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Some Characterizations of the Ellipsoid by Centroids of Concurrent Sections

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In [11] Meyer and Reisner proved the following result, generalizing a classical result due to Brunn [2]: If $K \subset \mathbb{R}^n$ is a convex body with the property that the centroids of every set of parallel sections, cut by parallel hyperplanes, are collinear, then K is an ellipsoid. In this paper we analyze the 3-dimensional analog of this result for the case of centroids of concurrent sections. For every line ℓ intersecting a convex body in $\mathbb{R}^3$, we consider the set of centroids of the sections of K, cut by planes through ℓ, and we assume the locus of these centroids determine a planar, differentiable simple closed curve with no segments. In such a case we prove that K is an ellipsoid. Furthermore, we prove that if for a convex body K there are two parallel planes such that for every line in any of these planes the associated locus of centroids is a circle, then K is a Euclidean ball.

Keywords: Euclidean ball, ellipsoid, centroids.

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1. Introduction

Let B denote the unit ball in the Euclidean space $\mathbb{R}^n$, for $n \geq 2$. If $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a non-singular affine transformation we say that $E = T(B)$ is an n-dimensional ellipsoid. Consider any set of parallel chords of E, then it is known that the midpoints of the chords are coplanar. Indeed, this property characterizes the ellipsoid among the convex bodies in $\mathbb{R}^n$ as was proved by Brunn [2].

Theorem 1.1. [Brunn] *If given a convex body $K \subset \mathbb{R}^n$, with $n \geq 2$, the midpoints of all sets of parallel chords are contained in a hyperplane, then K is an ellipsoid.*

A similar result can be obtained if this time we consider centroids of parallel sections of K, instead of parallel chords. This is a generalization of Brunn's Theorem, proved by Meyer and Reisner in [11] and it establishes the following.

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Theorem 1.2. [Meyer and Reisner] *Let $K \subset \mathbb{R}^n$ be a convex body such that given any hyperplane $H \subset \mathbb{R}^n$ the centroids of $(x+H) \cap K$ are collinear, for all $x \in \mathbb{R}^n$ such that $(x+H) \cap K \neq \emptyset$. Then K is an ellipsoid.*

The property that midpoints of parallel chords of any ellipse are collinear is just one of the many known properties of the ellipse. For example, for any given ellipse E and any point p in the plane, the locus of the midpoints of chords contained in lines concurrent at p is an elliptical arc. If the point of concurrency p moves away from the ellipse E, the lines become progressively closer to being parallel (see Figure 1.1). We could say that in the case of parallel chords, the point of concurrency is a point at infinity, that is, the point is infinitely far away from the body. The same applies to a family of parallel planes, concurrent at a line at infinity.

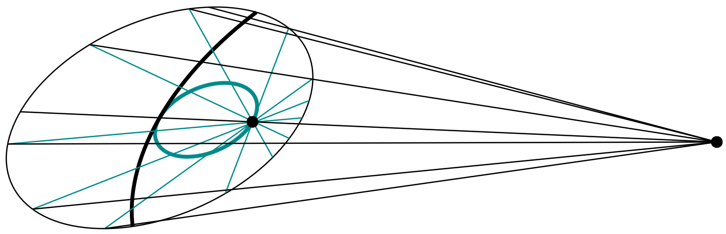


Figure 1.1: The locus of midpoints of concurrent chords is an elliptical arc.

This motivates us to wonder about characterizations of the ellipsoid through the locus of centroids of sections contained in planes concurrent at a line. This is the problem that will be analyzed in this article. For the two-dimensional case there are some results in the literature:

Theorem 1.3. [Jerónimo-Castro] *Let $K \subset \mathbb{R}^2$ be a convex body and let γ be a differentiable closed convex curve in $\mathbb{R}^2$. If for every point $p \in \operatorname{int} K$ we have that the locus of midpoints of chords of K through p is a curve directly homothetic to γ, then K is an ellipse.*

For the case of points exterior to K the following was proved:

Theorem 1.4. [Jerónimo-Castro] *Let $K \subset \mathbb{R}^2$ be a convex body, with $\partial K \subset C^2_+$, and let γ be a differentiable closed convex curve in $\mathbb{R}^2$. If for every point $p \in \mathbb{R}^2 \backslash K$ we have that the locus of midpoints of chords of K cut by lines through p is an arc contained in a directly homothetic copy of γ, then K is an ellipse.*

In order to state the problem and the results of this article we need the following notation: For any $u \in \mathbb{S}^2$ we denote $\mathbb{S}^2 \cap u^\perp$ as $S(u)$. Given a line ℓ and $v \in \mathbb{S}^2$, $H(\ell, v)$ denotes the plane that contains ℓ and that is orthogonal to v and $c(\ell, v)$ denotes the centroid of the section $H(\ell, v) \cap K$ when $H(\ell, v) \cap K \neq \emptyset$ (see Figure 1.2). If $u \in \mathbb{S}^2$ is parallel to ℓ, then the locus of centroids of sections of $H(\ell, v) \cap K$ is given by

$$\Omega(\ell) = \{c(\ell, v) : v \in S(u)\}.$$

We conjecture the following characterization of ellipsoids.

Conjecture 1.5. *Let K be a convex body in $\mathbb{R}^3$. If there exists a plane H such that for any line $\ell \subset H$ we have that $\Omega(\ell)$ is an elliptical arc, then K is an ellipsoid. Furthermore, if $\Omega(\ell)$ is a circular arc, then K is a Euclidean ball.*

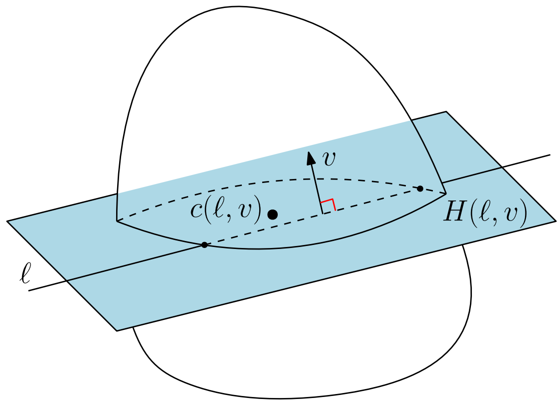


Figure 1.2: There is a centroid $c(\ell, u)$ for every plane that contains ℓ.

It is not difficult to prove that all ellipsoids satisfy the property mentioned in Conjecture 1.5. Thus, the conjecture, if true, would give a characterization of an ellipsoid.

Now we introduce more notation. Given a line ℓ we will say that ℓ possesses the (X) property if one of the following two cases occur:

1. $\Omega(\ell)$ can be parameterized as a planar, differentiable simple closed curve with no segments.
2. $\Omega(\ell)$ can be parameterized as an arc of a curve satisfying the properties in the previous case.

Notice that if ℓ possesses the (X) property and $\ell \cap K \neq \emptyset$, where K is strictly convex, then Case 1 occurs, and if $\ell \cap K = \emptyset$, then Case 2 occurs. In any case, we will denote the plane containing $\Omega(\ell)$ by $P(\ell)$.

The results we were able to obtain with respect to the above conjecture are the following:

Theorem 1.6. *Let $K \subset \mathbb{R}^3$ be a convex body. Every line ℓ that intersects K possesses the (X) property if and only if K is an ellipsoid.*

Theorem 1.7. *Let $K \subset \mathbb{R}^3$ be a strictly convex body. Assume that P_0 and P_1 are parallel planes that intersect K but do not contain the centroid of K. If for every line $\ell \subset P_0 \cup P_1$, $\Omega(\ell)$ is a circular arc, then K is a Euclidean ball.*

Theorem 1.6 is proved in Section 2 and Theorem 1.7 is proved in Section 3.

2. A characterization of the ellipsoid

In this section we prove Theorem 1.6. We begin with some lemmas that limit the possible behavior of $\Omega(\ell)$.

Lemma 2.1. *Let ℓ be a line intersecting a convex body $K \subset \mathbb{R}^3$ and satisfying the (X) property. Suppose that $\Omega(\ell)$ is different from a point. Then $\Omega(\ell)$ and ℓ are not coplanar.*

Proof. Suppose ℓ and $\Omega(\ell)$ are contained in the plane $P(\ell)$. Let H be any plane through ℓ other than $P(\ell)$. Since $H \cap P(\ell) = \ell$, the centroid of $H \cap K$ lies in ℓ. Since H was arbitrary, using continuity, the centroid of $P(\ell) \cap K$ lies in ℓ too. It follows that $\Omega(\ell)$ is contained in ℓ and since it has no segments, it must be a point. $\square$

Let ℓ be a line with the (X) property and such that $\ell \cap int K \neq \emptyset$. Consider $[a, b] = \ell \cap K$, where $[a, b]$ denotes the segment with endpoints a and b.

Lemma 2.2. *There exists a plane H such that $[a, b] \subset H$ and the centroid of $H \cap K$ is contained in the chord $[a, b]$. Moreover, ℓ intersects $\Omega(\ell)$ at exactly one point and if $\Omega(\ell)$ is not a point, then H is the unique plane through ℓ that is tangent to $\Omega(\ell)$ at the centroid of $H \cap K$.*

Proof. Without loss of generality, we may assume that the origin is contained in the chord $[a, b]$. Let $u \in \mathbb{S}^2$ be a direction parallel to ℓ. We define the function $f \colon S(u) \to \mathbb{R}$ by $f(v) = \langle c(\ell, v), (u \times v)\rangle$ (see Figure 2.1), in other words, $f(v)$ is the length (with sign) of the projection of $c(\ell, v)$ over the line generated by $u \times v$.

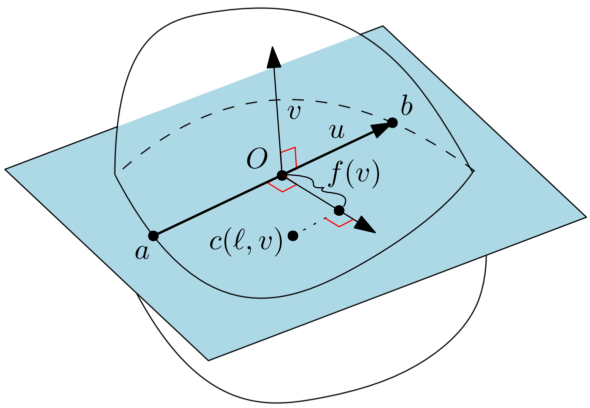


Figure 2.1: Geometric representation of $f(v)$.

First, note that f is an odd function since

$$f(-v) = \langle c(\ell, -v), u \times (-v)\rangle = \langle c(\ell, v), -(u \times v)\rangle = -f(v).$$

Applying the Borsuk-Ulam Theorem we conclude that there exists a vector $v_0 \in S(u)$ such that $f(v_0) = 0$. Since $c(\ell, v_0)$ and $u \times v_0$ are contained in the plane $H(\ell, v_0)$, we have that $c(\ell, v_0) \in [a, b]$.

Suppose that $H(\ell, v_0)$ intersects $\Omega(\ell)$ at a point $y \neq c(\ell, v_0)$ corresponding to the centroid of a section $H' \cap K$, where H' is a plane through ℓ. Since $H' \cap H(\ell, v_0) = \ell$, y lies in ℓ and we must have $\ell \subset P(\ell)$. By Lemma 2.1, $\Omega(\ell)$ must be a point. The uniqueness of $H(\ell, v_0)$ follows from the differentiability of $\Omega(\ell)$ (see Figure 2.2). $\square$

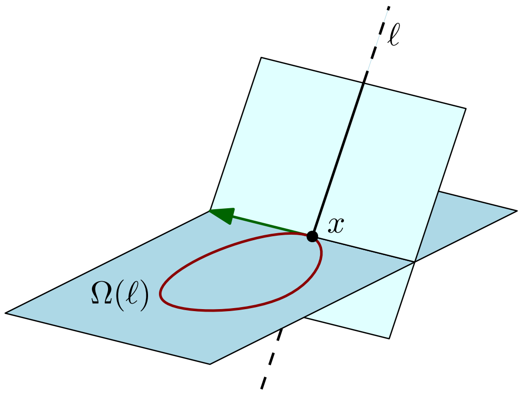


Figure 2.2: There is a unique plane containing ℓ and tangent to $\Omega(\ell)$.

Given any vector $v \in \mathbb{S}^2$ we denote by $Sb(K, v)$ the shadow boundary of K in direction v, i.e., $Sb(K, v) \equiv \{x \in \partial K : \text{there is a line parallel to } v \text{ intersecting } K \text{ at } x\}$. For the proof of the main result in this section we need the following characterization of the ellipsoid due to W. Blaschke [2].

Theorem 2.3. [Blaschke] *Let $K \subset \mathbb{R}^3$ be a convex body such that for every $v \in \mathbb{S}^2$ the shadow boundary $Sb(K, v)$ is a planar curve. Then K is an ellipsoid.*

Proof of Theorem 1.6. Suppose first that K is an ellipsoid and let ℓ be an arbitrary line intersecting K. Consider a non singular affine transformation T that maps K onto the unit Euclidean ball. Since $T(\ell)$ is a line that intersects the ball, it is not difficult to see that $\Omega(T(\ell))$ is a circle. We know that T maps centroids of sections to centroids of sections and circles to ellipses. Thus $T^{-1}(\Omega(T(\ell))) = \Omega(\ell)$ is an ellipse. Therefore ℓ has the (X) property.

Now we prove sufficiency.

Assertion 1. Let ℓ be any line intersecting K. Then the centroid of $P(\ell) \cap K$ is in $\Omega(\ell)$.

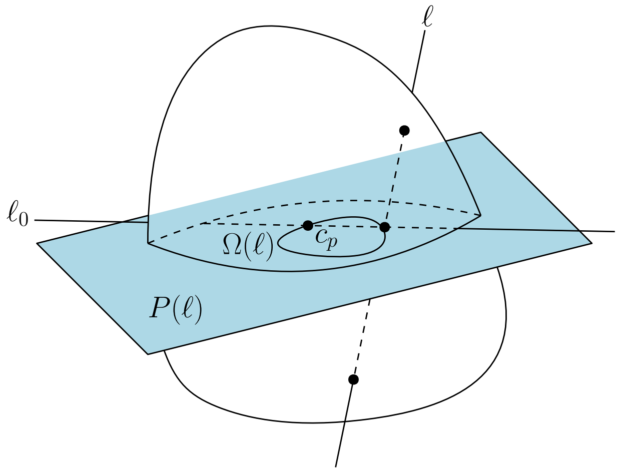


Figure 2.3: The centroid of the section that contains $\Omega(\ell)$ is contained in the curve.

Let c_p be the centroid of $P(\ell) \cap K$ and let $H(\ell, v_0)$ be the plane that contains ℓ and c_p. Also let ℓ_0 be the line that contains c_p and $c(\ell, v_0)$.

Since the planes $P(\ell)$ and $H(\ell, v_0)$ both contain ℓ_0, we have that c_p and $c(\ell, v_0)$ are both contained in $\Omega(\ell_0)$. If c_p and $c(\ell, v_0)$ were different, then $\Omega(\ell_0)$ would intersect ℓ_0 in at least two points (see Figure 2.3). This is a contradiction, and therefore $c_p = c(\ell, v_0)$, i.e., c_p is in $\Omega(\ell)$.

Assertion 2. The convex body K is centrally symmetric.

Let ℓ_1 be a line tangent to K on some point $x \in \partial K$. From Assertion 1, the centroid of $P(\ell_1) \cap K$ is in $\Omega(\ell_1)$. Then, this centroid can be written as $c(\ell_1, u)$ for some $u \in \mathbb{S}^2$. Consider the line passing through $c(\ell_1, u)$ and x. Since $c(\ell_1, u)$ and x are both contained in the planes $H(\ell_1, u)$ and $P(\ell_1)$, this line is given by $\ell = H(\ell_1, u) \cap P(\ell_1)$. From Lemma 2.2, $H(\ell_1, u)$ and $P(\ell_1)$ are both tangent to $\Omega(\ell)$ on $c(\ell_1, u)$, but since the curve is C^1, the two planes are different, and the curve is not coplanar with either, we have that $\Omega(\ell)$ is a point. Since $\Omega(\ell)$ is a point, we conclude from Assertion 1 that $\Omega(\ell)$ is the centroid of any section that contains $\Omega(\ell)$. It follows by a theorem proved by W. Blaschke in [3] that K is centrally symmetric.

Suppose that the center of symmetry of K is at the origin O. Let ℓ_0 and $-\ell_0$ be lines parallel to some vector $u \in \mathbb{S}^2$ and tangent to K at diametrically opposite points x_0 and $-x_0$, respectively. Since K has center of symmetry at O and every section of K through O has its centroid at O, we have that $\Omega(-\ell_0) = -\Omega(\ell_0)$. Since O is in $\Omega(\ell_0) \cap \Omega(-\ell_0)$, it follows that $\Omega(\ell_0)$ and $\Omega(-\ell_0)$ are contained in the same plane, that is, $P(\ell_0) = P(-\ell_0)$.

Let $x \in Sb(K, u)$ be different from x_0 and $-x_0$ and let ℓ be the line parallel to u and tangent to K at x. Since ℓ and ℓ_0 are coplanar, their respective curves $\Omega(\ell)$ and $\Omega(\ell_0)$ intersect at some point $x_1 \neq O$. Similarly, there exists a point $x_2 \in \Omega(\ell) \cap \Omega(-\ell_0)$ such that $x_2 \neq x_1$ and $x_2 \neq O$. Since $\Omega(\ell)$ is a planar curve we have that x_1, x_2, x, and O are coplanar. Then x is contained in $P(\ell_0)$. But x and u were arbitrary, and so $Sb(K, u)$ is a planar curve for all directions u. Now we apply Theorem 2.3 and conclude that K is an ellipsoid. $\square$

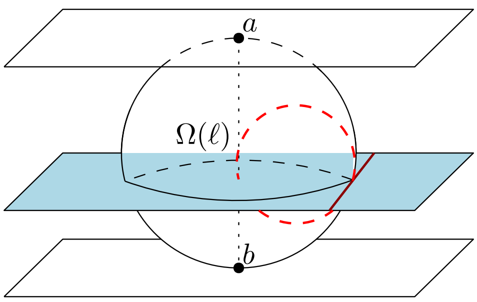


Figure 2.4: Case in which $\Omega(\ell)$ is a circle.

A particular example of a planar and convex differentiable simple closed curve, is the circle. If we assume that $\Omega(\ell)$ is a circle in the previous result, we reach a characterization for the Euclidean ball (see Figure 2.4).

Corollary 2.4. *Suppose that K satisfies the hypothesis of Theorem 1.6. If there exists a plane P that intersects K such that for every line ℓ contained in P we have that $\Omega(\ell)$ is a circle, then K is a Euclidean ball.*

Proof. From Theorem 3, K is an ellipsoid. Consider a line ℓ contained in P that is tangent to K and let $u \in \mathbb{S}^2$ be a vector parallel to ℓ. Since all the sections of K are ellipses and the centroid of an ellipse is the midpoint of any of its affine diameters, we have that $\Omega(\ell)$ and $Sb(K,u)$ are coplanar and homothetic. Since $\Omega(\ell)$ is a circle, $Sb(K,u)$ must also be so.

Consider the two supporting planes parallel to P and let a and b be the two points of tangency. Since given any vector w parallel to P we have that $Sb(K,w)$ is tangent to the supporting planes at a and b, the chord $[a,b]$ is an affine diameter of $Sb(K,w)$. If v is orthogonal to P, then for any $x \in \partial K$ there exists $w_0 \in S(v)$ such that $x \in Sb(K,w_0)$. Thus, any point on the boundary of K is equidistant from $\frac{a+b}{2}$, which makes ∂K a sphere. □

3. A characterization of the Euclidean ball

In this section we prove Theorem 1.7 using the following lemmas. Nonetheless, it is important to notice that the proofs of these lemmas do not require $\Omega(\ell)$ to be a circular arc. Similarly to the previous section, we begin with a lemma that restrains the possible behavior of $\Omega(\ell)$. Unlike previously, this lemma is reserved for lines that are either tangent to K or do not intersect K. Its proof is inspired in an already known generalization of Pappus's Theorem (see [7]).

Lemma 3.1. *Let $K \subset \mathbb{R}^3$ be a strictly convex body. If a line ℓ disjoint to K or tangent to K possesses the (X) property, then the centroid of K is coplanar with $\Omega(\ell)$.*

Proof. Let the centroid of K be $c = (c_x, c_y, c_z)$. We define the vectors

$$e_1(t) = (\cos(t), \sin(t), 0), \quad e_2(t) = (-\sin(t), \cos(t), 0), \quad e_3(t) = (0,0,1).$$

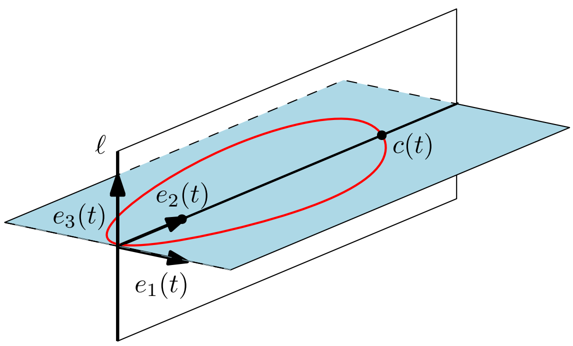


Figure 3.1: Centroid curve parameterization with respect to t.

Suppose first that ℓ is tangent to K. We may assume that $\Omega(\ell)$ is contained in a plane $\{(t,x,0) \in \mathbb{R}^3 : t, x \in \mathbb{R}\}$ and that the point of intersection of $\Omega(\ell)$ and ℓ is the origin. Also, by using an appropriate affine transformation we may suppose that ℓ is generated by the vector $e_3(t)$.

Given that $e_1(t) \times e_2(t) = e_3(t)$, the vectors $e_1(t), e_2(t)$ and $e_3(t)$ are orthonormal and oriented positively. Let us suppose that $e_1(d)^\perp$ is the plane tangent to K that contains ℓ and that $e_1(d)$ and K are contained in the same half-space bounded by $e_1(d)^\perp$. If $c(t)\colon [d, d+\pi] \to \mathbb{R}^3$ is the centroid of $e_1(t)^\perp \cap K$, we define the function $F\colon [d, d+\pi] \times \mathbb{R}^2 \to \mathbb{R}^3$ by

$$F(t, x, y) = c(t) + xe_2(t) + ye_3(t).$$

Then the function F is onto. Let $S_t = e_1(t)^\perp \cap K$ and define $\tilde{S}_t = F^{-1}(S_t)$ and $\tilde{K} = F^{-1}(K)$.

Since $$c_y = \frac{1}{V(K)} \int_K y\, dV \quad \text{and} \quad J_F = c'(t) \cdot e_1(t) - x$$

we have $$V(K)c_y = \int_K y\, dV = \int_{\tilde{K}} (y \circ F) J_F dV$$

$$= \int_{\tilde{K}} y[c'(t) \cdot e_1(t) - x] dV = \int_{\tilde{K}} yc'(t) \cdot e_1(t) \mathrm{d}V - \int_{\tilde{K}} xy\, dV$$

On the other hand, the first integral in the last expression can be written as

$$\int_{\tilde{K}} yc'(t) \cdot e_1(t) dV = \int_d^{d+\pi} c'(t) \cdot e_1(t) \int_{\tilde{S}_t} y\, dx\, dy\, dt.$$

Since affine transformations take centroids to centroids, we have that

$$F\left(t, \int_{\tilde{S}_t} x\, \mathrm{d}x\, \mathrm{d}y, \int_{\tilde{S}_t} y\, \mathrm{d}x\, \mathrm{d}y\right) = c(t) = F(t, 0, 0).$$

For t fixed, $(x, y) \mapsto F(t, x, y)$ is injective and therefore

$$\int_{\tilde{S}_t} y\, \mathrm{d}x\, \mathrm{d}y = 0,$$

from which we can see that $\quad V(K)c_y = \int_{\tilde{K}} (-xy) dV.$

Now, let $U\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $U(t, x, y) = (t, x, -y)$. Then $J_U = -1$ and it follows that

$$V(K)\, c_y = \int_{\tilde{K}} (-xy) dV = \int_{U^{-1}(\tilde{K})} [(-xy) \circ U] J_U dV = \int_{U^{-1}(\tilde{K})} (-xy) dV,$$

while $$-V(K)\, c_y = \int_{\tilde{K}} xy\, dV = \int_{U^{-1}(\tilde{K})} [(xy) \circ U] J_U dV = \int_{U^{-1}(\tilde{K})} xy\, dV.$$

We conclude that $V(K)c_y = 0$ and that the centroid of K is then contained in the plane $\{(t, x, 0) \in \mathbb{R}^3 \colon t, x \in \mathbb{R}\}$.

Suppose now that ℓ does not intersect K. We may assume that $\Omega(\ell)$ is contained in a plane $\{(t,x,0)\in\mathbb{R}^3 : t,x\in\mathbb{R}\}$, that the origin is at the point of intersection of the plane and ℓ, and that ℓ is generated by the vector $e_3(t)$. Suppose that $e_1(d_1)^\perp$ and $e_1(d_2)^\perp$ are the planes tangent to K that contain ℓ where $0<d_2-d_1\le\pi$. We now define the function $F\colon [d_1,d_2]\times\mathbb{R}^2\to\mathbb{R}^3$ as

$$F(t,x,y)=c(t)+xe_2(t)+ye_3(t).$$

We then proceed as in the previous case to obtain $c_y=0$. □

Lemma 3.2. *Let $K\subset\mathbb{R}^3$ be a strictly convex body. Assume that P_0 and P_1 are two parallel planes that intersect K but do not contain the centroid of K. If every line $\ell\subset P_0\cup P_1$ possess the (X) property, then, $Sb(u)$ is flat for all directions u parallel to P_0 (or P_1).*

Proof. Let c_0 be the centroid of $P_0\cap K$ and c be the centroid of K. Given $u\in\mathbb{S}^2$ parallel to P_0 and $d\in\mathbb{R}$, we denote the orthogonal line to u whose signed distance from c_0 is d by $\ell_{d,u}$. Recall that the plane that contains $\Omega(\ell_{d,u})$ is denoted by $P(\ell_{d,u})$. We also denote

$$\Omega_\infty=\left\{x\in\mathbb{R}^3\ :\ \begin{array}{l}x \text{ is the centroid of } P\cap K \text{ where } P\\ \text{is parallel to } P_0 \text{ and } P\cap K\neq\emptyset\end{array}\right\}.$$

We first prove that Ω_∞ is the line segment determined by c and c_0.

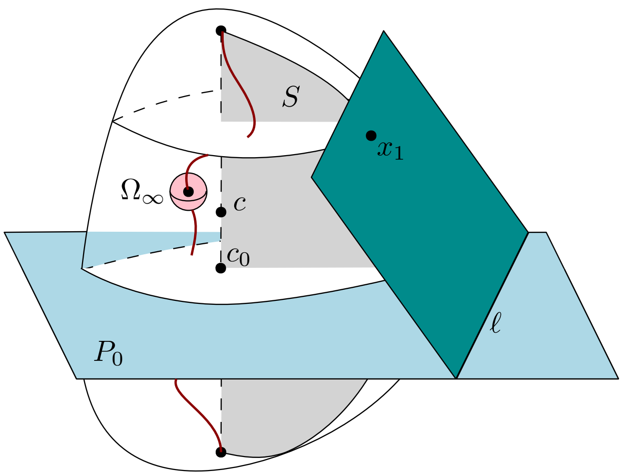


Figure 3.2: The neighborhood around $\Omega_\infty(x_0)$ is outside S.

Proceeding by contradiction, consider $x_0\in\Omega_\infty$ such that x_0 is not contained in the line determined by c and c_0. Let ϵ be sufficiently small so that the ball $B(x_0,\epsilon)$ does not intersect the line defined by c and c_0. By continuity, for every $u\in\mathbb{S}^2$ parallel to P_0, there exists $D_u>0$ so that for all $|d|>D_u$, we have that $\Omega(\ell_{d,u})$ intersects $B(x_0,\epsilon)$. We will see that this leads to a contradiction. Let S be a section of K that contains $[c,c_0]$ and does not intersect the neighborhood of x_0. Let us also consider a section S' parallel to P_0, sufficiently close to one of the tangent planes to K that are parallel to P_0. If $x_1 \in \partial K \cap S \cap S'$, then the tangent plane to K at the point x_1 intersects P_0 in a line ℓ that is sufficiently far from c_0 (see Figure 3.2). Since x_1, c_0 and c are non-collinear points in S and $P(\ell)$ we have that $P(\ell) \cap K = S$, contradicting the fact that $\Omega(\ell)$ intersect $B(x_0, \epsilon)$. It follows that Ω_∞ is a line segment.

Now we prove that the for every $|d|$ sufficiently large, the planes $P(\ell_{d,u})$ are determined only by u. That is, we prove that for every $d \in \mathbb{R}$ such that $\ell_{d,u}$ is tangent to K or does not intersect K, we have $P(\ell_{d,u}) = P(\ell_{0,u})$. Since Ω_∞ is a line segment containing c and c_0, we know that $P(\ell_{0,u})$ contains the centroid of $P_1 \cap K$. From here there are two cases:

Case 1. $\Omega(\ell_{0,u})$ is not a point.

Let $\ell \subset P_1$ be a line orthogonal to u such that $\ell \cap K = \emptyset$. Using Lemma 3.1, we know that $P(\ell)$ coincides on three points with $P(\ell_{0,u})$: on c, on the centroid of $P_1 \cap K$, and on the centroid of the intersection of the plane that contains $\ell_{0,u}$ and ℓ with K. Thus, $P(\ell) = P(\ell_{0,u})$. Now consider the line $\ell_{d,u}$ with $d \in \mathbb{R}$ such that $\ell_{d,u}$ does not intersect the interior of K. Since $P(\ell_{d,u})$ coincides with $P(\ell)$ on c, c_0, and the centroid of the intersections of the plane that contains $\ell_{d,u}$ and ℓ with K, we can determine that $P(\ell_{d,u}) = P(\ell) = P(\ell_{0,u})$. Notice that a similar result can be obtained for lines in P_1.

Case 2. $\Omega(\ell_{0,u})$ is a point.

Consider the line ℓ_2 in P_1 that is orthogonal to u and passes through the centroid of $P_1 \cap K$. Since the centroid of the intersection of the plane that contains ℓ_2 and $\ell_{0,u}$ with K is $\Omega(\ell_{0,u})$, we have that $\Omega(\ell_2)$ intersects both ℓ_2 and $\Omega(\ell_{0,u})$ and therefore cannot be a point. Thus, we can proceed as in the first case, with respect to P_1.

Finally, let $u \in \mathbb{S}^2$ parallel to P_0. Given $x \in Sb(u)$, the intersection of P_0 with the tangent plane to K at x is necessarily either a line $\ell_{d,v}$ where v is orthogonal to u or an empty set. In the first case, x is contained in the centroid curve $\Omega(\ell_{d,v})$ and in $P(\ell_{d,v}) = P(\ell_{0,v})$. In the second case, the plane tangent to K at x is parallel to P_0 and x is then contained in Ω_∞. Hence, x is contained $P(\ell_{0,v})$. It follows that $Sb(u)$ is a subset of $P(\ell_{0,v})$ and hence it is flat. $\square$

The following lemma is independent from the previous hypotheses, but it allows us to extend the previous result. Its proof is essentially given in the proof of Lemma 4.7 in the paper by G. R. Burton and P. Mani [6].

Lemma 3.3. *Let K be a convex body. If there exists a direction η such that for all u contained in $S(\eta)$ we have that $Sb(u)$ is a planar curve, then there exists a line ℓ such that all sections of K orthogonal to η are homothetic and centrally symmetric with center in ℓ.*

In case of having a pair of parallel planes that meet the requirements of Lemma 4, we have that K is the union of homothetic centrally symmetric sections. These hypotheses, however, are not enough to guarantee K is an ellipsoid. Actually, even if the hypothesis of central symmetry is added, K is not necessarily an ellipsoid.

Now we are interested in the following question: what hypotheses would be sufficiently strong to guarantee K is a sphere? Theorem 1.7 gives an answer. We present now its proof.

Proof of Theorem 1.7. We may assume that the xy-plane is tangent to K at the origin and parallel to P_0. By Lemmas 3.2 and 3.3, all sections parallel to P_0 are homothetic, centrally symmetric and have centroids in a line ℓ_0 parameterized as $t \mapsto tm$, where $m \in \mathbb{S}^2$.

Assertion 1. For any plane P containing ℓ_0, the centroid of $P \cap K$ is a fixed point and lies in ℓ_0.

In order to prove this assertion, we may assume, after applying a linear transformation, that ℓ_0 is the z-axis. Let P be a plane containing the z-axis and intersecting K. Since any chord of $P \cap K$ is a chord of $(P_0 + (0,0,t)) \cap K$ that passes through $(0,0,t)$ for some $t \in \mathbb{R}$ and $(0,0,t)$ is the center of symmetry of $(P_0 + (0,0,t)) \cap K$, we conclude that $P \cap K$ is symmetric with respect to the z-axis. Then the centroid of P lies in the z-axis.

Let P_1 and P_2 be two planes containing the z-axis and intersecting K. By the previous argument, for $i = 1,2$, $P_i \cap K$ is the union of chords parallel to P_0 whose midpoints are in the z-axis. Moreover, the quotient of the lengths of any two such chords in $P_1 \cap K$ equals the quotient of the lengths of the corresponding chords in $P_2 \cap K$. It follows that $P_1 \cap K$ and $P_2 \cap K$ differ by a linear transformation T that fixes all points in the z-axis. Since T maps centroids to centroids, we conclude that the centroids of $P_1 \cap K$ and $P_2 \cap K$ are the same and Assertion 1 follows.

Now we continue with the proof of the theorem. By Assertion 1, there exists $t_c \in \mathbb{R}$ such that for any plane P containing ℓ, the centroid of $P \cap K$ is $t_c m$. Just as in the proof of Lemma 3.2, given $u \in \mathbb{S}^2$ parallel to P_0 and $d \in \mathbb{R}$, we denote the orthogonal line to u whose signed distance from the centroid of $P_0 \cap K$ is d by $\ell_{d,u}$.

Assertion 2. The point $t_c m$ is $\Omega(\ell_{d,u})$ for any given $d \in \mathbb{R}$ and any $u \in S(e_3)$.

To prove this assertion, denote the point $\ell_{0,u} \cap \Omega(\ell_{0,u})$ by A, the centroid of $P_1 \cap K$ by B and denote $t_c m$ by C. Consider a line ℓ' parallel to u and contained in P_1, denote the intersection of ℓ' with $P(\ell_{0,u})$ by D, and finally denote the centroid of the section coplanar to ℓ' and $\ell_{0,u}$ by E. Let $F \in P(\ell_{0,u}) \cap P_0$ be a point such that $\angle FAC$ covers the same arc as $\angle AEC$. From Lemma 4 these points are coplanar. Since $\angle FAC$ and $\angle AEC$ cover the same arc, we have that $\angle FAC = \angle AEC$.

By properties of parallel lines we have that $\angle FAC = \angle ABD$. Depending on the order of the points, it follows that either $\angle CBD + \angle CED = \pi$ or $\angle CBD = \angle CED$. In both cases, the quadrilateral $EDBC$ is cyclic and $t_c m = C$ is an element of $\Omega(\ell')$ (see Figure 3.3). Assertion 2 now follows.

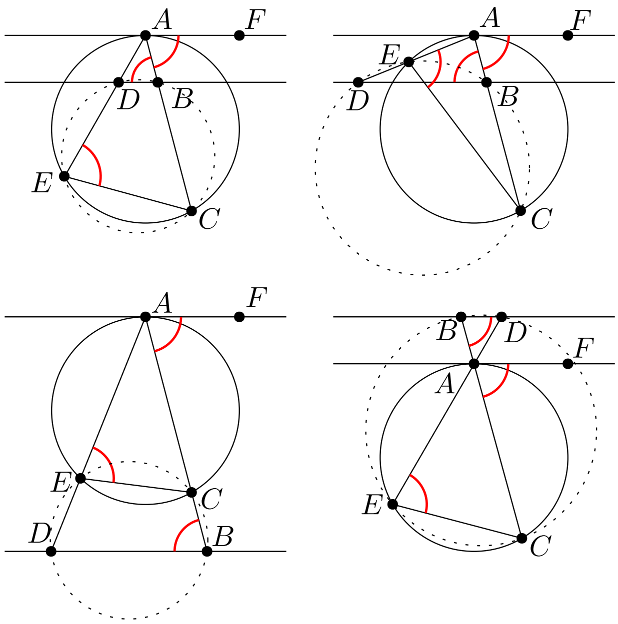
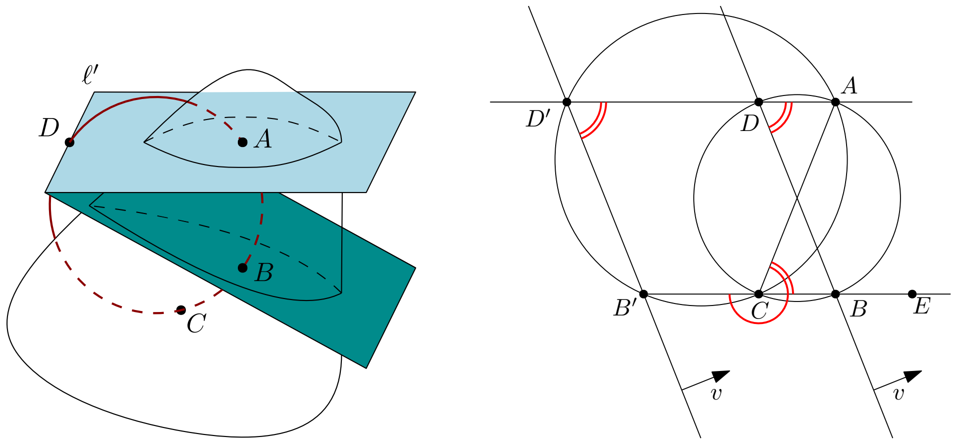


Figure 3.3: The different cases for the order of the points and lines.

Proceeding with the proof, consider $v \in \mathbb{S}^2$ with $v \neq e_3$ and some plane orthogonal to v. Denote the intersection of the plane with P_0 by ℓ' and the centroid of the intersection of the plane with K by B. If we define A, C and D as in the previous assertion, we have a cyclic quadrilateral.

Let E be a point outside K and on the line BC. Considering that the angles are measured in an anticlockwise manner we have that $\angle ABC + \angle CBA = 2\pi$. Depending on the order of A, B, C and D, we have that either $\angle BCA = \angle BDA$ or $\angle BCA + \angle BDA = 2\pi$. Since in both cases we have that $\angle ECA = \angle BDA$, which is constant for any plane orthogonal to v, then the centroids of parallel planes are collinear and are contained on the line with angle $\angle ECA$ with respect to CA. By Theorem 1.2 and Corollary 2.4 it follows that K is a Euclidean ball. $\square$



(a) Tridimensional view. (b) Bidimensional view.

Figure 3.4: Angles formed between centroids of some curve $\Omega(\ell')$.

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