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On the Convergence and Regularity of Aumann-Pettis Integrable Multivalued Martingales

Mohammed El Allali
Faculty of Sciences and Technology, University Sidi Mohamed Ben Abdellah, Fez, Morocco
elallalimohammed1@gmail.com

M'hamed El-Louh
Faculty of Sciences and Technology, University Sidi Mohamed Ben Abdellah, Fez, Morocco
ellouh.mhamed@gmail.com

Fatima Ezzaki
Faculty of Sciences and Technology, University Sidi Mohamed Ben Abdellah, Fez, Morocco
fatimaezzaki@yahoo.fr

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We prove a representation of Aumann-Pettis integrable multivalued martingales by Pettis integrable martingale selectors. Regularity of Aumann-Pettis integrable multivalued martingales and their convergence in Mosco sense, Wijsman topology, and linear topology are established.

Keywords: Pettis multivalued martingales, regularity, representation theorem, Mosco convergence, linear topology.

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1. Introduction

There is a rich literature describing integrability of multifunctions with values in closed convex and bounded (resp. convex weakly compact) subsets of a separable Banach space. Most of the papers concern the Bochner type integration, but there is also quite a number of them dealing with the Pettis integrability. The purpose of this work is to partially reduce this gap by developing new results of martingales consisting of Aumann-Pettis integrable multifunctions.

The notion of multivalued martingales in Bochner type integration has been studied by several authors. This study is based on measurability and inegrability of multifunctions, on the existence of the conditional expectation of integrable multifunctions and on set convergence.

The extension of this notion to Pettis integrable multivalued martingales first requires the existence of the conditional expectation of Pettis integrable multifunctions. But unlike the conditional expectation of a Bochner integrable random variable, the conditional expectation of a Pettis integrable function does not generally exist.

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Recently some papers have been published in which the authors have proved the existence of the conditional expectation of a Pettis integrable function as Egghe [10], and of Pettis integrable functions and Pettis integrable multifunctions as Akhiat, Castaing and Ezzaki [1], EL harami and Ezzaki [12], Akhiat, El Harami and Ezzaki [2] and others.
The main question raised by martingales consisting of Pettis integrable multifunctions is the existence of their Pettis integrable martingales selectors and their representation by martingales selectors. This problem has been studied for integrably bounded and integrable multivalued martingales by many authors, see Luu [15], Hess [13], Choukairi [9], Zhen-Peng Wang and Xing-Hong Xue [21] and others. To our knowledge, this problem still remains open for Pettis integrable multivalued martingales.
The purpose of the present paper is to prove a new representation theorem of Aumann-Pettis integrable multivalued martingales by their Pettis integrable martingales selectors and to prove regularity of Aumann-Pettis integrable multivalued martingales and their convergence in several modes.
Some preliminaries are given in Section 2. In Section 3 we study the representation of Aumann-Pettis integrable multivalued martingales by their Pettis integrable martingales selectors. The last section presents applications of Sections 3 and 4 to the regularity of Aumann-Pettis integrable multivalued martingales and their convergence in Mosco, Wijsman and linear topologies.

2. Preliminaries and background

Throughout this paper $(\Omega, \mathcal{F}, P)$ is a complete probability space, $(\mathcal{F}_n)_{n\geq 1}$ is an increasing sequence of sub σ-algebras of $\mathcal{F}$ such that $\mathcal{F}$ is the σ-algebra generated by $\cup_{n\geq 1}\mathcal{F}_n$, E is a separable Banach space, E^* is its topological dual space and B^* is the closed unit ball of E^*.

A Banach space E is said to have the *Radon-Nikodym property*, abreviated by RNP, if for every vector measure $\lambda\colon \mathcal{F} \to E$ of σ-bounded variation, which is P-absolutely continuous, there exists a P-Bochner integrable function f such that $\lambda(A) = \int_A f dP$ for each $A \in \mathcal{F}$.

We denote by $cl(E)$ the family of all closed non-empty subsets of E. The following subfamilies will be considered:

$cc(E)$: the class of all closed convex non-empty subsets of E.

$ccb(E)$: the class of all closed bounded convex non-empty subsets of E.

$cwk(E)$: the class of all convex weakly compact non-empty subsets of E.

$lw^*(E)$: the class of all members of $cc(E)$, that are weakly locally compact and do not contain any line.

For $A \in 2^E \setminus \emptyset$, we denote by $\overline{A}$ and $\overline{co}A$ the *closure* and the *closed convex hull* of A respectively, and define $|A| = \sup\{\|x\|, x \in A\}$; the *support function* and the *distance function* associated with A are defined respectively by

$$s(x^*, A) = \sup\{\langle x^*, x\rangle, x \in A\},\ x^* \in E^*; \quad d(x, A) = \inf\{\|x-y\|, y \in A\},\ x \in E.$$

For $(A_n)_{n\geq 1} \subset cc(E)$ and $A \in cc(E)$, let

$$s - li\ A_n = \{x \in E, \exists\, x_k \xrightarrow{s} x : x_k \in A_k,\ k \geq 1\},$$

$$w - ls\ A_n = \{x \in E, \exists\, x_k \xrightarrow{w} x : x_k \in A_{n_k},\ k \geq 1\}.$$

s and w are respectively the norm topology and the weak topology of E.

We say that $(A_n)_{n\geq 1}$ is *Mosco convergent* to A, and we denote $M - lim\ A_n = A$ if

$$s - li\ A_n = A = w - ls\ A_n.$$

$(A_n)_{n\geq 1}$ is said to be *Wijsman convergent* to A if

$$\lim_{n\to\infty} d(x, A_n) = d(x, A), \qquad \forall\ x \in E.$$

We say that $(A_n)_{n\geq 1}$ is *convergent to A in the linear topology* and we write $\tau_L - \underset{n\to\infty}{lim} A_n = A$ if

$$\lim_{n\to\infty} s(x^*, A_n) = s(x^*, A),\ \forall\ x^* \in E^*, \quad \lim_{n\to\infty} d(x, A_n) = d(x, A),\ \forall\ x \in E.$$

Let $f\colon \Omega \longrightarrow E$. f is said to be *scalarly integrable* if the function $\langle x^*, f\rangle$ is integrable for each $x^* \in E^*$. We say that f is *Pettis integrable* if it is scalarly integrable and for each $A \in \mathcal{F}$, there exists $x_A \in E$ such that

$$\langle x^*, x_A\rangle = \int_A \langle x^*, f\rangle\, dP, \qquad \forall\ x^* \in E^*.$$

x_A is called the *Pettis integral* of f over A, and it is denoted by $\int_A f\ dP$.

We denote by $L^1(E)$ (resp, $\mathcal{P}_E$) the space of all Bochner integrable functions, endowed with the norm $\|f\|_1 = \int_\Omega |f(w)|\, dP$ (resp, the space of all Pettis integrable functions, endowed with the norm $\|f\|_{pe} = \sup_{x^*\in B^*}\ \|\langle x^*, f\rangle\|_1$).

An equivalent norm to the norm $\|.\|_{pe}$ is given by $|||f|||_{Pe} = \sup\,\{\|\int_A f\ dP\|,$ $A \in \mathcal{F}\}$, see [18, p.198].

A multifunction $F\colon \Omega \longrightarrow cl(E)$ is said to be *$\mathcal{F}$-measurable* if there exists a sequence $(f_n)_{n\geq 1}$ of $\mathcal{F}$-measurable functions defined from Ω to E such that

$$F(w) = \overline{\{f_n(w), n \geq 1\}}, \qquad \forall w \in \Omega,$$

such a sequence is called a *Castaing representation* of F. We denote the family of all $\mathcal{F}$-measurable multifunctions with values in $cl(E)$ by $\mathcal{M}\,[\Omega, \mathcal{F}, E]$. A $\mathcal{F}$-measurable function $f\colon \Omega \longrightarrow E$ is called a *selector* of F if $f(w) \in F(w)$ for each $w \in \Omega$, the set of all selectors of F is denoted by S_F . Let

$$S^1_F = \{f \in L^1(E),\ f(w) \in F(w)\ a.e\}, \quad S^{Pe}_F = \{f \in \mathcal{P}_E,\ f(w) \in F(w)\ a.e\}.$$

Let $F \in \mathcal{M}\,[\Omega, \mathcal{F}, E]$ with values in $cc(E)$, F is said to be

(a) *scalarly integrable* if for each $x^* \in B^*$, $s(x^*, F(.))$ is integrable.

(b) *Pettis integrable* if it is scalarly integrable and for each $A \in \mathcal{F}$ there exists $C_A \in cc(E)$ such that $s(x^*, C_A) = \int_A s(x^*, F(.))\, dP$, $\forall x^* \in E^*$. C_A is called the *Pettis integral* of F over A.

(c) *Aumann-Pettis integrable* if $S_F^{Pe} \neq \emptyset$. The Aumann-Pettis integral of F over A is defined by $Pe - \int_A F\, dP = \{\int_A f\, dP, f \in S_F^{Pe}\}$.

(d) *uniformly scalarly integrable* if the set $\{s(x^*, F_n(.)),\ x^* \in B^*, n \geq 1\}$ is uniformly integrable.

Every scalarly integrable and Aumann-Pettis integrable multifunction is Pettis integrable, see [11, theorem 3.7]. If F is Aumann-Pettis integrable then $C_A(F) = \overline{co}\ Pe - \int_A F\, dP$ for each $A \in \mathcal{F}$, with $C_A(F)$ is the Pettis integral of F over A, see [11, Corollary 3.10].

If F is uniformly scalarly integrable, then every selector of F is Pettis integrable, see [12, Proposition 2.3] and [18, Theorem 5.2].

Let F be a Pettis integrable multifunction, we define

$$\|F\|_{pe} = \sup_{x^* \in B^*} \|s(x^*, F(w))\|_1.$$

If F is a Pettis integrable multifunction with values in $ccb(E)$, then

$$\|F\|_{pe} < \infty,$$

see [11, Remark 3.4, (b)] and [17, P. 836].

Lemma 2.1. [20] *Let F be an Aumann-Pettis integrable multifunction with values in cc(E), then there exists a sequence $(f_n)_{n \geq 1} \subset S_F^{Pe}$ such that $\forall w \in \Omega$, $F(w) = \overline{\{f_n(w), n \geq 1\}}$.*

Remark 2.2. It follows from lemma 2.1 that if $F_1, F_2 \in \mathcal{M}\,[\Omega, \mathcal{F}, E]$ such that $S_{F_1}^{Pe} = S_{F_2}^{Pe} \neq \emptyset$, then $F_1 = F_2$ a.e.

3. Representation theorem for Aumann-Pettis integrable multivalued martingales

The study of set convergence requires the identification of the limit. The resolution of this problem has been studied by several methods. The best known are those using the compactness theorems. But the most natural are those using the selectors of multifunctions if they exist.

In this section, we give a representation theorem for Pettis integrable martingale with values in $ccb(E)$ by its Pettis integrable martingales selectors.

We begin by proving the existence of Pettis integrable selectors for a Pettis integrable multivalued martingale.

Let $\mathcal{B}$ be a sub σ-algebra of $\mathcal{F}$. We define

$$S_F^{Pe}(\mathcal{B}) = \{f \in S_F^{Pe},\ f\ is\ \mathcal{B}-measurable\},\ S_F(\mathcal{B}) = \{f \in S_F,\ f\ is\ \mathcal{B}-measurable\}.$$

Let $f \in \mathcal{P}_E$, the conditional expectation of f relative to $\mathcal{B}$, if it exists, it is denoted $E(f|\mathcal{B})$, and it is the unique $\mathcal{B}$-measurable function such that $\forall A \in \mathcal{B}$, $Pe - \int_A g\, dP = Pe - \int_A f\, dP$.

For the existence of the conditional expectation for Pettis integrable functions, we refer the reader to [1], [2] and [12] .

Definition 3.1. Let $\mathcal{C}$ be a family of closed convex subsets of E, and let D be a countable subset of B^*, we say that $\mathcal{C}$ is *countably supported with respect to* D or D*-countably supported* if for each $C \in \mathcal{C}$ the following equality holds

$$C = \cap_{x^* \in D}\{x \in E, \langle x^*, x\rangle \leq s(x^*, C)\}.$$

Remark 3.2. The family $lw^*(E)$ and all its subspaces such as $cwk(E)$ and $ck(E)$ are countably supported with respect to any countable subset $\{x_n^*, n \geq 1\}$ of B^* such that $x_n^* = \frac{y_n^*}{\|y_n^*\|}$ where $(y_n^*)_{n\geq 1}$ is a dense sequence in E^* with respect to the Mackey topology $\tau(E^*, E)$, see [8, Lemma III.34]. If E^* is separable then the family $ccb(E)$ is countably supported with respect to any countable dense subset D of B^* with respect to the strong topology. If a multifunction $F \in \mathcal{M}[\Omega, \mathcal{F}, E]$ is countably supported a.e. (i.e., there exists a countable subset $D = \{x_n^*, n \geq 1\}$ of B^* such that $F(w) = \cap_{x^* \in D}\{x \in E, \langle x^*, x\rangle \leq s(x^*, F(w))\}$ a.e.), then S_F^{Pe} is closed in $\mathcal{P}_E$, see [11, Proposition 4.2]).

Let $\mathcal{K}$ be a countably supported family of $cc(E)$, and let F be an Aumann-Pettis integrable multifunction with values in $\mathcal{K}$. we say that the conditional expectation of F relative to $\mathcal{B}$ exists if

(a) For each $f \in S_F^{Pe}$, $E(f|\mathcal{B})$ exists.

(b) There exists an Aumann-Pettis integrable and $\mathcal{B}$-measurable multifunction G with values in $\mathcal{K}$ such that

$$\overline{Pe - \int_A G\, dP} = \overline{Pe - \int_A F\, dP}, \qquad \forall A \in \mathcal{B}.$$

By [11, Proposition 4.1], such a multifunction G is unique if it exists. G is called the *conditional expectation* of F relative to $\mathcal{B}$, and it is denoted by $E(F|\mathcal{B})$.
By [20, Theorem 4.1], $E(F|\mathcal{B})$ is the unique $\mathcal{B}$-measurable multifunction satisfying the following equality:

$$S_{E(F|\mathcal{B})}^{Pe}(\mathcal{B}) = \overline{\{E(f|\mathcal{B}), f \in S_F^{Pe}\}}^{\|\cdot\|_{Pe}}.$$

For the existence of the conditional expectation for Pettis integrable multifunctions, we refer the reader to [1], [2], [12] and [20].

Definition 3.3. An adapted sequence $(F_n, \mathcal{F}_n)_{n\geq 1}$ of Aumann-Pettis integrable multifunctions with values in $cc(E)$ is said to be a *martingale* if $E(F_k|\mathcal{F}_n)$ exists $\forall k > n \geq 1$, and

$$E(F_{n+1}|\mathcal{F}_n) = F_n, \qquad \forall n \geq 1. \tag{1}$$

By remark 2.2, (1) holds if and only if

$$S_{F_n}^{Pe}(\mathcal{F}_n) = S_{E(F_{n+1}|\mathcal{F}_n)}^{Pe}(\mathcal{F}_n) = \overline{\{E(f|\mathcal{F}_n), f \in S_{F_{n+1}}^{Pe}(\mathcal{F}_{n+1})\}}^{\|\cdot\|_{Pe}}, \quad \forall n \geq 1.$$

Definition 3.4. Let $(F_n, \mathcal{F}_n)_{n\geq 1}$ be an Aumann-Pettis integrable multivalued martingale with values in $cc(E)$, a Pettis integrable martingale $(f_n, F_n)_{n\geq 1}$ is called a *martingale selector* of $(F_n, \mathcal{F}_n)_{n\geq 1}$ if $f_n \in S^{Pe}_{F_n}$, $\forall n \geq 1$. The set of all Pettis martingale selectors of $(F_n, \mathcal{F}_n)_{n\geq 1}$ will be denoted by $PMS(F_n, \mathcal{F}_n)$.

Definition 3.5. An adapted sequence $(f_n, \mathcal{F}_n)_{n\geq 1}$ of Pettis integrable random variables with values in E is called a *Pettis quasi-martingale* if

$$\sum_{i=1}^{\infty} \|f_n - E(f_{n+1}|\mathcal{F}_n)\|_{Pe} < \infty.$$

Proposition 3.6. *Assume that E has RNP, and let $(F_n, \mathcal{F}_n)_{n\geq 1}$ be a martingale of Aumann-Pettis integrable multifunctions with values in $cc(E)$ such that there exists a partition $(A_k)_{k\geq 1}$ of Ω in $\mathcal{F}_1$ which satisfies*

$$\sup_{n\geq 1} \int_{A_k} |F_n| \, dP < \infty, \qquad \forall k \geq 1.$$

Then

(3.6.1) *$(F_n, \mathcal{F}_n)_{n\geq 1}$ admits a martingale selector.*

(3.6.2) *Each martingale selector of $(F_n, \mathcal{F}_n)_{n\geq 1}$ is a.e. convergent.*

(3.6.3) *If $\sup_{n\geq 1} \|F_n\|_{pe} < \infty$, then each martingale selector of $(F_n, \mathcal{F}_n)_{n\geq 1}$ is a.e. convergent to a Pettis integrable function.*

Proof. (3.6.1) If $(F_n, \mathcal{F}_n)_{n\geq 1}$ is an integrable martingale with values in $cc(E)$, then it admits a martingale selector by virtue of a result of Luu, see [16], see also Choukairi [9, Proposition 2.6]. Combining the result on the convergence in Pettis norm of quasimartingale, see [10, Theorem 4.1] and a careful adaptation of the techniques of Luu developed in [16, Proposition 2.3], (3.6.1) can be obtained. In fact, Since $(F_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale, for each $n \geq 1$,

$$S^{Pe}_{F_n}(\mathcal{F}_n) = \overline{\{E(f|\mathcal{F}_n), f \in S^{Pe}_{F_{n+1}}(\mathcal{F}_{n+1})\}}^{\|\cdot\|_{Pe}}.$$

Let $f_1 \in S^{Pe}_{F_1}(\mathcal{F}_1)$, there exists $f_2 \in S^{Pe}_{F_2}(\mathcal{F}_2)$ such that

$$\|f_1 - E(f_2|(\mathcal{F}_1)\|_{Pe} < \frac{1}{2}.$$

By induction we get a sequence $(f_n)_{n\geq 1}$ such that for all $n \geq 1$

$$f_n \in S^{Pe}_{F_n}(\mathcal{F}_n), \quad \text{and} \quad \|f_n - E(f_{n+1}|\mathcal{F}_n)\|_{Pe} < 2^{-n}.$$

Then $(f_n)_{n\geq 1}$ is a Pettis quasi-martingale. By [10, Theorem 4.1] there exists a Pettis integrable martingale $(h_n)_{n\geq 1}$ such that

$$\lim_{m\to\infty} \| f_m - h_m\|_{Pe} = 0.$$

Since for each $m \geq n \geq 1$,

$$\|E(f_m|\mathcal{F}_n) - h_n\|_{Pe} = \|E(f_m - h_m|\mathcal{F}_n)\|_{Pe} \leq \| f_m - h_m\|_{Pe},$$

we have for each $n \geq 1$: $\lim_{m\to\infty} \| E(f_m|\mathcal{F}_n) - h_n\|_{Pe} = 0.$

On the other hand $E(f_m|\mathcal{F}_n) \in S^{Pe}_{F_n}(\mathcal{F}_n)$ for each $m \geq n$ and $S^{Pe}_{F_n}(\mathcal{F}_n)$ is closed in $\mathcal{P}_E$, then $h_n \in S^{Pe}_{F_n}(\mathcal{F}_n)$.

(3.6.2) Let $(f_n)_{n\geq 1}$ be a martingale selector of $(F_n, \mathcal{F}_n)_{n\geq 1}$. For each $k \geq 1$, $(f_n 1_{A_k})_{n\geq 1}$ is an $L^1(E)$ bounded martingale. Since E has RNP, by Chatterji's theorem, $(f_n 1_{A_k})_{n\geq 1}$ is a.e. convergent to a function f_k. Hence $(f_n)_{n\geq 1}$ is a.e. convergent to $\sum_{i=1}^{\infty} f_k \, 1_{A_k}$.

(3.6.3) Let $(f_n)_{n\geq 1}$ be a martingale selector of $(F_n, \mathcal{F}_n)_{n\geq 1}$. By (3.6.2) $(f_n)_{n\geq 1}$ is a.e convergent to a function f. Let us prove that f is Pettis integrable. For each $x^* \in B^*$ we have

$$|\langle x^*, f_n(w)\rangle| \leq |s(x^*, F_n(w))| + |s(-x^*, F_n(w))| \quad a.e, \forall n \geq 1.$$

Then by Fatou's lemma, for each $x^* \in B^*$,

$$\|\langle x^*, f\rangle\|_1 \leq \sup_{n\geq 1} \|f_n\|_{Pe} \leq 2 \sup_{n\geq 1} \|F_n\|_{pe} < \infty.$$

Then f is scalarly integrable. By [18, Theorem 5.4], f is Pettis integrable. □

Theorem 3.7. *Assume that E has RNP and E^* is separable, and let $(F_n, \mathcal{F}_n)_{n\geq 1}$ be an Aumann-Pettis integrable multivalued martingale with values in ccb(E) such that there exists a partition $(A_k)_{k\geq 1}$ of Ω in $\mathcal{F}_1$ which satisfies*

$$\sup_{n\geq 1} \int_{A_k} |F_n| \, dP < \infty, \qquad \forall k \geq 1.$$

Then there exists a sequence $(\sigma_n^i)_{i\geq 1, n\geq 1}$ such that

(3.7.1) $\forall i \geq 1, (\sigma_n^i, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n)$.

(3.7.2) $\forall n \geq 1,\ F_n = \overline{co}\{\ \sigma_n^i, \quad i \geq 1\}$.

Proof. Step 1: For a fixed $k \geq 1$ and $\epsilon > 0$. If $g \in S^{Pe}_{F_k}(\mathcal{F}_k)$, then starting in the proof of Proposition 3.6 with $f_k = g$ instead of f_1 and replacing 2^{-n} by $\epsilon\ 2^{-n}$, we construct a sequence $(f_n)_{n\geq k}$ and a martingale $(g_n)_{n\geq k}$ such that

$$\lim_{m\to\infty} \|\ E(f_m|\mathcal{F}_n) - g_n\|_{Pe} = 0 \ \text{ and } \ \|g - g_k\|_{Pe} < \epsilon.$$

Setting $g_n = E(g_k|\mathcal{F}_n)$ if $n < k$, we obtain $(g_n, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n)$ with $\|g - g_k\|_{Pe} < \epsilon$. It follows that for each $i \geq 1$ and $\epsilon = 1/i$, there exists a sequence $(g_n^i)_{i\geq 1, n\geq 1}$ such that

$$(g_n^i, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n),\ \forall i \geq 1, \ \text{ and } \lim_{i\to\infty} \|g_k^i - g\|_{Pe} \leq \lim_{i\to\infty} \frac{1}{i} = 0.$$

Step 2: For each $k \geq 1$, let $(f^{k,i})_{i\geq 1}$ be a Castaing representation of F_k in $S^{Pe}_{F_k}(\mathcal{F}_k)$. By step 1, for each $i \geq 1$, there exists a sequence $(g_n^{k,i,j})_{n\geq 1, j\geq 1}$ such that

$$(g_n^{k,i,j}, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n),\ \forall j \geq 1, \ \text{ and } \lim_{j\to\infty} \|g_k^{k,i,j} - f^{k,i}\|_{Pe} = 0.$$

Without loss of generality, we can suppose that $\forall w \in \Omega$, $g_n^{k,i,j}(w) \in F_n(w)$ for each $k \geq 1$, $i \geq 1$, $j \geq 1$, $n \geq 1$. Let $(x_m^*)_{m\geq 1}$ be a dense sequence in B^*.

For each $m \geq 1$, there exists a subsequence $(h_k^{k,i,j})_{j\geq 1}$ of $(g_k^{k,i,j})_{j\geq 1}$ such that $x_m^* h_k^{k,i,j}(w) \underset{j\to+\infty}{\longrightarrow} x_m^* f^{k,i}(w)$ for each $w \in \Omega \setminus B_m$ with $B_m \in \mathcal{F}_k$ and $P(B_m) = 0$.

Set
$$\begin{aligned} g_n^{\prime k,i,j} &= h_k^{k,i,j} 1_{\Omega\setminus\cup_m B_m} + f^{k,i} 1_{\cup_m B_m}, \qquad && n = k.\\ g_n^{\prime k,i,j} &= h_n^{k,i,j}, && n \neq k. \end{aligned}$$

$g_n^{\prime k,i,j} = g_n^{k,i,j}$ a.e., then

$$(g_n^{\prime k,i,j}, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n), \qquad j \geq 1,\ i \geq 1,\ k \geq 1.$$

On the other hand,

$$F_n(w) = \overline{co}\{g_n^{\prime k,i,j}(w), i \geq 1, j \geq 1, k \geq 1\}, \qquad \forall w \in \Omega.$$

Indeed, it is clear that $\forall w \in \Omega$, $\overline{co}\{g_n^{\prime k,i,j}(w), i \geq 1, j \geq 1, k \geq 1\} \subset F_n(w)$. Set

$$H_n(w) = \overline{co}\{g_n^{\prime n,i,j}(w), i \geq 1, j \geq 1\}, \qquad w \in \Omega.$$

Since H_n is with values in $ccb(E)$, then it is $((x_m^*)_{m\geq 1})$-countably supported. On the other hand, for each $m \geq 1$, $\langle x_m^*, f^{n,i}(w)\rangle \leq s(x_m^*, H_n(w))$ for each $w \in \Omega$, then $f^{n,i}(w) \in H_n(w)$, $\forall w \in \Omega$. Hence $F_n(w) \subset H_n(w)$, $\forall w \in \Omega$.
To complete the proof, it suffices to denote $(g_n^{\prime k,i,j})_{k\geq 1, i\geq 1, j\geq 1}$ by $(\sigma_n^l)_{l\geq 1}$. □

Remark 3.8. If the Pettis integrable martingale is with values in $cwk(E)$, then the assumption E^* is separable can be omitted. In fact, we have the following corollary.

Corollary 3.9. *Assume that E has RNP, and let $(F_n, \mathcal{F}_n)_{n\geq 1}$ be an Aumann-Pettis integrable multivalued martingale with values in $cwk(E)$ such that there exists a partition $(A_k)_{k\geq 1}$ of Ω in $\mathcal{F}_1$ which satisfies*

$$\sup_{n\geq 1} \int_{A_k} |F_n|\, dP < \infty, \qquad \forall k \geq 1.$$

Then there exists a sequence $(\sigma_n^i)_{i\geq 1, n\geq 1}$ such that

(3.9.1) $\forall i \geq 1, (\sigma_n^i, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n)$.

(3.9.2) $\forall n \geq 1,\ F_n = \overline{co}\{\ \sigma_n^i,\quad i \geq 1\}$.

Proof. it is sufficient to consider in step 2 a sequence $\{x_n^*, n \geq 1\}$ of B^* such that $x_n^* = \frac{y_n^*}{\|y_n^*\|}$ where $(y_n^*)_{n\geq 1}$ is a dense sequence in E^* with respect to the Mackey topology $\tau(E^*, E)$ instead of a dense sequence $(x_m^*)_{m\geq 1}$ in B^* with respect to the strong topology. □

4. Regularity and convergence of Aumann-Pettis integrable multivalued martingales

By using the non-emptiness of $PMS(F_n, \mathcal{F}_n)$ and the representation theorem of a Pettis integrable martingale proved in Section 3, we state some applications to Mosco convergence and convergence in linear and Wijsman topologies of Pettis integrable multivalued martingales. The regularity of Pettis integrable multivalued martingales is also established.

Definition 4.1. A set M of measurable functions is called *decomposable* (with respect to $\mathcal{F}$) if for each $A \in \mathcal{F}$ and $f, g \in M$, $1_A f + 1_{\Omega \setminus A} g \in M$.

Theorem 4.2. [20, Theorem 4.2] *Let D be a non empty subset of $\mathcal{P}_E$. Suppose that D is decomposable with respect to $\mathcal{F}$. Then there exists $F \in \mathcal{M}[\Omega, \mathcal{F}, E]$ and a sequence $(f_n)_{n\geq 1} \subset D$ such that*

(1) $F(w) = \overline{\{f_n(w), n \geq 1\}}, \quad \forall w \in \Omega,$

(2) $\overline{S_F^{Pe}}^{\|\cdot\|_{Pe}} = \overline{D}^{\|\cdot\|_{Pe}}.$

Remark 4.3. (1) It is clear from the proof of [20, Theorem 4.2] that it is still correct when $\overline{D}^{\|\cdot\|_{Pe}}$ is decomposable with respect to $\mathcal{F}$.

(2) If D is convex, then $F(w) = \overline{\{f_n(w), n \geq 1\}} = \overline{co}\{f_n(w), n \geq 1\}$ a.e.

Lemma 4.4. *Let $\mathcal{A}$ be an algebra, and let $\mathcal{B}$ be the σ-algebra generated by $\mathcal{A}$, a closed non-empty set M in $\mathcal{P}_E$ is decomposable (with respect to $\mathcal{B}$) iff for each $A \in \mathcal{A}$ and $f, g \in M$, $1_A f + 1_{\Omega \setminus A} g \in M$.*

Proof. Assume that for each $A \in \mathcal{A}$ and $f, g \in M$, $1_A f + 1_{\Omega \setminus A} g \in M$. Let $f, g \in M$, and let $A \in \mathcal{B}$, there exists a sequence $(A_n)_{n\geq 1}$ in $\mathcal{A}$ such that $1_{A_n} \underset{n\to+\infty}{\longrightarrow} 1_A$ a.e, then $f1_{A_n} + g1_{\Omega\setminus A_n} \underset{n\to+\infty}{\longrightarrow} f1_A + g1_{\Omega\setminus A}$ a.e. On the other hand, $\forall n \geq 1$, $\forall x^* \in B^*$, $|\langle x^*, f1_{A_n} + g1_{\Omega\setminus A_n}\rangle| \leq |\langle x^*, f\rangle| + |\langle x^*, g\rangle|$. Since $f, g \in \mathcal{P}_E$, the set $\{|\langle x^*, f\rangle| + |\langle x^*, g\rangle|, x^* \in B^*\}$ is uniformly integrable. Hence the set $\{\langle x^*, f1_{A_n} + g1_{\Omega\setminus A_n}\rangle, n \geq 1, x^* \in B^*\}$ is uniformly integrable. It follows from [4, theorem 2.4] that $\|(f1_{A_n} + g1_{\Omega\setminus A_n}) - (f1_A + g1_{\Omega\setminus A})\|_{Pe} \underset{n\to+\infty}{\longrightarrow} 0$. Thus $f1_A + g1_{\Omega\setminus A} \in M$. $\square$

Proposition 4.5. *Let $F\colon \Omega \to ccb(E)$ be an Aumann-Pettis integrable multifunction. Then F is scalarly integrable iff for each $x^* \in B^*$, the set $\{\langle x^*, f\rangle, f \in S_F^{Pe}\}$ is bounded in $L^1_{\mathbb{R}}$.*

Proof. For each $x^* \in B^*$, and for each $f \in S_F^{Pe}$,

$$|\langle x^*, f(w)\rangle| \leq |s(x^*, F(w))| + |s(-x^*, F(w))| \quad a.e.$$

Then the condition is necessary. Let $x^* \in B^*$. For each $A \in \mathcal{F}$,

$$\begin{aligned} |\int_A s(x^*, F)\, dP| &= |s(x^*, Pe - \int_A F\, dP)| = |\sup \{\int_A \langle x^*, f\rangle\, dP, f \in S_F^{Pe}\}| \\ &\leq \sup_{f\in S_F^{Pe}} \|\langle x^*, f(w)\rangle\|_1. \end{aligned}$$

For the left equality, see [11, Theorem 3.9]. Hence

$$\int_\Omega |s(x^*, F)|\, dP \leq 2 \sup_{A\in\mathcal{F}} |\int_A s(x^*, F)\, dP| \leq 2 \sup_{f\in S_F^{Pe}} \|\langle x^*, f(w)\rangle\|_1 < \infty. \qquad \square$$

Theorem 4.6. *Assume that E has RNP and E^* is separable. Let $(F_n, \mathcal{F}_n)_{n\geq 1}$ be a scalarly integrable martingale of Aumann-Pettis integrable multifunctions with values in $ccb(E)$ such that*

(1) $\sup_{n\geq 1} \|F_n\|_{pe} < \infty$.

(2) *There exists a partition $(A_m)_{m\geq 1}$ of Ω in $\mathcal{F}_1$ such that*

$$\sup_{n\geq 1} \int_{A_m} |F_n|\, dP < \infty, \qquad \forall m \geq 1.$$

Then there exists a scalarly integrable and Aumann-Pettis integrable multifunction F with values in $ccb(E)$ such that

(4.6.1) *For each $m \geq 1$, $\int_{A_m} |F|\, dP < \infty$.*

(4.6.2) $M - \lim_{n\to\infty} F_n(w) = F(w)$ *a.e.*

(4.6.3) *If $\sup_{n\geq 1} |F_n(w)| < \infty$ a.e., then $\tau_L - \lim_{n\to\infty} F_n(w) = F(w)$ a.e.*

Proof. Set $D = \{f \in \mathcal{P}_E, \exists (f_n, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n)$ such that $f_n \longrightarrow f\ a.e\}$.

Step 1: We claim that D is non-empty and convex, and $\overline{D}^{\|\cdot\|_{Pe}}$ is decomposable with respect to $\mathcal{F}$. By Proposition 3.6, D is non-empty. It is clear that D is convex. Let $A \in \cup_{n\geq 1}\mathcal{F}_n$, and let $f_1, f_2 \in D$. By definition of D, there exist two sequences $(f_{1n}, \mathcal{F}_n)_{n\geq 1}$ and $(f_{2n}, \mathcal{F}_n)_{n\geq 1}$ in $PMS(F_n, \mathcal{F}_n)$ such that

$$\lim_{n\to\infty} \|f_{1n} - f_1\| = 0 \text{ a.e., and } \lim_{n\to\infty} \|f_{2n} - f_2\| = 0 \text{ a.e.}$$

Let $m \geq 1$ such that $A \in \mathcal{F}_m$, and let

$$\begin{cases} h_n = 1_A f_{1n} + 1_{\Omega\setminus A} f_{2n}, & n \geq m, \\ h_n = E(h_{n+1}|\mathcal{F}_n), & n < m. \end{cases}$$

Then $(h_n, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n)$, and

$$\lim_{n\to\infty} \|h_n - (1_A f_{1n} + 1_{\Omega\setminus A} f_{2n})\| = 0 \text{ a.e.}$$

Then $1_A f_1 + 1_{\Omega\setminus A} f_2 \in D$. Hence by Lemma 4.4, $\overline{D}^{\|\cdot\|_{Pe}}$ is decomposable with respect to $\mathcal{F}$.

Step 2: We claim that for each sequence $(f^i)_{i\geq 1} \subset D$, we have

$$\forall m \geq 1, \int_{A_m} |H(w)|\, dP < \infty$$

where $H(w) = \overline{\{f^i(w), i \geq 1\}}$, $\forall \in w \in \Omega$.

Let $(f^i)_{i\geq 1} \subset D$, and let $m \geq 1$. We shall prove that there exists $M > 0$ such that for each $l \geq 1$, and each partition $\{B_1, ..., B_l\}$ of A_m we have

$$\|\sum_{i=1}^{l} 1_{B_i} f^i\|_1 \leq M.$$

By the definition of D, for each $i \geq 1$, there exists a sequence $(f_n^i)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n)$ such that $\lim_{n\to\infty} \|f_n^i - f^i\| = 0$ a.e. Let

$$g_n = \sum_{i=1}^{l} 1_{B_i} f_n^i, \qquad n \geq 1.$$

By Fatou's lemma, we have

$$\int_{A_m} \liminf_{n\to\infty} \|g_n\| \, dP \leq \liminf_{n\to\infty} \int_{A_m} \|g_n\| \, dP \leq \sup_{n\geq 1} \int_{A_m} |F_n| \, dP.$$

Then $\|\sum_{i=1}^{l} 1_{B_i} f^i\|_1 \leq M$ with $M = \sup_{n\geq 1} \int_{A_m} |F_n| \, dP$.

Let $H^{'} = H_{|A_m}$, and let $f \in S^1_{H'}$. Since $H^{'}(w) = \overline{\{f^i{}_{|A_m}(w), i \geq 1\}}$ for each $w \in A_m$ and $(f^i{}_{|A_m})_{i\geq 1} \subset S^1_{H'}$, by [14, Lemma 1.3], there exist $l \geq 1$ and a partition $\{B_1, ..., B_l\}$ of A_m such that $\|f - \sum_{i=1}^{l} 1_{B_i} f^i\|_1 \leq 1$. Then

$$\|f\|_1 \leq 1 + \| \textstyle\sum_{i=1}^{l} 1_{B_i} f_1^i\|_1 \leq M + 1.$$

Hence by [14, Theorem 3.2],

$$\int_{A_m} |H(w)| \, dP = \int_{A_m} |H^{'}(w)| \, dP < \infty.$$

Step 3: We claim the existence of a scalarly integrable and Aumann-Pettis integrable multifunction F with values in $ccb(E)$ and satisfies (4.6.1).

Since D is non empty and $\overline{D}^{\|\cdot\|_{Pe}}$ is decomposable with respect to $\mathcal{F}$, by Theorem 4.2 and Remark 4.3, there exists a multifunction $H \in \mathcal{M}[\Omega, \mathcal{F}, E]$ such that

$$H(w) = \overline{\{f^i(w), i \geq 1\}} = \overline{co}\{f^i(w), i \geq 1\} \text{ a.e.}$$

with $(f^i)_{i\geq 1} \subset D$, and $\overline{S_H^{Pe}}^{\|\cdot\|_{Pe}} = \overline{D}^{\|\cdot\|_{Pe}}$.

Let $(x_n^*)_{n\geq 1}$ be a dense sequence in B^*. By Step 2, the multifunction H is bounded a.e. Then H is $\{x_n^*, n \geq 1\}$-countably supported a.e. Hence $\overline{S_H^{Pe}}^{\|\cdot\|_{Pe}} = S_H^{Pe}$. By step 2, we also have

$$\int_{A_m} |H| \, dP < \infty, \quad \forall m \geq 1.$$

Since D is non empty, H is Aumann-Pettis integrable.

Let us prove that H is scalarly integrable. Set $M^{'} = \sup_{n\geq 1} \|F_n\|_{pe} < \infty$, and let $g \in \overline{D}$, there exists $f \in D$ such that $\|g - f\|_{Pe} \leq 1$. By definition of D, there exists a sequence $(f_{1n}, \mathcal{F}_n)_{n\geq 1}$ in $PMS(F_n, \mathcal{F}_n)$ such that $\lim\limits_{n\to\infty} \|f_n - f\| = 0$ a.e.

Then for each $x^* \in B^*$,

$$\|\langle x^*, g\rangle\|_1 \leq \|\langle x^*, f\rangle\|_1 + 1 \leq \liminf_{n\to\infty} \|\langle x^*, f_n\rangle\|_1 + 1 \leq 2M^{'} + 1.$$

Hence by Proposition 4.5, H is scalarly integrable. Let A be negligible set in $\mathcal{F}$ such that $H(w) \in ccb(E)$ for each $w \in \Omega \setminus A$. Set

$$F(w) = \begin{cases} H(w), & w \in \Omega \setminus A, \\ \{0\}, & w \in A. \end{cases}$$

F is a scalarly integrable and Aumann-Pettis integrable multifunction with values in $ccb(E)$ and satisfies 4.6.1. Moreover

$$S_F^{Pe} = \overline{D}^{\|\cdot\|_{Pe}}, \ \textit{and}\ F(w) = \overline{\{f^i(w), i \geq 1\}}\ \textit{a.e.}$$

with $(f^i)_{i\geq 1} \subset D$.

Step 4: We claim that $M - \underset{n\longrightarrow\infty}{lim} F_n(w) = F(w)$ a.e.

For this, it suffices to prove that

$$w - ls\, F_n(w) \subset F(w) \subset s - li\, F_n(w)\ a.e.$$

Let $(f^i)_{i\geq 1} \subset D$ such that $F(w) = \overline{\{f^i(w), i \geq 1\}}$ a.e. For each $i \geq 1$, there exists a sequence $(f_n^i, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n)$ such that $\lim\limits_{n\to\infty} \|f_n^i - f^i\| = 0$ a.e. Then for each $i \geq 1$,

$$\lim_{n\to\infty} d(f^i(w), F_n(w)) \leq \lim_{n\to\infty} \|f_n^i - f^i\| = 0\ \ a.e.$$

Then $f^i(w) \in s - li\, F_n(w)$ a.e. Hence $F(w) \subset s - li\ F_n(w)$ a.e.

For the inclusion $w - ls\, F_n(w) \subset F(w)$. By applying the part (b) of the proof of [9, Theorem 2.8], for each $m \geq 1$, $w - ls\, F_{n|A_m}(w) \subset F_{|A_m}(w)$ a.e. Hence $w - ls\, F_n(w) \subset F(w)$ a.e.

Moreover, it follows from the same proof that if $(x_k^*)_k$ is a countable sequence in E^*, then there exists a negligible set $A \in \mathcal{F}$ such that for each $w \in \Omega \setminus A$,

$$\lim_{n\to\infty}\ s(x_k^*, F_n(w)) \leq s(x_k^*, F(w)), \qquad \forall k \geq 1. \tag{2}$$

Step 5: We claim that if $\sup\limits_{n\geq 1} |F_n(w)| < \infty$ a.e., then $\tau_L - \underset{n\to\infty}{lim} F_n(w) = F(w)$ a.e.

Let $(f^i)_{i\geq 1} \subset D$ such that $F(w) = \overline{\{f^i(w), i \geq 1\}}$ a.e.

For each $i \geq 1$, there exists a sequence $(f_n^i, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n)$ such that $\lim\limits_{n\to\infty} \|f_n^i - f^i\| = 0$ a.e. Let $(x_k^*)_{k\geq 1}$ be a dense sequence in E^*. For each $k \geq 1$,

$$s(x_k^*, F(w)) = \sup_{i\geq 1} \langle x_k^*, f^i(w)\rangle = \sup_{i\geq 1} \lim_{n\to\infty} \langle x_k^*, f_n^i(w)\rangle \leq \lim_{n\to\infty} s(x_k^*, F_n(w)) \text{ a.e.}$$

It follows from (2) that there exists a negligible set $B \in \mathcal{F}$ such that for each $w \in \Omega \setminus B$,

$$\lim_{n\to\infty} s(x_k^*, F_n(w)) = s(x_k^*, F(w)), \qquad \forall k \geq 1. \tag{3}$$

Since $\sup\limits_{n\geq 1} |F_n(w)| < \infty$ a.e., for almost every $w \in \Omega$, the set $\{s(., F_n(w)),\ n \geq 1\}$ is equicontinuous. Hence it follows from (3) that there exists a negligible set $C \in \mathcal{F}$ such that for each $w \in \Omega \setminus C$,

$$\lim_{n\to\infty} s(x^*, F_n(w)) = s(x^*, F(w)), \qquad \forall x^* \in E^*. \tag{4}$$

For each $i \geq 1$, there exists a sequence $(f_n^i, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n)$ such that $\lim\limits_{n\to\infty} \|f_n^i - f^i\| = 0$ a.e. Then for each $i \geq 1$,

$$\limsup_{n\to\infty} d(x, F_n(w)) \leq \lim_{n\to\infty} \|x - f_n^i(w)\| = \|x - f^i(w)\| \text{ a.e.}$$

Hence
$$\limsup_{n\to\infty} d(x, F_n(w)) \leq d(x, F(w)) \text{ a.e.} \tag{5}$$

On the other hand, by [6, Corollary 1.5.3], for each non empty convex subset C of E, we have

$$d(., C) = \sup_{x^*\in B^*} (\langle x^*, .\rangle - s(x^*, C)).$$

Let $B \in ccb(E)$. Since the function $x^* \mapsto s(x^*, B)$ is continuous with respect to the norm topology on E^*. Then for every dense sequence $(x_m^*)_{n\geq 1}$ in B^*, we have

$$d(., B) = \sup_{m\geq 1} (\langle x_m^*, .\rangle - s(x_m^*, B)).$$

Let $(x_m^*)_{m\geq 1}$ be a dense sequence in B^*. Then

$$\begin{aligned}
\liminf_{n\to\infty} d(x, F_n(w)) &= \liminf_{n\to\infty} \sup_{m\geq 1} (\langle x_m^*, x\rangle - s(x_m^*, F_n(w))) \\
&\geq \sup_{m\geq 1} \liminf_{n\to\infty} (\langle x_m^*, x\rangle - s(x_m^*, F_n(w))) \qquad (6)\\
&= \sup_{m\geq 1} (\langle x_m^*, x\rangle - s(x_m^*, F(w))) = d(x, F(w)), \text{ a.e.}
\end{aligned}$$

□

Theorem 4.7. *Assume that E has RNP. Let $(F_n, \mathcal{F}_n)_{n\geq 1}$ be a scalarly integrable martingale of Aumann-Pettis integrable multifunctions with values in $cwk(E)$ with*

(1) $\sup\limits_{n\geq 1} \|F_n\|_{pe} < \infty$.

(2) *There exists a partition $(A_m)_{m\geq 1}$ of Ω in $\mathcal{F}_1$ such that*

$$\sup_{n\geq 1} \int_{A_m} |F_n|\, dP < \infty, \qquad \forall m \geq 1.$$

(3) $\overline{\bigcup\limits_{n\geq 1} F_n(.)} \in cwk(E)$ *a.e.*

Then there exists a scalarly integrable and Aumann-Pettis integrable multifunction F with values in $cwk(E)$ such that

(4.7.1) *For each $m \geq 1$, $\int_{A_m} |F|\, dP < \infty$.*

(4.7.2) $M - \underset{n\to\infty}{lim} F_n(w) = F(w)$ *a.e.*

(4.7.3) $\tau_L - \underset{n\to\infty}{lim} F_n(w) = F(w)$ *a.e.*

Proof. Set $D = \{f \in \mathcal{P}_E, \exists (f_n, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n)$ *such that* $f_n \longrightarrow f$ *a.e*$\}$.

Step 1: We claim the existence of a scalarly integrable and Aumann-Pettis integrable multifunction F with values in $ccb(E)$ which satisfies (4.7.1).

Following the proof of Theorem 4.6, there exists a multifunction $H \in \mathcal{M}\,[\Omega, \mathcal{F}, E]$ such that

$$H(w) = \overline{\{f^i(w), i \geq 1\}} = \overline{co}\{f^i(w), i \geq 1\} \text{ a.e.}$$

with $(f^i)_{i\geq 1} \subset D$, and $\overline{S_H^{Pe}}^{\|\cdot\|_{Pe}} = \overline{D}^{\|\cdot\|_{Pe}}$.

Since $H(w) \in cwk(E)$ a.e., we have $S_H^{Pe} = \overline{S_H^{Pe}}^{\|\cdot\|_{Pe}} = \overline{D}^{\|\cdot\|_{Pe}}$.

As in Theorem 4.6, we prove that H is a scalarly integrable and Aumann-Pettis integrable multifunction which satisfies (4.7.1).

Let A be negligible set in $\mathcal{F}$ such that $H(w) \in cwk(E)$ for each $w \in \Omega \setminus A$. Set

$$F(w) = \begin{cases} H(w), & w \in \Omega \setminus A, \\ \{0\}, & w \in A. \end{cases}$$

F is a scalarly integrable and Aumann-Pettis integrable multifunction with values in $cwk(E)$ and satisfies 4.7.1. Moreover

$$S_F^{Pe} = \overline{D}^{\|\cdot\|_{Pe}}, \quad F(w) = \overline{\{f^i(w), i \geq 1\}} \ a.e.$$

with $(f^i)_{i\geq 1} \subset D$, and 4.7.1 holds.

Step 2: We claim that $M - \underset{n\to\infty}{lim} F_n(w) = F(w)$ a.e.

As in Step 4 of Theorem 4.6, we prove that $F(w) \subset s - li\ F_n(w)$ a.e.

For the inclusion $w - ls\, F_n(w) \subset F(w)$, we apply the corresponding part of the proof of [9, Theorem 2.11] for each A_m separately.

Step 3: We claim that $\tau_L - \underset{n\to\infty}{lim} F_n(w) = F(w)$ a.e.

Let M^* be a countable dense sequence in E^* with respect to the Mackey topology. As it is in the proof of Theorem 3.7 Step 5, one proves that for each $x^* \in M^*$,

$$\lim_{n\to\infty} s(x^*, F_n(w)) = s(x^*, F(w)) \text{ a.e.} \tag{7}$$

We fix w such that $\overline{\underset{n\geq 1}{\cup} F_n(w)} \in cwk(E)$ for each $x^* \in M^*$. Then fix $x^* \in E^*$, $\epsilon > 0$. Since $\overline{\underset{n\geq 1}{\cup} F_n(w)} \in cwk(E)$, the set $\{s(., F_n(w)),\ n \geq 1\}$ is equicontinuous with respect to the Mackey topology.

Then there exists $x^*_\beta \in M^*$ such that

$$|s(x^*, F_n(w)) - s(x^*_\beta, F_n(w))| < \frac{\epsilon}{3}, \quad \forall n \geq 1,$$

and
$$|s(x^*, F(w)) - s(x^*_\beta, F(w))| < \frac{\epsilon}{3}.$$

By (7), there exists $n_0 \geq 1$ such that for each $n \geq n_0$

$$|s(x^*_\beta, F_n(w)) - s(x^*_\beta, F(w))| < \frac{\epsilon}{3}.$$

Hence for each $n \geq n_0$,

$$\begin{aligned}|s(x^*, F_n(w)) - s(x^*, F(w))| &\leq |s(x^*, F_n(w)) - s(x^*_\beta, F_n(w))| \\ &+ |s(x^*_\beta, F_n(w)) - s(x^*_\beta, F(w))| + |s(x^*_\beta, F(w)) - s(x^*, F(w))| < \epsilon.\end{aligned}$$

By [6, Corollary 1.5.3], for each non empty convex subset C of E, we have

$$d(.,C) = \sup_{x^* \in B^*} (\langle x^*, .\rangle - s(x^*, C)).$$

Let $B \in cwk(E)$. Since the function $x^* \mapsto s(x^*, B)$ is continuous with respect to the Mackey topology on E^*. Then for every dense sequence $(x^*_m)_{m\geq 1}$ in B^* with respect to the Mackey topology, we have

$$d(.,B) = \sup_{m\geq 1} (\langle x^*_m, .\rangle - s(x^*_m, B)).$$

Let $(x^*_m)_{m\geq 1}$ be a dense sequence in B^* with respect to the Mackey topology.

Then
$$\begin{aligned}\liminf_{n\to\infty} d(x, F_n(w)) &= \liminf_{n\to\infty} \sup_{m\geq 1} (\langle x^*_m, x\rangle - s(x^*_m, F_n(w))) \\ &\geq \sup_{m\geq 1} \liminf_{n\to\infty} (\langle x^*_m, x\rangle - s(x^*_m, F_n(w))) = \sup_{m\geq 1} (\langle x^*_m, x\rangle - s(x^*_m, F(w))) \\ &= d(x, F(w)), \quad \text{a.e.} \qquad (8)\end{aligned}$$

As in Theorem 4.6, we prove that

$$\limsup_{n\to\infty} d(x, F_n(w)) \leq d(x, F(w)) \ \text{ a.e.} \qquad \square \quad (9)$$

Theorem 4.8. *Assume that E has RNP and E^* is separable. Let $(F_n, \mathcal{F}_n)_{n\geq 1}$ be a scalarly integrable martingale of Aumann-Pettis integrable multifunctions with values in $ccb(E)$ such that*

(1) $\{s(x^*, F_n(.)),\ x^* \in B^*, n \geq 1\}$ *is uniformly integrable.*

(2) *There exists a partition $(A_m)_{m\geq 1}$ of Ω in $\mathcal{F}_1$ such that*

$$\sup_{n\geq 1} \int_{A_m} |F_n| \, dP < \infty, \qquad \forall m \geq 1.$$

Then there exists a scalarly integrable and Aumann-Pettis integrable multifunction F with values in $ccb(E)$ such that

(4.8.1) *For each $A \in \mathcal{F}$, $\tau_L - \lim_{n\to\infty} \overline{Pe - \int_A F_n \, dP} = \overline{Pe - \int_A F \, dP}$.*

(4.8.2) *For each $n \geq 1$, $E(F|\mathcal{F}_n) = F_n$.*

Proof. By Theorem 4.6, there exists a multifunction F with values in $ccb(E)$ such that $S_F^{Pe} = \overline{D}$ where

$$D = \{f \in \mathcal{P}_E, \exists (f_n, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n) \ \ such\, that \ \ f_n \longrightarrow f \ a.e\}.$$

Let $A \in \mathcal{F}$. By the proof of Theorem 4.6, we have

$$\lim_{n\to\infty} s(x^*, F_n(w)) = s(x^*, F(w)) \ \ a.e., \quad \forall x^* \in E^*.$$

Since $\{s(x^*, F_n(.)), \ x^* \in B^*, n \geq 1\}$ is uniformly integrable, for each $x^* \in E^*$,

$$\begin{aligned}\lim_{n\to\infty} s(x^*, Pe - \int_A F_n \, dP) &= \lim_{n\to\infty} \int_A s(x^*, F_n) \, dP \\ &= \int_A s(x^*, F) \, dP = s(x^*, Pe - \int_A F \, dP).\end{aligned}$$

On the other hand, it follows from [6, Corollary 1.5.3] that for each $x \in E$,

$$\liminf_{n\to\infty} d(x, Pe - \int_A F_n \, dP) \geq d(x, Pe - \int_A F \, dP). \tag{10}$$

Let $f \in D$. By the definition of D and [4, Theorem 2.4], there exists a sequence $(f_n, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n)$ such that $\lim\limits_{n\to\infty} \|f_n - f\|_{Pe} = 0$. Then for each $x \in E$,

$$\limsup_{n\to\infty} d(x, Pe - \int_A F_n \, dP) \leq \lim_{n\to\infty} \|x - \int_A f_n \, dP\| = \|x - \int_A f \, dP\|.$$

Hence for each $x \in E$,

$$\limsup_{n\to\infty} d(x, Pe - \int_A F_n \, dP) \leq d(x, Pe - \int_A F \, dP). \tag{11}$$

It follows from (10) and (11) that for each $x \in E$,

$$\lim_{n\to\infty} d(x, Pe - \int_A F_n \, dP) = d(x, Pe - \int_A F \, dP).$$

Let $n \geq 1$, and let $A \in \mathcal{F}_n$. For each $m \geq n$,

$$\overline{Pe - \int_A F_n \, dP} = \overline{Pe - \int_A F_m \, dP}.$$

Then by 4.8.1.,

$$\overline{Pe - \int_A F_n \, dP} = \tau_L - \lim_{m\to\infty} \overline{Pe - \int_A F_m \, dP} = \overline{Pe - \int_A F \, dP}.$$

Hence $E(F|\mathcal{F}_n) = F_n$. $\square$

Theorem 4.9. *Assume that E has RNP. Let $(F_n, \mathcal{F}_n)_{n\geq 1}$ be a scalarly integrable martingale of Aumann-Pettis integrable multifunctions with values in $cwk(E)$ such that*

(1) $\{s(x^*, F_n(.)),\ x^* \in B^*,\ n \geq 1\}$ *is uniformly integrable.*

(2) *There exists a partition $(A_m)_{m\geq 1}$ of Ω in $\mathcal{F}_1$ such that*

$$\sup_{n\geq 1} \int_{A_m} |F_n|\, dP < \infty, \qquad \forall m \geq 1.$$

(3) $\overline{\underset{n\geq 1}{\cup} F_n(.)} \in cwk(E)$ *a.e.*

Then there exists a scalarly integrable and Aumann-Pettis integrable multifunction F with values in $cwk(E)$ such that

(4.9.1) *For each $A \in \mathcal{F}$, $\tau_L - \lim\limits_{n\to\infty} \overline{Pe - \int_A F_n\ dP} = \overline{Pe - \int_A F\ dP}$.*

(4.9.2) *For each $n \geq 1$, $E(F|\mathcal{F}_n) = F_n$.*

Proof. By Theorem 4.7, there exists a multifunction F with values in $cwk(E)$ such that $S_F^{Pe} = \overline{D}$ where

$$D = \{f \in \mathcal{P}_E, \exists (f_n, \mathcal{F}_n)_{n\geq 1} \in PMS(F_n, \mathcal{F}_n)\ \ such\, that\ \ f_n \longrightarrow f\ a.e\}.$$

The rest of the proof is similar to the proof of Theorem 4.8. □

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