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Hausdorff Stability and Error Estimates for Compensated Convexity Based Methods for Approximation and Interpolation for Functions in $\mathbb{R}^n$

Maryam Alatawi
Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia
msoalatawi@ut.edu.sa

Kewei Zhang
School of Math. Sciences, University of Nottingham, Nottingham NG7 2RD, UK
kewei.zhang@nottingham.ac.uk

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We establish error estimates and Hausdorff stability for approximations and interpolations for sampled functions in $\mathbb{R}^n$ by using compensated convex transforms introduced previously by K. Zhang [*Compensated convexity and its applications*, Ann. l'Institut H. Poincaré (C), Non Linear Analysis 25(4) (2008) 743–771]. We generalize the sharp error estimates obtained recently by K. Zhang, E. Crooks, and A. Orlando [*Compensated convexity methods for approximations and interpolations of sampled functions in euclidean spaces: Theoretical foundations*, SIAM J. Math. Analysis 48(6) (2016) 4126–4154] to cases when the underlying functions are in the class of C^α or $C^{1,\beta}$. We also establish Hausdorff stability when the functions involved are just assumed to be bounded. The stability of our approximations in this paper is with respect to the Hausdorff distance between graphs of the sampled functions.

Keywords: Compensated convex transforms, compact samples, complement of bounded open sets interpolation, approximation, inpainting, bounded functions, C^α-functions, $C^{1,\beta}$-functions, error estimates, Hausdorff stability, Hausdorff distance, convex density radius.

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1. Introduction and main results

In this paper we establish error estimates for average approximations defined by compensated convex transforms [23] for bounded functions $f\colon \mathbb{R}^n \to \mathbb{R}$ when either $K = \mathbb{R}^n \backslash \Omega$ where $\Omega \subset \mathbb{R}^n$ is a bounded open set or $K \subset \mathbb{R}^n$ is a compact set. We extend the approximation theorems obtained in [23] for uniformly continuous, Lipschitz, and $C^{1,1}$ functions to C^α, $C^{1,\beta}$-functions $f\colon \mathbb{R}^n \to \mathbb{R}$ as the underlying functions to be approximated. We present asymptotically sharp error estimates for approximations for such functions. Error estimates are important issues for approximation/interpolation and image inpainting (see e.g., [4]). Furthermore, we present a stability result for the average approximation of sampled functions under the only assumption that the functions are bounded by using

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the Hausdorff distance between graphs of sampled functions. This result generalizes the Hausdorff-Lipschitz continuity property for the average approximation for bounded uniformly continuous functions established in [23] using the Hausdorff distance between the sampled sets K.

Throughout the paper, we denote by $\mathbb{R}^n$ the standard n-dimensional Euclidean space with the standard inner product $x \cdot y$ and norm $|x|$ for $x,\ y \in \mathbb{R}^n$. We denote by $B(x_0; r)$ and $\overline{B}(x_0; r)$ the open and closed balls centered at x_0 with radius $r > 0$ respectively. Also $\mathbf{co}[K]$ denotes the convex hull of K [11, 13] which is the smallest convex set containing K. For a given function $f\colon \mathbb{R}^n \to \mathbb{R}$ bounded below, the convex envelope $\mathbf{Co}[f]$ of f is the largest convex function whose graph lies below f [11, 13] .

Let us first recall the notions of quadratic compensated convex transforms for bounded functions in $\mathbb{R}^n$ defined in [20, 21, 22], followed by definitions of upper, lower, and the average compensated convex approximation [23]. For the definition of compensated convex transforms for functions under more general growth conditions, see [20, Definition 1.1].

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ satisfies the growth condition $|f(x)| \le C_f|x|^2 + C_1$ for some constants $C_f > 0,\ C_1 > 0$ and for all $x \in \mathbb{R}^n$.

The *quadratic lower compensated convex transform* for f (lower transform for short) is defined by

$$\mathcal{C}^l_\lambda(f)(x) = \mathbf{Co}[\lambda|\cdot|^2 + f(\cdot)](x) - \lambda|x|^2, \quad x \in \mathbb{R}^n,\ \text{for } \lambda > C_f, \tag{1}$$

where $\mathbf{Co}[\lambda|\cdot|^2 + f(\cdot)](x)$ is the value of the convex envelope of the function $y \mapsto f(y) + \lambda|y|^2$ at $x \in \mathbb{R}^n$ [11, 13].

The *quadratic upper compensated convex transform* for f (upper transform for short) is defined by

$$\mathcal{C}^u_\lambda(f)(x) := -\mathbf{Co}[\lambda|\cdot|^2 - f(\cdot)](x) + \lambda|x|^2, \quad x \in \mathbb{R}^n,\ \text{for } \lambda > C_f. \tag{2}$$

The two *quadratic mixed compensated convex transforms* for f (mixed transforms for short) are defined, respectively, by $\mathcal{C}^u_\lambda(\mathcal{C}^l_\tau(f))$, and $\mathcal{C}^l_\lambda(\mathcal{C}^u_\tau(f))$, $x \in \mathbb{R}^n$, when $\lambda,\ \tau > C_f$. It is easy to see under our growth condition for f that we have $\mathcal{C}^u_\lambda(f)(x) = -\mathcal{C}^l_\lambda(-f)(x)$.

We only assume in this paper, unless otherwise specified that , $K \subset \mathbb{R}^n$ is either a compact set or the complement of a bounded open set $\Omega \subset \mathbb{R}^n$. We call $f\colon \mathbb{R}^n \to \mathbb{R}$ the underlying function to be approximated. We define $f_K\colon K \subset \mathbb{R}^n \to \mathbb{R}$, our sampled function as $f_K(x) = f(x)$ for all $x \in K$, and $\mathit{\Gamma}_{f_K} := \{(x, f(x)) : x \in K)\}$ its graph. The aim for general interpolation / approximation for f_K is to define another function in $\mathbf{co}[K]$, which is equal to f on K or at least approximates f on K.

Let $K \subset \mathbb{R}^n$ be a nonempty closed set and suppose that for some constant $A_0 > 0$, $|f_K(x)| \le A_0$ for all $x \in K$. Given a positive number $M > 0$, we define two functions extending f_K to $\mathbb{R}^n \backslash K$, as follows [23]:

$$f_K^M(x) = \begin{cases} f_K(x), & x \in K, \\ M, & x \in \mathbb{R}^n \setminus K; \end{cases} \quad \text{and} \quad f_K^{-M}(x) = \begin{cases} f_K(x), & x \in K, \\ -M, & x \in \mathbb{R}^n \setminus K. \end{cases} \tag{3}$$

Now we present the definitions of the lower, the upper, and the average compensated convex approximations for a fixed $M > 0$ [23].

Definition 1.1. For $M > 0$, the *lower compensated convex approximation* (the lower approximation for short) with scale $\lambda > 0$ for the sampled function $f_K : K \to \mathbb{R}$ is defined by

$$L_\lambda^M(f_K)(x) = \mathcal{C}_\lambda^l(f_K^M)(x), \ x \in \mathbb{R}^n. \tag{4}$$

The *upper compensated convex approximation* (the upper approximation for short) with scale $\lambda > 0$ for the sampled function $f_K \colon K \to \mathbb{R}$ is defined by

$$U_\lambda^M(f_K)(x) = \mathcal{C}_\lambda^u(f_K^{-M})(x), \ x \in \mathbb{R}^n. \tag{5}$$

The *average compensated convex approximation* (the average approximation for short) with scale $\lambda > 0$ for the sampled function $f_K \colon K \to \mathbb{R}$ is defined by

$$A_\lambda^M(f_K)(x) = \frac{1}{2}\left(\mathcal{C}_\lambda^l(f_K^M)(x) + \mathcal{C}_\lambda^u(f_K^{-M})(x)\right), \ x \in \mathbb{R}^n. \tag{6}$$

In addition, we can set $M = +\infty$, then we can consider the following functions.

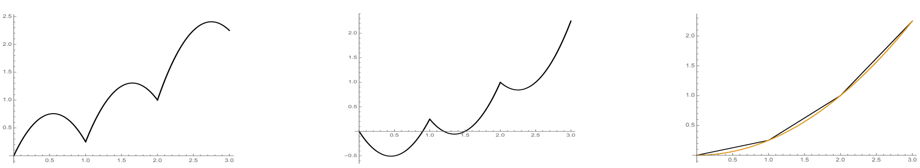
$$f_K^{+\infty}(x) = \begin{cases} f_K(x), & x \in K, \\ +\infty, & x \in \mathbb{R}^n \setminus K; \end{cases} \quad \text{and} \quad f_K^{-\infty}(x) = \begin{cases} f_K(x), & x \in K, \\ -\infty, & x \in \mathbb{R}^n \setminus K. \end{cases}$$

and define the corresponding average approximation of $A_\lambda^\infty(f_K)(x)$ as follows:

$$A_\lambda^\infty(f_K)(x) = \frac{1}{2}\left(\mathcal{C}_\lambda^l(f_K^{+\infty})(x) + \mathcal{C}_\lambda^u(f_K^{-\infty})(x)\right).$$

The following is a simple one-dimensional prototype example illustrating the geometric behaviour of the average approximation.

Example 1.2. Let $f(x) = x^2/4$ for $x \in \mathbb{R}$ with $K = \{0, 1, 2, 3\}$ and $\lambda = 10$.



(a) Lower approx. (b) Upper approx. (c) Average approx.

Figure 1.1: In 1d the average approximation for a finite set is just the affine interpolation of a given function when $M > 0$ is sufficiently large.

The present work is motivated by [23]. Our main results generalise those obtained in [23]. We focus on two important problems, both are related to the robustness of our method, namely Error Estimates and Stability. We first establish a general approximation theorem when either $K = \mathbb{R}^n \backslash \Omega$ where $\Omega \subset \mathbb{R}^n$ is a bounded open set (Theorem 1.5) or $K \subset \mathbb{R}^n$ is a compact set (Theorem 1.6). We generalize the approximation theorems obtained in [23] to C^α, $C^{1,\beta}$-functions $f: \mathbb{R}^n \to \mathbb{R}$ as the underlying functions to be approximated. We show that when $M > 0$ is sufficiently large, $A_\lambda^M(f_K)(x)$ approaches f_K in K as $\lambda \to \infty$. For a point x in $\mathbf{co}[K]$, we use the notion of convex density radius $r_c(x)$ [23] which is the smallest radius $r \geq 0$ of closed ball $\overline{B}(x;r)$ such that x is in the convex hull of $K \cap \overline{B}(x;r)$. We use $r_c(x)$ to bound the errors of our approximation $\left|A_\lambda^M(f_K)(x) - f(x)\right|$. We prove also a similar error estimate for $|A_\lambda^\infty(f_K)(x) - f(x)|$. We then prove a general Hausdorff stability result for $\Gamma_{f_K} \to A_\lambda^M(f_K)(x)$ by using the Hausdorff distance between graphs of sampled functions, showing that as long as the graph of two sampled functions are close to each other, then the output of approximation will be close to each other. Our approach is rather indirect by using a distance-like function introduced in [23]. The main result for this part is Theorem 1.8 which gives a very general Hausdorff stability result of the upper transform result for bounded functions. As a direct consequence, the Hausdorff stability property of the average approximation can be established.

Now we briefly discuss the backgrounds on approximation and interpolation methods for sampled functions. There are many methods of approximations and interpolations for functions by polynomials and other smooth functions in one-dimension [6, 16]. Reviews on classical interpolation methods for functions such as Lagrange, Newton and other polynomial based interpolation can be found in, for example [6, 12] and the basic idea of most of them is to find a polynomial passing through the given sample points. Many one-dimensional interpolation methods cannot be extended directly to higher dimensional cases such as scattered data. For higher dimensional problems, radial basis functions [2, 8, 18], thin spline plate methods [9], kernel and convolution methods [19] are widely used .

Based on geometric partial differential equations (PDEs) [17, 15, 3] and variational methods using total variation (TV) models [5, 14], many techniques have been introduced to interpolate and approximate functions in image processing [5] for different purposes such as image segmentation, denoising, modeling or representation, restoration, salt & pepper noise removal, and image inpainting [5]. We mainly focus on inpainting which is a problem in image processing. In general, let Ω be the image domain, and $f: \Omega \to \mathbb{R}$ be the original image. We can consider $D \subset \Omega$ as the area to be inpainted from $\Omega \setminus D$ to Ω. This means that D is the region with missing information. The main aim is to reconstruct or repair the damaged image by using the existing information of other parts of the image $f: \Omega \setminus D \to \mathbb{R}$ [23]. The most well-known PDE based inpainting method is the total variation regularization based model introduced by Rudian, Osher and Fatemi (ROF model for short) [14]. Many of the PDE based methods give impressive experimental results.

However, for functions to be interpolated or approximated, many methods are not flexible with respect to types of sample sets such as scattered data, that is, the set is finite and the points are isolated [23] or level set interpolation by using the infinity Laplacian (or AMLE) requires the sampled set to be unions of finitely many sets of isolated point and Jordan curves [3] with output $C^{1,\alpha}$- functions in 2d [7]. Also, few known inpainting methods have been proved to be Hausdorff stable. On the other hand, our compensated convexity based approximation can be easily applied to data sets of more general types [23] such as level sets, edges, finite sets, compact sets and the complement of bounded open sets. Moreover, the basic average approximation $A_\lambda^M(f_K)(x)$ has the Hausdorff stability property, that means, if f is continuous and there is another sample sets $E \subset \mathbb{R}^n$ close to K , then $A_\lambda^M(f_E)(x)$ is close to $A_\lambda^M(f_K)(x)$ [23].

As far as we know, many existing methods for inpainting, error estimates between the original function and inpainted area are very difficult to obtain. The TV-based inpainting methods [5] is one of such examples for which extra geometric assumptions on solutions are required. In [4] an error estimate for the harmonic inpainting was obtained . For this model, Chan and Kang solved the Dirichlet problem for the Laplacian equation $\Delta u = 0$ in $D \subset \Omega$ satisfying $u = f$ on ∂D was solved by assuming that $f \in C^2(\overline{\Omega})$ and ∂D is smooth. They established the following error estimate

$$|u(x) - f(x)| \leq \frac{N}{8}\beta^2, \ x \in \Omega,$$

where $|\Delta f| \leq N$ in $\overline{\Omega}$ and β is the minor diameter of an ellipse covering the domain D to be inpainted. They also presented [4] numerical experiments by harmonic inpainting. They commented that the error estimate should depend on the 'width' of the inpainting domain rather than β.

On the other hand our approach does not require any smoothness assumptions of the boundary. Under suitable assumptions on the continuity of the underling function f, quantitative error estimates depending on the width of the domain to be inpainted can be established [23]. Our method measures the difference between the approximation of the sampled function and the original function in the missing area. Applications of this method to scattered data interpolation and inpainting are presented in [24].

Before we state our main results, we first introduce the notion of convex density radius $r_c(x)$ [23, Definition 3.5] which is needed in Theorem 1.5, and Theorem 1.6.

Definition 1.3. Suppose $K \subset \mathbb{R}^n$ is a non-empty closed set and for $x \in \mathbf{co}[K]$ consider the balls $B(x;r)$ such that $x \in \mathbf{co}[\overline{B}(x;r) \cap K]$. The *convex density radius of x with respect to K* is defined by

$$r_c(x) = \inf\{r \geq 0, x \in \mathbf{co}[\overline{B}(x;r) \cap K]\},$$

whereas the *convex density radius of K in $\mathbf{co}[K]$* is defined as

$$r_c(K) = \sup\{r_c(x), x \in \mathbf{co}[K]\}.$$

Next we introduce the definition of Hausdorff continuity with respect to closed sampled sets for transforms of continuous bounded real functions from $\mathbb{R}^n$ to $\mathbb{R}$

as follows [23, Definition 4.1]. For the definition of Hausdorff distance, we refer to Definition 2.4 in Section 2.

Definition 1.4. A transform $\mathrm{T}\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *Hausdorff continuous* with respect to closed sample sets at a continuous function f if the mapping $K \to \mathrm{T}(f\chi_K)$ is Hausdorff continuous at each non empty closed set $K_0 \subset \mathbb{R}^n$, in the sense that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\mathrm{T}(f\chi_K)(x) - \mathrm{T}(f\chi_{K_0})(x)| < \epsilon,$$

for all $x \in \mathbb{R}^n$ whenever K is a non-empty closed set with $\mathrm{dist}_{\mathcal{H}}(K, K_0) < \delta$.

We generalize the approximation theorems obtained in [23] for a compact set $K \in \mathbb{R}^n$ [23, Theorem 3.6] and for $K = \mathbb{R}^n \backslash \Omega$ with Ω a bounded open set [23, Thereom 3.7] for uniformly continuous, Lipschitz, and $C^{1,1}$ functions to C^α and $C^{1,\beta}$ functions. Now we state our main results of the error estimates for the average compensated convex approximation for C^α and $C^{1,\beta}$ functions, respectively. First we consider the case when $K = \Omega^c$ is the complement of a bounded open set $\Omega \subset \mathbb{R}^n$ and M is finite. The estimates are expressed in terms of the convex density radius.

Theorem 1.5. *Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is a bounded continuous function, satisfying $|f(x)| \leq A_0$ for some constant $A_0 > 0$ and for all $x \in \mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $K = \Omega^c$. Denote by* $\mathrm{diam}(\Omega)$ *the diameter of* Ω *(for the definition of the diameter see Section* 2*).*

(i) *If f is a C^α-function (Hölder continous) in $\mathbb{R}^n$ for some $0 < \alpha < 1$ with constant $L > 0$ satisfying $|f(x) - f(y)| \leq L|x-y|^\alpha$ for all $x,\ y \in \mathbb{R}^n$, then for $\lambda > 0$, $M > A_0 + \mathrm{diam}(\Omega)^2$ and $x \in \mathbb{R}^n$, we have, for $x \in \mathbb{R}^n$ that*

$$\left|A_\lambda^M(f_K)(x) - f(x)\right| \leq \frac{L}{2}(r_c(x))^\alpha + \frac{L}{2}\left(\frac{2}{\lambda}\left(\lambda r_c^2(x) + L(r_c(x))^\alpha + C(\alpha,\lambda)\right)\right)^{\frac{\alpha}{2}},$$

where $r_c(x) \geq 0$ and $C(\alpha,\lambda) = \left(\frac{2-\alpha}{2\alpha}\right)\lambda\left(\frac{\alpha L}{\lambda}\right)^{\frac{2}{2-\alpha}}$.

(ii) *If f is a $C^{1,\beta}$-function in $\mathbb{R}^n$ for some $0 < \beta < 1$, with constant $L > 0$, satisfying $|Df(x) - Df(y)| \leq L|x-y|^\beta$ for all $x, y \in \mathbb{R}^n$, then for $\lambda > 0$, $M > A_0 + \lambda\ \mathrm{diam}(\Omega)^2$ and $x \in \mathbb{R}^n$, we have, for $x \in \mathbb{R}^n$ that*

$$\begin{aligned}\left|A_\lambda^M(f_K)(x) - f(x)\right| &\leq \frac{L}{2(\beta+1)}(r_c(x))^{\beta+1} \\ &\quad + \frac{L}{2(\beta+1)}\left(\frac{2}{\lambda}\left(\lambda r_c^2(x) + \frac{L}{\beta+1}(r_c(x))^{\beta+1} + C(\beta,\lambda)\right)\right)^{\frac{\beta+1}{2}},\end{aligned}$$

where $C(\beta,\lambda) = \left(\frac{1-\beta}{2(1+\beta)}\right)\lambda\left(\frac{L}{\lambda}\right)^{\frac{2}{1-\beta}}$.

Then we consider the case when K is compact and $M = +\infty$. We have the following error estimate for our average approximation.

Theorem 1.6. *Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is a bounded continuous function satisfying $|f(x)| \le A_0$ for some constant $A_0 > 0$ and all $x \in \mathbb{R}^n$, and let $K \subset \mathbb{R}^n$ be a nonempty compact set.*

(i) *If f is a C^α-function (Hölder continous) in $\mathbb{R}^n$ for some $0 < \alpha < 1$ with constant $L > 0$ satisfying $|f(x) - f(y)| \le L|x-y|^\alpha$ for all $x,\ y \in \mathbb{R}^n$, then for all $\lambda > 0$, and $x \in \boldsymbol{co}[K]$, we have an asymptotic estimate depending only on $(r_c(x))^\alpha$, as below*

$$\limsup_{\lambda\to\infty} |A_\lambda^\infty(f_K)(x) - f(x)| \le L(r_c(x))^\alpha.$$

(ii) *If f is a $C^{1,\beta}$-function in $\mathbb{R}^n$ for some $0 < \beta < 1$ with constant $L > 0$ satisfying $|Df(x) - Df(y)| \le L|x-y|^\beta$ for all $x,\ y \in \mathbb{R}^n$, then for all $\lambda > 0$, and $x \in \boldsymbol{co}[K]$, we have an asymptotic estimate depending only on $(r_c(x))^{\beta+1}$, as below*

$$\limsup_{\lambda\to\infty} |A_\lambda^\infty(f_K)(x) - f(x)| \le \frac{L}{(\beta+1)}(r_c(x))^{\beta+1}.$$

Next we extend the Hausdorff -Lipschitz continuity result for the upper transform of bounded uniformly continuous functions [23, Theorem 4.10] to the case of general bounded functions. The following is our main result on Hausdorff stability of the upper compensated convex transform at a bounded upper semicontinous function.

Theorem 1.7. *Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is a bounded upper semicontinuous function such that for some constants $m, M > 0$, $0 < m \le f \le M$ for all $x \in \mathbb{R}^n$. Let $K, E \subset \mathbb{R}^n$ be compact sets with* $\mathrm{dist}_{\mathcal{H}}(\mathit{\Gamma}_{f_K}, \mathit{\Gamma}_{f_E}) < +\infty$. *Then for all $x \in \mathbb{R}^n$,*

$$|\mathcal{C}_\lambda^u(f\chi_K)(x) - \mathcal{C}_\lambda^u(f\chi_E)(x)| \le \left(2\sqrt{M\lambda + \frac{M}{4m}}\right) \mathrm{dist}_{\mathcal{H}}(\mathit{\Gamma}_{f_K}, \mathit{\Gamma}_{f_E}). \tag{7}$$

Thus we can easily state Hausdorff stability for general bounded function.

Theorem 1.8. *Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is a bounded function such that for some constants $m, M > 0$, $0 < m \le f \le M$ for all $x \in \mathbb{R}^n$. Let $K, E \subset \mathbb{R}^n$ be compacts sets with* $\mathrm{dist}_{\mathcal{H}}(\mathit{\Gamma}_{f_K}, \mathit{\Gamma}_{f_E}) < +\infty$. *Then for all $x \in \mathbb{R}^n$,*

$$|\mathcal{C}_\lambda^u(f\chi_K)(x) - \mathcal{C}_\lambda^u(f\chi_E)(x)| \le \left(2\sqrt{M\lambda + \frac{M}{4m}}\right) \mathrm{dist}_{\mathcal{H}}(\mathit{\Gamma}_{f_K}, \mathit{\Gamma}_{f_E}). \tag{8}$$

The stable ridge transform was introduced in [22] where the Hausdorff stability property is proved for characteristic function.

Definition 1.9. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is a bounded function such that for some constants $m, M>0$, $0<m\le f \le M$ for all $x\in\mathbb{R}^n$. Let K be a non empty compact subset of $\mathbb{R}^n$. We define the *stable ridge transform* for $f\chi_K$, $x\in\mathbb{R}^n$, $\lambda, \tau > 0$ by

$$SR_{\lambda,\tau}(f\chi_K)(x) = R_\tau(\mathcal{C}_\lambda^u(f\chi_K))(x) = \mathcal{C}_\lambda^u(f\chi_K)(x) - \mathcal{C}_\tau^l(\mathcal{C}_\lambda^u(f\chi_K)(x)). \qquad \square$$

As an immediate consequence of the Theorem 1.8, we obtain the following

Corollary 1.10. *Assume $\lambda > 0, \tau > 0$ and under assumptions of Lemma* 3.2, *we have, for all $x \in \mathbb{R}^n$ that*

$$|SR_{\lambda,\tau}(f\chi_K)(x) - SR_{\lambda,\tau}(f\chi_E)(x)| \leq \left(4\sqrt{M\lambda + \frac{M}{4m}}\right) \mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K}, \Gamma_{f_E}). \qquad (9)$$

The plan of the rest of the paper is as follows. Section 2 will present some preliminary results which are needed for the proofs. We present the proofs of our main results in Section 3.

2. Preliminary results

We first collect some basic results from convex analysis which will be used in the following. The reader is referred to [11, 13] for further details and proofs. We often use the following characterization of convex envelope in the proofs.

Proposition 2.1. [22, Propostion 2.1] *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be bounded below and $x_0 \in \mathbb{R}^n$. Then*

(i) *The convex envelope $\boldsymbol{Co}[f](\cdot)$ of f at $x_0 \in \mathbb{R}^n$ is given by*

$$\boldsymbol{Co}[f](x_0) = \inf_{i=1,\dots,n+1} \left\{ \sum_{i=1}^{n+1} \lambda_i f(x_i) : \sum_{i=1}^{n+1} \lambda_i = 1, \sum_{i=1}^{n+1} \lambda_i x_i = x_0, \lambda_i \geq 0, x_i \in \mathbb{R}^n \right\}. \qquad (10)$$

Further, if f is lower semicontinous and coercive in the sense that $f(x)/|x| \to \infty$ as $|x| \to \infty$, then the infimum can be attained by some (λ_i^*, x_i^*) for $0 < \lambda_i^* \leq 1$, $x_i^* \in \mathbb{R}^n$, for $i = 1, \dots, k$ with $1 \leq i \leq k \leq n+1$, satisfy $\sum_{i=1}^k \lambda_i^* = 1$, and $\sum_{i=1}^k \lambda_i^* x_i^* = x_0$, such that

$$\boldsymbol{Co}[f](x_0) = \sum_{i=1}^k \lambda_i^* f(x_i^*).$$

(ii) *The value $\boldsymbol{Co}[f](\cdot)$ for f taking only finite values can also be obtained by an affine functions as follows* [20, 22]:

$$\boldsymbol{Co}[f](x_0) = \sup \left\{ \ell(x_0) : \ \ell \text{ affine and } \ell(y) \leq f(y) \text{ for all } y \in \mathbb{R}^n \right\}.$$

with the sup *attained by an affine function $\ell^* \in \mathbb{R}^n$.*

Next we list some basic properties of compensated convex transforms [20, 21, 22] which will be used in the proofs. The lower and upper transforms are related to each other when $\lambda > 0$ by the following relation

$$\mathcal{C}_\lambda^l(f)(x) = -\mathcal{C}_\lambda^u(-f)(x),$$

when both can be defined. Furthermore, the ordering property of compensated convex transform holds for $x \in \mathbb{R}^n$, $0 < \lambda < \tau$,

$$\mathcal{C}_\lambda^l(f)(x) \leq \mathcal{C}_\tau^l(f)(x) \leq f(x) \leq \mathcal{C}_\tau^u(f)(x) \leq \mathcal{C}_\lambda^u(f)(x), \qquad (11)$$

and for $f \le g$ in $\mathbb{R}^n$, we have

$$\mathcal{C}^l_\lambda(f)(x) \le \mathcal{C}^l_\lambda(g)(x), \text{and} \quad \mathcal{C}^u_\lambda(f)(x) \le \mathcal{C}^u_\lambda(g)(x). \tag{12}$$

In addition, the compensated convex transforms enjoy the property [22] that

$$\mathcal{C}^l_\lambda(f+\ell)(x) = \mathcal{C}^l_\lambda(f)(x) + \ell, \quad \mathcal{C}^l_\lambda(f+\ell)(x) = \mathcal{C}^u_\lambda(f)(x) + \ell, \tag{13}$$

where ℓ is any affine function and we also have [22, Theorem 2.1(iii)]

$$\mathcal{C}^u_\tau(\mathcal{C}^u_\lambda(f)) = \begin{cases} \mathcal{C}^u_\lambda(f), & \text{if } \tau \ge \lambda, \\ \mathcal{C}^u_\tau(f), & \text{if } \tau \le \lambda; \end{cases} \quad \text{and} \quad \mathcal{C}^l_\tau(\mathcal{C}^l_\lambda(f)) = \begin{cases} \mathcal{C}^l_\lambda(f), & \text{if } \tau \ge \lambda, \\ \mathcal{C}^l_\tau(f), & \text{if } \tau \le \lambda. \end{cases} \tag{14}$$

We will use the following translation -invariance property in our proofs [22, Propsition 2.10] in order to allow us to consider the point $x_0 = 0$ without loss of generality.

Proposition 2.2. (Translation-invariance property) *For any $f\colon \mathbb{R}^n \to \mathbb{R}$ bounded below and for affine function $\ell\colon \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{Co}[f+\ell] = \boldsymbol{Co}[f] + \ell$. Consequently, both $\mathcal{C}^l_\lambda(f)(x)$ and $\mathcal{C}^u_\lambda(f)(x)$ are translation invariant against the weight function, that is:*

$$\mathcal{C}^l_\lambda(f)(x) = \boldsymbol{Co}[\lambda|(\cdot) - x_0|^2 + f(\cdot)](x) - \lambda|x - x_0|^2,$$

$$\mathcal{C}^u_\lambda(f)(x) = -\boldsymbol{Co}[\lambda|(\cdot) - x_0|^2 - f(\cdot)](x) + \lambda|x - x_0|^2.$$

for all $x \in \mathbb{R}^n$ and for every fixed $x_0 \in \mathbb{R}^n$. In particular, at x_0

$$\mathcal{C}^l_\lambda(f)(x_0) = \boldsymbol{Co}[\lambda|(\cdot) - x_0|^2 + f(\cdot)](x_0), \ \mathcal{C}^u_\lambda(f)(x_0) = -\boldsymbol{Co}[\lambda|(\cdot) - x_0|^2 - f(\cdot)](x_0).$$

As usual we denote the class of globally *Hölder* continuous functions by C^α, and the class of functions whose gradient is *Hölder continous* by $C^{1,\beta}$ [10] and the upper/lower semicontinous envelope of f by $\overline{f}(x), \underline{f}(x)$ see [11, 13].

The following properties of compensated convex transforms in [22, Proposition 3.1] will also be used:

$$\mathcal{C}^u_\lambda(\overline{f})(x) = \mathcal{C}^u_\lambda(f)(x), \quad \mathcal{C}^u_\lambda(\underline{f})(x) = \mathcal{C}^l_\lambda(f)(x). \tag{15}$$

Next we state the definition of Hausdorff distance between two non empty sets [1, 22]. We first need the notation of δ-neighborhood of a set [22], and also define the diameter of a set.

Definition 2.3. Given a non-empty subset $E \subset \mathbb{R}^n$ and $\delta > 0$, we define the *δ-neighborhood* E^δ of E by

$$E^\delta = \{x \in \mathbb{R}^n : \operatorname{dist}(x; E) < \delta\},$$

where $\operatorname{dist}(x; E) = \inf\{|y - x| : y \in E\}$ and the *diameter* of E by

$$\operatorname{diam}(E) := \sup\{|x - y|, x, y \in E\}. \qquad \square$$

Definition 2.4. Let E, F be non-empty bounded closed subsets of $\mathbb{R}^n$. The *Hausdorff distance* between E and F is defined by

$$\mathrm{dist}_{\mathcal{H}}(E,F) = \inf\{\delta > 0 : F \subset E^\delta \text{ and } E \subset F^\delta\}. \tag{16}$$

This definition is equivalent to saying that

$$\mathrm{dist}_{\mathcal{H}}(E,F) = \max\Big\{\sup_{x\in E}\mathrm{dist}(x;F),\ \sup_{x\in F}\mathrm{dist}(x;E)\Big\}. \qquad \square$$

Motivated by Definition 4.4 in [23], we introduce a notion of distance-like function.

Definition 2.5. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ with $0 < f \leq M$. We define the following distance-like functions for a closed set $K \subset \mathbb{R}^n$:

$$d_{\lambda,f_K}(x,\mathit{\Gamma}_{f_K}) = \inf_{y\in K}\left\{|y-x| - \sqrt{\frac{f_K(y)}{\lambda}}\right\}, \quad x \in \mathbb{R}^n, \tag{17}$$

$$D_{\lambda,f_K}(x,\mathit{\Gamma}_{f_K}) = -\sqrt{\lambda}\inf\{0, d_{\lambda,f_K}(x,\mathit{\Gamma}_{f_K})\}. \tag{18}$$

The last function can be reduced to $D_\lambda(x,K) = \max\{0, 1-\sqrt{\lambda}\ \mathrm{dist(x;K)}\}$ if $f \equiv 1$ (see [22, Definition 5.1]).

We conclude this section by providing a few preliminary results which are needed to prove the Hausdorff stability results.

Remark 2.6. Let $f\colon K \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded below, lower semicontinous and coercive function in the sense that if $|x_n| \to +\infty$, then $f(x_n) \to +\infty$. Then $\inf f(x)$ can be reached. $\square$

The following result should be obvious. We state it here for convenience.

Corollary 2.7. *Suppose that $f(x) = \sqrt{x}$ and $g(x)$ is a non-negative upper semicontinuous, then $f \circ g$ is upper semicontinuous function.*

The following is a simple property for $d_{\lambda,f_K}(x,\mathit{\Gamma}_{f_K})$.

Corollary 2.8. *Let f be a non-negative upper semicontinuous function on $\mathbb{R}^n$ and K be closed, then the mapping $x \to d_{\lambda,f_K}(x,\mathit{\Gamma}_{f_K})$ is lower semicontinuous function in $\mathbb{R}^n$.*

3. Proofs of the main results

We first prove Theorem 1.5 and Theorem 1.6.

Proof of Theorem 1.5. Part (i): By the translation invariance property for the lower and upper compensated convex transforms, assume without loss of generality that $x = 0$. Since both $y \to \lambda|y|^2 + f_K^M(y)$ and $y \to \lambda|y|^2 - f_K^{-M}(y)$ are coercive and lower semicontinuous, by Proposition 2.1 in [22], we have the following representations by convex combinations:

$$\mathcal{C}^l_\lambda(f^M_K)(0)=\mathbf{Co}[f^M_K+\lambda|\cdot|^2](0)=\sum_{j=1}^{k_l}\lambda^l_j(\lambda|x^l_j|^2+f^M_K(x^l_j)), \tag{19}$$

$$-\mathcal{C}^u_\lambda(f^{-M}_K)(0)=\mathbf{Co}[\lambda|\cdot|^2-f^{-M}_K](0)=\sum_{j=1}^{k_u}\lambda^u_j(\lambda|x^u_j|^2-f^M_K(x^u_j)), \quad \text{where}$$

$1\le k_l,\ k_u\le n+1,\ \lambda^l_j>0,\ x^l_j\in\mathbb{R}^n,\ j=1,...,k_l,\ \sum_{j=1}^{k_l}\lambda^l_j=1,\ \sum_{j=1}^{k_l}\lambda^l_jx^l_j=0,$ $\lambda^u_j>0,\ x^u_j\in\mathbb{R}^n$, $j=1,\dots,k_u,\ \sum_{j=1}^{k_u}\lambda^u_j=1,\ \sum_{j=1}^{k_u}\lambda^u_jx^u_j=0.$

Equality (19) implies that there is an affine function $\ell(y)$ such that

$$\ell(y)\le f^M_K(y)+\lambda|y|^2,\ \text{for } y\in\mathbb{R}^n, \quad \text{and} \quad \ell(x^l_j)=f^M_K(x^l_j)+\lambda|x^l_j|^2. \tag{20}$$

We first show that $x^l_j\in K$ for $j=1,2,\dots,k_l$. If we assume $M>A_0+\lambda\ \mathrm{diam}(\Omega)^2$ with $|f|\le A_0$, we can show by the definition of the convex envelope and a contradiction argument that in the above , all $x^l_j\in K$. Assume that $x^l_j\notin K$, then there is some $1\le j_0\le k_l$ such that $x^l_{j_0}\in\Omega$. Since $\ell(x^l_{j_0})=f^M_K(x^l_{j_0})+\lambda|x^l_{j_0}|^2$, $\ell(y)$ is an affine function of $M+\lambda|y|^2$ at $x^l_{j_0}$ and hence is the unique tangent plane of the function $M+\lambda|y|^2$. Thus $\ell(y)=M+\lambda|x^l_{j_0}|^2+2\lambda x^l_{j_0}\cdot y$. If $x^l_{j_0}=0$, then $\ell(y)=M$ which is a contradiction with the assumption that $M>A_0+\lambda\ \mathrm{diam}(\Omega)^2$. If $x^l_{j_0}\neq 0$ and $x^l_{j_0}\in\Omega$, then since Ω is bounded, there are two points $x^{l'}_{j_0},x^{l''}_{j_0}\in\partial\Omega$ and some $0<\gamma<1$, such that $x^l_{j_0}=\gamma x^{l'}_{j_0}+(1-\gamma)x^{l''}_{j_0}$. By using the fact that $x^{l'}_{j_0},x^{l''}_{j_0}\in\partial\Omega$ and $0\in\Omega$, so $|x^{l'}_{j_0}-0|\le\mathrm{diam}(\Omega)$ and $|x^{l''}_{j_0}-0|\le\mathrm{diam}(\Omega)$, we also get

$$\begin{aligned}\gamma(f^M_K(x^{l'}_{j_0})&+\lambda|x^{l'}_{j_0}|^2)+(1-\gamma)(f^M_K(x^{l''}_{j_0})+\lambda|x^{l''}_{j_0}|^2)\\ &=\gamma(f(x^{l'}_{j_0})+\lambda|x^{l'}_{j_0}|^2)+(1-\gamma)(f(x^{l''}_{j_0})+\lambda|x^{l''}_{j_0}|^2)\\ &\le A_0+\lambda\ \mathrm{diam}(\Omega)^2<M\le M+\lambda|x^l_{j_0}|^2=f^M_K(x^l_{j_0})+\lambda|x^l_{j_0}|^2.\end{aligned}$$

Thus
$$1=\sum_{j=1}^{k_l}\lambda^l_j=\left(\sum_{j=1,j\neq j_0}^{k_l}\lambda^l_j\right)+\gamma x^l_{j_0}+(1-\gamma)x^l_{j_0}, \quad \text{and}$$

$$0=\sum_{j=1}^{k_l}\lambda^l_jx^l_j=\left(\sum_{j=1,j\neq j_0}^{k_l}\lambda^l_jx^l_j\right)+\gamma\lambda^l_{j_0}x^{l'}_{j_0}+(1-\gamma)\lambda^l_{j_0}x^{l''}_{j_0},$$

and hence

$$\begin{aligned}\mathbf{Co}[f^M_K+\lambda|\cdot|^2](0)=\sum_{j=1}^{k_l}\lambda^l_j(\lambda|x^l_j|^2+f^M_K(x^l_j))>\left(\sum_{j=1,\ j\neq j_0}^{k_l}\lambda^l_j(\lambda|x^l_j|^2+f^M_K(x^l_j))\right)+\\ +\gamma\lambda^l_{j_0}(\lambda|x^{l'}_j|^2+f^M_K(x^{l'}_j))+(1-\gamma)\lambda^l_{j_0}(\lambda|x^{l''}_j|^2+f^M_K(x^{l''}_j)).\end{aligned}$$

Thus, there is a contradiction with the definition of convex envelope. Hence $x^l_j\in K$, for all $j=1,2,\dots,k_l$.

Now we give an estimate of $\mathcal{C}^l_\lambda(f^M_K)(0)$. We first define

$$B_0 = \min\left\{\sum_{k=1}^{n+1}\lambda_k|x_k|^2, \lambda_k>0, x_k\in K, k=1,...,n+1, \sum_{k=1}^{n+1}\lambda_k=1, \sum_{k=1}^{n+1}\lambda_k x_k=0\right\}$$
$$= \sum_{k=1}^{m^*}\lambda^*_k|x^*_k|^2,$$

for some $2\le m^*\le n+1$, $\lambda^*_k>0$, $x^*_k\in K$ for $k=1,\dots,m^*$, $\sum_{k=1}^{m^*}\lambda^*_k=1$, and $\sum_{k=1}^{m^*}\lambda^*_k x^*_k=0$, and let

$$C_0 = \min\left\{\sum_{k=1}^{n+1}\lambda_k|x_k|^2,\ \lambda_k>0,\ x_k\in \overline{B}(0;r_c(0))\cap K,\ 1\le k\le n+1,\right.$$
$$\left.\sum_{k=1}^{n+1}\ \lambda_k=1,\ \sum_{k=1}^{n+1}\lambda_k x_k=0\right\}.$$

Furthermore, $C_0=\sum_{k=1}^{n+1}\lambda^r_k|x^r_k|^2$. Clearly $C_0\le r^2_c(0)$ and by definition, we have

$$B_0\le C_0\le r^2_c(0).$$

Now using the Cauchy-Schwarz inequality, it follows that

$$B_0=\sum_{k=1}^{m^*}\lambda^*_k|x^*_k|^2=\sum_{k=1}^{m^*}\lambda^*_k\sum_{k=1}^{m^*}\lambda^*_k|x^*_k|^2\ge\left(\sum_{k=1}^{m^*}\sqrt{\lambda^*_k}\left(\sqrt{\lambda^*_k}|x^*_k|\right)\right)^2,$$

so that
$$\sum_{k=1}^{m^*}\lambda^*_k|x^*_k|\le r_c(0).$$

$$\mathcal{C}^l_\lambda(f^M_K)(0)\le\sum_{k=1}^{m^*}\lambda^*_k(\lambda|x^*_k|^2+f(x^*_k))=\lambda\sum_{k=1}^{m^*}\lambda^*_k|x^*_k|^2+\sum_{k=1}^{m^*}\lambda^*_k f(x^*_k)$$
$$=\lambda B_0+f(0)+\sum_{k=1}^{m^*}\lambda^*_k(f(x^*_k)-f(0))\le\lambda B_0+f(0)+\sum_{k=1}^{m^*}\lambda^*_k|f(x^*_k)-f(0)|$$
$$\le\lambda B_0+f(0)+\sum_{k=1}^{m^*}L\lambda^*_k|x^*_k|^\alpha.$$

Here we have used that f is C^α-function. By the Hölder's inequality we get

$$\sum_{k=1}^{m^*}\lambda^*_k|x^*_k|^\alpha=\sum_{k=1}^{m^*}(\lambda^*_k)^{1-\alpha}(\lambda^*_k)^\alpha|x^*_k|^\alpha=\sum_{k=1}^{m^*}(\lambda^*_k)^{1-\alpha}(\lambda^*_k|x^*_k|)^\alpha$$
$$\le\left(\sum_{k=1}^{m^*}\lambda^*_k\right)^{1-\alpha}\left(\sum_{k=1}^{m^*}\lambda^*_k|x^*_k|\right)^\alpha.$$

Then by (3) we have $\sum_{k=1}^{m^*}\lambda^*_k|x^*_k|^\alpha\le(r_c(0))^\alpha$.

Furthermore, we have an upper bound of $\mathcal{C}^l_\lambda(f^M_K)(0)$ as follows:

$$\mathcal{C}^l_\lambda(f^M_K)(0) \le \lambda B_0 + f(0) + L(r_c(0))^\alpha. \tag{21}$$

Now we give an estimate of lower bound of $\mathcal{C}^l_\lambda(f^M_K)(0)$. From (19), we have

$$\begin{aligned}\mathcal{C}^l_\lambda(f^M_K)(0) &= \sum_{j=1}^{k_l} \lambda^l_j(\lambda|x^l_j|^2 + f(x^l_j)) = f(0) + \sum_{j=1}^{k_l} \lambda^l_j \left(\lambda|x^l_j|^2 + (f(x^l_j) - f(0))\right)\\ &\ge f(0) + \lambda \sum_{j=1}^{k_l} \lambda^l_j|x^l_j|^2 - \sum_{j=1}^{k_l} \lambda^l_j|f(x^l_j) - f(0)|\\ &\ge f(0) + \lambda \sum_{j=1}^{k_l} \lambda^l_j|x^l_j|^2 - L\sum_{j=1}^{k_l} \lambda^l_j|x^l_j|^\alpha. \end{aligned}\tag{22}$$

Assume that $t = |x^l_j|$ and $h(t) = \lambda t^2 - Lt^\alpha = \frac{\lambda t^2}{2} + (\frac{\lambda t^2}{2} - Lt^\alpha)$, and $g(t) = \frac{\lambda t^2}{2} - Lt^\alpha$. We need to find the minimum of $g(t)$ in order to obtain the lower bound of (22). Since $g'(t) = 0$ at $t = \left(\frac{\alpha L}{\lambda}\right)^{\frac{1}{2-\alpha}}$ and $g''\left(\left(\frac{\alpha L}{\lambda}\right)^{\frac{1}{2-\alpha}}\right)$ is positive, then $g(t)$ has a minimum value at $t = \left(\frac{\alpha L}{\lambda}\right)^{\frac{1}{2-\alpha}}$, that means,

$$g(t) = \frac{\lambda t^2}{2} - Lt^\alpha \ge -\left(\frac{2-\alpha}{2\alpha}\right)\lambda\left(\frac{\alpha L}{\lambda}\right)^{\frac{2}{2-\alpha}} = -C(\alpha,\lambda), \tag{23}$$

where $C(\alpha,\lambda) > 0$. Hence we have

$$\mathcal{C}^l_\lambda(f^M_K)(0) \ge f(0) + \frac{\lambda}{2}\sum_{j=1}^{k_l} \lambda^l_j|x^l_j|^2 - C(\alpha,\lambda). \tag{24}$$

By comparing (21) with (24), it follows that

$$f(0) + \frac{\lambda}{2}\sum_{j=1}^{k_l} \lambda^l_j|x^l_j|^2 - C(\alpha,\lambda) \le \lambda B_0 + f(0) + L(r_c(0))^\alpha,$$

and hence

$$\begin{aligned}\sum_{j=1}^{k_l} \lambda^l_j|x^l_j|^2 &\le \frac{2}{\lambda}\left(\lambda B_0 + L(r_c(0))^\alpha + C(\alpha,\lambda)\right)\\ &\le \frac{2}{\lambda}\left(\lambda r_c^2(0) + L(r_c(0))^\alpha + C(\alpha,\lambda)\right), \end{aligned}\tag{25}$$

which gives the upper bound of $\sum_{j=1}^{k_l} \lambda^l_j|x^l_j|^2$. Again by using the Hölder's inequality and (25), we obtain

$$\begin{aligned}\sum_{j=1}^{k_l} \lambda^l_j|x^l_j|^\alpha &= \sum_{j=1}^{k_l}(\lambda^l_j)^{1-\frac{\alpha}{2}}(\lambda^l_j)^{\frac{\alpha}{2}}|x^l_j|^\alpha = \sum_{j=1}^{k_l}(\lambda^l_j)^{1-\frac{\alpha}{2}}(\lambda^l_j|x^l_j|^2)^{\frac{\alpha}{2}}\\ &\le \left(\sum_{j=1}^{k_l}\lambda^l_j\right)^{1-\frac{\alpha}{2}}\left(\sum_{j=1}^{k_l}\lambda^l_j|x^l_j|^2\right)^{\frac{\alpha}{2}} \le \left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + L(r_c(0))^\alpha + C(\alpha,\lambda)\right)\right)^{\frac{\alpha}{2}}. \end{aligned}\tag{26}$$

Now by substituting the estimate (26) in (22), we have

$$\mathcal{C}^l_\lambda(f^M_K)(0) \geq f(0) + \lambda B_0 - L\left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + L(r_c(0))^\alpha + C(\alpha,\lambda)\right)\right)^{\frac{\alpha}{2}}, \qquad (27)$$

and by combining (27) and (21), we obtain

$$f(0) + \lambda B_0 - L\left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + L(r_c(0))^\alpha + C(\alpha,\lambda)\right)\right)^{\frac{\alpha}{2}} \leq \mathcal{C}^l_\lambda(f^M_K)(0) \leq \lambda B_0 + f(0) + L(r_c(0))^\alpha.$$

Similarly, we can show that

$$f(0) - \lambda B_0 - L(r_c(0))^\alpha \leq \mathcal{C}^u_\lambda(f^M_K)(0) \leq f(0) - \lambda B_0 + L\left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + L(r_c(0))^\alpha + C(\alpha,\lambda)\right)\right)^{\frac{\alpha}{2}}.$$

Furthermore, we obtain

$$\begin{aligned} 2f(0) - L\left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + L(r_c(0))^\alpha + C(\alpha,\lambda)\right)\right)^{\frac{\alpha}{2}} - L(r_c(0))^\alpha \\ \leq \mathcal{C}^l_\lambda(f^M_K)(0) + \mathcal{C}^u_\lambda(f^M_K)(0) \\ \leq 2f(0) + L\left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + L(r_c(0))^\alpha + C(\alpha,\lambda)\right)\right)^{\frac{\alpha}{2}} + L(r_c(0))^\alpha, \end{aligned}$$

and thus

$$\left|A^M_\lambda(f_K)(0) - f(0)\right| \leq \frac{L}{2}(r_c(0))^\alpha + \frac{L}{2}\left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + L(r_c(0))^\alpha + C(\alpha,\lambda)\right)\right)^{\frac{\alpha}{2}}.$$

Proof of Theorem 1.5. Part (ii): The proof is similar to that of part i. We can use a similar argument to show that all x^l_j are in K and then we have

$$\begin{aligned} \mathcal{C}^l_\lambda(f^M_K)(0) \leq & \sum_{k=1}^{m^*} \lambda^*_k(\lambda|x^*_k|^2 + f(x^*_k)) = \lambda\sum_{k=1}^{m^*}\lambda^*_k|x^*_k|^2 + \sum_{k=1}^{m^*}\lambda^*_k f(x^*_k) \\ = & \lambda B_0 + f(0) + \sum_{k=1}^{m^*}\lambda^*_k(f(x^*_k) - f(0)). \end{aligned}$$

From the definition of B_0, we have $\sum_{k=1}^{m^*}\lambda^*_k x^*_k = 0$. Furthermore, we also have

$$\sum_{k=1}^{m^*}\lambda^*_k\ Df(0)\cdot x^*_k = 0.$$

Thus

$$\begin{aligned} \mathcal{C}^l_\lambda(f^M_K)(0) & \leq \lambda B_0 + f(0) + \sum_{k=1}^{m^*}\lambda^*_k\left(f(x^*_k) - f(0) - Df(0)\cdot x^*_k\right) \\ & \leq \lambda B_0 + f(0) + \sum_{k=1}^{m^*}\lambda^*_k|f(x^*_k) - f(0) - Df(0)\cdot x^*_k|. \end{aligned}$$

Following the Fundamental Theorem of Calculus, we have

$$\mathcal{C}_\lambda^l(f_K^M)(0) \le \lambda B_0 + f(0) + \frac{L}{\beta+1}\sum_{k=1}^{m^*}\lambda_k^*|x_k^*|^{\beta+1}.$$

Thus by the Hölder's inequality we have

$$\begin{aligned}\sum_{k=1}^{m^*}\lambda_k^*|x_k^*|^{\beta+1} &= \sum_{k=1}^{m^*}(\lambda_k^*)^{\frac{1-\beta}{2}}(\lambda_k^*)^{\frac{\beta+1}{2}}|x_k^*|^{\beta+1} = \sum_{k=1}^{m^*}(\lambda_k^*)^{\frac{1-\beta}{2}}(\lambda_k^*|x_k^*|^2)^{\frac{\beta+1}{2}}\\ &\le \left(\sum_{k=1}^{m^*}\lambda_k^*\right)^{\frac{1-\beta}{2}}\left(\sum_{k=1}^{m^*}\lambda_k^*|x_k^*|^2\right)^{\frac{\beta+1}{2}} \le (r_c(0))^{\beta+1}.\end{aligned}$$

Then we obtain

$$\mathcal{C}_\lambda^l(f_K^M)(0) \le \lambda B_0 + f(0) + \frac{L}{\beta+1}(r_c(0))^{\beta+1}, \tag{28}$$

and also from (19), we have

$$\begin{aligned}\mathcal{C}_\lambda^l(f_K^M)(0) &= \sum_{j=1}^{k_l}\lambda_j^l(\lambda|x_j^l|^2 + f(x_j^l)) \qquad (29)\\ &= f(0) + \sum_{j=1}^{k_l}\lambda_j^l\left(\lambda|x_j^l|^2 + \left(f(x_j^l) - f(0) - Df(0)\cdot x_j^l\right)\right)\\ &\ge f(0) + \lambda\sum_{j=1}^{k_l}\lambda_j^l|x_j^l|^2 - \sum_{j=1}^{k_l}\lambda_j^l|f(x_j^l) - f(0) - Df(0)\cdot x_j^l|\\ &\ge f(0) + \lambda\sum_{j=1}^{k_l}\lambda_j^l|x_j^l|^2 - \frac{L}{\beta+1}\sum_{j=1}^{k_l}\lambda_j^l|x_j^l|^{\beta+1}. \qquad (30)\end{aligned}$$

Assume that $t = |x_j^l|$ and $h(t) = \lambda t^2 - \frac{L}{\beta+1}t^{\beta+1} = \frac{\lambda t^2}{2} + (\frac{\lambda t^2}{2} - \frac{L}{\beta+1}t^{\beta+1})$, and $g(t) = \frac{\lambda t^2}{2} - \frac{L}{\beta+1}t^{\beta+1}$. We need to find the minimum of $g(t)$ in order to obtain the lower bound of (30). Since $g'(t) = 0$ at $t = \left(\frac{L}{\lambda}\right)^{\frac{1}{1-\beta}}$ and $g''\left(\left(\frac{L}{\lambda}\right)^{\frac{1}{1-\beta}}\right)$ is positive, then $g(t)$ has a minimum value at $t = \left(\frac{L}{\lambda}\right)^{\frac{1}{1-\beta}}$, that means,

$$g(t) = \frac{\lambda t^2}{2} - \frac{L}{\beta+1}t^{\beta+1} \ge -\left(\frac{1-\beta}{2(1+\beta)}\right)\lambda\left(\frac{L}{\lambda}\right)^{\frac{2}{1-\beta}} = -C(\beta,\lambda), \tag{31}$$

where $C(\beta,\lambda) > 0$. Hence

$$\mathcal{C}_\lambda^l(f_K^M)(0) \ge f(0) + \frac{\lambda}{2}\sum_{j=1}^{k_l}\lambda_j^l|x_j^l|^2 - C(\beta,\lambda). \tag{32}$$

By comparing (28) with (32), we obtain

$$f(0) + \frac{\lambda}{2}\sum_{j=1}^{k_l}\lambda_j^l|x_j^l|^2 - C(\beta,\lambda) \le \lambda B_0 + f(0) + \frac{L}{\beta+1}(r_c(0))^{\beta+1},$$

and thus
$$\sum_{j=1}^{k_l} \lambda_j^l |x_j^l|^2 \le \frac{2}{\lambda}\left(\lambda B_0 + \frac{L}{\beta+1}\left(r_c(0)\right)^{\beta+1} + C(\beta,\lambda)\right)$$
$$\le \frac{2}{\lambda}\left(\lambda r_c^2(0) + \frac{L}{\beta+1}\left(r_c(0)\right)^{\beta+1} + C(\beta,\lambda)\right), \qquad (33)$$

which gives the upper bound of $\sum_{j=1}^{k_l} \lambda_j^l |x_j^l|^2$. Again by using the Hölder's inequality and (33), we obtain

$$\sum_{j=1}^{k_l} \lambda_j^l |x_j^l|^{\beta+1} = \sum_{j=1}^{k_l} (\lambda_j^l)^{\frac{1-\beta}{2}} (\lambda_j^l)^{\frac{\beta+1}{2}} |x_j^l|^{\beta+1} = \sum_{j=1}^{k_l} (\lambda_j^l)^{\frac{1-\beta}{2}} (\lambda_j^l |x_j^l|^2)^{\frac{\beta+1}{2}}$$
$$\le \left(\sum_{j=1}^{k_l} \lambda_j^l\right)^{\frac{1-\beta}{2}} \left(\sum_{j=1}^{k_l} \lambda_j^l |x_j^l|^2\right)^{\frac{\beta+1}{2}}$$
$$\le \left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + \frac{L}{\beta+1}(r_c(0))^{\beta+1} + C(\beta,\lambda)\right)\right)^{\frac{\beta+1}{2}}. \qquad (34)$$

By substituting (34) in (30), we have

$$\mathcal{C}_\lambda^l(f_K^M)(0) \ge f(0) + \lambda B_0 - \frac{L}{\beta+1}\left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + \frac{L}{\beta+1}(r_c(0))^{\beta+1} + C(\beta,\lambda)\right)\right)^{\frac{\beta+1}{2}},$$

and by combining the last inequality and (28), we obtain

$$f(0) + \lambda B_0 - \frac{L}{\beta+1}\left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + \frac{L}{\beta+1}(r_c(0))^{\beta+1} + C(\beta,\lambda)\right)\right)^{\frac{\beta+1}{2}}$$
$$\le \mathcal{C}_\lambda^l(f_K^M)(0) \le \lambda B_0 + f(0) + \frac{L}{\beta+1}(r_c(0))^{\beta+1}.$$

Similarly, we can show that

$$f(0) - \lambda B_0 - \frac{L}{\beta+1}(r_c(0))^{\beta+1} \le \mathcal{C}_\lambda^u(f_K^M)(0)$$
$$\le f(0) - \lambda B_0 + \frac{L}{\beta+1}\left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + \frac{L}{\beta+1}(r_c(0))^{\beta+1} + C(\beta,\lambda)\right)\right)^{\frac{\beta+1}{2}}.$$

Furthermore, we obtain

$$2f(0) - \frac{L}{\beta+1}\left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + \frac{L}{\beta+1}(r_c(0))^{\beta+1} + C(\beta,\lambda)\right)\right)^{\frac{\beta+1}{2}} - \frac{L}{\beta+1}(r_c(0))^{\beta+1}$$
$$\le \mathcal{C}_\lambda^l(f_K^M)(0) + \mathcal{C}_\lambda^u(f_K^M)(0)$$
$$\le 2f(0) + \frac{L}{\beta+1}\left(\frac{2}{\lambda}\left(\lambda r_c^2(0) + \frac{L}{\beta+1}(r_c(0))^{\beta+1} + C(\beta,\lambda)\right)\right)^{\frac{\beta+1}{2}} + \frac{L}{\beta+1}(r_c(0))^{\beta+1}.$$

Thus $\quad \left|A_\lambda^M(f_K)(0)-f(0)\right| \leq$

$$\leq \frac{L}{2(\beta+1)}(r_c(0))^{\beta+1}+\frac{L}{2(\beta+1)}\left(\frac{2}{\lambda}\left(\lambda r_c^2(0)+\frac{L}{\beta+1}(r_c(0))^{\beta+1}+C(\beta,\lambda)\right)\right)^{\frac{\beta+1}{2}},$$

which completes the proof. $\square$

Remark 3.1. Now we consider the asymptotic estimate

$$\limsup_{\lambda\to\infty}\limsup_{M\to\infty}\left|A_\lambda^M(f_K)(0)-f(0)\right|.$$

Since for fixed $\lambda>0$ when $M>\lambda$ is sufficiently large A_λ^M is independent of M, we can see that

$$\lim_{M\to\infty}A_\lambda^M=A_\lambda^\infty.$$

Next we give an estimate $|A_\lambda^\infty-f|$ hence obtain the following asymptotic error estimate

$$\limsup_{\lambda\to\infty}|A_\lambda^\infty(f_K)-f(0)|\leq L(r_c(0))^\alpha. \qquad \square$$

Proof of Theorem 1.6. The proof is very similar to that of Theorem 1.5 (i), (ii). The only difference is in the infinite case that the convex envelope of $f_K^\infty(x)+\lambda|x|^2$ does not need touch the ceiling $M=+\infty$. For the case $M=+\infty$, the trick is to split the terms as follows $\lambda t^2-Lt^\alpha=(1-\epsilon)\lambda t^2+(\epsilon\lambda t^2-Lt^\alpha)$ and find the minimum of the second term. The rest of the proof of Part (i) and Part (ii) then follows from a similar argument to that of Part (i), (ii) of Theorem 1.5, respectively, and we obtain another estimate depending only on λ and ϵ. In order to obtain the asymptotic estimate, we first take the upper limit for $\lambda\to\infty$, then we take the upper limit for $\epsilon\to 0+$. Then we obtain an asymptotic estimates depending only on $(r_c(0))^\alpha$. $\square$

We need the following preparations for the proof of our main Hausdorff stability results.

Lemma 3.2. *Suppose $f\colon\mathbb{R}^n\to\mathbb{R}$ is a bounded upper semicontinuous function such that for some constants $m,M>0$, $0<m\leq f\leq M$, for all $x\in\mathbb{R}^n$. If $K,E\subset\mathbb{R}^n$ are compacts sets with $\operatorname{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E})<+\infty$ where $\Gamma_{f_K}:=\{(x,f(x)):x\in K\}$, then*

$$|d_{\lambda,f_K}(x,\Gamma_{f_K})-d_{\lambda,f_E}(x,\Gamma_{f_E})|\leq\left(\sqrt{1+\frac{1}{4\lambda m}}\right)\operatorname{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E}). \tag{35}$$

Corollary 3.3. *By relaxing the lower bound $0<m\leq f\leq M$, to $0<f\leq M$, for all $x\in\mathbb{R}^n$ in Lemma 3.2, then we have,*

$$|d_{\lambda,f_K}(x,\Gamma_{f_K})-d_{\lambda,f_E}(x,\Gamma_{f_E})|\leq\left(\sqrt{1+\frac{1}{\lambda}}\right)\operatorname{dist}_{\mathcal{H}}(\Gamma_{\sqrt{f_K}},\Gamma_{\sqrt{f_E}}).$$

Lemma 3.4. *Under the assumption of Lemma* 3.2, *we have*

$$\left|D^2_{\lambda,f_K}(x,\Gamma_{f_K}) - D^2_{\lambda,f_E}(x,\Gamma_{f_E})\right| \le \left(2\sqrt{M\lambda + \frac{M}{4m}}\right) \mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E}). \tag{36}$$

The following lemma in [23, Lemma 4.8] is important when considering the value of the upper transform at a single point.

Lemma 3.5. *Suppose $f > 0$ is upper semicontinous function and $x_0 \in \mathbb{R}^n$, then for $\lambda > 0$,*

$$\mathcal{C}^u_\lambda(f\chi_{\{x_0\}})(x) = \begin{cases} \lambda\left(|x-x_0| - \sqrt{\frac{f_K(x_0)}{\lambda}}\right)^2, & \textit{for } |x-x_0| \le \sqrt{\frac{f_K(x_0)}{\lambda}}, \\ \\ 0, & \textit{for } |x-x_0| \ge \sqrt{\frac{f_K(x_0)}{\lambda}}. \end{cases}$$

The following result is a generalization of Lemma 4.9 in [23]

Lemma 3.6. *Suppose f satisfies the assumptions of Lemma* 3.2 *and $K \subset \mathbb{R}^n$ is closed . Then for $\lambda > 0$ and for all $x \in \mathbb{R}^n$, we have*

$$\mathcal{C}^u_\lambda(f\chi_K)(x) = \mathcal{C}^u_\lambda(D^2_{\lambda,f_K}(\cdot,\Gamma_{f_K}))(x). \tag{37}$$

We can generalize Lemma 3.2 to the case of bounded function as follows.

Lemma 3.7. *Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is a bounded function such that for some constants $m, M > 0$, $0 < m \le f \le M$, for all $x \in \mathbb{R}^n$. If $K, E \subset \mathbb{R}^n$ are compacts sets with $\mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E}) < +\infty$, then*

$$|d_{\lambda,f_K}(x,\Gamma_{f_K}) - d_{\lambda,f}(x,\Gamma_{f_E})| \le \left(\sqrt{1+\frac{1}{4\lambda m}}\right) \mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E}). \tag{38}$$

Next we prove our main Hausdorff stability results starting with the proofs of the lemmas above.

Proof of Lemma 3.2: Fix $x \in \mathbb{R}^n$. For every $\delta > \mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E})$, by Remark (2.6), there is some $x^K \in K$ such that

$$d_{\lambda,f_K}(x,\Gamma_{f_K}) = |x^K - x| - \sqrt{\frac{f_K(x^K)}{\lambda}}.$$

For $(x^K, f_K(x^K)) \in \Gamma_{f_K}$, there is some $(\overline{x}^E, f_E(\overline{x}^E)) \in \Gamma_{f_E}$ such that

$$\left\|(\overline{x}^E, f_E(\overline{x}^E)) - (x^K, f_K(x^K))\right\|_2 < \delta. \tag{39}$$

Thus

$$\begin{aligned} d_{\lambda,f_E}(x,\Gamma_{f_E}) - d_{\lambda,f_K}(x,\Gamma_{f_K}) &= d_{\lambda,f_E}(x,\Gamma_{f_E}) - \left(|x^K - x| - \sqrt{\frac{f_K(x^K)}{\lambda}}\right) \\ &\le |\overline{x}^E - x| - \sqrt{\frac{f_E(\overline{x}^E)}{\lambda}} - |x^K - x| + \sqrt{\frac{f_K(x^K)}{\lambda}}. \end{aligned}$$

By using the triangle inequality, the conjugate, and since $m \leq f$, we have

$$d_{\lambda,f_E}(x,\Gamma_{f_E}) - d_{\lambda,f_K}(x,\Gamma_{f_K}) \leq |\overline{x}^E - x^K| + \frac{1}{2\sqrt{m\lambda}} \left| f_E(\overline{x}^E) - f_K(x^K) \right|.$$

By using the Cauchy-Schwarz inequality and (39), we also have

$$\begin{aligned} d_{\lambda,f_E}(x,\Gamma_{f_E}) - d_{\lambda,f_K}(x,\Gamma_{f_K}) &\leq \sqrt{1+\frac{1}{4\lambda m}} \sqrt{|\overline{x}^E - x^K|^2 + \left| f_E(\overline{x}^E) - f_K(x^K) \right|^2} \\ &\leq \sqrt{1+\frac{1}{4\lambda m}} \left\| (\overline{x}^E, f_E(\overline{x}^E)) - (x^K, f_K(x^K)) \right\|_2 \leq \sqrt{1+\frac{1}{4\lambda m}}\ \delta, \end{aligned}$$

for all $\delta > \mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K}, \Gamma_{f_E})$. Then by letting δ tend to $\mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K}, \Gamma_{f_E})$, we obtain

$$d_{\lambda,f_E}(x,\Gamma_{f_E}) - d_{\lambda,f_K}(x,\Gamma_{f_K}) \leq \sqrt{1+\frac{1}{4\lambda m}}\ \mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K}, \Gamma_{f_E}).$$

Similarly, we can show that

$$d_{\lambda,f_K}(x,\Gamma_{f_K}) - d_{\lambda,f_E}(x,\Gamma_{f_E}) \leq \sqrt{1+\frac{1}{4\lambda m}}\ \mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K}, \Gamma_{f_E}),$$

and the conclusion then follows. □

Proof of Corollary 3.3: The proof is very similar to that of Lemma 3.2. The only difference is we apply the Cauchy Schwarz inequality directly. □

Proof of Lemma 3.4: The proof is very similar to the proof of Lemma 4.7 in [23]. We have

$$\begin{aligned} &\left| D^2_{\lambda,f_K}(x,\Gamma_{f_K}) - D^2_{\lambda,f_E}(x,\Gamma_{f_E}) \right| \\ &\qquad \leq \left(|D_{\lambda,f_K}(x,\Gamma_{f_K})| + |D_{\lambda,f_E}(x,\Gamma_{f_E})| \right) |D_{\lambda,f_K}(x,\Gamma_{f_K}) - D_{\lambda,f_E}(x,\Gamma_{f_E})|. \end{aligned}$$

By definition of $D_{\lambda,f_K}(x,\Gamma_{f_K})$, we then have if

$$\inf_{y\in K} \left(|y-x| - \sqrt{\frac{f_K(y)}{\lambda}} \right) > 0,\ \ |D_{\lambda,f_K}(x,\Gamma_{f_K})| = 0.$$

Similarly, we have $|D_{\lambda,f_E}(x,\Gamma_{f_E})| = 0$, and if

$$d_{\lambda,f_K}(x,\Gamma_{f_K}) = \inf_{y\in K} \left(|y-x| - \sqrt{\frac{f_K(y)}{\lambda}} \right) = |x^K - x| - \sqrt{\frac{f_K(x^K)}{\lambda}} < 0$$

for some $x^K \in K$, then

$$\begin{aligned} |D_{\lambda,f_K}(x,\Gamma_{f_K})| &= \left| -\sqrt{\lambda}\ d_{\lambda,f_K}(x,\Gamma_{f_K}) \right| = \sqrt{\lambda} \left| |x^K - x| - \sqrt{\frac{f_K(x^K)}{\lambda}} \right| \\ &\leq \sqrt{f_K(x^K)} \leq \sqrt{M}. \end{aligned}$$

Similarly, we have $|D_{\lambda,f_E}(x,\Gamma_{f_E})| \leq \sqrt{M}$.

Next, by the formula $\min\{0,a\}=(a-|a|)/2$ for $a\in\mathbb{R}$, we then have

$$
\begin{aligned}
&|D_{\lambda,f_K}(x,\Gamma_{f_K})-D_{\lambda,f_E}(x,\Gamma_{f_E})| \\
&\quad=\frac{\sqrt{\lambda}}{2}\left|d_{\lambda,f_K}(x,\Gamma_{f_K})-d_{\lambda,f_E}(x,\Gamma_{f_E})\right. \left.+|d_{\lambda,f_E}(x,\Gamma_{f_E})|-|d_{\lambda,f_K}(x,\Gamma_{f_K})|\right| \\
&\quad\le\sqrt{\lambda}\left|d_{\lambda,f_K}(x,\Gamma_{f_K})-d_{\lambda,f_E}(x,\Gamma_{f_E})\right|.
\end{aligned}
$$

By Lemma 3.2, we have

$$
|D_{\lambda,f_K}(x,\Gamma_{f_K})-D_{\lambda,f_E}(x,\Gamma_{f_E})|\le\sqrt{\lambda}\left(\sqrt{1+\frac{1}{4\lambda m}}\right)\mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E}).
$$

Hence (36) follows. □

Proof of Lemma 3.6: The proof is very similar to the proof of Lemma 4.9 in [23]. We need first to show that

$$
\overline{f\chi_K}(x)\le D^2_{\lambda,f_K}(\cdot,\Gamma_{f_K})(x), \tag{40}
$$

for all $x\in\mathbb{R}^n$, where $\overline{f\chi_K}(x)$ is upper semicontinuous envelope of $f\chi_K$. So that by ordering property of the upper transform, we have

$$
\mathcal{C}^u_\lambda(\overline{f\chi_K})(x)\le\mathcal{C}^u_\lambda(D^2_{\lambda,f_K}(\cdot,\Gamma_{f_K}))(x), \tag{41}
$$

for all $x\in\mathbb{R}^n$. If $x\notin K$, then $f\chi_K(x)=0\le D^2_{\lambda,f_K}(x,\Gamma_{f_K})$. If $x\in K$, since

$$
d_{\lambda,f_K}(x,\Gamma_{f_K})=\inf_{y\in K}\left(|y-x|-\sqrt{\frac{f_K(y)}{\lambda}}\right)\le|x-x|-\sqrt{\frac{f_K(x)}{\lambda}}=-\sqrt{\frac{f_K(x)}{\lambda}}<0,
$$

we then have

$$
D_{\lambda,f_K}(x,\Gamma_{f_K})=-\sqrt{\lambda}\inf_{y\in K}\{0,d_{\lambda,f_K}(x,\Gamma_{f_K})\}=-\sqrt{\lambda}\,d_{\lambda,f_K}(x,\Gamma_{f_K})\ge\sqrt{f_K(x)},
$$

and thus $D^2_{\lambda,f_K}(x,\Gamma_{f_K})\ge f_K(x)$. Therefore, (40) holds for all $x\in\mathbb{R}^n$. Then by using the order property (12), (41) will follow.

Next, we show that the opposite inequality also holds,

$$
\mathcal{C}^u_\lambda(\overline{f\chi_K})(x)\ge\mathcal{C}^u_\lambda(D^2_{\lambda,f_K}(\cdot,\Gamma_{f_K}))(x). \tag{42}
$$

We will consider two cases as follows.

Case 1: If $d_{\lambda,f_K}(x,\Gamma_{f_K})>0$, then by definition, $D_{\lambda,f_K}(x,\Gamma_{f_K})=0$ and hence $D^2_{\lambda,f_K}(x,\Gamma_{f_K})=0$. Therefore we show in this case

$$
\mathcal{C}^u_\lambda(D^2_{\lambda,f_K}(\cdot,\Gamma_{f_K}))(x)=0\le\mathcal{C}^u_\lambda(\overline{f\chi_K})(x).
$$

Consider the function $z\mapsto\lambda|z-x|^2-D^2_{\lambda,f_K}(z,\Gamma_{f_K})$ for $z\in\mathbb{R}^n$. We show that the value of the convex envelope of this function at x is zero.

Let $\ell(z) = 0$ for all $z \in \mathbb{R}^n$ be the affine function and show that

$$0 = \ell(x) = \big(\lambda|z-x|^2 - D^2_{\lambda,f_K}(z, \Gamma_{f_K})\big)|_{z=x}, \tag{43}$$

and
$$0 = \ell(z) \le \lambda|z-x|^2 - D^2_{\lambda,f_K}(z, \Gamma_{f_K}), z \in \mathbb{R}^n. \tag{44}$$

Equality (43) holds since $\big(\lambda|z-x|^2 - D^2_{\lambda,f_K}(z, \Gamma_{f_K})\big)|_{z=x} = -D^2_{\lambda,f_K}(x, \Gamma_{f_K}) = 0$. Now we prove (44), that is, $\lambda|z-x|^2 - D^2_{\lambda,f_K}(z, \Gamma_{f_K}) \ge 0$, is equivalent to

$$D^2_{\lambda,f_K}(z, \Gamma_{f_K}) \le \lambda|z-x|^2, z \in \mathbb{R}^n. \tag{45}$$

If $d_{\lambda,f_K}(z, \Gamma_{f_K}) > 0$, then $D^2_{\lambda,f_K}(z, \Gamma_{f_K}) = 0$ hence (45) holds for all $z \in \mathbb{R}^n$. If $d_{\lambda,f_K}(z, \Gamma_{f_K}) < 0$, then $D^2_{\lambda,f_K}(z, \Gamma_{f_K}) = \lambda\, d^2_{\lambda,f_K}(z, \Gamma_{f_K})$. We need to show that

$$\lambda\, d^2_{\lambda,f_K}(z, \Gamma_{f_K}) = \lambda \left(\inf_{y\in K}\left(|y-x| - \sqrt{\frac{f_K(y)}{\lambda}}\right)\right)^2 \le \lambda|z-x|^2,$$

which is equivalent to $\displaystyle -\inf_{y\in K}\left(|y-z| - \sqrt{\frac{f_K(y)}{\lambda}}\right) \le |z-x|,$

which is equivalent to $\displaystyle \inf_{y\in K}\left(|y-z| - \sqrt{\frac{f_K(y)}{\lambda}}\right) + |z-x| \ge 0.$

Since $\inf(f+a) = a + \inf f$, then we have

$$\inf_{y\in K}\left(|z-x| + |y-z| - \sqrt{\frac{f_K(y)}{\lambda}}\right) \ge 0.$$

By using the triangle inequality, we have

$$\begin{aligned}\inf_{y\in K}\left(|z-x| + |y-z| - \sqrt{\frac{f_K(y)}{\lambda}}\right) &\ge \inf_{y\in K}\left(|y-x| - \sqrt{\frac{f_K(y)}{\lambda}}\right)\\ &= d_{\lambda,f_K}(x, \Gamma_{f_K}) > 0.\end{aligned}$$

Thus (44) holds. Therefore,

$$0 = \mathbf{Co}[\lambda|(\cdot)-x|^2 - D^2_{\lambda,f_K}(\cdot, \Gamma_{f_K})](x) = -\mathcal{C}^u_\lambda(D^2_{\lambda,f_K}(\cdot, \Gamma_{f_K}))(x),$$

which implies

$$D^2_{\lambda,f_K}(\cdot, \Gamma_{f_K})](x) \le \mathcal{C}^u_\lambda(D^2_{\lambda,f_K}(\cdot, \Gamma_{f_K}))(x) = 0 \le \mathcal{C}^u_\lambda(\overline{f\chi_K})(x). \tag{46}$$

Case 2: Finally we consider the case when

$$d_{\lambda,f_K}(x, \Gamma_{f_K}) = \inf_{y\in K}\left(|y-x| - \sqrt{\frac{f_K(y)}{\lambda}}\right) = |x^K - x| - \sqrt{\frac{\overline{f_K}(x^K)}{\lambda}} < 0.$$

By Lemma 3.5, we have

$$\mathcal{C}^u_\lambda(\overline{f\chi_{\{x^K\}}})(y) = \begin{cases} \lambda\left(|y-x^K| - \sqrt{\frac{\overline{f_K}(x^K)}{\lambda}}\right)^2, & \text{for } |y-x^k| \le \sqrt{\frac{\overline{f_K}(x^K)}{\lambda}}, \\ 0, & \text{for } |y-x^K| \ge \sqrt{\frac{\overline{f_K}(x^K)}{\lambda}}. \end{cases}$$

Since $x^K \in K$, then we have $\overline{f\chi_{\{x^K\}}}(y) \le \overline{f\chi_K}(y)$ for all $y \in \mathbb{R}^n$. So that by the ordering property of upper transforms,

$$\mathcal{C}^u_\lambda\left(\overline{f\chi_{\{x^K\}}}\right)(y) \le \mathcal{C}^u_\lambda\left(\overline{f\chi_K}\right)(y), \text{ for all } y \in \mathbb{R}^n.$$

Since $d_{\lambda,f_K}(x, \mathit{\Gamma}_{f_K}) < 0$, then we also have $|x^K - x| < \sqrt{\frac{\overline{f_K}(x^K)}{\lambda}}$. Thus

$$\mathcal{C}^u_\lambda(\overline{f\chi_{\{x^K\}}})(x) = \lambda\left(|x^K - x| - \sqrt{\frac{\overline{f_K}(x^K)}{\lambda}}\right)^2 = \lambda\, d^2_{\lambda,f_K}(x, \mathit{\Gamma}_{f_K}) = D^2_{\lambda,f_K}(x, \mathit{\Gamma}_{f_K}).$$

So, in this case, $D^2_{\lambda,f_K}(x, \mathit{\Gamma}_{f_K}) = \mathcal{C}^u_\lambda(\overline{f\chi_{\{x^K\}}})(x) \le \mathcal{C}^u_\lambda\left(\overline{f\chi_K}\right)(x)$. By combining this case and (46), we have for all $x \in \mathbb{R}^n$,

$$D^2_{\lambda,f_K}(x, \mathit{\Gamma}_{f_K}) \le \mathcal{C}^u_\lambda\left(\overline{f\chi_K}\right)(x).$$

So by using the ordering property (12) and the property of compensated convex transforms (14) in the previous inequality, we have

$$\mathcal{C}^u_\lambda(D^2_{\lambda,f_K}(\cdot, \mathit{\Gamma}_{f_K}))(x) \le \mathcal{C}^u_\lambda\left(\mathcal{C}^u_\lambda(\overline{f\chi_K})\right)(x) = \mathcal{C}^u_\lambda(\overline{f\chi_K})(x)\ \forall x \in \mathbb{R}^n.$$

Since the opposite inequality also holds in (41), we have

$$\mathcal{C}^u_\lambda(\overline{f\chi_K})(x) = \mathcal{C}^u_\lambda(D^2_{\lambda,f_K}(\cdot, \mathit{\Gamma}_{f_K}))(x), \tag{47}$$

for all $x \in \mathbb{R}^n$. By using the property of upper transform (the first equation in (15)), then we obtain

$$\mathcal{C}^u_\lambda(f\chi_K) = \mathcal{C}^u_\lambda(D^2_{\lambda,f_K}(\cdot, \mathit{\Gamma}_{f_K}))(x) \quad \forall x \in \mathbb{R}^n,$$

which completes the proof. □

Proof of Theorem 1.7: The proof is very similar to the proof of Theorem 4.10 in [23]. By Lemma 3.4, we have for all $x \in \mathbb{R}^n$,

$$\left|D^2_{\lambda,f_K}(x, \mathit{\Gamma}_{f_K}) - D^2_{\lambda,f_E}(x, \mathit{\Gamma}_{f_E})\right| \le \left(2\sqrt{M\lambda + \frac{M}{4m}}\right)\operatorname{dist}_{\mathcal{H}}(\mathit{\Gamma}_{f_K}, \mathit{\Gamma}_{f_E}).$$

Thus

$$D^2_{\lambda,f_E}(x,\Gamma_{f_E}) - \left(2\sqrt{M\lambda+\frac{M}{4m}}\right)\mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E})$$
$$\le D^2_{\lambda,f_K}(x,\Gamma_{f_K}) \le D^2_{\lambda,f_E}(x,\Gamma_{f_E}) + \left(2\sqrt{M\lambda+\frac{M}{4m}}\right)\mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E}).$$

By taking the upper transform in the above inequality and using the ordering property (12), and the affine covariant property of the upper transform (13), we get

$$\mathcal{C}^u_\lambda(D^2_{\lambda,f_E}(x,\Gamma_{f_E})) - \left(2\sqrt{M\lambda+\frac{M}{4m}}\right)\mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E})$$
$$\le \mathcal{C}^u_\lambda(D^2_{\lambda,f_K}(x,\Gamma_{f_K})) \le \mathcal{C}^u_\lambda(D^2_{\lambda,f_E}(x,\Gamma_{f_E})) + \left(2\sqrt{M\lambda+\frac{M}{4m}}\right)\mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E}).$$

This is equivalent to

$$\left|\mathcal{C}^u_\lambda(D^2_{\lambda,f_K}(x,\Gamma_{f_K})) - \mathcal{C}^u_\lambda(D^2_{\lambda,f_E}(x,\Gamma_{f_E}))\right| \le \left(2\sqrt{M\lambda+\frac{M}{4m}}\right)\mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E}).$$

By Lemma 3.6, we have for all $x\in\mathbb{R}^n$,

$$\mathcal{C}^u_\lambda(f\chi_K)(x) = \mathcal{C}^u_\lambda(D^2_{\lambda,f}(\cdot,\Gamma_{f_K}))(x).$$

Hence the result then follows. □

Proof of Corollary 1.10: From the definition of stable ridge transform, we have

$$SR_{\lambda,\tau}(f\chi_K)(x) - SR_{\lambda,\tau}(f\chi_E)(x)$$
$$= \mathcal{C}^u_\lambda(f\chi_K)(x) - \mathcal{C}^u_\lambda(f\chi_E)(x) + \mathcal{C}^l_\tau(\mathcal{C}^u_\lambda(f\chi_E)(x)) - \mathcal{C}^l_\tau(\mathcal{C}^u_\lambda(f\chi_K)(x)).$$

By using the triangle inequality, we obtain

$$|SR_{\lambda,\tau}(f\chi_K) - SR_{\lambda,\tau}(f\chi_E)|$$
$$\le |\mathcal{C}^u_\lambda(f\chi_K) - \mathcal{C}^u_\lambda(f\chi_E)| + \left|\mathcal{C}^l_\tau(\mathcal{C}^u_\lambda(f\chi_E)(x)) - \mathcal{C}^l_\tau(\mathcal{C}^u_\lambda(f\chi_K)(x))\right|.$$

By taking the lower transform in (7) and using the ordering property of the lower compensated convex transforms, we have

$$\left|\mathcal{C}^l_\tau(\mathcal{C}^u_\lambda(f\chi_E)(x)) - \mathcal{C}^l_\tau(\mathcal{C}^u_\lambda(f\chi_K)(x))\right| \le \left(2\sqrt{M\lambda+\frac{M}{4m}}\right)\mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E}).$$

Thus, we obtain

$$|SR_{\lambda,\tau}(f\chi_K) - SR_{\lambda,\tau}(f\chi_E)| \le \left(4\sqrt{M\lambda+\frac{M}{4m}}\right)\mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E}),$$

which completes the proof. □

Proof of Lemma 3.7: Fix $x\in\mathbb{R}^n$. For every $\delta > \mathrm{dist}_{\mathcal{H}}(\Gamma_{f_K},\Gamma_{f_E})$ we have, by definition, $d_{\lambda,f_K}(x,\Gamma_{f_K}) = \inf_{y\in K}\left(|y-x| - \sqrt{\frac{f_K(y)}{\lambda}}\right)$.

Then for all $x^K \in K$, $\quad d_{\lambda,f_K}(x,\varGamma_{f_K}) \le |x^K - x| - \sqrt{\dfrac{f_K(x^K)}{\lambda}}$.

Also from the definition of $d_{\lambda,f_E}(x,\varGamma_{f_E})$ we have that for all $\epsilon > 0$ there exists $x^E \in E$ such that

$$|x^E - x| - \sqrt{\frac{f_E(x^E)}{\lambda}} < d_{\lambda,f_E}(x,\varGamma_{f_E}) + \epsilon.$$

Then for $(x^E, f_E(x^E)) \in \varGamma_{f_E}$, there exists $(\overline{x}^K, f_K(\overline{x}^K)) \in \varGamma_{f_K}$ such that

$$\big\|(\overline{x}^K, f_K(\overline{x}^K)) - (x^E, f_E(x^E))\big\|_2 < \delta. \tag{48}$$

Thus

$$\begin{aligned} d_{\lambda,f_K}(x,\varGamma_{f_K}) - d_{\lambda,f_E}(x,\varGamma_{f_E}) &\le |\overline{x}^K - x| - \sqrt{\frac{f_K(\overline{x}^K)}{\lambda}} - \left(|x^E - x| - \sqrt{\frac{f_E(x^E)}{\lambda}} - \epsilon\right), \\ &\le |\overline{x}^K - x^E| + \frac{1}{2\sqrt{m\lambda}}\left|f_K(\overline{x}^K) - f_E(x^E)\right| + \epsilon. \end{aligned}$$

By using the Cauchy-Schwarz inequality and the inequality (48), we have

$$\begin{aligned} d_{\lambda,f_K}(x,\varGamma_{f_K}) - d_{\lambda,f_E}(x,\varGamma_{f_E}) &\le \sqrt{1+\frac{1}{4\lambda m}}\ \sqrt{|\overline{x}^K - x^E|^2 + \left|f_K(\overline{x}^K) - f_E(x^E)\right|^2} + \epsilon \\ &\le \sqrt{1+\frac{1}{4\lambda m}}\ \big\|(\overline{x}^K, f_K(\overline{x}^K)) - (x^E, f_E(x^E))\big\|_2 + \epsilon \le \sqrt{1+\frac{1}{4\lambda m}}\ \delta + \epsilon, \end{aligned}$$

for all $\epsilon > 0$ and for all $\delta > \mathrm{dist}_{\mathcal{H}}(\varGamma_{f_K}, \varGamma_{f_E})$. Then by letting $\epsilon \to 0+$ and $\delta \to \mathrm{dist}_{\mathcal{H}}(\varGamma_{f_K}, \varGamma_{f_E})$ in the inequalities above, we have

$$d_{\lambda,f_K}(x,\varGamma_{f_K}) - d_{\lambda,f_E}(x,\varGamma_{f_E}) \le \sqrt{1+\frac{1}{4\lambda m}}\ \mathrm{dist}_{\mathcal{H}}(\varGamma_{f_K}, \varGamma_{f_E}).$$

Similarly, we can show that

$$d_{\lambda,f_E}(x,\varGamma_{f_E}) - d_{\lambda,f_K}(x,\varGamma_{f_K}) \le \sqrt{1+\frac{1}{4\lambda m}}\ \mathrm{dist}_{\mathcal{H}}(\varGamma_{f_K}, \varGamma_{f_E}),$$

and conclusion then follows. □

Proof of Theorem 1.8 : The proof is the same as the proof of Theorem 1.7. □

The next example shows that we do need the Hausdorff distance between graphs of sampled functions when the underlying function is not continuous. If the function is uniformly continuous [23, Theorem 4.10] we can use the Hausdorff distance between the sampled sets to measure stability. If a function is not continuous, we cannot use the Hausdorff distance between sets only to measure the difference of the upper transforms obtained from two different samples.

Example 3.8. Let $\quad f(x) = \begin{cases} 2, & \text{for } x \notin \mathbb{Q}, \\ 1, & \text{for } x \in \mathbb{Q}. \end{cases}$

We will take two samples sets in $\mathbb{R}$ depending on n, say E_n and F_n such that for any $C > 0$, when n is large $C\mathrm{dist}_{\mathcal{H}}(E_n, F_n) < |\mathcal{C}^u_\lambda(f\chi_{F_n})(x) - \mathcal{C}^u_\lambda(f\chi_{E_n})(x)|$.

Therefore for the discontinuous case we cannot use the Hausdorff distance between sets to bound the difference of the upper transforms taking from two sampled sets. Consider E_n be the set of rational numbers $E_n = \{\frac{k}{n} : k = 0, \cdots, n\}$ and F_n be the set of irrational numbers $F_n = \{\frac{k}{n} + \frac{\sqrt{2}}{4n} : k = 0, \cdots, n-1\}$. Then we conclude that $\mathrm{dist}_{\mathcal{H}}(E_n, F_n) \leq \frac{1}{n\sqrt{2}}$. Since the E_n and F_n are closed to each other, then $\mathrm{dist}_{\mathcal{H}}(E_n, F_n)$ tends to zero as $n \to \infty$. Next we use the upper transforms of characteristic function of $\{\frac{-1}{2n}, \frac{1}{2n}\}$ to calculate the upper transform for $f\chi_{E_n}$ in $[0,1]$ and for $f\chi_{F_n}$ in $[\frac{\sqrt{2}}{4n}, \frac{n-1}{n} + \frac{\sqrt{2}}{4n}]$. It is easy to see that

$$\mathcal{C}^u_\lambda\left(\chi_{\{\frac{-1}{2n}, \frac{1}{2n}\}}\right)(0) = 1 - \frac{\lambda}{4n^2}.$$

For $x \in [0,1]$, $\mathcal{C}^u_\lambda(f\chi_{F_n})(x) \geq 2 - \frac{\lambda}{4n^2}$ in $[\frac{\sqrt{2}}{4n}, \frac{n-1}{n} + \frac{\sqrt{2}}{4n}]$, $\mathcal{C}^u_\lambda(f\chi_{E_n})(x) \leq 1$ in $[0,1]$, we then have the following difference

$$\mathcal{C}^u_\lambda(f\chi_{F_n})(x) - \mathcal{C}^u_\lambda(f\chi_{E_n})(x) \geq 2 - \frac{\lambda}{4n^2} - 1 = 1 - \frac{\lambda}{4n^2}.$$

However for $C > 0$ when n is large, as the Hausdorff distance $\mathrm{dist}_{\mathcal{H}}(E_n, F_n) \leq \frac{1}{n\sqrt{2}}$, we have

$$C\mathrm{dist}_{\mathcal{H}}(E_n, F_n) \leq \frac{C}{n\sqrt{2}} < 1 - \frac{\lambda}{4n^2} \leq \left|\mathcal{C}^u_\lambda(f\chi_{F_n})(x) - \mathcal{C}^u_\lambda(f\chi_{E_n})(x)\right|.$$

Therefore, we use the Hausdorff distance of the graph of the sampled function defined on E_n and F_n, that means, $\Gamma(f_{E_n}) = \{(\frac{k}{n}, 1)\}$, $\Gamma(f_{F_n}) = \{(\frac{k}{n} + \frac{\sqrt{2}}{4n}, 2)\}$, $\mathrm{dist}_{\mathcal{H}}(\mathit{\Gamma}_{f_{F_n}}, \mathit{\Gamma}_{f_{E_n}}) \geq 1$. Then the estimate (8) in Theorem 1.8 using the Hausdorff distance between graphs is necessary as follows:

$$|\mathcal{C}^u_\lambda(f\chi_{F_n})(x) - \mathcal{C}^u_\lambda(f\chi_{E_n})(x)| \leq \left(2\sqrt{2\lambda + \frac{1}{2}}\right)\mathrm{dist}_{\mathcal{H}}(\mathit{\Gamma}_{f_{F_n}}, \mathit{\Gamma}_{f_{E_n}}).$$

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