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MVT, Integration, and Monotonicity of a Generalized Subdifferential in Locally Convex Spaces*

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We extend Zagrodny's Mean Value Theorem (MVT) to the class of weakly lower semicontinuous epi-pointed functions defined in locally convex spaces. With this at hand we extend the integration of generalized subdifferentials and the characterization of convexity via the monotonicity of the subdifferential.

Keywords: Approximate Mean Value Theorem, integration of subdifferentials, monotonicity of the subdifferential, epi-pointed functions, generalized subdifferentials.

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1. Introduction

One of the most important results in the theory of subdifferentials is Zagrodny's Mean Value Theorem. This theorem can be used to prove many abstract and important results in nonsmooth analysis in Banach spaces. Having such a powerful tool outside this framework makes it possible to extend many fruitful results of the theory to locally convex spaces (lcs). In this line, we extend Zagrodny's Mean Value Theorem to the class of weakly lower semicontinuous (lsc) epi-pointed functions defined in a linear space with a locally convex topology, which is generated by a family of Banach spaces. With this we extend two other important results, which are the integration of generalized subdifferentials and the characterization of convexity via the monotonicity of the subdifferential mapping.

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Convex analysis is one of the most important topics in nonsmooth analysis, in fact it is the starting point. The development of this theory has been fruitful, with multiple results in different areas of applied mathematics, especially in nonsmooth optimization, where the creation of the convex subdifferential provides many powerful techniques to study the behavior of minima of constrained optimization problems. However, new mathematical models have imposed the necessity to consider nonconvex data, leading many authors to propose new kinds of subdifferentials, such as the *Clarke*, the *approximate*, the *Ioffe*, the *limiting*, the *Dini-Hadamard*, the *Fréchet*, the *proximal subdifferentials* among others. Since the discovery of variational principles in Banach spaces, and examples which show the vacuity of the subdifferential of a proper convex lsc function defined in a lcs (see, e.g., [3]), the theory of subdifferentials has centered its attention on the setting of Banach spaces (and for some of the subdifferentials) with appropriate smooth properties for each subdifferential.

The aim of this paper is to explore *subdifferential techniques* to work outside the framework of Banach spaces. The example mentioned above of proper convex lsc functions with empty subdifferential everywhere shows that this task is impossible for all lsc functions. So, our efforts are concentrated on providing these techniques for the class of convex and nonconvex epi-pointed functions in the settings of arbitrary locally convex spaces.

To do this, we focus our attention on the subdifferentials previously mentioned; namely, the construction of the approximate subdifferential of Ioffe (see, for example, [15, 16, 17, 18]), and the ε-enlargement of the Fréchet subdifferential (see also [17] for the ε-enlargement of the subdifferential of Dini-Hadamard), in order to introduce our notion of an abstract subdifferential adopted in this work. The notion of ε-enlargements led to the definition of the limiting subdifferential, which obeys many suitable calculus rules in the setting of arbitrary Banach space (not necessarily an Asplund space) see for example [23, §I]. The Approximate and the limiting subdifferentials motivate us to introduce the concept of families of subdifferentials that are the main ingredient of this work, and which are shown to include most of classical subdifferentials.

This work is organized as follows. In Section 2 we give classical notation of convex analysis used in this work. The main results of the paper are established in Section 3, including the definition of the family of subdifferentials. Before the extension of Zagrodny's MVT, given in Theorem 3.8, we present two lemmas that can be seen as variational principles for epi-pointed lsc functions defined in locally convex spaces. As a simple consequence, we give in Corollary 3.10 a Brøndsted-Rockafellar-like Theorem for our family of subdifferentials, outside the framework of Banach spaces. In the same spirit, we obtain a generalization of Thibault-Zagrodny's integration theorem, which was given previously in [32, Theorem 2.1] in the framework of Banach spaces. To show the potential applications of the last result, we finish the paper by giving a short proof of three results available in the literature, two of which are recent results referring to integration of subdifferentials (see [22, Theorem 1.2] and [10, Theorem 4.8.]) and the last one is a characterization of the convexity by means of the monotonicity of the subdifferential (see [12, Theorem 2.2 and 2.4]).

2. Notation

In the following, X and X^* will be two (separated) locally convex spaces (lcs) in duality by the bilinear symmetric form $\langle\cdot,\cdot\rangle\colon X^*\times X\to\mathbb{R}$, given by $\langle x^*,x\rangle=\langle x,x^*\rangle=x^*(x)$. The weak topology in X and the weak* topology in X^* are denoted by $w(X,X^*)$ and $w(X^*,X)$, respectively (w and w^*, for short), while the Mackey topology is denoted by $\tau(X,X^*)$ and $\tau(X^*,X)$, respectively; the initial topology in X is denoted by τ_X, or simply τ (hence, $w(X,X^*)\subseteq\tau\subseteq\tau(X,X^*)$).

The associated convergence is written $\to^{\tau}$, while $\xrightarrow{f}{}^{\tau}$ (or, simply, $\xrightarrow{f}$ where no confusion occurs) means the graphical convergence. For a point $x\in X$ ($x^*\in X^*$, resp.), $\mathcal{N}_x(\tau_X)$ ($\mathcal{N}_{x^*}(\tau_{X^*})$, resp.) represents the neighbourhoods system of x (resp. x^*) with respect to the topology τ_X (resp. τ_{X^*}); we omit the symbols τ_X and τ_{X^*} when there is no confusion. Given a *pre-ordered* set $(\Re,\preceq)$, a subset $\Im$ of $\Re$ is said to be *cofinal* if for every $r\in\Re$ there exist $s\in\Im$ such that $r\preceq s$.

For a set $A\subseteq X$ (or X^*), we denote by $\operatorname{Int}(A)$, $\overline{A}$, $\operatorname{co}(A)$ and $\overline{\operatorname{co}}(A)$, the interior, the closure, the *convex hull* and the *closed convex hull* of A, respectively. The *polar* of A is the set $A^o:=\{x^*\in X^*\mid\langle x^*,x\rangle\le 1,\forall x\in A\}$. Given a convex function $\rho\colon X\to\mathbb{R}_+$, $x\in X$ and $r\ge 0$, we denote $\mathbb{B}_\rho:=\{y\in X:\rho(y)\le 1\}$. For two seminorms ρ_1 and ρ_2 on X, the inequality $\rho_1\le\rho_2$ means that $\rho_1(x)\le\rho_2(x)$ for all $x\in X$. For a seminorm ρ and a set $A\subset X$, we define the distance from x to A with respect to ρ as $d_A^\rho(x):=\inf_{z\in A}\rho(x-z)$.

Given a function $f\colon X\to\mathbb{R}\cup\{+\infty\}$, the (effective) *domain* of f is the set $\operatorname{dom} f:=\{x\in X\mid f(x)<+\infty\}$. We say that f is *proper* if $\operatorname{dom} f\neq\emptyset$ and $f>-\infty$, and *inf-compact* if for every $\lambda\in\mathbb{R}$ the sublevel set $\{x\in X\mid f(x)\le\lambda\}$ is compact. The *conjugate* of a proper function f is the function $f^*\colon X^*\to\mathbb{R}\cup\{+\infty\}$ defined by $f^*(x^*):=\sup_{x\in X}\{\langle x^*,x\rangle-f(x)\}$, and the *biconjugate* of f is $f^{**}:=(f^*)^*\colon X\to\mathbb{R}\cup\{+\infty\}$. The *inf-convolution* of two functions $f,g\colon X\to\mathbb{R}\cup\{+\infty\}$ is the function $f\square g:=\inf_{z\in X}\{f(z)+g(\cdot-z)\}$. The *indicator* and the *support* functions of a set A ($\subset X,X^*$) are, respectively,

$$\delta_A(x):=\begin{cases}0 & x\in A\\ +\infty & x\notin A,\end{cases}\qquad\qquad \sigma_A:=\delta_A^*.$$

The *ε-subdifferential* of f at a point $x\in\operatorname{dom} f$, for $\varepsilon\ge 0$, is the set

$$\partial_\varepsilon f(x):=\{x^*\in X^*\mid\langle x^*,y-x\rangle\le f(y)-f(x)+\varepsilon\ \forall y\in X\};$$

if $f(x)$ is not finite, we set $\partial_\varepsilon f(x):=\emptyset$. The special case $\varepsilon=0$ corresponds to the classical convex subdifferential $\partial f:=\partial_0 f$, aslo knwon as the *Fenchel-Moreau-Rockafellar subdifferential*. For a set $A\subseteq X$ (or X^*) and a point $x\in X$, the normal cone to A at x is defined by $N_A(x):=\partial\delta_A(x)$. The directional derivative of a convex lower semi-continuous (lsc, for short) function f at $x\in\operatorname{dom} f$ in the direction u is given by $f'(x;u) := \inf_{s>0}(f(x+su)-f(x))/s$. Recall that $f'(x;\cdot)$ is a positively homogeneous convex function and $\partial f(x) = \partial f'(x;\cdot)(0)$. We denote by $\Gamma_0(X)$, the set of proper, convex and lower semicontinuous functions. For a multifunction $M : X \rightrightarrows X^*$, we denote its domain by $\operatorname{dom} M := \{x \in X : M(x) \neq \emptyset\}$ and its graph by $\operatorname{gph} M := \{(x,x^*) \in X \times X^* : x^* \in M(x)\}$. The inverse of M is the multifunction $M^{-1} : X^* \rightrightarrows X$ given by $M^{-1}(x^*) := \{x \in X : x^* \in M(x)\}$. A multifunciton M is said to be *monotone* if for all $(x,x^*),(y,y^*) \in \operatorname{gph} M$, $\langle x^* - y^*, x - y\rangle \geq 0$.

Finally, we present the definition of an epi-pointed function in lcs (for more details about this class of functions, see [6, 10, 7, 9, 8, 26, 27, 1] and the references therein).

Definition 2.1. A function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is said to be *epi-pointed* if f^* is proper and $\tau(X^*, X)$-continuous at some point of its domain. □

3. Subdifferentials on locally convex spaces

In this section, X and X^* are as before two (separated) locally convex spaces (lcs) in duality. We introduce a new family of quasi-presubdifferentials, that adapts and extends the one in [30]. This familiy is motivated by the definition of the approximate subdifferential given by Ioffe [16], where the author uses reductions to finite-dimensional subspaces and to suitable subspaces with nice properties for the Dini-Hadammard subdifferential (weak trustworthy spaces), together with ε-enlargements of the Dini-Hadammard subdifferential. The ε-enlargements are also used in the definition of the limiting/Mordukhovich subdifferential in the settings of arbitrary Banach spaces (see, for example, [23, §I]).

We will simply call this object subdifferential instead of quasi-presubdifferential, as in [30]. In the following definition, given directed set $(\mathbb{I}, \preceq)$ and directed (by inclusion) family of spaces $\mathcal{L}$ of X, the set $\mathbb{I} \times \mathcal{L}$ is a directed set with respect to the preorder $(i_1, L_1) \leq (i_2, L_2)$ iff $i_1 \preceq i_2$ and $L_1 \subseteq L_2$.

Definition 3.1. Assume that the family of spaces $\mathcal{L}$ covers X. Given a proper function $f\colon X \to \mathbb{R} \cup \{+\infty\}$, we say that a net of multi-valued operators $\{\hat{\partial}_{i,L} : i \in \mathbb{I},\ L \in \mathcal{L}\}$ from X to X^* is a *family of subdifferentials for f* if:

(a) $\hat{\partial}_{i,L} f(x) = \emptyset$ for all $x \notin \operatorname{dom} f$, $i \in \mathbb{I}$ and $L \in \mathcal{L}$,

(b) for every $L \in \mathcal{L}$ and every Lipschitz convex function $g\colon X \to \mathbb{R}$, if x_0 is a local minimum of $f + g$ relative to L, then there exist net $(x_\alpha, x_\alpha^*)_{\alpha \in \mathbb{D}} \in X \times X^*$ and subnet $(\hat{\partial}_{i_\alpha,L})_{\alpha \in \mathbb{D}}$ of $(\hat{\partial}_{i,L})_{i \in \mathbb{I}}$ such that $x_\alpha \in L$, $x_\alpha^* \in \hat{\partial}_{i_\alpha,L} f(x_\alpha)$, $x_\alpha \xrightarrow{f} x_0$, $x_\alpha^* \xrightarrow{w^*} x_0^*$, $x_0^* \in -\partial g(x_0) + L^\perp$ and $\liminf_\alpha \langle x_\alpha^*, x_0 - x_\alpha\rangle \geq 0$.
When $\hat{\partial}_{i,L}$ is independent of i and L, say $\hat{\partial}_{i,L} \equiv \hat{\partial}$, we simply say that $\hat{\partial}$ is a subdiferential of the function f. □

Property (b) is a kind of fuzzy controlled sum rule (see [19] and the reference therein for more details and discussions about these properties). It is not difficult to see that all the subdifferentials mentioned in the introduction satisfy Definition 3.1, when the function f is defined in a Banach space with appropriate smooth properties; for example, the G-subdifferential of Ioffe and the Clarke subdifferential in all Banach spaces, the Fréchet subdifferential and the limiting/Mordukovich subdifferential in Asplund spaces, the Dini-Hadammard subdifferential in Banach spaces with Gâteaux differentiable renorm, the proximal subdifferential in Hilbert spaces, etc.

Example 3.2. (i) The convex subdifferential and the ε-subdifferential are subdifferentials for every proper convex lsc function in any lcs.

(ii) Recall that the Clarke subdifferential of an lsc function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\partial_C f(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq f^{\uparrow}(x; u),\ \forall h \in X\},$$

where $f^{\uparrow}(x; u)$ is the Clarke-Rockafellar directional derivative or the lower subderivative of f at x with respect to u (see, e.g., [29]),

$$f^{\uparrow}(x; v) = \limsup_{\substack{x' \xrightarrow{f} x \\ t \to 0^+}} \inf_{v' \to v} \frac{f(x' + tv') - f(x')}{t}.$$

Then ∂_C is a subdifferential for every proper lsc in any lcs.

(iii) The approximate subdifferential, introduced by Ioffe (see, e.g., [15, 16, 17]), is given by

$$\partial_A f(x) = \bigcap_{L \in \mathcal{F}} \limsup_{u \xrightarrow{f} x} \partial^- f_{u+L}(u),$$

where $f_{u+L} := f + \delta_{u+L}$, $\mathcal{F}$ denotes the collection of all finite-dimensional subspaces of X, ∂^- denotes the Dini-subdifferential, and the lim sup is with respect to the weak* topology in X^*. So, for any proper lsc function defined in a Banach space, ∂_A is a subdifferential of f, and $\{\partial^-_L\}_{L\in\mathcal{F}}$ defined as $\partial^-_L f(x) := \partial^- f_{x+L}(x)$, is a family of subdifferentials of f. Also, for any admissible family $\mathcal{L}$ of weak trustworthy spaces of X (see, e.g., [16]), the family $\{\partial^-_{\varepsilon,L} : \varepsilon > 0; L \in \mathcal{F}\}$ defined as $\partial^-_{\varepsilon,L} f(x) := \partial^-_{\varepsilon} f_{x+L}(x)$, with ∂^-_{ε} being the ε-Dini subdifferential, is a family of subdifferentials for every proper lsc function in any lcs.

(iv) Given a subdifferential ∂ in the sense of [18, Definition 2.1.], and a family of subspaces $\mathcal{L}$ of X such that ∂ is trusted on $\mathcal{L}$ (see [18, Definition 2.12.]) and $\mathcal{L}$ covers X (for example $\mathcal{L} = \mathcal{F}$), the family $\{\partial_L : L \in \mathcal{F}\}$, defined as $\partial_L f(x) := \partial f_{x+L}(x)$, is a family of subdifferentials.

The next result is a kind of a diagonal argument but for generalized sequences.

Lemma 3.3. [20, §2, Theorem 4, p. 69] *Given directed sets D and E_d, $d \in D$, we denote $F := D \times \Pi_{d\in D} E_d$. If $\{S_{(d,e)} : d \in D,\ e \in E_d\} \subset U$ for some topological space U, then $\lim_{(d,f)\in F} S_{(d,f(d))} = \lim_{d\in D} \lim_{e\in E_d} S(d, e)$ whenever this iterated limit exists.*

Remark 3.4. Observe that the previous argument is also valid for upper and lower limits: namely, if $S_{(d,e)}$ are real numbers such that we have for every $d \in D$,

$\liminf_{e \in E_d} S_{(d,e)} \geq S_d$ for some $S_d \in R$, then $\liminf_{(d,f)\in F} S_{(d,f(d))} \geq \liminf_d S_d$. This fact can be easily proved using the lower topology on $\mathbb{R} \cup \{+\infty\}$, which is generated by the base of sets of the form $A_a := \{r : r > a\}$ with $a \in [-\infty, +\infty)$. Then it is easy to prove that for a net $(s_\alpha)_\alpha$ we have that $s_\alpha \to s$ if and only if $\liminf s_\alpha \geq s$ (see, e.g., [20, §3, Exercise F]).

The following lemma is known (e.g. [2]).

Lemma 3.5. *Consider a convex and closed set $C \subseteq X$ and a continuous seminorm $p\colon X \to \mathbb{R}_+$. Then the function $d_C^p\colon X \to [0, +\infty)$ defined as $d_C^p(x) := \inf_{z\in C} p(x-z)$, is convex, and p-Lipschitz; that is, $d_C^p(x) - d_C^p(y) \leq p(x-y)$ for all $x, y \in X$. In addition, if for a given $x \in X$ there exists $\bar{z} \in C$ such that $d_C^p(x) = p(x-\bar{z})$, then $\partial d_C^p(x) \subseteq \partial p(x-\bar{z}) \cap N_C(\bar{z}) \subseteq \mathbb{B}_p^\circ \cap N_C(\bar{z})$.*

Two important tools are needed to develop subdifferential theory; one of them is an appropriate definition of such subdifferentials, and the second one is the validity of some variational-like principles. In the setting of Banach spaces, these techniques have been well studied in the last fifty years for lsc functions, such as Ekeland's variational principle, Borwein-Preiss's variational principle, Deville-Godefroy-Zizler's variational principle and the smooth variational principle in Asplund spaces (see, e.g., [2, 23, 13]). Outside the setting of Banach spaces it is not possible to generalize these tools for all lsc functions. For this reason, the family of w-lsc epi-pointed functions appears to be promising to develop techniques of variational analysis in lcs spaces, (see [7], for an extension of classical results of convex analysis from Banach spaces to epi-pointed functions in lcs spaces).

Lemma 3.7 below is a kind of a variational principle for the class of w-lsc epi-pointed functions, which is valid in general locally convex spaces. We need the following technical result.

Lemma 3.6. *A lsc proper convex function $g\colon X^* \to \mathbb{R} \cup \{+\infty\}$ is $\tau(X^*, X)$-continuous at $x^* \in \operatorname{dom} g$ if and only if $g^* - x^*$ is w-inf-compact. Consequently, for every w-lsc epi-pointed function f, the functions $f - x^*$, $x^* \in \operatorname{Int}(\operatorname{dom} f^*)$, are w-inf-compact.*

Proof. The first part is a classical result in convex analysis (see, e.g., [21, 24]). The second part is an easy application of the first statement: if f is w-lsc and epi-pointed, then the function $f^{**} - x^*$ is $w(X, X^*)$-inf-compact for every $x^* \in \operatorname{Int}(\operatorname{dom} f^*)$. So, since $f^{**} \leq f$, we deduce that $\{f - x^* \leq \lambda\} \subseteq \{f^{**} - x^* \leq \lambda\}$ for every $\lambda \in \mathbb{R}$, which proves that the function $f - x^*$ is w-inf-compact. $\square$

Lemma 3.7. *Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a w-lsc and epi-pointed function, $a, b \in X$, and $x_0^* \in X^*$. Then there exists a continuous seminorm ρ_0 such that, for all seminorm $\rho \geq \rho_0$ and all $M \geq 1$, the function $H\colon X \to \mathbb{R} \cup \{+\infty\}$ defined as $H(x) := f(x) - \langle x_0^*, x\rangle + M d_{[a,b]}^\rho(x)$ is w-inf-compact. Consequently, if x is an ε^2-minimun of the function H, then for every nonnegative lsc convex function g there exists $x_\varepsilon \in X$, a minimum of the function $H(\cdot) + \varepsilon g(\cdot - x_\varepsilon)$ such that $g(x_\varepsilon - x) \leq \varepsilon$.*

Proof. Given $M \geq 1$, we choose $y_0^* \in \mathrm{Int}(\mathrm{dom}\, f^*)$ together with the the seminorm $\rho_0(x) := |\langle y_0^* - x_0^*, x\rangle|$. So, for all $c, x \in X$

$$\begin{aligned} f(x) - \langle x_0^*, x\rangle + M\rho_0(x-c) &\geq f(x) - \langle x_0^*, x\rangle + \rho_0(x-c) \\ &\geq f(x) - \langle x_0^*, x\rangle + \langle x_0^* - y_0^*, x - c\rangle = f(x) - \langle y_0^*, x\rangle - \langle x_0^* - y_0^*, c\rangle. \end{aligned}$$

Therefore, for every $c \in X$ the function $f - \langle x_0^*, \cdot\rangle + M\rho_0(\cdot - c)$ is w-inf-compact, by Lemma 3.6; hence, for all seminorm $\rho \geq \rho_0$, $f - \langle x_0^*, \cdot\rangle + M\rho(\cdot - c)$ is w-inf-compact. Now, we show that the function H is w-inf-compact. Indeed, consider $\alpha \in \mathbb{R}$ and net $(x_d)_{d\in D} \subset \{x \in X : H(x) \leq \alpha\}$. Since $[a, b]$ is compact, for all $d \in D$ there exists $u_d \in [a, b]$ such that $d^{\rho}_{[a,b]}(x_d) = \rho(x_d - u_d)$. Moreover, by taking a subnet if necessary, we can assume that $u_d \to u \in [a, b]$. We pick $d_0 \in D$ such that, for all $d \geq d_0$, $\rho(u_d - u) \leq 1$. Then for all such d we have that $x_d \in \{x \in X : f(x) - \langle x_0^*, x\rangle + M\rho(x - u_d) \leq \alpha\} \subset \{x \in X : f(x) - \langle x_0^*, x\rangle + M\rho(x - u) \leq \alpha + M\}$. Since this last set is w-compact, as we have showed above, (x_d) possesses a weak accumulation point that belongs to the set $\{x \in X : H(x) \leq \alpha\}$; in other words, H is w-inf-compact.

The second part of the lemma follows from the fact that the function $H(\cdot) + \varepsilon g(\cdot - c)$ is also w-lsc and w-inf-compact, so that $H(\cdot) + \varepsilon g(\cdot - x)$ attains its minimum and we obtain that $g(x_\varepsilon - x) \leq \varepsilon$. □

In the next result, we extend the Zagrodny's Mean Value Theorem of [25] to the setting of w-lsc and epi-pointed functions defined in arbitrary locally convex spaces, by using the current subdifferential. The proof of is an adaptation of [30], which is itself an adaptation of the original work of Zagrodny [34].

Theorem 3.8. *Consider a family of subdifferentials $\{\hat{\partial}_{i,L} : i \in \mathbb{I},\ L \in \mathcal{L}\}$ for a given proper lsc function f. Assume that one of the following conditions holds:*

(a) *The topology on X is generated by a family of seminorms $(\rho_L)_{L\in\mathcal{L}}$, where for every $L \in \mathcal{L}$, (L, ρ_L) is a Banach space, and $\rho_M \leq \rho_L$ for all $M \subseteq L$.*

(b) *f is a w-lsc and epi-pointed function.*

Then, for every $a, b \in X$ with $a \in \mathrm{dom}\, f$ and $a \neq b$, $r \in \mathbb{R}$ with $r \leq f(b)$ and every continuous seminorm p such that $p(a - b) \neq 0$, there exist $c \in [a, b[$ and $(x_\alpha, x_\alpha^*)_{\alpha\in\mathbb{A}} \in X \times X^*$ such that $\{(i_\alpha, L_\alpha)\}_{\alpha\in\mathbb{A}}$ is cofinal in $\mathbb{I} \times \mathcal{L}$, $(x_\alpha)_{\alpha\in\mathbb{A}} \xrightarrow{f} c$, $x_\alpha^* \in \hat{\partial}_{i_\alpha, L_\alpha} f(x_\alpha)$ with $x_\alpha \in L_\alpha$, and

(i) $r - f(a) \leq \liminf_\alpha \langle b - a, x_\alpha^*\rangle$, (ii) $0 \leq \liminf_\alpha \langle c - x_\alpha, x_\alpha^*\rangle$,

(iii) $\frac{p(b-c)}{p(b-a)}(r - f(a)) \leq \liminf_\alpha \langle b - x_\alpha, x_\alpha^*\rangle$, (iv) $\frac{p(b-a)}{p(c-a)}(f(c) - f(a)) \leq (r - f(a))$.

Moreover, if $c \neq a$, then one has $\lim_\alpha \langle b - a, x_\alpha^*\rangle = 0$.

Proof. Take $x^* \in X^*$ such that $\langle b - a, x^*\rangle = r - f(a)$, and consider the function $h = f - x^*$; hence, $h(a) \leq h(b)$. Because h is lsc, there exists $c \in [a, b[$ such that $h(c) \leq h(x)$, for all $x \in [a, b]$. Take $\bar{\lambda} \in [0, 1[$ such that $c = (1 - \bar{\lambda})a + \bar{\lambda} b$. It follows that $c - a = \bar{\lambda}(b - a)$ and $p(c - a) = \bar{\lambda} p(b - a)$; hence, (iv) follows from

$$f(c) - f(a) = h(c) - h(a) + \langle c - a, x^*\rangle \leq \bar{\lambda}(r - f(a)).$$

Let $\gamma < h(c)$, so that from the lower semicontinuity of h and the compactness of the closed segment $[a,b]$, we can find (a convex, closed and balanced neighborhood) $U_\gamma \in \mathcal{N}_0$ such that $\gamma < h(x)$ for all $x \in [a,b] + U_\gamma$.

In what follows, we choose ρ_0 as in Lemma 3.7 (associated with f, a, b and x^*) if f is a w-lsc and epi-pointed function, and $\rho_0 = 0$, otherwise. From the previous paragraph, there are $U_n \in \mathcal{N}_0$, $n \geq 1$, such that

$$U_{n+1} \subseteq U_n \subset \mathbb{B}_{p+\rho_0} = \{x \in X : p(x) + \rho_0(x) \leq 1\}$$

(without loss of generality) and

$$h(c) - \frac{1}{n^2} < h(x) \quad \forall n \geq 1 \text{ and } \forall x \in [a,b] + U_n.$$

We denote $\rho_n := \sigma_{U_n^\circ}$ $(\geq \rho_0)$ and $d^n_{[a,b]}(x) := d^{\rho_n}_{[a,b]}(x)$, so that

$$h(c) \leq h(x) + d^n_{[a,b]}(x) + \frac{1}{n^2}, \quad \forall x \in U := [a,b] + U_1; \tag{1}$$

this inequality trivially holds if $x \in [a,b] + U_n$, while for $x \in U \setminus ([a,b] + U_n)$, we have that $d^n_{[a,b]}(x) \geq 1$ and, so,

$$h(x) + d^n_{[a,b]}(x) \geq h(c) - 1 + 1 \geq h(c) - \frac{1}{n^2}.$$

Now, we consider the functions $H_n(x) := h(x) + d^n_{[a,b]}(x) + \delta_U(x)$, $n \geq 1$, which satifies $H_n(c) \leq \inf_U H_n + \frac{1}{n^2}$, thanks to (1). To continue, we proceed by analyzing the alternative of the current theorem:

1. If we are in case (a), we may assume without loss of generality that the ρ_L's are such that $\mathbb{B}_{\rho_L} \subseteq \operatorname{Int}(U_1)$, $\forall L \in \mathcal{L}$.
   Then, for each $n \geq 1$ and $L \in \mathcal{L}$, by applying the Ekeland variational principle in the Banach space (L, ρ_L) we find a minimum $u \in L$ of the function $H_n(\cdot) + n^{-1}\rho_L(\cdot - u)$ on L such that $\rho_L(c-u) \leq 1/n$.
2. In the case of (b), we choose a family of seminorms $\{\rho_j\}_{\in \mathbb{J}}$ that generates the topology on X and such that $\mathbb{B}_{\rho_j} \subseteq \operatorname{Int}(U_1)$, $\forall j \in \mathbb{J}$, where $\mathbb{J}$ is an ordered set satisfying $\rho_{j_1} \leq \rho_{j_2}$ whenever $j_1 \leq j_2$. Then, by applying Lemma 3.7, for each $n \geq 1$, $j \in \mathbb{J}$, and $L \in \mathcal{L}$, we find a minimum $u \in L$ of the function $H_n(\cdot) + n^{-1}\rho_j(\cdot - c) + \delta_L$ such that $\rho_j(c-u) \leq 1/n$.

Consequently, in each one of the two cases above there exist an ordered set $\mathbb{J}$, a cofinal set $\mathbb{D} \subseteq \mathbb{N} \times \mathbb{J} \times \mathcal{L}$ and nets $(u_{n,j,L})_{(n,j,L)\in\mathbb{D}}$, $(v_{n,j,L})_{(n,j,L)\in\mathbb{D}}$ such that

$$p_j(u_{n,j,L} - c) \leq \frac{1}{n}, \; H_n(u_{n,j,L}) \leq H_n(c),$$

where $u_{n,j,L}$ is a minimum of $H_n + n^{-1}p_j(\cdot - v_{n,j,L})$; hence,

$$u_{n,j,L} \to c. \tag{2}$$

Moreover, since $\mathbb{B}_{\rho_j} \subseteq \operatorname{Int}(U_1)$, we obtain that $(u_{n,j,L}) \subset \operatorname{Int}(U)$, and this yields that each $u_{n,j,L}$ is a local minimum of the function $f - x^* + d^n_{[a,b]} + n^{-1}p_j(\cdot - v_{n,j,L})$ over L. Thus, by Definition 3.1, for each $\omega := (n,j,L) \in \mathbb{D}$ there exists a net $(x_{\alpha,\omega}, x^*_{\alpha,\omega})_{\alpha\in\Lambda_\omega} \subseteq X \times X^*$, together with elements

$$v^*_\omega \in \partial\, d^n_{[a,b]}(u_{n,j,L}) \text{ and } b^*_{n,j,L} \in (\mathbb{B}_{p_j})^\circ, \tag{3}$$

such that:

$$(A1)\;\; x^*_{\alpha,\omega} \in \hat{\partial}_{i_\alpha,L}\, f(x_{\alpha,\omega}), \quad (A2)\;\; u^*_{n,j,L} \in x^* - (v^*_{n,j,L} + \frac{1}{n} b^*_{n,j,L}) + L^\perp,$$

$$(A3)\;\; x_{\alpha,\omega} \xrightarrow{f} u_{n,j,L}, \quad (A4)\;\; x^*_{\alpha,\omega} \xrightarrow{w^*} u^*_{n,j,L}, \quad (A5)\;\; \liminf_\alpha \langle x^*_{\alpha,\omega}, u_{n,j,L} - x_{\alpha,\omega}\rangle \geq 0.$$

Observe that, due to (A4) and (A5)

$$\begin{aligned}\liminf_\alpha \langle c - x_{\alpha,\omega}, x^*_{\alpha,\omega}\rangle &\geq \liminf_\alpha \langle u_{n,j,L} - x_{\alpha,\omega}, x^*_{\alpha,\omega}\rangle + \liminf_\alpha \langle c - u_{n,j,L}, x^*_{\alpha,\omega}\rangle \\ &\geq \liminf_\alpha \langle c - u_{n,j,L}, x^*_{\alpha,\omega}\rangle = \langle c - u_{n,j,L}, u^*_{n,j,L}\rangle. \end{aligned}\tag{4}$$

Let $y_{n,j,L} \in [a,b]$ be such that $d^n_{[a,b]}(u_{n,j,L}) = \rho_n(u_{n,j,L} - y_{n,j,L})$; because $u_{n,j,L} \to_{n,j,L} c$ and $c \neq b$, we may suppose without loss of generality that $y_{n,j,L} \neq b$ for all $(n,j,L) \in \mathbb{D}$, and so we write $y_{n,j,L} = (1-\lambda_{n,j,L})a + \lambda_{n,j,L}b$ for some $\lambda_{n,\nu,L} \in [0,1[$. Moreover, by Lemma 3.5 we have that

$$\partial\, d^n_{[a,b]}(u_{n,j,L}) \subseteq \partial\,\rho_n(u_{n,j,L} - y_{n,j,L}) \cap N_{[a,b]}(y_{n,j,L}) \subseteq \mathbb{B}^\circ_{\rho_n} \cap N_{[a,b]}(y_{n,j,L}). \tag{5}$$

Then due to (3), $v^*_{n,j,L}$ satisfies

$$(\lambda - \lambda_{n,\nu,L})\langle b - a, v^*_{n,j,L}\rangle = \langle (1-\lambda)a + \lambda b - y_{n,j,L}, v^*_{n,j,L}\rangle \leq 0 \;\text{ for all } \lambda \in [0,1],$$

which yields

$$\langle b - a, v^*_{n,j,L}\rangle \leq 0; \tag{6}$$

moreover, when $y_{n,j,L} \neq a$, we have $\lambda_{n,\nu,L} \in]0,1[$ and the last inequality implies

$$\langle b - a, v^*_{n,j,L}\rangle = 0. \tag{7}$$

In particular, since $c \in [a,b[$ we also have (recall the first inclusion in (5))

$$\begin{aligned}\langle c - u_{n,j,L}, v^*_{n,j,L}\rangle &= \langle c - y_{n,j,L}, v^*_{n,j,L}\rangle - \langle u_{n,j,L} - y_{n,j,L}, v^*_{n,j,L}\rangle \\ &\leq \langle y_{n,j,L} - u_{n,j,L}, v^*_{n,j,L}\rangle \leq -\rho_n(u_{n,j,L} - y_{n,j,L}) \leq 0. \end{aligned}\tag{8}$$

We may assume that $a, b \in L$ for all $L \in \mathcal{L}$. Then, on the one hand, using (A2) and (6) we get

$$\begin{aligned}\langle b - a, u^*_{n,j,L}\rangle &= \langle b - a, x^*\rangle - \langle b - a, v^*_{n,j,L}\rangle - \frac{1}{n}\langle b - a, b^*_{n,j,L}\rangle \\ &\geq \langle b - a, x^*\rangle - \frac{1}{n}p_j(b-a) = r - f(a) - \frac{1}{n}p_j(b-a), \end{aligned}\tag{9}$$

and so, because $\lim \frac{1}{n} p_j(b-a) = 0$ (without loss of generality), we deduce that

$$\liminf_{n,j,L} \langle b-a, u^*_{n,j,L}\rangle \geq r - f(a). \tag{10}$$

On the other hand, by using again (A2) and the fact that $a, b, u_{n,j,L} \in L$, together with (8), we get

$$\begin{aligned}\langle c - u_{n,j,L}, u^*_{n,j,L}\rangle &= \langle c - u_{n,j,L}, x^*\rangle - \langle c - u_{n,j,L}, v^*_{n,j,L}\rangle - \frac{1}{n}\langle c - u_{n,j,L}, b^*_{n,j,L}\rangle \\ &\geq \langle c - u_{n,j,L}, x^*\rangle - \frac{1}{n} p_j(c - u_{n,j,L}) \geq \langle c - u_{n,j,L}, x^*\rangle - \frac{1}{n^2},\end{aligned}$$

which gives us

$$\liminf_{n,j,L} \langle c - u_{n,j,L}, u^*_{n,j,L}\rangle \geq 0. \tag{11}$$

Also, from the inequality $H_n(u_{n,j,L}) \leq H_n(c)$, we get

$$f(u_{n,j,L}) \leq f(c) - \langle c - u_{n,j,L}, x^*\rangle,$$

which implies that

$$f(c) \leq \liminf_{n,j,L} f(u_{n,j,L}) \leq \limsup_{n,j,L} f(u_{n,j,L}) \leq f(c),$$

and this together with (2) prove that $u_{n,j,L} \xrightarrow{f} c$.

Finally, we consider the set $\mathbb{A} = \mathbb{D} \times \Pi_{(n,j,L)\in\mathbb{D}} \Lambda(n,j,L)$ that we order by the relation

$$(\nu_1, T_1) \preceq (\nu_2, T_2) \text{ if and only if } \nu_1 \leq \nu_2 \text{ and } T_1(q) \leq T_1(q) \text{ for all } q \in D,$$

and define the net $\left(x_{q,T}, x^*_{q,T}\right)_{(q,T)\in\mathbb{A}}$ as $x_{q,T} := x_{T(q)}$ and $x^*_{q,T} := x^*_{T(q)}$. We are going to check that the previous net satisfies the conclusions of the theorem. By taking into account Lemma 3.3, from the paragraph above the following statements hold true:

(i) $x^*_{q,T} \in \hat{\partial}_{i_{T(q),L}} f(x_{q,T})$, due to (A1),

(ii) $x_{q,T} \xrightarrow{f}_{q,T} c$, thanks to (2) and (A3),

(iii) $\liminf_{q,T} \langle b-a, x^*_{q,T}\rangle \geq r - f(a)$, by (10) and (A4) (see, also, Remark 3.4),

(iv) $\liminf_{q,T} \langle c - x_{q,T}, x^*_{q,T}\rangle \geq 0$, by (4) and (11) (see Remark 3.4).

Moreover, since that $b - c = (1-\bar\lambda)(b-a)$, by using the last statements (iii) and (iv) we obtain, for every $p \geq p_0$,

$$\begin{aligned}\liminf_{q,T} \langle b - x_{q,T}, x^*_{q,T}\rangle &= \liminf_{q,T} \left((1-\bar\lambda)\langle b-a, x^*_{q,T}\rangle + \langle c - x_{q,T}, x^*_{q,T}\rangle\right) \\ &\geq (1-\bar\lambda) \liminf_{q,T} \langle b-a, x^*_{q,T}\rangle + \liminf_{q,T} \langle c - x_{q,T}, x^*_{q,T}\rangle \\ &\geq (1-\bar\lambda)(r - f(a)) = \frac{p(b-c)}{p(b-a)}(r - f(a)).\end{aligned}$$

The main statement of the theorem is proved. If $c \neq a$, then $y_{n,j,L} \neq a$ and by (7) we obtain that $\langle b-a, v^*_{n,j,L}\rangle = 0$, which by (9) gives us

$$\langle b-a, u^*_{n,j,L}\rangle \to r - f(a).$$

Then, using again Lemma 3.3 we also may suppose that $\lim_{q,T}\langle b-a, x^*_{q,T}\rangle = 0$. □

Remark 3.9. When the space X is metrizable and the sets $\mathcal{L}$ and $\mathbb{I}$ have countable cofinal sets, we can take sequences instead of nets in Theorem 3.8.

We obtain in the following corollary a Brøndsted-Rockafellar-like density theorem for our subdifferential. The counterpart to epi-pointed lsc convex functions has been proved in [7].

Corollary 3.10. *In the setting of Theorem* 3.8, *we fix* $x \in \operatorname{dom} f$. *Then there exists a net* $(x_\alpha, x^*_\alpha)_{\alpha\in\mathbb{A}} \subseteq X \times X^*$ *such that* $x^*_\alpha \in \hat{\partial}_{i_\alpha, L_\alpha} f(x_\alpha)$ $(\forall\alpha)$, $x_\alpha \xrightarrow{f} x$ *and* $\liminf_\alpha \langle x - x_\alpha, x^*_\alpha\rangle \geq 0$. *In particular,* $\operatorname{dom} f \subseteq \limsup_{i,L}(\operatorname{dom}\hat{\partial}_{i,L} f)$ *in the sense of Painlevé-Kuratowski.*

Proof. First, assume that x is a local minimum of f. Then, for each $L \in \mathcal{L}$, x is a local minimum of f relative to L and, so, there exist $x^*_L \in L^\perp$, net $(x_\beta, x^*_\beta)_{\beta\in\mathbb{D}_L} \subset X \times X^*$ and subnet of operators $(\partial_{i_\beta,L})_{\beta\in\mathbb{D}_L}$ of $(\partial_{i,L})_{i\in\mathbb{I}}$, such that

$$x^*_\beta \in \partial_{i_\beta,L} f(x_\beta),\ x_\beta \xrightarrow{f} x,\ x^*_\beta \xrightarrow{w^*} x^*_L \ \text{ and } \ \liminf_\beta \langle x - x_\beta, x^*_\beta\rangle \geq 0.$$

We denote $\mathbb{A} = \mathcal{L} \times \Pi_{L\in\mathcal{L}}\mathbb{D}_L$ and consider the nets $(y_{L,T}, y^*_{L,T})_{(L,T)\in\mathbb{A}}$ defined as $y_{L,T} = x_{T(L)}$ and $y^*_{L,T} = x^*_{T(L)}$. Then the conclusion follows by Lemma 3.3.

Now, we assume that x is not a local minimum of f. Then, because f is lsc, we can find a net $(x_\nu)_{\nu\in\mathcal{D}} \xrightarrow{f} x$ with $f(x_\nu) < f(x)$. By Theorem 3.8 applied with x_ν, x, $r = f(x)$ and a seminorm ρ_ν such that $\rho_\nu(x_\nu - x) \neq 0$, for each ν there exist $c_\nu \in [x_\nu, x[$ and net $(z_\beta, z^*_\beta)_{\beta\in\mathbb{D}(\nu)}$ such that $z^*_\beta \in \hat{\partial}_{L_\beta, i_\beta} f(z_\beta)$, $z_\beta \xrightarrow{f} c_\nu$, $0 < \frac{\rho_\nu(x-c_\nu)}{\rho_\nu(x-x_\nu)}(f(x) - f(x_\nu)) \leq \liminf_\beta \langle x - z_\beta, z^*_\beta\rangle$ and (because $\frac{\rho_\nu(c_\nu - x_\nu)}{\rho_\nu(x-x_\nu)} \leq 1$)

$$f(c_\nu) \leq f(x_\nu) + \frac{\rho_\nu(c_\nu - x_\nu)}{\rho_\nu(x - x_\nu)}(f(x) - f(x_\nu)) \leq f(x).$$

So, $\limsup_\nu f(c_\nu) \leq f(x)$ and by the lower semicontinuity of f and the fact that for every continuous seminorm p we have $p(x - c_\nu) \leq p(x - x_\nu) \to 0$, we conclude that $c_\nu \xrightarrow{f} x$. Finally, using Lemma 3.3, the conclusion follows by taking the ordered set $\mathbb{A} = \mathcal{D} \times \Pi_{\nu\in\mathcal{D}}\mathbb{D}(\nu)$, and defining $w_{\nu,T} = z_{T(\nu)}$, $w^*_{\nu,T} = z^*_{T(\nu)}$. □

The following result is due to Thibault-Zagrodny in the setting of Banach spaces [32, Theorem 2.1] (see, also, [31] and [33, Theorem 3.2] for more general approaches dealing with nonconvex functions g). We adapt the proof of the classical statement ([32, Theorem 2.1]) to our framework.

Theorem 3.11. *In the setting of Theorem* 3.8*, consider a function* $g \in \Gamma_0(X)$ *and a continuous seminorm* ρ*. Suppose there exist* $\varepsilon \geq 0$*, a net* $(\varepsilon_i)_{i\in I} \downarrow \varepsilon$ *and an open convex set* C *with* $C \cap \operatorname{dom} f \neq \emptyset$ *such that*

$$\hat{\partial}_{i,L} f(x) \subseteq \partial g(x) + \varepsilon_i \mathbb{B}_\rho^\circ + L^\perp, \ \ \forall x \in C, \ \forall i \in \mathbb{I}, \forall L \in \mathcal{L}.$$

Then $C \cap \operatorname{dom} f = C \cap \operatorname{dom} g$ *and for all* $x \in C$*,* $y \in C \cap \operatorname{dom} f$ *one has*

$$g(x) - g(y) - \varepsilon\rho(x-y) \leq f(x) - f(y) \leq g(x) - g(y) + \varepsilon\rho(x-y). \tag{12}$$

First, we prove the following two lemmas.

Lemma 3.12. *In the setting of Theorem* 3.11*, let* $y \in C \cap \operatorname{dom} f \cap \operatorname{dom} g$ *and* $x \in C \cap \operatorname{dom} g$ *such that* $x \neq y$*. Consider* $\gamma > \varepsilon$ *and a continuous seminorm* $\bar{\rho}$ *such that* $\bar{\rho}(x-y) \neq 0$*. Consider* $u := \frac{x-y}{\bar{\rho}(x-y)}$ *and* $h \colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$*, given by* $h(t) := g(y+tu)$*. We assume that* $g'(y,u) \in \mathbb{R}$*. Then* $h \in \Gamma_0(\mathbb{R})$*,* h *is continuous on* $\operatorname{dom} h \supseteq [0, \bar{\rho}(x-y)]$*,* $h'_+(t) := \lim\limits_{t' \downarrow t} \frac{h(t')-h(t)}{t'-t} = g'(y+tu, u)$ *for all* $t \in \operatorname{dom} h$*, and there exists* $x_0 = y + t_0 u$ *with* $t_0 \in (0, \bar{\rho}(x-y))$ *such that*

$$f(x_0) - f(y) \leq g(x_0) - g(y) + \gamma\bar{\rho}(x_0 - y).$$

Proof. Clearly, $h \in \Gamma_0(\mathbb{R})$ and $[0, \bar{\rho}(x-y)] \subseteq \operatorname{dom} h$, so that (see, e.g., [28, Theorems 10.1 and Theorem 24.1]) h is continuous on $\operatorname{dom} h$, $h'_+(t) = g'(y+tu, u)$ for all $t \in \operatorname{dom} h$, and

$$\lim_{t\downarrow 0} g'(y+tu, u) = \lim_{t \downarrow 0} h'_+(t) = h'_+(0) = g'(y,u)$$

(as $g'(y,u)$ is finite). Therefore, there exists $t_0 \in (0, \bar{\rho}(x-y))$ such that we have $g'(y+t_0u, u) \leq g'(y,u) + \frac{1}{2}(\gamma - \varepsilon)$. Define $x_0 := y + t_0 u \in \ ]y, x[$; hence,

$$g'(x_0, u) \leq \frac{g(y+t_0u) - g(y)}{t_0} + \frac{1}{2}(\gamma - \varepsilon) = \frac{g(x_0) - g(y)}{\bar{\rho}(x_0 - y)} + \frac{1}{2}(\gamma - \varepsilon).$$

We claim that $f(x_0) - f(y) \leq g(x_0) - g(y) + \gamma\bar{\rho}(x_0 - y)$. Indeed, if this was not the case; that is, $f(x_0) - f(y) > g(x_0) - g(y) + \gamma\bar{\rho}(x_0-y)$, then we choose $r \in \mathbb{R}$ such that $r \leq f(x_0)$ and $r - f(y) > g(x_0) - g(y) + \gamma\bar{\rho}(x_0 - y)$. So, by Theorem 3.8 there exist $x_\alpha \to z \in [y, x_0[$ and $x^*_\alpha \in \hat{\partial}_{i_\alpha, L_\alpha} f(x_\alpha)$ such that $x_\alpha \in L_\alpha$ and $\bar{\rho}(x_0 - z)(r - f(y)) \leq \bar{\rho}(x_0 - y) \liminf \langle x_0 - x_\alpha, x^*_\alpha \rangle$. We may assume without loss of generality that $x_0 \in L_\alpha$ for all α. Since $\bar{\rho}(x_0 - x_\alpha) \to \bar{\rho}(x_0 - z)$, we have

$$\frac{r - f(y)}{\bar{\rho}(x_0 - y)} \leq \liminf \left\langle \frac{x_0 - x_\alpha}{\bar{\rho}(x_0 - x_\alpha)}, x^*_\alpha \right\rangle,$$

and so, there exists an α_0 such that for all $\alpha \geq \alpha_0$

$$\left\langle \frac{x_0 - x_\alpha}{\bar{\rho}(x_0 - x_\alpha)}, x^*_\alpha \right\rangle > \frac{g(x_0) - g(y)}{\bar{\rho}(x_0 - y)} + \gamma.$$

Since $x_\alpha \to z \in [y, x_0[\ \subseteq C$, we may assume that $x_\alpha \in C$ for all $\alpha \geq \alpha_0$. Hence, by our assumption we can assume that $x_\alpha^* = z_\alpha^* + \varepsilon_{i_\alpha} u_\alpha^* + \lambda_\alpha^*$ with $z_\alpha^* \in \partial g(x_\alpha)$, $u_\alpha^* \in \mathbb{B}_\rho^\circ$ and $\lambda_\alpha^* \in {L_\alpha}^\perp$ for all $\alpha \geq \alpha_0$. Thus,

$$\frac{g(x_0) - g(x_\alpha)}{\bar\rho(x_0 - x_\alpha)} \geq \langle \frac{x_0 - x_\alpha}{\bar\rho(x_0 - x_\alpha)}, z_\alpha^* \rangle > \frac{g(x_0) - g(y)}{\bar\rho(x_0 - y)} + \gamma - \varepsilon_{i_\alpha} \ \ \forall \alpha \geq \alpha_0,$$

and, taking into account the lower semi-continuity of g,

$$\frac{g(x_0) - g(z)}{\bar\rho(x_0 - z)} \geq \frac{g(x_0) - g(y)}{\bar\rho(x_0 - y)} + \gamma - \varepsilon \geq g'(x_0, u) + \frac{1}{2}(\gamma - \varepsilon) > g'(x_0, u). \qquad (13)$$

Since $z \in [y, x_0[$, we have $z = y + su$ for some $s \in [0, t_0[$, and so, using the convexity of h, we get

$$\frac{g(x_0) - g(z)}{\bar\rho(x_0 - z)} = \frac{h(t_0) - h(s)}{t_0 - s} \leq h'_-(t_0) \leq h'_+(t_0) = g'(x_0, u),$$

which contradicts (13). Therefore we necessarily have that

$$f(x_0) - f(y) \leq g(x_0) - g(y) + \gamma\bar\rho(x_0 - y). \qquad \square$$

Lemma 3.13. *In the setting of Lemma* 3.12*, let* $y \in C \cap \operatorname{dom} f \cap \operatorname{dom} \partial g$ *and* $x \in C \cap \operatorname{dom} g$ *such that* $x \neq y$*. Consider* $\gamma > \varepsilon$ *and a continuous seminorm* ρ *such that* $\bar\rho(x - y) \neq 0$*. Then* $f(x) - f(y) \leq g(x) - g(y) + \gamma\bar\rho(x - y)$*.*

Proof. We define $u := \frac{x-y}{\bar\rho(x-y)}$, so that $\frac{g(x)-g(y)}{\bar\rho(x-y)} \geq g'(y, u) > -\infty$ (as $g \in \Gamma_0(X)$); that is, $g'(y, u) \in \mathbb{R}$. As a consequence of Lemma 3.12 there exists $x_0 = y + t_0 u$ with $t_0 \in (0, \bar\rho(x - y))$ such that

$$f(x_0) - f(y) \leq g(x_0) - g(y) + \gamma\bar\rho(x_0 - y).$$

Now, define $T = \{t \in (0, \bar\rho(x - y)] : f(y + tu) - f(y) \leq g(y + tu) - g(y) + \gamma t\}$ and $\bar t := \sup T$. Because f is lsc and h is continuous on $[0, \bar\rho(x - y)]$ (see Lemma 3.12), it follows that $\bar t \in T$. Let us verify that $\bar t = \bar\rho(x - y)$. For otherwise, when $\bar t < \bar\rho(x - y)$, we define $\bar y := y + \bar t u$, so that $u = \frac{x-\bar y}{\bar\rho(x-\bar y)}$. Then, since $\bar y \in C \cap \operatorname{dom} f \cap \operatorname{dom} g$, we get

$$\frac{h(\bar\rho(x - y)) - h(\bar t)}{\bar\rho(x - y) - \bar t} = \frac{g(x) - g(\bar y)}{\bar\rho(x - \bar y)} \geq g'(\bar y, u) = h'_+(\bar t) \geq h'_+(0) = g'(y, u) > -\infty.$$

Hence, $g'(\bar y, u) \in \mathbb{R}$, and so, again by Lemma 3.12, there exists $x_0 = \bar y + s_0 u$ with $s_0 \in (0, \bar\rho(x - \bar y))$ such that

$$f(\bar y + s_0 u) - f(\bar y) \leq g(\bar y + s_0 u) - g(\bar y) + \gamma\bar\rho(x_0 - \bar y) = g(\bar y + s_0 u) - g(\bar y) + \gamma s_0.$$

Hence, by adding $f(\bar y) - f(y) \leq g(\bar y) - g(y) + \bar t\gamma$ to this last inequality we obtain

$$f(y + (\bar t + s_0)u) - f(y) \leq g(y + (\bar t + s_0)u) - g(y) + (\bar t + s_0)\gamma,$$

which contradicts the definition of $\bar t$. Consequently, $\bar t = \bar\rho(x - y)$ and the conclusion of the lemma follows. $\square$

Proof of Theorem 3.11 First, we assume that $y \in C \cap \operatorname{dom} f \cap \operatorname{dom} \partial g$, $x \in C \cap \operatorname{dom} g$ and $x \neq y$, and choose a seminorm ρ_0 such that $\rho_0(x-y) \neq 0$. Define $\bar{\rho}_n := \rho + n^{-1}\rho_0$. Then, by Lemma 3.13, $f(x) - f(y) \leq g(x) - g(y) + \gamma\bar{\rho}_n(x-y)$ for all $\gamma > \varepsilon$, and $n \geq 1$. Therefore, for all $x \in C \cap \operatorname{dom} g$ and all $y \in C \cap \operatorname{dom} f \cap \operatorname{dom} \partial g$,

$$f(x) - f(y) \leq g(x) - g(y) + \varepsilon\rho(x-y). \tag{14}$$

Now, taking $x \in C \cap \operatorname{dom} g$ and (if there exists) $y \in C \cap \operatorname{dom} f$ such that $x \neq y$, by Corollary 3.10 there exists $(y_\alpha, y_\alpha^*) \subseteq \operatorname{gph} \hat{\partial}_{i_\alpha, L_\alpha} f$ such that $y_\alpha \xrightarrow{f} y$. Because $y \in C$, C is open and $y_\alpha \to y$, we can suppose that $y_\alpha \in C$ and $y_\alpha \neq x$ for all α; moreover, from our assumptions $x_\alpha \in \operatorname{dom} f \cap \operatorname{dom} \partial g$. Then by (14) we get $f(x) - f(x_\alpha) \leq g(x) - g(x_\alpha) + \varepsilon\rho(x - x_\alpha)$. Hence, from the lower semicontinuity of g we conclude that, for all $x \in C \cap \operatorname{dom} g$ and all $y \in C \cap \operatorname{dom} f$,

$$f(x) - f(y) \leq g(x) - g(y) + \varepsilon\rho(x-y). \tag{15}$$

So, on the one hand, using the latter inequality and the fact that $C \cap \operatorname{dom} f \neq \emptyset$, we easily get $C \cap \operatorname{dom} g \subseteq C \cap \operatorname{dom} f$. On the other hand, take $y \in \operatorname{dom} f \cap C$. Then by Corollary 3.10 there exists $(y_\alpha, y_\alpha^*) \subseteq \operatorname{gph} \hat{\partial}_{i_\alpha, L_\alpha} f$ such that $y_\alpha \xrightarrow{f} y$, and so by our hypothesis there exists some α_0 such that $y_\alpha \in C \cap \operatorname{dom} \partial g \subseteq C \cap \operatorname{dom} g$. If $y_\alpha = y$ for all $\alpha \geq \alpha_0$, then we get $y \in C \cap \operatorname{dom} g$. Otherwise, we can assume that $y_\alpha \neq y$ for some y_α, and then (15) implies that $y \in C \cap \operatorname{dom} g$. This proves that $\operatorname{dom} f \cap C = \operatorname{dom} g \cap C$. Therefore, interchanging the roles of x and y in (14), we obtain (12). □

Corollary 3.14. *In the setting of Theorem* 3.11*, assume that* $\varepsilon = 0$ *and* $C = X$*. Then there exists* $c \in \mathbb{R}$ *such that* $f = g + c$*.*

Proof. By Theorem 3.11 we have $\operatorname{dom} g = \operatorname{dom} f$ and $f(x) - f(y) = g(x) - g(y)$ for all x, y. Therefore, fixing $x \in \operatorname{dom} f$, we have $f(y) = g(y) + f(x) - g(x)$, for all $y \in X$. □

To show the potential applications of Theorem 3.11 we give short proofs of two results from the literature referent to integration theorems. It is worth mentioning that the original proofs of the two result do not follow the argument presented in this paper and they cannot even be covered by the original statement of [32, Theorem 2.1]; in other words, Theorem 3.11 allows us to include three results, which appear to be independent, in the same framework.

The following corollary is due to Marcellin-Thibault [22, Theorem 1.2] (see also [4, Corollary 10] for a nonconvex approach). In this paper the authors use the ε-subdifferential of a proper lsc convex function f to establish conditions to recover a function from its subdifferential in an arbitrary locally convex space. More precisely, we have the following statement.

Corollary 3.15. *Let* $f, g \colon X \to \mathbb{R} \cup \{+\infty\}$ *be two proper lsc convex functions where for every* $x \in X$ *there exists* $\delta(x) > 0$ *such that* $\partial_\varepsilon f(x) \subseteq \partial_\varepsilon g(x)$ *for all* $\varepsilon < \delta(x)$*. Then there exists* $c \in \mathbb{R}$ *such that* $f = g + c$*.*

Proof. Fix $x \in \operatorname{dom} f$ and consider $y \in \operatorname{dom} g$ different from x, consider L as the subspace generated by x, y and let U be a closed neighbourhood of zero such that $x, y \in \operatorname{Int}(U)$ and $B = L \cap U$ is compact (relative to L). Then the functions $\tilde{f} = f + \delta_B$ and $\tilde{g} = g + \delta_B$ are w-lsc and epi-pointed, since for every $w \in X$,

$$\partial \tilde{f}(w) = \bigcap_{\varepsilon>0} \overline{\partial_\varepsilon f(w) + \partial_\varepsilon \delta_B(w)}^{w^*} \quad \text{and} \quad \partial \tilde{g}(w) = \bigcap_{\varepsilon>0} \overline{\partial_\varepsilon g(w) + \partial_\varepsilon \delta_B(w)}^{w^*}$$

(see, e.g., [14, Theorem 2.1]). Therefore, for every $w \in X$,

$$\partial \tilde{f}(w) = \bigcap_{\varepsilon>0} \overline{\partial_\varepsilon f(w) + \partial_\varepsilon \delta_B(w)}^{w^*} \subseteq \overline{\partial_\varepsilon g(w) + \partial_\varepsilon \delta_B(w)}^{w^*} = \partial \tilde{g}(w).$$

Then, by Corollary 3.14, we get $x \in \operatorname{dom} \tilde{g} \subseteq \operatorname{dom} g$, $y \in \operatorname{dom} \tilde{f} \subseteq \operatorname{dom} f$ and $f(x) - f(y) = g(x) - g(y)$. □

The following corollary is a particular case of a more general result given in [10, Theorem 4.8] (see also [4] for another approach). For the sake of simplicity, in our statement we avoid the introduction of more sophisticated notions of convex and non-smooth analysis.

Corollary 3.16. *Let $f \colon X \to \mathbb{R} \cup \{+\infty\}$ be a w-lsc epi-pointed function and let $g \colon X \to \mathbb{R}$ be an arbitrary function such that $\partial f(x) \subseteq \partial g(x)$ for all $x \in X$. Then there exists a constant $c \in \mathbb{R}$ such that $f^{**} = g^{**} \Box \sigma_{\operatorname{dom} f^*} + c$.*

Proof. Consider $x^* \in \partial f(x)$. Then

$$x \in \partial f^*(x^*) = \overline{\operatorname{co}}^{w^*} \left(\partial f\right)^{-1}(x^*) + N_{\operatorname{dom} f^*}(x^*),$$

by [5, Corollary 6]. Hence, there exist $y, \lambda \in X$, with $\lambda \in N_{\operatorname{dom} f^*}(x^*)$, and nets $y_i = \sum_{k=1}^{n_i} \alpha_i^k w_i^k$ such that $x = y + \lambda$, $y = \lim y_i$, $\alpha_i \geq 0$, $\sum_{k=1}^{n_i} \alpha_i^k = 1$ and $x^* \in \partial f(w_i^k)$; thus, $x^* \in \partial g(w_i^k)$ and so $w_i^k \in \partial g^*(x^*)$. Using the fact that $\partial g^*(x^*)$ is convex and closed, we get $y \in \partial g^*(x^*)$. Consequently,

$$x \in \partial g^*(x^*) + N_{\operatorname{dom} f^*}(x^*) \subseteq \partial(g^* + \delta_{\overline{\operatorname{dom} f^*}})(x^*).$$

Therefore, by Corollary 3.14, and using the fact that f is epi-pointed, there exists a constant $c \in \mathbb{R}$ such that $f^{**} = (g^* + \delta_{\overline{\operatorname{dom} f^*}})^* + c = g^{**} \Box \sigma_{\operatorname{dom} f^*} + c$. □

The following theorem is given in [12, Theorem 2.2 and 2.4] for lsc functions defined in a Banach space X (see also [11]). We extend this result to weakly lsc epi-pointed functions defined in an arbitrary locally convex space.

Theorem 3.17. *In the setting of Theorem* 3.8*, assume that $\hat{\partial}$ is a subdifferential and consider the following statements:*

(i) *f is convex,* (ii) *$\hat{\partial} f$ is monotone,* (iii) *$\hat{\partial} f(x) \subseteq \partial f(x)$, for all $x \in X$.*

Then (ii) *and* (iii) *are equivalent and each one implies* (i). *In addition, if $\hat{\partial} g$ coincides with the convex subdifferential for any convex function g, then the three statements are equivalent.*

Proof. (ii) $\Rightarrow$ (iii). First, if $\operatorname{dom} f = \emptyset$, then $f = +\infty$, and the theorem is true. If $\operatorname{dom} f \neq \emptyset$, then take $a \in \operatorname{dom} f$, $x \in \operatorname{dom} \hat{\partial} f$ with $x \neq a$ (if there are, recall Corollary 3.10), a continuous seminorm ρ such that $\rho(a-x) \neq 0$, $x^* \in \hat{\partial} f(x)$ and $r \in \mathbb{R}$ with $r \leq f(x)$. Then, by Theorem 3.8, there exist net $(x_\alpha, x_\alpha^*) \in \operatorname{gph} \hat{\partial} f$ and $c \in [a, x[$ such that $x_\alpha \to c$ and

$$\begin{aligned} r - f(a) &\leq \frac{\rho(x-a)}{\rho(x-c)} \liminf \langle x_\alpha^*, x - x_\alpha \rangle \\ &= \frac{\rho(x-a)}{\rho(x-c)} \liminf \big(\langle x_\alpha^* - x^*, x - x_\alpha \rangle + \langle x^*, x - x_\alpha \rangle \big) \\ \text{(by monotonicity:)} \qquad &\leq \frac{\rho(x-a)}{\rho(x-c)} \liminf \langle x^*, x - x_\alpha \rangle = \langle x^*, x - a \rangle. \end{aligned}$$

Therefore, $x \in \operatorname{dom} f$ and $x^* \in \partial f(x)$.

The implication (iii) $\Rightarrow$ (ii) follows from the monotonicity of the ∂f. Finally, to prove that (iii) $\Rightarrow$ (i), we observe that by (iii) it holds that

$$\hat{\partial} f(x) \subseteq \partial f(x) \subseteq \partial f^{**}(x), \quad \text{for all } x \in X.$$

Then, by Corollary 3.14, we get $f = f^{**}$. The final implication is only a matter of using the extra hypothesis. □

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