

Non-Occurrence of a Gap Between Bounded and Sobolev Functions for a Class of Nonconvex Lagrangians

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We consider the classical functional of the Calculus of Variations of the form

$$I(u) = \int_{\Omega} F(x, u(x), \nabla u(x)) \, dx$$

where Ω is a bounded open subset of $\mathbb{R}^n$ and $F: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a given Carathéodory function; the admissible functions u coincide with a given Lipschitz function on $\partial\Omega$. We formulate some conditions under which a given function in $\phi + W_0^{1,p}(\Omega)$ with $I(u) < +\infty$ can be approximated by a sequence of functions $u_k \in \phi + W_0^{1,p}(\Omega) \cap L^\infty$ converging to u in the norm of $W^{1,p}$, and such that $I(u_k) \rightarrow I(u)$. The problem is strictly related with the non occurrence of the Lavrentiev gap.

Keywords: Lavrentiev, Lavrentieff, approximation, bounded functions, regularity.

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1. Introduction

Consider the classical functional of the Calculus of Variations of the form

$$I(u) = \int_{\Omega} F(x, u(x), \nabla u(x)) \, dx$$

where Ω is a bounded open subset of $\mathbb{R}^n$ and $F: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a given Carathéodory function. We also consider a prescribed boundary function ϕ that we will assume to be Lipschitz.

The existence of a minimizer of I among the functions that share the same boundary datum is well established in the Sobolev spaces $W_\phi^{1,p}(\Omega) := \phi + W_0^{1,p}(\Omega)$, $p \geq 1$,

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under some suitable assumptions. For practical purposes, e.g., numerical approximations of the minimizer and of the minimum level of I , it may be useful to approximate a given function u such that $F(x, u(x), \nabla u(x)) \in L^1(\Omega)$ with a sequence of functions $(u_k)_k$ of a dense subset $X \cap W_\phi^{1,p}(\Omega)$ of $W_\phi^{1,p}(\Omega)$, both in the norm of $W^{1,p}$ and in energy, i.e., $\lim_{k \rightarrow +\infty} I(u_k) = I(u)$: when this happens we say that F is X -regular at u in $W_\phi^{1,p}(\Omega)$. When $X = W_\phi^{1,\infty}(\Omega)$, the validity of such an approximation is referred to as the non occurrence of the Lavrentiev gap at u . Its occurrence represents a difficulty in estimating the value of $I(u)$ via standard numerical methods.

In [2] we considered the autonomous functional

$$I(v) = \int_{\Omega} F(v, \nabla v) \, dx,$$

with $F(s, \xi)$ convex, and we proved that this phenomenon does not occur, whenever $I(u)$ is finite. A first and crucial step in the proof of [2, Theorem 1] is to show that there exists a sequence $(u_k)_k$ of *bounded* functions approximating u in $W_\phi^{1,p}(\Omega)$ such that $\lim_{k \rightarrow +\infty} I(u_k) = I(u)$, so that, using the definition above, F is L^∞ -regular.

It is worth mentioning that, on the one hand, a key point in the proof of [2, Theorem 1] is the convexity of the Lagrangian and, on the other hand, no examples of the occurrence of the Lavrentiev phenomenon are known in the autonomous case. This remark suggests that it is of some interest to try to detect classes of Lagrangians, larger than the class of convex ones, that are $W^{1,\infty}$ -regular.

In this paper we make a first step in this direction studying the problem of enlarging the class of functionals that are at least L^∞ -regular. This problem is interesting also for numerical approximations of the minimum.

We emphasize the fact that, in the definition of X -regularity, we require the sequence of approximating functions to share the same boundary datum ϕ . Building approximating functions without this constraint is much simpler. For instance, the minimizer of the celebrated example of the occurrence of the Lavrentiev phenomenon in [1] may be easily approximated in norm and in energy by a sequence of Lipschitz functions if one does not take care of the boundary datum.

Approximating a given function both in $W^{1,1}(\Omega)$ and in energy with a sequence of bounded functions of $W^{1,1}(\Omega)$ is not always possible; a counterexample is given in Example 3.3.

We formulate here in Theorem 3.5 a condition, other than convexity, that ensures the L^∞ -regularity for F at a given $u \in W_\phi^{1,p}(\Omega)$ and give several examples of Lagrangians for which the assumptions of Theorem 3.5 are satisfied. In particular we introduce various classes of Lagrangians of the form

$$F(u, \nabla u) = a(u)g(\nabla u) + b(u)$$

that are L^∞ -regular at any $u \in W_\phi^{1,p}(\Omega)$ for which $I(u)$ is finite. Our methods are inspired by those of [4, 5, 2] and on Stampacchia truncation method.

Notation

- The *scalar product* of x, y in $\mathbb{R}^n$ is denoted by $\langle x, y \rangle$.
- The *pointwise maximum* (resp. *minimum*) of two functions u, v is denoted by $u \vee v$ (resp. $u \wedge v$), $u^+ = u \vee 0$ (resp. $u^- = (-u) \vee 0$) is the positive (resp. negative) part of u .
- The *subdifferential* in the sense of Moreau-Rockafellar of a (non necessarily convex) function $g: \mathbb{R}^m \rightarrow \mathbb{R}$ at $\xi_0 \in \mathbb{R}^m$ is the set

$$\partial g(\xi_0) := \{\nu \in \mathbb{R}^m : g(\xi) - g(\xi_0) \geq \langle \nu, \xi - \xi_0 \rangle \quad \forall \xi \in \mathbb{R}^m\}.$$

- For $E \subseteq \mathbb{R}^n$, $\lambda(E)$ is the n -dimensional Lebesgue measure of E .
- $\mathbf{1}_E$ is the indicator function of a set E , i.e.

$$\mathbf{1}_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{otherwise.} \end{cases}$$

2. Assumptions

- $F: \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, $(x, s, \xi) \mapsto F(x, s, \xi)$ is a Carathéodory function, bounded below by $\langle \alpha(x), \xi \rangle + \beta(x)$ for some $\alpha \in L^1(\Omega; \mathbb{R}^n)$, $\beta \in L^1(\Omega)$.
- $\Omega \subset \mathbb{R}^n$ is an open and bounded set.
- We define $I(u) := \int_{\Omega} F(x, u, \nabla u) dx$ (the “energy”).
- ϕ is a Lipschitz function on $\overline{\Omega}$.

3. L^∞ -regularity

Definition 3.1. Let X be a set of functions. Let $u \in W_\phi^{1,p}(\Omega)$ be such that $F(x, u, \nabla u) \in L^1(\Omega)$. The Lagrangian F is said to be X -regular at u (in $W_\phi^{1,p}(\Omega)$) if there is a sequence $(u_k)_k$ in $X \cap W_\phi^{1,p}(\Omega)$ such that $(u_k)_k$ converges in norm and in energy to u , namely

$$(i) \quad \lim_{k \rightarrow +\infty} \|u_k - u\|_{W^{1,p}} = 0; \quad \text{and} \quad (ii) \quad \lim_{k \rightarrow +\infty} I(u_k) = I(u).$$

F is said to be X -regular if it is X -regular at every $u \in W_\phi^{1,p}(\Omega)$ with the property $F(x, u, \nabla u) \in L^1(\Omega)$. $\square$

Remark 3.2. When $X = W_\phi^{1,\infty}(\Omega)$, the X -regularity of F is equivalent to the fact that, for every $u \in W^{1,p}(\Omega)$, there is no Lavrentiev gap at u . We emphasize the fact that the sequence of approximating functions is required to satisfy the *same* boundary condition as u . When $X = L^\infty(\Omega)$ we will simply refer to $L^\infty(\Omega)$ -regularity as to L^∞ -regularity.

Example 3.3. (A Lagrangian that is not L^∞ -regular) Let Ω be the unit disk of $\mathbb{R}^2$. Let $\phi \in W^{1,1}(\Omega) \setminus L^3(\Omega)$ be Lipschitz in a neighbourhood of $\partial\Omega$.

Then ϕ minimizes

$$I(u) = \int_{\Omega} |u - \phi|^3 dx$$

in $W_{\phi}^{1,1}(\Omega)$. However, $I(u) = +\infty$ for any bounded function u . Indeed, if we have $I(u) < +\infty$, then $u - \phi \in L^3(\Omega)$, so that if u is bounded then $\phi \in L^3(\Omega)$, a contradiction.

Example 3.4. When the Lagrangian $F(v, \nabla v)$ is autonomous and *convex* in the joint variables, there is $W^{1,\infty}$ regularity for every $u \in W_{\phi}^{1,p}(\Omega)$ with the property $F(u, \nabla u) \in L^1(\Omega)$ (see [2]).

When F is non autonomous, a celebrated example [1] shows a polynomial $F(x, s, \xi)$ in $\mathbb{R}^3$ that is convex and superlinear in ξ , that is not $W^{1,\infty}$ -regular in the space of the absolutely continuous functions.

Theorem 3.5. *Consider two sequences of affine functions*

$$\varphi_{k,-}(x) := \langle \xi_-, x \rangle + \tau_{k,-}, \quad \varphi_{k,+}(x) := \langle \xi_+, x \rangle + \tau_{k,+} \quad (k \in \mathbb{N})$$

where ξ_-, ξ_+ are given vectors in $\mathbb{R}^n$ and $(\tau_{k,-})_k$ and $(\tau_{k,+})_k$ are monotonic sequences satisfying

$$\lim_{k \rightarrow +\infty} \tau_{k,-} = -\infty, \quad \lim_{k \rightarrow +\infty} \tau_{k,+} = +\infty \quad (1)$$

Let $u \in W_{\phi}^{1,p}(\Omega)$ be such that $F(x, u, \nabla u) \in L^1(\Omega)$ and

$$\lim_{k \rightarrow +\infty} \int_{\{u > \varphi_{k,+}\}} F(x, \varphi_{k,+}, \nabla \varphi_{k,+}) dx = 0, \quad (2)$$

$$\lim_{k \rightarrow +\infty} \int_{\{u < \varphi_{k,-}\}} F(x, \varphi_{k,-}, \nabla \varphi_{k,-}) dx = 0. \quad (3)$$

Then F is L^{∞} -regular at u in $W^{1,p}(\Omega)$.

The next elementary result, based on the Convergence Dominated Theorem, will be widely used in the proof of Theorem 3.5.

Lemma 3.6. *Let $u, h \in L^1(\Omega)$ and let $(\psi_k)_k$ be a sequence of measurable functions such that*

$$\lim_{k \rightarrow +\infty} \psi_k(x) = +\infty \quad \text{a.e. } x \in \Omega.$$

Then

$$\lim_{k \rightarrow +\infty} \int_{\{u > \psi_k\}} |h| dx = 0.$$

Proof of Theorem 3.5. The proof is based on the truncation method. Let

$$u_k := \begin{cases} \varphi_{k,-} & \text{if } u < \varphi_{k,-}, \\ u & \text{if } \varphi_{k,-} \leq u \leq \varphi_{k,+}, \\ \varphi_{k,+} & \text{if } u > \varphi_{k,+}. \end{cases}$$

Since $(\varphi_{k,+})_k$ (resp. $(\varphi_{k,-})_k$) converges uniformly to $+\infty$ (resp. $-\infty$), then $u_k = \phi$ on $\partial\Omega$ as soon as k is such that

$$-\|\phi\|_\infty > \max\{\varphi_{k,-}(x) : x \in \Omega\}, \quad \|\phi\|_\infty < \min\{\varphi_{k,+}(x) : x \in \Omega\}.$$

It follows that the sequence $(u_k)_k$ converges to u in $W^{1,p}(\Omega)$. Indeed, for k big enough in such a way that $\varphi_{k,+} > 0$ and $\varphi_{k,-} < 0$, from Lemma 3.6 we obtain

$$\begin{aligned} \int_{\Omega} |u_k - u|^p dx &= \int_{\{u < \varphi_{k,-}\}} |u - \varphi_{k,-}|^p dx + \int_{\{u > \varphi_{k,+}\}} |u - \varphi_{k,+}|^p dx \\ &\leq \int_{\{u < \varphi_{k,-}\}} |u|^p dx + \int_{\{u > \varphi_{k,+}\}} |u|^p dx \rightarrow 0 \quad \text{as } k \rightarrow +\infty. \end{aligned}$$

Analogously,

$$\begin{aligned} \int_{\Omega} |\nabla u_k - \nabla u|^p dx &= \int_{\{u < \varphi_{k,-}\}} |\nabla \varphi_{k,-}|^p dx + \int_{\{u > \varphi_{k,+}\}} |\nabla \varphi_{k,+}|^p dx \\ &= \int_{\{u < \varphi_{k,-}\}} |\xi_- - \nabla u|^p dx + \int_{\{u > \varphi_{k,+}\}} |\xi_+ - \nabla u|^p dx \\ &\leq 2^p \left(\int_{\{u < \varphi_{k,-}\}} |\xi_-|^p + |\nabla u|^p dx + \int_{\{u > \varphi_{k,+}\}} |\xi_+|^p + |\nabla u|^p dx \right) \end{aligned}$$

tends to 0 as $k \rightarrow +\infty$. Moreover

$$\begin{aligned} I(u_k) &= \int_{\varphi_{k,-} \leq u \leq \varphi_{k,+}} F(x, u, \nabla u) dx + \\ &\quad + \int_{\{u < \varphi_{k,-}\}} F(x, \varphi_{k,-}, \nabla \varphi_{k,-}) dx + \int_{\{u > \varphi_{k,+}\}} F(x, \varphi_{k,+}, \nabla \varphi_{k,+}) dx. \end{aligned} \quad (4)$$

It follows from (2) and (3) that

$$\lim_{k \rightarrow +\infty} I(u_k) = \lim_{k \rightarrow +\infty} \int_{\varphi_{k,-} \leq u \leq \varphi_{k,+}} F(x, u, \nabla u) dx = I(u),$$

due to the integrability of $F(x, u, \nabla u)$. □

In [2] we showed that if $F(s, \xi)$ is autonomous, then F is $W^{1,\infty}$ regular in $W_{\phi}^{1,p}(\Omega)$ at every $u \in W_{\phi}^{1,p}(\Omega)$ such that $I(u)$ is finite. Here is a condition, other than convexity, that ensures the L^∞ -regularity of F .

Proposition 3.7. *Let $F(s, \xi) = a(s)g(\xi)$ where $a: \mathbb{R} \rightarrow [0, +\infty[$, $g: \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous, and $g(\xi) \geq c|\xi|$ for some $c > 0$.*

Then F is L^∞ -regular at any $u \in W^{1,1}(\Omega)$ such that $F(u, \nabla u) \in L^1(\Omega)$.

Proof. Let $u \in W_\phi^{1,1}(\Omega)$ be such that $F(u, \nabla u) \in L^1(\Omega)$. We show the existence of sequences $(\tau_{k,-})_k$ and $(\tau_{k,+})_k$ that satisfy (1) and (2) with $\xi_+ = \xi_- = 0$. Notice that

$$\int_{\{u \geq t\}} F(t, 0) dx = \lambda(\{u \geq t\})a(t)g(0) \quad \forall t \geq 0. \quad (5)$$

Denoting $A(t) := \int_0^t a(s) ds$, we have

$$\begin{aligned} \int_0^{+\infty} \lambda(\{u \geq t\})a(t) dt &= \int_0^{+\infty} a(t) \int_{\Omega} \mathbf{1}_{\{u \geq t\}} dx dt \\ &= \int_{\{u \geq 0\}} \int_0^{u(x)} a(t) dt dx = \int_{\{u \geq 0\}} A(u(x)) dx. \end{aligned} \quad (6)$$

We claim that $A(u) \in L^1(\{u \geq 0\})$. (7)

Since $F(u, \nabla u) = a(u)g(\nabla u) \in L^1(\Omega)$ then the growth assumption on g implies that $a(u)|\nabla u| \in L^1(\Omega)$. Let

$$a_k := a \mathbf{1}_{\{a \leq k\}}, \quad A_k(t) := \int_0^t a_k(s) ds \quad \forall k \in \mathbb{N}.$$

Since $a_k \in L^\infty(\Omega)$ and $A_k(0) = 0$ then, from [3, Theorem 1.74],

$$A_k(u) \in W^{1,1}(\Omega), \quad \nabla A_k(u) = a_k(u) \mathbf{1}_{\{a(u) \leq k\}} \nabla u.$$

The Sobolev inequality then yields

$$\|A_k(u) - A_k(\phi)\|_1 \leq C \|a_k(u) \nabla u\|_1 \leq C \|a(u) \nabla u\|_1,$$

for some constant C depending just on Ω : in particular

$$\|A_k(u)\|_1 \leq \|A_k(\phi)\|_1 + C \|a(u) \nabla u\|_1 \leq \|A(\phi)\|_1 + C \|a(u) \nabla u\|_1$$

for all $k \geq \|\phi\|_\infty$. Now, for $u(x) \geq 0$,

$$A_k(u(x)) = \int_0^{u(x)} a_k(s) ds \uparrow_k \int_0^{u(x)} a(s) ds = A(u(x)).$$

Beppo Levi's monotonic convergence Theorem implies that $A(u \mathbf{1}_{\{u \geq 0\}}) \in L^1(\Omega)$, proving (7).

It follows from (7) and (6) that $\lambda(\{u \geq t\})a(t) \in L^1([0, +\infty[)$; thus

$$\liminf_{t \rightarrow +\infty} \lambda(\{u \geq t\})a(t) = 0.$$

As a consequence, there exists a sequence $(\tau_{k,+})_k$ satisfying

$$\lim_{k \rightarrow +\infty} \lambda\{u \geq \tau_{k,+}\}a(\tau_{k,+}) = 0,$$

so that, by (5),
$$\lim_{k \rightarrow +\infty} \int_{\{u > \tau_{k,+}\}} F(\tau_{k,+}, 0) dx = 0.$$

Analogously, one obtains a sequence $(\tau_{k,-})_k$ with the desired properties. Theorem 3.5 yields the conclusion. $\square$

4. The non-oscillatory condition at infinity (NOC)

We introduce here Condition (NOC) that will ensure in Section 5 the L^∞ -regularity of a Lagrangian F . It is a property that is inspired by convex, autonomous Lagrangians. If $F(s, \xi)$ is autonomous and convex let, for every $k \in \mathbb{N}_{\geq 1}$, $(q_k, \zeta_{k,+}) \in \partial F(k, 0)$. Then

$$\forall s \in \mathbb{R} \quad F(s, \xi) - F(k, 0) \geq q_k(s - k) + \langle \zeta_{k,+}, \xi \rangle. \quad (8)$$

Thus, if $(\bar{q}, \bar{\zeta}) \in \partial F(0, 0)$, the monotonicity of the convex subdifferential gives

$$0 \leq (q_k - \bar{q})(k - 0) + (\zeta_k - \bar{\zeta})(0 - 0) = (q_k - \bar{q})k,$$

so that $q_k \geq \bar{q}$ and (8) yields

$$\forall s \geq k \quad F(s, \xi) - F(k, 0) \geq \bar{q}(s - k) + \langle \zeta_{k,+}, \xi \rangle \geq -|\bar{q}|s^{p^*} + \langle \zeta_{k,+}, \xi \rangle. \quad (9)$$

Analogously, it turns out that there are $\zeta_{k,-} \in \mathbb{R}^n$ and $d \geq 0$ such that

$$\forall s \leq -k \quad F(s, \xi) - F(-k, 0) \geq -d|s|^{p^*} + \langle \zeta_{k,-}, \xi \rangle. \quad (10)$$

The non-oscillatory condition (NOC) imposes more general conditions than (9)–(10) and is satisfied by a wider class of functions than autonomous and convex ones. If $1 < p < n$, we set $p^* = \frac{np}{n-p}$; recall that the embedding $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ is compact. If $p = 1$ we set $p^* = p = 1$.

Definition 4.1. (Non-oscillatory condition at infinity (NOC)) The Lagrangian

$$F: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$$

satisfies the non-oscillatory condition (NOC) if there are:

- (1) sequences $(\varphi_{k,-})_k$ and $(\varphi_{k,+})_k$ of affine functions of the form

$$\varphi_{k,-}(x) := \langle \xi_-, x \rangle + \tau_{k,-}, \quad \varphi_{k,+}(x) := \langle \xi_+, x \rangle + \tau_{k,+} \quad \forall x \in \Omega;$$

where ξ_+, ξ_- are prescribed vectors in $\mathbb{R}^n$ and $(\tau_{k,+})_k$ and $(\tau_{k,-})_k$ are real monotonic sequences with

$$\lim_{k \rightarrow +\infty} \tau_{k,-} = -\infty, \quad \lim_{k \rightarrow +\infty} \tau_{k,+} = +\infty, \quad (11)$$

- (2) $a \in L^\infty_{loc}(\mathbb{R})$, $c > 0$, $d \geq 0$;

- (3) sequences $(\zeta_{k,+})_k$ and $(\zeta_{k,-})_k$ in $\mathbb{R}^n$

such that, for all $k \in \mathbb{N}$, the following conditions hold:

- (a) The maps $x \mapsto F(x, \varphi_{k,-}, \nabla \varphi_{k,-})$, $x \mapsto F(x, \varphi_{k,+}, \nabla \varphi_{k,+})$ are summable;

(b) for all $x \in \Omega$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$,

$$F(x, s, \xi) \geq \langle \zeta_{k,+}, a(s)\xi - a(\varphi_{k,+})\xi_+ \rangle + cF(x, \varphi_{k,+}, \nabla \varphi_{k,+}) - d|s|^{p^*},$$

$$s \geq \varphi_{k,+}(x), \quad (12)$$

$$F(x, s, \xi) \geq \langle \zeta_{k,-}, a(s)\xi - a(\varphi_{k,-})\xi_- \rangle + cF(x, \varphi_{k,-}, \nabla \varphi_{k,-}) - d|s|^{p^*},$$

$$s \leq \varphi_{k,-}(x). \quad (13)$$

Remark 4.2. When $\xi_- = \xi_+ = 0$, points (1) and (2) in (NOC) become:

(1) The maps $x \mapsto F(x, \tau_{k,-}, 0)$, $x \mapsto F(x, \tau_{k,+}, 0)$ are in $L^1(\Omega)$;

(2) For all $x \in \Omega$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$,

$$F(x, s, \xi) \geq a(s)\langle \zeta_{k,+}, \xi \rangle + cF(x, \tau_{k,+}, 0) - d|s|^{p^*} \quad \forall s \geq \tau_{k,+}, \quad (14)$$

$$F(x, s, \xi) \geq a(s)\langle \zeta_{k,-}, \xi \rangle + cF(x, \tau_{k,-}, 0) - d|s|^{p^*} \quad \forall s \leq \tau_{k,-}. \quad (15)$$

There are several cases in which an autonomous Lagrangian is allowed to satisfy (NOC).

Proposition 4.3. (Validity of (NOC)) *Condition (NOC) holds with $\xi_+ = \xi_- = 0$ whenever*

$$F: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad (s, \xi) \mapsto F(s, \xi)$$

is autonomous and one of the following conditions is fulfilled:

(1) *The map F is continuous. There exist real numbers α_+, α_- and sequences $(\tau_{k,-})_k, (\tau_{k,+})_k$ as in (11) such that, for all k ,*

$$\begin{aligned} \exists q_{k,+} \geq \alpha_+, \quad (q_{k,+}, \zeta_k^+) &\in \partial F(\tau_{k,+}, 0), \\ \exists q_{k,-} \leq \alpha_-, \quad (q_{k,-}, \zeta_{k,-}) &\in \partial F(\tau_{k,-}, 0). \end{aligned} \quad (16)$$

(2) *The map $(s, \xi) \mapsto F(s, \xi)$ is convex.*

(3) *There exist continuous $a, b: \mathbb{R} \rightarrow \mathbb{R}$, $a \geq 0$, $L: \mathbb{R}^n \rightarrow \mathbb{R}$ with $\partial L(0) \neq \emptyset$ and*

$$F(s, \xi) = L(a(s)\xi) + b(s) \quad \forall s \in \mathbb{R}, \xi \in \mathbb{R}^n. \quad (17)$$

(4) *$F(s, \xi) = a(s)g(\xi) + b(s)$ for some $a, b: \mathbb{R} \rightarrow \mathbb{R}$ continuous, $a \geq 0$, and $g: \mathbb{R}^n \rightarrow \mathbb{R}$ continuous. Moreover $\partial g(0) \neq \emptyset$ and there are $c > 0, d \geq 0$ and $(\tau_{k,-})_k, (\tau_{k,+})_k$ as in (11) such that, setting $\sigma(s) := a(s)g(0) + b(s)$,*

$$\begin{aligned} \forall k \in \mathbb{N} \quad \sigma(s) &\geq c\sigma(\tau_{k,+}) - d|s|^{p^*} \quad \forall s \geq \tau_{k,+}, \\ \forall k \in \mathbb{N} \quad \sigma(s) &\geq c\sigma(\tau_{k,-}) - d|s|^{p^*} \quad \forall s \leq \tau_{k,-}. \end{aligned}$$

Proof. Notice first that the continuity of F ensures the validity of point (1) of Condition (NOC). It remains to prove that of point (2) of Definition 4.1; we restrict ourselves to the proof of (12), that of (13) can be obtained by following the same path. In what follows we consider $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

(1) Assume (16). For every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ and $k \in \mathbb{N}$,

$$F(s, \xi) \geq F(\tau_{k,+}, 0) + q_{k,+}(s - \tau_{k,+}) + \langle \zeta_{k,+}, \xi \rangle.$$

Since $q_{k,+} \geq \alpha_+$, for $s \geq \max\{\tau_{k,+}, 1\}$ we obtain

$$\begin{aligned} F(s, \xi) &\geq F(\tau_{k,+}, 0) + \alpha_+(s - \tau_{k,+}) + \langle \zeta_{k,+}, \xi \rangle \\ &\geq F(\tau_{k,+}, 0) - |\alpha_+| |s|^{p^*} + \langle \zeta_{k,+}, \xi \rangle, \end{aligned}$$

showing the validity of (14), and thus of point (2) of Condition (NOC).

(2) The validity of (12) was shown in (9). It is nevertheless a consequence of (1).

Indeed let $\tau_{k,+} := k + 1$. Then if $(q_k, \zeta_k) \in \partial F(\tau_k, 0)$, and $(\alpha_+, \zeta_*) \in \partial F(0, 0)$, the monotonicity of the convex subdifferential gives

$$0 \leq (q_k - \alpha_+)(k + 1) + \langle \zeta_k - \zeta_*, 0 \rangle,$$

from which one deduces $q_k \geq \alpha_+$.

(3) Let $\zeta \in \partial L(0)$. Since $L(a(s)\xi) \geq a(s)\langle \zeta, \xi \rangle + L(0)$, then, for any $k \in \mathbb{N}$

$$\begin{aligned} F(s, \xi) &= L(a(s)\xi) + b(s) \geq a(s)\langle \zeta, \xi \rangle + L(0) + b(s) \\ &= F(k, 0) + a(s)\langle \zeta, \xi \rangle, \end{aligned}$$

so that (12) holds true, with $\tau_{k,+} := k$ and $\xi_+ = 0$.

(4) If $\zeta \in \partial g(0)$ then, for all $k \in \mathbb{N}$ and $s \geq \tau_{k,+}$,

$$\begin{aligned} F(s, \xi) &= a(s)g(\xi) + b(s) \geq a(s)(g(0) + \langle \zeta, \xi \rangle) + b(s) \\ &\geq (a(s)g(0) + b(s)) + a(s)\langle \zeta, \xi \rangle \\ &\geq (c(a(\tau_{k,+})g(0) + b(\tau_{k,+})) - d|s|^{p^*}) + a(s)\langle \zeta, \xi \rangle \\ &= cF(\tau_{k,+}, 0) + a(s)\langle \zeta, \xi \rangle - d|s|^{p^*}, \end{aligned}$$

proving the validity of (12). □

Remark 4.4. Point (4) of Proposition 4.3 was expressed in a less general setting in [2, (1.4a)–(1.4b)].

Example 4.5. Let $a: \mathbb{R} \rightarrow \mathbb{R}$ be continuous, $a \geq 0$. Then $F(s, \xi) = a(s)|\xi|^p$ satisfies the conditions expressed in (3) of Proposition 4.3 with $L(z) = |z|^p$, $b(s) = 0$.

In the case where $F(s, \xi) = a(s)g(\xi) + b(s)$, point (4) of Proposition 4.3 is fulfilled under the following circumstances.

Proposition 4.6. Assume that $F(s, \xi) = a(s)g(\xi) + b(s)$ for some $a, b: \mathbb{R} \rightarrow \mathbb{R}$ continuous, $a \geq 0$, $g: \mathbb{R}^n \rightarrow \mathbb{R}$ continuous. Then F fulfils the non-oscillatory Condition (NOC) if at least one of the following conditions hold:

- (1) $F(s, \xi) = a(s)g(\xi)$ for some $a: \mathbb{R} \rightarrow \mathbb{R}$ convex, $a \geq 0$, $g: \mathbb{R}^n \rightarrow \mathbb{R}$ continuous. Moreover $g(0) \geq 0$ and $\partial g(0) \neq \emptyset$.
- (2) The function g has a nonempty convex subdifferential at 0 and

$$\sigma(s) := a(s)g(0) + b(s) = \sigma_1(s) + \sigma_2(s)$$

where, for some $r > 0$ and $D > 0$,

- The map σ_1 is decreasing on $] -\infty, -r]$ and increasing on $[r, +\infty[$;
- the map σ_2 is C^1 and satisfies $|\sigma'_2(s)| \leq D|s|^{p^*-1}$ for every $|s| \geq r$.

Proof. (1) If a is convex, let $p(s) \in \partial a(s)$ for all $s \in \mathbb{R}$ and set $\sigma(s) := a(s)g(0)$. For all $k \in \mathbb{N}$ and $s \geq k$ we have

$$\sigma(s) - \sigma(k) = (a(s) - a(k))g(0) \geq p(k)(s - k)g(0) \geq p(0)(s - k)g(0),$$

so that
$$\sigma(s) \geq \sigma(k) - |p(0)g(0)||s|^{p^*} \quad \forall s \geq \max\{k, 1\}.$$

Analogously one obtains that

$$\sigma(s) \geq \sigma(-k) - |p(0)g(0)||s|^{p^*} \quad \forall s \leq -\max\{k, 1\}.$$

Therefore, the conditions of point (4) are fulfilled, with $\tau_{k,+} := \max\{k, 1\}$, $\tau_{k,-} := -\tau_{k,+}$, $c = 1$, $d = |p(0)g(0)|$.

(2) Let $k \in \mathbb{N}$, $k \geq r$. For all $s \geq \tau_{k,+}$,

$$\begin{aligned} \sigma(s) - \sigma(k) &= (\sigma_1(s) - \sigma_1(k)) + (\sigma_2(s) - \sigma_2(k)) \\ &\geq \sigma_2(s) - \sigma_2(k) \geq -C|s - k|^{p^*-1}(s - k) \geq -C|s|^{p^*}. \end{aligned}$$

Analogously, for $s \leq -k \leq -r$, $\sigma(s) \geq \sigma(-k) - C|s|^{p^*}$. Therefore, σ satisfies the conditions of point (4) of Proposition 4.3, proving the claim. $\square$

Example 4.7. $F(s, \xi) = a(s)g(\xi) + b(s)$ satisfies the (NOC) condition if the functions $a, b: \mathbb{R} \rightarrow \mathbb{R}$ are continuous, $a \geq 0$, $g: \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous, $\partial g(0) \neq \emptyset$ and, moreover, a, b are decreasing on $] -\infty, -r]$, increasing on $[r, +\infty[$ for some $r \geq 0$. Indeed, in this case $\sigma(s) = a(s)g(0) + b(s)$ satisfies the monotonicity condition expressed in point (2) of Proposition 4.6, with $\sigma_1 = \sigma, \sigma_2 = 0$.

5. L^∞ -regularity under the (NOC)

The non-oscillatory condition (NOC) is sufficient for the L^∞ -regularity.

Theorem 5.1. Assume that F satisfies (NOC). Then F is L^∞ -regular.

Proof. Let $u \in W_\phi^{1,p}(\Omega)$ with $F(x, u, \nabla u) \in L^1(\Omega)$. From Theorem 3.5 it is enough to show that (2) holds.

(a) We prove that

$$\limsup_{k \rightarrow +\infty} \int_{\{u > \varphi_{k,+}\}} F(x, \varphi_{k,+}, \nabla \varphi_{k,+}) dx \leq 0. \quad (18)$$

Let us first remark that $\langle \zeta_{k,+}, a(u)\nabla u - a(\varphi_{k,+})\xi_+ \rangle$ is bounded above by a summable function. Indeed from (12) we know that, if $u > \varphi_{k,+}$,

$$\langle \zeta_{k,+}, a(u)\nabla u - a(\varphi_{k,+})\xi_+ \rangle \leq F(x, u, \nabla u) - cF(x, \varphi_{k,+}, \nabla \varphi_{k,+}) + d|u|^{p^*},$$

and point (1) of Definition 4.1 ensures the integrability of $F(x, \varphi_{k,+}, \nabla \varphi_{k,+})$. It follows from (12) that

$$\begin{aligned} \int_{\{u > \varphi_{k,+}\}} F(x, u, \nabla u) dx &\geq \int_{\{u > \varphi_{k,+}\}} \langle \zeta_{k,+}, a(u) \nabla u - a(\varphi_{k,+}) \xi_+ \rangle dx + \\ &\quad + c \int_{\{u > \varphi_{k,+}\}} F(x, \varphi_{k,+}, \nabla \varphi_{k,+}) dx - d \int_{\{u > \varphi_{k,+}\}} |u|^{p^*} dx, \end{aligned}$$

so that

$$\begin{aligned} \int_{\{u > \varphi_{k,+}\}} F(x, \varphi_{k,+}, \nabla \varphi_{k,+}) dx &\leq \frac{1}{c} \int_{\{u > \varphi_{k,+}\}} F(x, u, \nabla u) dx + \\ &\quad + \frac{d}{c} \int_{\{u > \varphi_{k,+}\}} |u|^{p^*} dx - \frac{1}{c} \int_{\{u > \varphi_{k,+}\}} a(u) \langle \zeta_{k,+}, \nabla u - \xi_+ \rangle dx. \end{aligned} \quad (19)$$

Of course

$$\lim_{k \rightarrow +\infty} \int_{\{u > \varphi_{k,+}\}} F(x, u, \nabla u) dx = \lim_{k \rightarrow +\infty} \int_{\{u > \varphi_{k,+}\}} |u|^{p^*} dx = 0. \quad (20)$$

Claim: There is k_ϕ depending only on ϕ such that

$$\int_{\{u > \varphi_{k,+}\}} \langle \zeta_{k,+}, a(u) \nabla u - a(\varphi_{k,+}) \xi_+ \rangle dx = 0 \quad \forall k > k_\phi. \quad (21)$$

Let $v \in W_\phi^{1,p}(\Omega) \cap L^\infty(\Omega)$. For each $s \in \mathbb{R}$ we set $A(s) := \int_0^s a(r) dr$. Then

$$A(v) \in W^{1,p}(\Omega), \quad \nabla A(v) = a(v) \nabla v.$$

From (11) we may choose k_ϕ , depending only on ϕ , to be large enough in such a way that

$$\|\phi\|_\infty < \min\{\varphi_{k,+}(x) : x \in \Omega\} \quad \forall k > k_\phi.$$

In this case we have

$$v \vee \varphi_{k,+} = \varphi_{k,+} \text{ on } \partial\Omega, \quad A(v \vee \varphi_{k,+}) = A(\varphi_{k,+}) \text{ on } \partial\Omega.$$

It follows that $\int_\Omega \langle \nabla(A(v \vee \varphi_{k,+}) - A(\varphi_{k,+})), \zeta_{k,+} \rangle dx = 0$,

or, equivalently,

$$\int_{\{v > \varphi_{k,+}\}} \langle \zeta_{k,+}, a(v) \nabla v - a(\varphi_{k,+}) \xi_+ \rangle dx = 0 \quad \forall v \in W_\phi^{1,p}(\Omega) \cap L^\infty(\Omega), \quad \forall k \geq k_\phi.$$

Fix $i \in \mathbb{N}$. Applying the above equality to $v := u^+ \wedge i \in W_\phi^{1,p}(\Omega) \cap L^\infty(\Omega)$ we get

$$\int_{\{i \geq u > \varphi_{k,+}\}} \langle \zeta_{k,+}, a(v) \nabla v - a(\varphi_{k,+}) \xi_+ \rangle dx = 0, \quad \forall k \geq k_\phi. \quad (22)$$

By Fatou's Lemma, from (22), for $k \geq k_\phi$ we obtain

$$0 = \limsup_{i \rightarrow +\infty} \int_{\{i \geq u > \varphi_{k,+}\}} \langle \zeta_{k,+}, a(u) \nabla u - a(\varphi_{k,+}) \xi_+ \rangle dx \leq \int_{\{u > \varphi_{k,+}\}} \langle \zeta_{k,+}, a(u) \nabla u - a(\varphi_{k,+}) \xi_+ \rangle dx,$$

from which we deduce that

$$\int_{\{u > \varphi_{k,+}\}} \langle \zeta_{k,+}, a(u) \nabla u - a(\varphi_{k,+}) \xi_+ \rangle dx \geq 0 \quad \forall k \geq k_\phi.$$

We deduce (21) by applying the latter inequality both to the functions $a(s)$ and to its opposite $-a(s)$.

The validity of (18) now follows from (19), together with (20) and (21).

(b) Let $k \in \mathbb{N}$. By applying (12), replacing s with $\varphi_{k,+}$, ξ with ξ_+ and k with 0 and taking into account the monotonicity of $(\varphi_{k,+}(x))_k$ we obtain

$$F(x, \varphi_{k,+}, \nabla \varphi_{k,+}) \geq cF(x, \varphi_{0,+}, \nabla \varphi_{0,+}) - d(\varphi_{k,+})^{p^*},$$

thus

$$\int_{\{u > \varphi_{k,+}\}} F(x, \varphi_{k,+}, \nabla \varphi_{k,+}) dx \geq c \int_{\{u > \varphi_{k,+}\}} F(x, \varphi_{0,+}, \nabla \varphi_{0,+}) dx - d \int_{\{u > \varphi_{k,+}\}} |u|^{p^*} dx.$$

The integrability of $F(x, \varphi_{0,+}, \nabla \varphi_{0,+})$ and of $|u|^{p^*}$ together with Lemma 3.6 yield

$$\liminf_{k \rightarrow +\infty} \int_{\{u > \varphi_{k,+}\}} F(x, \varphi_{k,+}, \nabla \varphi_{k,+}) dx \geq 0,$$

which, together with (18), give the first equality in (2); the second one follows similarly. Theorem 3.5 yields the conclusion. $\square$

6. An example

The procedure followed here is based on the truncation method illustrated in Theorem 3.5. It may happen that the latter does not apply, and nevertheless that the Lagrangian is L^∞ -regular.

Example 6.1. Consider the functional

$$I(v) = \int_{B_1} \left| |\nabla v| - \frac{1}{2} |v|^3 \right|^p dx \quad v \in 1 + W_0^{1,1}(B_1), \quad (23)$$

where B_1 is the unit disk in $\mathbb{R}^2$. The function $u(x) = |x|^{-1/2} \in 1 + W_0^{1,1}(B_1)$ is a minimizer of I . Indeed $I(v) \geq 0$, for every $v \in 1 + W_0^{1,1}(B_1)$ and $I(u) = 0$.

We consider the truncation of u with the sequence of functions

$$\begin{aligned} \varphi_{k,-} &\equiv -k, \quad \varphi_{k,+} \equiv k, \quad k \in \mathbb{N}, \\ \text{i.e.,} \quad u_k(x) &= \begin{cases} k & \text{if } u(x) > k, \text{ i.e., } |x| < \frac{1}{k^2}, \\ |x|^{-1/2} & \text{if } u(x) < k \text{ i.e., } \frac{1}{k^2} \leq |x| < 1. \end{cases} \end{aligned} \quad (24)$$

We remark that u_k converges to u in $W^{1,1}(B_1)$ and

$$I(u_k) = \frac{1}{2^p} k^{3p} \lambda \left(B_{\frac{1}{k^2}} \right) = \frac{\pi}{2^p} k^{3p-4}. \quad (25)$$

However, for $p \geq 3/4$, the truncated sequence $(u_k)_k$ is not a minimizing one.

Nevertheless it is quite easy to check that F is $W^{1,\infty}$ -regular (and therefore L^∞ -regular) at u , so that the Lavrentiev phenomenon does not occur. Let us consider the sequence $(v_k)_k$ of functions in $1 + W_0^{1,\infty}(B_1)$ defined by

$$v_k(x) = \begin{cases} \left(|x| + \frac{1}{k}\right)^{-1/2} & \text{if } |x| < 1 - \frac{1}{k}, \\ 1 & \text{if } 1 - \frac{1}{k} \leq |x| < 1. \end{cases} \quad (26)$$

Then $(v_k)_k$ converges to u in $W^{1,p}(\Omega)$ and $I(v_k) = \frac{\pi}{2^p} \left(1 - \left(1 - \frac{1}{k}\right)^2\right) \rightarrow 0$ as $k \rightarrow +\infty$, proving the claim. This example suggests that, in dealing with Lagrangians that are not convex in both the variables u and ∇u , there may be approximating sequences, different from those used in the proof of Theorem 3.5, that may in any case lead to the L^∞ -regularity, or even to the non occurrence of the Lavrentiev gap.

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