni} = \sum\limits_{j=1}^{m} x_j.\mathbf{1}_{(W_{ni}=x_j)}$, $EW_{ni} = \sum\limits_{j=1}^{m} x_j.P(W_{ni} = x_j)$ for all $n \geq 1$, $1 \leq i \leq n$ and we have

$$\begin{aligned}
\Big\|\frac{1}{n}\sum_{i=1}^{n}(f_{ni} - Ef_{ni})\Big\| &\leq \frac{1}{n}\sum_{i=1}^{n}\|f_{ni} - g_{ni}\| + \frac{1}{n}\sum_{i=1}^{n}\|g_{ni} - W_{ni}\| \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\|W_{ni} - EW_{ni}\| + \frac{1}{n}\sum_{i=1}^{n}\|EW_{ni} - Eg_{ni}\| + \frac{1}{n}\sum_{i=1}^{n}\|Eg_{ni} - Ef_{ni}\| \\
&:= J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

By (5), we obtain $\displaystyle \frac{1}{n}\sum_{i=1}^{n} E\|h_{ni}\| = \frac{1}{n}\sum_{i=1}^{n} E\{\|f_{ni}\|\mathbf{1}_{(f_{ni}(.)\notin K)}\} < \varepsilon,$ and so

$$\frac{1}{n}\sum_{i=1}^{n} E\|h_{ni}\| \to 0 \text{ as } n \to \infty. \tag{6}$$

Since $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent random elements, $\{\|h_{ni}\| - E\|h_{ni}\| : n \geq 1, 1 \leq i \leq n\}$ is also a triangular array of rowwise independent random variables with mean 0. Moreover, using increasing of the function $\psi(|t|)$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\big(\psi(|\|h_{ni}\| - E\|h_{ni}\||)\big)}{\psi(n)} &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\big(\psi(\|h_{ni}\| + E\|h_{ni}\|)\big)}{\psi(n)} \\
&\leq C_1\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\big(\psi(\|h_{ni}\|) + \psi(E\|h_{ni}\|)\big)}{\psi(n)} \quad \text{(by (2))}
\end{aligned}$$

$$\leq 2C_1 \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\big(\psi(\|h_{ni}\|)}{\psi(n)} \quad \text{(by Jensen's inequality)}$$

$$\leq 2C_1 \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\big(\psi(\|f_{ni}\|)}{\psi(n)} < \infty \quad \text{(by (3))}.$$

On the other hand, by the c_r-inequality and (4),

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} E\Big| \frac{\|h_{ni}\| - E\|h_{ni}\|}{n} \Big|^2 \right)^{2k} \leq \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} E\Big(\frac{\|h_{ni}\| + E\|h_{ni}\|}{n} \Big)^2 \right)^{2k}$$

$$\leq 4^{2k} \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} E\Big(\frac{\|h_{ni}\|}{n} \Big)^2 \right)^{2k} \leq 4^{2k} \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} E\Big\| \frac{f_{ni}}{n} \Big\|^2 \right)^{2k} < \infty.$$

Hence, applying the SLLN for the triangular array of rowwise independent random variables $\{\|h_{ni}\| - E\|h_{ni}\| : n \geq 1, 1 \leq i \leq n\}$ (see [14, Theorem 2.1]), we get

$$\frac{1}{n} \sum_{i=1}^{n} \big(\|h_{ni}\| - E\|h_{ni}\|\big) \to 0 \text{ a.s. as } n \to \infty.$$

Combining this with (6), we obtain

$$J_1 = \frac{1}{n} \sum_{i=1}^{n} \|h_{ni}\| \to 0 \text{ a.s. as } n \to \infty.$$

For J_2, $\|g_{ni} - W_{ni}\| = \|f_{ni} - U_{ni}\| \mathbf{1}_{(f_{ni}(.) \in K)} < \varepsilon$. Therefore, $J_2 < \varepsilon$.

For J_4, $\|EW_{ni} - Eg_{ni}\| \leq E\|W_{ni} - g_{ni}\| < \varepsilon$. Then, $J_4 < \varepsilon$.

By (5), $J_5 \leq \frac{1}{n} \sum\limits_{i=1}^{n} E\|h_{ni}\| < \varepsilon$.

For J_3, $\displaystyle \frac{1}{n} \sum_{i=1}^{n} \|W_{ni} - EW_{ni}\| \leq \sum_{j=1}^{m} \|x_j\| \Big| \frac{1}{n} \sum_{i=1}^{n} \big(\mathbf{1}_{(W_{ni} = x_j)} - P(W_{ni} = x_j) \big) \Big|.$

Since $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent random elements, for each $j = 1, 2, \dots, m$, $\{\mathbf{1}_{(W_{ni} = x_j)} - P(W_{ni} = x_j) : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent random variables with mean 0 and for every $t > 0$,

$$P(|\mathbf{1}_{(W_{ni} = x_j)} - P(W_{ni} = x_j)| > t) \leq P(\mathbf{1}_{(W_{ni} = x_j)} + P(W_{ni} = x_j) > t)$$
$$\leq P(|f| > t), \text{ where } f(\omega) = 2 \text{ for all } \omega \in \Omega.$$

By applying the SLLN for the triangular array $\{\mathbf{1}_{(W_{ni} = x_j)} - P(W_{ni} = x_j) : n \geq 1, 1 \leq i \leq n\}$ of rowwise independent random variables (see [13, Theorem 2]), we get for each $j = 1, 2, \dots, m$,

$$\frac{1}{n} \sum_{i=1}^{n} \big(\mathbf{1}_{(W_{ni} = x_j)} - P(W_{ni} = x_j) \big) \to 0 \text{ completely as } n \to \infty,$$

Therefore, $J_3 \to 0$ completely as $n \to \infty$.

From the above arguments, we obtain

$$\Big\|\frac{1}{n}\sum_{i=1}^{n}(f_{ni}-Ef_{ni})\Big\| \to 0 \text{ a.s. as } n\to\infty. \tag{7}$$

For the case $r=1$, by applying the SLLN for the triangular array of rowwise independent random variables $\{\|h_{ni}\|-E\|h_{ni}\| : n\geq 1, 1\leq i\leq n\}$ (see [14, Theorem 2.2]) and by the same proof as above, we obtain (7).

Next, for the case $r=0$, by applying the SLLN for the triangular array of rowwise independent random variables $\{\|h_{ni}\|-E\|h_{ni}\| : n\geq 1, 1\leq i\leq n\}$ (see [14, Theorem 2.3]) and by the same proof as above, we also obtain (7). □

Remark 3.3. There are many functions satisfying the conditions (1) and (2). For example, the exponentiation functions $\psi(t) = a.|t|^{r+\varepsilon}$ and the functions $\psi(t) = a.|t|^r.\log_b(|t|+c)$, for some nonnegative integer r and the real numbers a, b, c, ε satisfy $a>0$, $b>1$, $c\geq e$, $0<\varepsilon<1$ (where e is the base of the natural logarithm). □

Remark 3.4. Similar as in [14], if the condition (1) is satisfied for $r=0$, $r=1$, then the conclusion of Lemma 3.2 is true without the condition (4). □

Beer-Tamaki's lemma for the multidimensional case with the convergence as $\mathbf{n}_{\min}\to\infty$ was provided in [8, Lemma 4.5]. In the following we get a similar result for the triangular array case.

Lemma 3.5. *Let $A\in c(\mathfrak{X})$ and $\{A_{ni} : n\geq 1, 1\leq i\leq n\}$ be a triangular array in $c(\mathfrak{X})$. If $(A_{ni})^r \to A^r$ as $n\to\infty$ in the sense of Painlevé-Kuratowski, relative to the topology s, for every $r>0$, then $A_{ni}\to A$ as $n\to\infty$ in the Wijsman topology.*

4. Main results

The first two theorems will establish SLLNs for triangular arrays of rowwise independent and CUI random sets with respect to the gap topology.

Theorem 4.1. *Let $\{F_{ni} : n\geq 1, 1\leq i\leq n\}$ be a triangular array of rowwise independent and CUI random sets in a separable Banach space. Let $\psi(t)$ be a positive, even, convex, continuous function and $\psi(t)$ satisfies* (1) *and* (2) *for some nonnegative integer r. If the following conditions are satisfied*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\big(\psi(\|F_{ni}\|)\big)}{\psi(n)}<\infty, \quad \text{and} \quad \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}E\Big\|\frac{F_{ni}}{n}\Big\|^2\right)^{2k}<\infty, \tag{8}$$

for some positive integer k and there exists $X\in k(\mathfrak{X})$ such that

$$X\subset s\text{-}\liminf_{n\to\infty}(\mathrm{cl}E(F_{ni}|\mathcal{A}_{F_{ni}})); \tag{9}$$

$$\limsup_{n\to\infty} s(x^*,\mathrm{cl}EF_{ni})\leq s(x^*,X) \quad \text{for all } x^*\in\mathfrak{X}^*, \tag{10}$$

then $\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n}F_{ni}(\omega)\to\overline{\mathrm{co}}X$ *almost surely, in the gap topology, as $n\to\infty$.*

Proof. For any $x \in \overline{\text{co}}X$ and $\varepsilon > 0$, there exists $x_1, \dots, x_m \in X$ such that

$$\Big\|\frac{1}{m}\sum_{j=1}^{m} x_j - x\Big\| < \varepsilon. \tag{11}$$

By condition (9), for each $x_j \in X$, $j = 1, \dots, m$, there exists $z_{ni}^j \in \text{cl}E(F_{ni}|\mathcal{A}_{F_{ni}})$ such that $\|z_{ni}^j - x_j\| \to 0$ as $n \to \infty$. Therefore, there exists $y_{ni}^j \in E(F_{ni}|\mathcal{A}_{F_{ni}})$ such that $\|y_{ni}^j - z_{ni}^j\| \to 0$ as $n \to \infty$.

Hence, there exists a triangular array $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ of $f_{ni} \in S^1_{F_{ni}}(\mathcal{A}_{F_{ni}})$ such that

$$\big\|Ef_{n,(i-1)m+j} - x_j\big\| \to 0, \text{ for each } j = 1, 2, \dots, m. \tag{12}$$

Put $y_{ni} = Ef_{ni}$ for all $n \geq 1, 1 \leq i \leq n$. If $n = (l-1)m + p$, where $1 \leq p \leq m$, then

$$\begin{aligned}\Big\|\frac{1}{n}\sum_{i=1}^{n} f_{ni}(\omega) - \frac{1}{m}\sum_{j=1}^{m} x_j\Big\| &\leq \Big\|\frac{1}{n}\sum_{i=1}^{n}\big(f_{ni}(\omega) - y_{ni}\big)\Big\| + \Big\|\frac{1}{n}\sum_{i=1}^{n} y_{ni} - \frac{1}{m}\sum_{j=1}^{m} x_j\Big\| \\ &\leq \Big\|\frac{1}{n}\sum_{i=1}^{n}\big(f_{ni}(\omega) - y_{ni}\big)\Big\| + \frac{l}{n}\sum_{j=1}^{m}\frac{1}{l}\sum_{i=1}^{l}\Big\|\big(y_{n,(i-1)m+j} - x_j\big)\Big\| \\ &\quad + \frac{1}{n}\sum_{j=p+1}^{m}\|y_{n,(l-1)m+j}\| + \Big(\frac{l}{n} - \frac{1}{m}\Big)\Big\|\sum_{j=1}^{m} x_j\Big\| := I_1 + I_2 + I_3 + I_4.\end{aligned}$$

For I_1, since $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent and CUI random sets, by Lemma 3.1 we get that $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent and CUI random elements.

Moreover, by (8) $\psi(|t|)$ is an increasing function of $|t|$, and we obtain

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\big(\psi(\|f_{ni}\|)\big)}{\psi(n)} \leq \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\big(\psi(\|F_{ni}\|)\big)}{\psi(n)} < \infty,$$

$$\sum_{n=1}^{\infty}\bigg(\sum_{i=1}^{n} E\Big\|\frac{f_{ni}}{n}\Big\|^2\bigg)^{2k} \leq \sum_{n=1}^{\infty}\bigg(\sum_{i=1}^{n} E\Big\|\frac{F_{ni}}{n}\Big\|^2\bigg)^{2k} < \infty,$$

for some positive integer k. By applying Lemma 3.2, we obtain

$$I_1 = \Big\|\frac{1}{n}\sum_{i=1}^{n}\big(f_{ni} - Ef_{ni}\big)\Big\| \to 0 \text{ a.s. as } n \to \infty.$$

For I_2 and I_3, by the CUI condition of the triangular array $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$, for each $j = 1, 2, \dots, m$, fix $\varepsilon_0 > 0$, there exists a compact subset K of $\mathfrak{X}$ with

$$\sup_{n,i} E\Big(\|F_{ni}\|\mathbf{1}_{(F_{ni}(.)\subset K)^c}\Big) < \varepsilon_0.$$

This implies

$$\begin{aligned}\|EF_{ni}\| \le E\|F_{ni}\| &= E\big(\|F_{ni}\|\mathbf{1}_{(F_{ni}(.)\subset K)}\big) + E\big(\|F_{ni}\|\mathbf{1}_{(F_{ni}(.)\subset K)^c}\big)\\ &\le \|K\| + \varepsilon_0 := C \quad \text{for all} \quad n \ge 1,\ 1 \le i \le n. \qquad (13)\end{aligned}$$

Therefore, for all $l \ge 1$, $1 \le i \le l$,

$$\begin{aligned}&\big\|y_{(l-1)m+p,(i-1)m+j} - x_j\big\| = \big\|Ef_{n,(i-1)m+j} - x_j\big\|\\ &\quad\le \big\|Ef_{n,(i-1)m+j}\big\| + \|x_j\| \le \big\|EF_{n,(i-1)m+j}\big\| + \|x_j\| \le C + \|x_j\| \qquad (14)\end{aligned}$$

Combining (12) and (14), we have that $\{\|y_{(l-1)m+p,(i-1)m+j} - x_j\| : l \ge 1, 1 \le i \le l\}$ satisfies all the assumptions of [15, Lemma 3.3]. Therefore, by applying this lemma for the triangular array $\{\|y_{(l-1)m+p,(i-1)m+j} - x_j\| : l \ge 1, 1 \le i \le l\}$ we get

$$\frac{1}{l}\sum_{i=1}^{l}\big\|y_{n,(i-1)m+j} - x_j\big\| \to 0 \text{ as } l \to \infty.$$

Hence, $I_2 \to 0$ as $n \to \infty$ and, as $n \to \infty$,

$$\begin{aligned}I_3 &= \frac{1}{n}\sum_{j=p+1}^{m}\Big\|\sum_{i=1}^{l} y_{n,(i-1)m+j} - \sum_{i=1}^{l-1} y_{n,(i-1)m+j}\Big\|\\ &= \frac{l}{n}\sum_{j=p+1}^{m}\Big\|\frac{1}{l}\sum_{i=1}^{l}\big(y_{n,(i-1)m+j} - x_j\big)\\ &\qquad\qquad - \frac{l-1}{l}.\frac{1}{l-1}\sum_{i=1}^{l-1}\big(y_{n,(i-1)m+j} - x_j\big) + \big(1 - \frac{l-1}{l}\big)x_j\Big\|\\ &\le \sum_{j=1}^{m}\Bigg(\frac{1}{l}\sum_{i=1}^{l}\Big\|y_{n,(i-1)m+j} - x_j\Big\|\\ &\qquad\qquad + \frac{l-1}{l}.\frac{1}{l-1}\sum_{i=1}^{l-1}\Big\|y_{n,(i-1)m+j} - x_j\Big\| + \frac{1}{l}\Big\|x_j\Big\|\Bigg) \to 0.\end{aligned}$$

Moreover, since $\left(\frac{l}{n} - \frac{1}{m}\right) \to 0$ as $n \to \infty$, $I_4 \to 0$ as $n \to \infty$. Therefore,

$$\Big\|\frac{1}{n}\sum_{i=1}^{n} f_{ni}(\omega) - \frac{1}{m}\sum_{j=1}^{m} x_j\Big\| \to 0 \text{ a.s. as } n \to \infty.$$

This implies that $\frac{1}{m}\sum\limits_{j=1}^{m} x_j \in s\text{-}\liminf\limits_{n\to\infty}\big(\frac{1}{n}\mathrm{cl}\sum\limits_{i=1}^{n} F_{ni}(\omega)\big)$ a.s. Combining this with (11), we get

$$\overline{\mathrm{co}}X \subset s\text{-}\liminf_{n\to\infty}\Big(\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega)\Big) \text{ a.s.}$$

Therefore, there is a null subset N_1 of Ω so that, for every $\omega \in \Omega \setminus N_1$ and $x \in \overline{\text{co}}X$, we can choose $x_n \in \frac{1}{n}\text{cl}\sum_{i=1}^n F_{ni}(\omega)$ such that $\lim_{n\to\infty} x_n = x$. Hence, for any $y \in r.B$, $\lim_{n\to\infty}(x_n + y) = x + y$, where B is the closed unit ball with respect to the fixed arbitrary equivalent norm of $\mathfrak{X}$ and $r > 0$. Thus, we obtain

$$\overline{\text{co}}X + r.B \subset s\text{-}\liminf_{n\to\infty}\Big(\frac{1}{n}\text{cl}\sum_{i=1}^n F_{ni}(\omega) + r.B\Big).$$

Since $X \in k(\mathfrak{X})$, by applying [16, Lemma 3.4], we have $\overline{\text{co}}X + r.B = \overline{\text{co}}(X + r.B)$. Combining this with the separation theorem,

$$\overline{\text{co}}X + r.B = \bigcap_{x^*\in S^*} \{x : \langle x^*, x\rangle \leq s(x^*, \overline{\text{co}}X + r.B)\}.$$

Since $\mathfrak{X}$ is separable, there exists a countable Mackey dense subset D^* of S^* such that

$$\overline{\text{co}}X + r.B = \bigcap_{x^*\in D^*} \{x : \langle x^*, x\rangle \leq s(x^*, \overline{\text{co}}X + r.B)\}.$$

Next, by (9) and [11, Lemma 3.1], we easily obtain

$$s(x^*, X) \leq \liminf_{n\to\infty} s(x^*, \text{cl}EF_{ni}).$$

Combining this with (10), we obtain

$$E(s(x^*, F_{ni})) = s(x^*, \text{cl}EF_{ni}) \to s(x^*, X) = s(x^*, \overline{\text{co}}X) < \infty \text{ as } n \to \infty.$$

On the other hand,

$$\begin{aligned}|E(s(x^*, F_{ni})) - s(x^*, \overline{\text{co}}X)| &\leq |E(s(x^*, F_{ni}))| + |s(x^*, \overline{\text{co}}X)| \\ &\leq E\|F_{ni}\| + |s(x^*, \overline{\text{co}}X)| \leq C + |s(x^*, \overline{\text{co}}X)|\end{aligned}$$

for all $n \geq 1$, $1 \leq i \leq n$. By applying [15, Lemma 3.3] for the triangular array $\{E(s(x^*, F_{ni})) - s(x^*, \overline{\text{co}}X) : n \geq 1, 1 \leq i \leq n\}$, we have

$$\frac{1}{n}\sum_{i=1}^n E\big(s(x^*, F_{ni})\big) \to s(x^*, \overline{\text{co}}X) \text{ as } n \to \infty. \tag{15}$$

Furthermore, since $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent random sets, $\{s(x^*, F_{ni}) : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent random variables. Therefore, $\{s(x^*, F_{ni}) - E\big(s(x^*, F_{ni})\big) : n \geq 1, 1 \leq i \leq n\}$ is also a triangular array of rowwise independent random variables with mean 0 and by (1), (2), and (8) we obtain

$$\sum_{n=1}^\infty \sum_{i=1}^n \frac{E\Big(\psi\big(\big|s(x^*, F_{ni}) - E\big(s(x^*, F_{ni})\big)\big|\big)\Big)}{\psi(n)}$$

$$\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\Big(\psi\big(\big|s(x^*,F_{ni})\big|+\big|E\big(s(x^*,F_{ni})\big)\big|\big)\Big)}{\psi(n)}$$

$$\leq C_1\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\Big(\psi\big(\big|s(x^*,F_{ni})\big|\big)+\psi\big(\big|E\big(s(x^*,F_{ni})\big)\big|\big)\Big)}{\psi(n)}$$

$$\leq 2C_1\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\Big(\psi\big(\big|s(x^*,F_{ni})\big|\big)\Big)}{\psi(n)}\leq 2C_1\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\Big(\psi\big(\big\|F_{ni}\big\|\big)\Big)}{\psi(n)}<\infty,$$

and, for some positive integer k,

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}E\left|\frac{s(x^*,F_{ni})-E\big(s(x^*,F_{ni})\big)}{n}\right|^2\right)^{2k}$$

$$\leq \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E\Big(\big|s(x^*,F_{ni})\big|+\big|E\big(s(x^*,F_{ni})\big)\big|\Big)^2}{n^2}\right)^{2k}$$

$$\leq \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{2^2E\big|s(x^*,F_{ni})\big|^2}{n^2}\right)^{2k}\leq 4^{2k}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E\big\|F_{ni}\big\|^2}{n^2}\right)^{2k}$$

$$=4^{2k}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}E\left\|\frac{F_{ni}}{n}\right\|^2\right)^{2k}<\infty.$$

Hence, the triangular array $\{s(x^*,F_{ni})-E\big(s(x^*,F_{ni})\big):n\geq 1,1\leq i\leq n\}$ satisfies all the conditions of Lemma 3.2. This implies that

$$\frac{1}{n}\sum_{i=1}^{n}\Big(s(x^*,F_{ni}(\omega))-E\big(s(x^*,F_{ni})\big)\Big)\to 0 \text{ a.s. as } n\to\infty. \tag{16}$$

Combining (15) with (16), we obtain

$$s\Big(x^*,\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n}F_{ni}(\omega)\Big)=\frac{1}{n}\sum_{i=1}^{n}s(x^*,F_{ni}(\omega))\to s(x^*,\overline{\mathrm{co}}X) \text{ a.s. as } n\to\infty.$$

Therefore, there is a null subset N_2 of Ω so that, for every $\omega\in\Omega\setminus N_2$,

$$s\Big(x^*,\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n}F_{ni}(\omega)\Big)=\frac{1}{n}\sum_{i=1}^{n}s(x^*,F_{ni}(\omega))\to s(x^*,\overline{\mathrm{co}}X) \text{ as } n\to\infty.$$

Fix any $\omega\in\Omega\setminus N_2$. If $z\in s\text{-}\limsup_{n\to\infty}\Big(\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n}F_{ni}(\omega)+r.B\Big)$, then there exists $x_s\in\frac{1}{n_s}\mathrm{cl}\sum_{i=1}^{n_s}F_{n_si}(\omega)$ and $y_s\in r.B$ for $s\geq 1$ such that $z=\lim_{s\to\infty}(x_s+y_s)$. And so

$$\begin{aligned}
\langle x^*, z\rangle &= \lim_{s\to\infty}\Big(\langle x^*, x_s\rangle + \langle x^*, y_s\rangle\Big)\\
&\le \limsup_{s\to\infty} s\Big(x^*, \frac{1}{n_s}\mathrm{cl}\sum_{i=1}^{n_s} F_{n_s i}(\omega)\Big) + \limsup_{s\to\infty} s(x^*, r.B)\\
&= s(x^*, \overline{\mathrm{co}}X) + s(x^*, r.B) = s(x^*, \overline{\mathrm{co}}X + r.B) \text{ for every } x^* \in D^*.
\end{aligned}$$

This implies $z \in \overline{\mathrm{co}}X + r.B$. Hence,

$$s\text{-}\limsup_{n\to\infty}\Big(\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega) + r.B\Big) \subset \overline{\mathrm{co}}X + r.B \text{ for all } \omega \in \Omega\setminus N_2.$$

Put $N = N_1 \cup N_2$. Then N has probability zero and for every $\omega \in \Omega \setminus N$,

$$\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega) + r.B \to \overline{\mathrm{co}}X + r.B,$$

in the Painlevé-Kuratowski sense, relative to the topology s, as $n \to \infty$, for all $r > 0$ and for B being the unit ball of any equivalent norm, in which the choice of N_1 and N_2 does not depend on B and r.

Therefore, by Lemma 3.5 and by the equality $A^r = A + r.B$,

$$\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega) \to \overline{\mathrm{co}}X \text{ as } n\to\infty,$$

in all the Wijsman topologies generated by equivalent norms, except on the set $\Omega\setminus N$. Combining this with [3, Theorem 5.2] and [4, Theorem 3.1], we obtain this convergence for the gap topology in the convex case.

Thus, in the convex case, we achieve convergence in the slice topology or the gap topology. It remains to prove that this convergence extends to the non-convex case. Let A be a non-empty bounded closed convex set. We must prove that

$$D(\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega), A) \to D(\overline{\mathrm{co}}X, A) \text{ a.s. as } n\to\infty$$

except on a null set simultaneously valid for all such A.

By the definition of the gap functional, we get

$$D(\overline{\mathrm{co}}X, A) = \inf_{x\in\overline{\mathrm{co}}X}\inf_{y\in A}\|x-y\| = \inf_{y\in A} d(y, \overline{\mathrm{co}}X).$$

Therefore,
$$D(\overline{\mathrm{co}}X, A) \le d(y, \overline{\mathrm{co}}X) \text{ for all } y \in A. \tag{17}$$

Moreover, for any $\varepsilon' > 0$, we can find $y \in A$ such that

$$|D(\overline{\mathrm{co}}X, A) - d(y, \overline{\mathrm{co}}X)| \le \varepsilon'. \tag{18}$$

Furthermore, by (17), we get

$$D\Big(\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n}\overline{\mathrm{co}}F_{ni}(\omega), A\Big) \le D\Big(\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n}F_{ni}(\omega), A\Big) \le d\Big(y, \frac{1}{n}\mathrm{cl}\sum_{i=1}^{n}F_{ni}(\omega)\Big). \quad (19)$$

For any $\omega \in \Omega \setminus N$, by virtue of our intermediate results for the gap and Wijsman topologies, we obtain

$$D\Big(\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n}\overline{\mathrm{co}}F_{ni}(\omega), A\Big) \to D(\overline{\mathrm{co}}X, A) \;\text{ as } n \to \infty, \quad (20)$$

and
$$\lim_{n\to\infty} d\Big(y, \frac{1}{n}\mathrm{cl}\sum_{i=1}^{n}F_{ni}(\omega)\Big) = d(y, \overline{\mathrm{co}}X) \;\text{ for all } y \in A. \quad (21)$$

Combining (18), (19), (20) and (21), we conclude that

$$\limsup_{n\to\infty}\Big|D\Big(\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n}F_{ni}(\omega), A\Big) - D(\overline{\mathrm{co}}X, A)\Big| \le \varepsilon'.$$

This with the arbitrariness of $\varepsilon' > 0$ yields the desired result irrespective of the nonempty closed bounded convex set A. □

In the proof of Theorem 4.1, we need the CUI condition of the triangular array $\{F_{ni} : n \ge 1, 1 \le i \le n\}$. The following example shows that the CUI condition is not deducible from the other conditions. Namely, we can choose a triangular array $\{F_{ni} : n \ge 1, 1 \le i \le n\}$ of random sets such that the CUI condition is not satisfied although all other conditions are satisfied.

Example 4.2. Consider the Banach space $\mathbb{X} = \mathbb{R}$. We can choose $\psi(t) = |t|^{5+\varepsilon}$, where ε is a certain real number satisfying $0 < \varepsilon < 1$. Then $\psi(t)$ satisfies the condition (1) for $r = 5$ and (2).

We define the random sets $F_{ni}(\omega) = [-n^{1/7}, n^{1/7}]$ for all $\omega \in \Omega$, $n \ge 1$, $1 \le i \le n$. Clearly, $\mathcal{A}_{F_{ni}} = \{\emptyset, \Omega\}$. Therefore, we infer that $\{F_{ni} : n \ge 1, 1 \le i \le n\}$ is a triangular array of rowwise independent random sets and for all $n \ge 1$, $1 \le i \le n$, $EF_{ni} = [-n^{1/7}, n^{1/7}]$. Hence, EF_{ni} is not bounded for all $n \ge 1$, $1 \le i \le n$. This implies that $\{F_{ni} : n \ge 1, 1 \le i \le n\}$ is not a triangular array of CUI random sets.

Moreover,
$$\|F_{ni}\|(\omega) = \sup\{\|f\|, f \in F_{ni}(\omega)\} = n^{1/7},$$

$$\Big\|\frac{F_{ni}}{n}\Big\|(\omega) = \sup\{\|f\|, f \in \frac{F_{ni}}{n}(\omega)\} = \frac{1}{n^{6/7}}.$$

Then

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\big(\psi(\|F_{ni}\|)\big)}{\psi(n)} = \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{n^{\frac{5+\varepsilon}{7}}}{n^{5+\varepsilon}} = \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{1}{n^{4+\frac{2+6\varepsilon}{7}}} = \sum_{n=1}^{\infty}\frac{1}{n^{3+\frac{2+6\varepsilon}{7}}} < \infty;$$

$$\sum_{n=1}^{\infty}\Bigg(\sum_{i=1}^{n}\Big\|\frac{F_{ni}}{n}\Big\|^2\Bigg)^{2k} = \sum_{n=1}^{\infty}\Big(\sum_{i=1}^{n}\frac{1}{n^{12/7}}\Big)^{2k} = \sum_{n=1}^{\infty}\frac{1}{n^{10k/7}} < \infty.$$

Thus, the triangular array $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ is rowwise independent and satisfies the conditions (1), (2), and (8), but the CUI condition does not hold for it. $\square$

Theorem 4.3. *Let* $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ *be a triangular array of rowwise independent and CUI random sets in a separable Banach space.*

If $\{\|F_{ni}\| : n \geq 1, 1 \leq i \leq n\}$ *are uniformly bounded by a random variable* ξ *with* $E\xi^2 < \infty$ *and there exists* $X \in k(\mathfrak{X})$ *satisfying* (9) *and* (10), *then*

$$\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega) \to \overline{\mathrm{co}}X$$

almost surely, in the gap topology, as $n \to \infty$.

Proof. For any $x \in \overline{\mathrm{co}}X$ and $\varepsilon > 0$, there exists $x_1, \ldots, x_m \in X$ satisfying (11). By the same argument as in the proof of Theorem 4.1, there exists a triangular array $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ of $f_{ni} \in S^1_{F_{ni}}(\mathbb{A}_{F_{ni}})$ satisfying (12).

Putting $y_{ni} = Ef_{ni}$ for all $n \geq 1, 1 \leq i \leq n$. If $n = (l-1)m+p$, where $1 \leq p \leq m$, then

$$\begin{aligned}
\Big\|\frac{1}{n}\sum_{i=1}^{n} f_{ni}(\omega) - \frac{1}{m}\sum_{j=1}^{m} x_j\Big\| &\leq \Big\|\frac{1}{n}\sum_{i=1}^{n}\big(f_{ni}(\omega) - y_{ni}\big)\Big\| \\
&+ \frac{l}{n}\sum_{j=1}^{m}\frac{1}{l}\sum_{i=1}^{l}\Big\|y_{n,(i-1)m+j} - x_j\Big\| + \frac{1}{n}\sum_{j=p+1}^{m}\|y_{n,(l-1)m+j}\| + (\frac{l}{n} - \frac{1}{m})\Big\|\sum_{j=1}^{m} x_j\Big\| \\
&:= I_5 + I_6 + I_7 + I_8.
\end{aligned}$$

For I_6, I_7, I_8, by the same argument as for I_2, I_3, I_4 of Theorem 4.1, we obtain

$$I_6 \to 0,\ I_7 \to 0 \text{ and } I_8 \to 0 \text{ as } n \to \infty.$$

For I_5, since $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent and CUI random sets, by Lemma 3.1 we have that $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent and CUI random elements. Therefore, $\{f_{ni} - Ef_{ni} : n \geq 1, 1 \leq i \leq n\}$ is also a triangular array of rowwise independent random elements with mean 0 and furthermore, for all $t > 0$,

$$\begin{aligned}
P(\|f_{ni} - Ef_{ni}\| > t) &\leq P(\|f_{ni}\| + \|Ef_{ni}\| > t) \\
&\leq P(\|F_{ni}\| + C > t) \leq P(|\xi| + C > t) \text{ with } E\xi^2 < \infty \quad \text{(by (13))}.
\end{aligned}$$

Therefore, by applying [13, Theorem 3], we have

$$I_5 = \Big\|\frac{1}{n}\sum_{i=1}^{n}\big(f_{ni}(\omega) - Ef_{ni}\big)\Big\| \to 0 \text{ a.s. as } n \to \infty.$$

From the above arguments, we obtain

$$\Big\|\frac{1}{n}\sum_{i=1}^{n} f_{ni}(\omega) - \frac{1}{m}\sum_{j=1}^{m} x_j\Big\| \to 0 \text{ a.s. as } n \to \infty.$$

This implies that $\overline{\mathrm{co}}X \subset s\text{-}\liminf_{n\to\infty}\Big(\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega)\Big)$ a.s. Similarly as in the proof of Theorem 4.1, we get

$$\overline{\mathrm{co}}X + r.B \subset s\text{-}\liminf_{n\to\infty}\Big(\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega) + r.B\Big) \text{ a.s.},$$

where B is the closed unit ball with respect to the fixed arbitrary equivalent norm of $\mathfrak{X}$ and $r > 0$. By (9), (10) and [11, Lemma 3.1], we obtain

$$E(s(x^*, F_{ni})) = s(x^*, \mathrm{cl}EF_{ni}) \to s(x^*, X) = s(x^*, \overline{\mathrm{co}}X) < \infty \quad \text{as } n \to \infty.$$

Thus,

$$\frac{1}{n}\sum_{i=1}^{n} E(s(x^*, F_{ni})) \to s(x^*, \overline{\mathrm{co}}X) \text{ as } n \to \infty.$$

On the other hand, since $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent random sets, $\{s(x^*, F_{ni}) : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent random variables. Therefore,

$$\{s(x^*, F_{ni}) - E\big(s(x^*, F_{ni})\big) : n \geq 1, 1 \leq i \leq n\}$$

is also a triangular array of rowwise independent random variables with mean 0 and for any real number $t > 0$, we get

$$\begin{aligned} P\Big(\Big|s(x^*, F_{ni}) - E\big(s(x^*, F_{ni})\big)\Big| > t\Big) &\leq P\Big(\big|s(x^*, F_{ni})\big| + \big|E\big(s(x^*, F_{ni})\big)\big| > t\Big) \\ &\leq P\Big(\|F_{ni}\| + E\|F_{ni}\| > t\Big) \leq P\Big(|\xi| + C > t\Big) \quad \text{(by (13))}. \end{aligned}$$

Therefore, by applying the SLLN for the triangular array of rowwise independent, CUI random variables (see [13, Theorem 2]), we get

$$\frac{1}{n}\sum_{i=1}^{n}\Big(s(x^*, F_{ni}(\omega)) - E\big(s(x^*, F_{ni})\big)\Big) \to 0 \quad \text{a.s. as } n \to \infty,$$

and so

$$s\Big(x^*, \frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega)\Big) = \frac{1}{n}\sum_{i=1}^{n} s(x^*, F_{ni}(\omega)) \to s(x^*, \overline{\mathrm{co}}X) \quad \text{a.s. as } n \to \infty.$$

Thus, there is a null subset N_2 of Ω so that, for every $\omega \in \Omega \setminus N_2$,

$$s\Big(x^*, \frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega)\Big) = \frac{1}{n}\sum_{i=1}^{n} s(x^*, F_{ni}(\omega)) \to s(x^*, \overline{\mathrm{co}}X) \quad \text{as } n \to \infty.$$

For the rest of the proof we proceed as in the proof of Theorem 4.1. $\square$

The theorem below will establish the SLLN for a triangular array of rowwise independent random sets in a separable Banach space of type p $(1 < p \leq 2)$ with respect to the gap topology, based on Terán's method [17] and the results of Quang and Giap in [15].

Theorem 4.4. *Let $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ be a triangular array of rowwise independent, bounded expectation random sets in a separable Banach space of type p $(1 < p \leq 2)$. Let $\{a_n\}$ be a sequence of positive real numbers such that $a_{n+1} > a_n$ and $\lim_{n\to\infty} a_n = +\infty$. Let $\psi(t)$ be a positive, even, convex, continuous function such that $\frac{\psi(|t|)}{|t|^r}$ is a monotone increasing function of $|t|$, $\frac{\psi(|t|)}{|t|^{r+p-1}}$ is a monotone decreasing function of $|t|$ for some nonnegative integer r, and there exists a positive constant C_1 so that $\psi(t)$ satisfies (2). If the following conditions are satisfied*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\big(\psi(\|F_{ni}\|)\big)}{\psi(n)} < \infty, \quad \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} E\Big\|\frac{F_{ni}}{n}\Big\|^p\right)^{p.k} < \infty,$$

for some positive integer k, and there exists $X \in k(\mathfrak{X})$ satisfying (9) and (10), then

$$\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega) \to \overline{\mathrm{co}}X$$

almost surely, in the gap topology, as $n \to \infty$.

Proof. By applying [15, Theorem 3.4], we get

$$\overline{\mathrm{co}}X \subset s\text{-}\liminf_{n\to\infty}\Big(\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega)\Big) \quad \text{a.s.}$$

Similarly as in the proof of Theorem 4.1, we obtain

$$\overline{\mathrm{co}}X + r.B \subset s\text{-}\liminf_{n\to\infty}\Big(\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega) + r.B\Big) \quad \text{a.s.},$$

where B is the closed unit ball with respect to the fixed arbitrary equivalent norm of $\mathfrak{X}$ and $r > 0$. By (9), (10) and [11, Lemma 3.1], we get

$$E(s(x^*, F_{ni})) = s(x^*, \mathrm{cl}EF_{ni}) \to s(x^*, X) = s(x^*, \overline{\mathrm{co}}X) < \infty \quad \text{as } n \to \infty.$$

Therefore,
$$\frac{1}{n}\sum_{i=1}^{n} E(s(x^*, F_{ni})) \to s(x^*, \overline{\mathrm{co}}X) \quad \text{as } n \to \infty.$$

Moreover, since $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent random sets, $\{s(x^*, F_{ni}) - E\big(s(x^*, F_{ni})\big) : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent random variables with mean 0 and

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\Big(\psi\big(\big|s(x^*,F_{ni})-E\big(s(x^*,F_{ni})\big)\big|\big)\Big)}{\psi(n)}$$

$$\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\Big(\psi\big(\big|s(x^*,F_{ni})\big|+\big|E\big(s(x^*,F_{ni})\big)\big|\big)\Big)}{\psi(n)} \quad (\text{by } \psi(|t|)\uparrow)$$

$$\leq C_1\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\Big(\psi\big(\big|s(x^*,F_{ni})\big|\big)+\psi\big(\big|E\big(s(x^*,F_{ni})\big)\big|\big)\Big)}{\psi(n)} \quad (\text{by } (2))$$

$$\leq C_1\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\Big(\psi\big(\big|s(x^*,F_{ni})\big|\big)+\psi\big(E\big|s(x^*,F_{ni})\big|\big)\Big)}{\psi(n)} \quad (\text{by } \psi(|t|)\uparrow)$$

$$\leq C_1\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\Big(\psi\big(\big|s(x^*,F_{ni})\big|\big)+E\big(\psi\big(\big|s(x^*,F_{ni})\big|\big)\big)\Big)}{\psi(n)} \quad (\text{convexity of } \psi(t))$$

$$= 2C_1\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\Big(\psi\big(\big|s(x^*,F_{ni})\big|\big)\Big)}{\psi(n)} \leq 2C_1\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\Big(\psi\big(\|x^*\|.\|F_{ni}\|\big)\Big)}{\psi(n)}$$

$$\leq 2C_1\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\Big(\psi\big(\|F_{ni}\|\big)\Big)}{\psi(n)} < \infty \quad (\text{by } \psi(|t|)\uparrow),$$

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}E\left|\frac{s(x^*,F_{ni})-E\big(s(x^*,F_{ni})\big)}{n}\right|^p\right)^{p.k}$$

$$\leq \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E\Big(\big|s(x^*,F_{ni})\big|+\big|E\big(s(x^*,F_{ni})\big)\big|\Big)^p}{n^p}\right)^{p.k}$$

$$\leq \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{2^{p-1}\Big(E\big(\big|s(x^*,F_{ni})\big|^p\big)+E\big|E\big(s(x^*,F_{ni})\big)\big|^p\Big)}{n^p}\right)^{p.k} (c_r\text{-inequality})$$

$$\leq \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{2^pE\Big(\big|s(x^*,F_{ni})\big|^p\Big)}{n^p}\right)^{p.k} \leq 2^{p^2k}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}E\left\|\frac{F_{ni}}{n}\right\|^p\right)^{p.k} < \infty,$$

for some positive integer k.

Hence, by applying the results of Bozorgnia et al. (see [5, Theorem 2.2] for $r \geq 2$, [5, Theorem 2.3] for $r = 1$ and [5, Theorem 2.4] for $r = 0$), we get

$$\frac{1}{n}\sum_{i=1}^{n}\Big(s(x^*,F_{ni}(\omega))-E\big(s(x^*,F_{ni})\big)\Big) \to 0 \text{ a.s. as } n\to\infty.$$

For the rest of proof, we proceed as in proof of Theorem 4.1. □

Acknowledgements. The research of the first author was funded by the Ministry of Education and Training of Vietnam, under grant number B2019-TDV-01. The research of the second and the fourth author was supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.03-2017.24, and by the Department of Science and Technology (HCMC-DOST) and the Institute for Computational Science and Technology (ICST) at Ho Chi Minh City, Vietnam, under grant number 14/2017/HD-KHCNTT.

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