Journal of Convex Analysis
Volume 27 (2020), No. 4, 1363–1374

Multiplicity of Positive Solutions for an Anisotropic Problem via Sub-Supersolution Method and Mountain Pass Theorem

Gelson C. G. dos Santos
Faculdade de Matemática, Universidade Federal do Pará, 66075-110 Belém-Pa, Brazil
cgelson@ymail.com

Giovany Figueiredo
Departamento de Matemática, Universidade de Brasília, 70910-900 Brasília-DF, Brazil
giovany@unb.br

Julio R. S. Silva
Universidade Federal do Pará, Campus Universitário de Cametá, 68.400-000 Cametá, Brazil
julioroberto@ufpa.br

Received: April 6, 2019
Accepted: April 1, 2020

We use the sub-supersolution method and the Mountain Pass Theorem in order to show existence and multiplicity of solution for an anisotropic problem given by

$$\begin{cases} -\Big[\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{pi-2}\frac{\partial u}{\partial x_i}\Big)\Big] = a(x)u + h(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \end{cases}$$

We also prove the uniqueness of the solution for the linear anisotropic problem, a Comparison Principle for the anisotropic operator and a regularity result.

Keywords: Anisotropic operator, sub-supersolution method, Mountain Pass Theorem.

2010 Mathematics Subject Classification: Primary 35J60; secondary 35J66.

1. Introduction

In this paper we are concerned with existence of solution to the class of nonlinear boundary value anisotropic problems given by

$$\begin{cases} -\Big[\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{pi-2}\frac{\partial u}{\partial x_i}\Big)\Big] = a(x)u + h(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u \in W_0^{1,\overrightarrow{p}}(\Omega), \end{cases} \tag{1}$$

where $\overrightarrow{p} = (p_1, ..., p_N)$, $p_i > 1$, $\sum_{i=1}^{N} \frac{1}{p_i} > 1$, $p_1 \leq p_2 \leq ... \leq p_N$ and $p^* := \frac{N\overline{p}}{N-\overline{p}}$.

ISSN 0944-6532 / $ 2.50

In this paper $\overline{p}$ denotes the harmonic mean $\overline{p} = \dfrac{N}{\sum_{i=1}^{N} \frac{1}{p_i}}$.

It is well known that $W_0^{1,\overrightarrow{p}}(\Omega)$, which is the completion of the space $\mathcal{D}(\Omega)$ with respect to the norm

$$\|u\| = \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i},$$

is a reflexive Banach space and is continuously embedded in $L^{p^*}(\Omega)$. Here $\|.\|_{p_i}$ is the usual norm in $L^{p_i}(\Omega)$.

Since Ω is a bounded domain of $\mathbb{R}^N$, from [16, Theorem 1], the continuity of the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^s(\Omega)$, for all $s \in [1, p^*]$ relies on a well-known Poincaré-type inequality. More precisely, denoting by $e_1, ..., e_n$ the canonical basis of $\mathbb{R}^N$, assume that Ω has width $b > 0$ in the direction of e_i, namely $\sup\limits_{x,y \in \Omega} (x - y, e_i) = b$.

Thus, for every $q \geq 1$, we have for all $u \in \mathcal{D}(\Omega)$

$$\|u\|_q \leq \frac{bq}{2} \left\| \frac{\partial u}{\partial x_i} \right\|_q. \tag{2}$$

By a solution for (1) we understand as a function $0 \leq u \in W_0^{1,\overrightarrow{p}}(\Omega)$ satisfying

$$\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx = \int_\Omega a(x) u \varphi \, dx + \int_\Omega h(x, u) \varphi \, dx, \tag{3}$$

for all $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega)$. The function a satisfies

(a₁) $a \in L^\infty(\Omega)$ with $a(x) > 0$ a.e in Ω.

In this paper h is a continuous function on $\Omega \times [0, \infty)$ satisfying

(h_1) There is $\delta > 0$ such that

$$h(x, t) \geq (1 - t) a(x), \text{ for every } 0 \leq t \leq \delta \text{ a.e in } \Omega.$$

(h_2) There are $1 < r < p^*$ such that

$$h(x, t) \leq a(x)(t^{r-1} + 1), \text{ for every } 0 \leq t \text{ and a.e in } \Omega.$$

The main results of this paper are the following:

Theorem 1.1. *Assume that conditions* (a_1), (h_1) *and* (h_2) *hold. Then, problem* (1) *has a positive weak solution if* $\|a\|_\infty$ *is small.*

In order to establish the existence of two solutions for problem (1), we also assume:

(h_3) There are $t_0 > 0$ and $\theta > p_N$ such that

$$0 < \theta H(x, t) = \theta \int_0^t h(x, s) ds \leq t h(x, t), \text{ for every } t_0 \leq t \text{ in } \Omega \text{ and a.e in } \Omega.$$

Theorem 1.2. *Assume that conditions* (a_1), $(h_1) - (h_3)$ *hold. Then, problem* (1) *has two positive weak solutions if* $\|a\|_\infty$ *is small.*

We would like to highlight that our theorems can be applied to the following nonlinear model.

Fixed $s_0 > 0$, the function

$$h(x,t) = \begin{cases} a(x)(1-t) & \text{if } t \in [0, s_0], \\ a(x)\big((1-s_0) + (t-s_0)^{r-1}\big) & \text{if } t \in (s_0, \infty). \end{cases}$$

satisfies (h_1) and (h_2) for $\delta \in (0, s_0]$ and $r > 1$. It also satisfies $(h_1), (h_2)$ and (h_3) for $t_0 > 0$ large and $r \in (p_N, p^*)$. Note that, for $s_0 > 1$, we have $h(x,t) < 0$ em $(1, s_0)$.

In the last years the main interest in this class of problems has been due to the fact that they arise from applications in biology and the operator has different orders of derivations in different directions. For instance, as can be seen in [26], it modelizes directionally dependent phenomena and appear in biology, see [7] and [8], as a model describing the spread of an epidemic disease in heterogeneous environments. They also emerge, see [4] and [6], from the mathematical description of the dynamics of fluids with different conductivities in different directions.

In [5], [14], [17], [19], [22], [25] the anisotropic case with variable exponent was studied.

The case in $\mathbb{R}^2$ with exponential nonlinearity and concentration results was studied in [29]. In [1] it was studied a priori estimates for solutions to anisotropic elliptic problems via symmetrization and a comparison result. Using an approximation approach, the authors of [18] proved the existence and regularity of the solutions to an anisotropic problem involving a singular nonlinearity.

For other interesting articles involving anisotropic operator, we refer to [2], [10], [11], [12], [13], [15], [16], [28] and [27]. In particular, in [2], at least to our knowledge, it was used in the first time(see Chapter 5 in that paper) the method of sub-supersolution for an anisitropic problem.

The present work is strongly influenced by [1] and [2]. Below we list what we believe that are the main contributions of our paper.

(i) In [2] the subsolution and the supersolution are constants. Here, motived by [20], the subsolution and the supersolution are not necessarily constant functions.

(ii) Unlike [1], in the present paper we show existence and unicity of solution for the anisotropic linear problem, a comparison principle of solutions and a regularity result without making use of arguments involving symmetrization.

This paper is organized as follows. In Section 2 we prove the unicity of solutions for the linear anisotropic problem, a Comparison Principle and a regularity result for solutions to this class of problems. In Section 3 we prove Theorem 1.1. Theorem 1.2 is proved in Section 4.

2. Technical results

We start by proving a result of unicity of solution to the linear problem and a Comparison Principle of the anisotropic operator.

Lemma 2.1. *There is a unique solution* $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ *for the problem*

$$\begin{cases} -\Big[\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial w}{\partial x_i}\Big|^{pi-2}\frac{\partial w}{\partial x_i}\Big)\Big] = a(x) \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega. \end{cases} \tag{4}$$

Proof: Consider the operator $T : W_0^{1,\overrightarrow{p}}(\Omega) \to (W_0^{1,\overrightarrow{p}}(\Omega))'$ such that $< Tu, \phi >$ is given by

$$\Big\langle Tu, \phi \Big\rangle = \sum_{i=1}^{N} \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i}\, dx.$$

Since the inequality

$$C_i\Big|\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\Big|^{p_i} \leq \Big\langle \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i} - \Big|\frac{\partial v}{\partial x_i}\Big|^{p_i-2}\frac{\partial v}{\partial x_i}, \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\Big\rangle \tag{5}$$

is true for some $C_i > 0$ and for all $i = 1, ..., N$, we have that

$$\Big\langle Tu - Tv, u - v \Big\rangle > 0 \text{ for all } u, v \in W_0^{1,\overrightarrow{p}}(\Omega) \text{ with } u \neq v.$$

Moreover, if $\|u\| \to +\infty$, then, without loss of generality, we can assume that

$$\int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i} dx \geq 1, \text{ for some } i = 1, 2, ..., N.$$

Hence, since $1 < p_1 \leq p_i$, for all $i = 1, 2, ..., N$, we have

$$\sum_{i=1}^{N} \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i} dx \geq \sum_{i=1}^{N}\Big(\int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i} dx\Big)^{\frac{p_1}{p_i}} \geq \frac{1}{N^{p_1-1}}\Big(\sum_{i=1}^{N}\Big|\frac{\partial u}{\partial x_i}\Big|_{L^{p_i}}\Big)^{p_1},$$

which implies
$$\lim_{\|u\|\to\infty} \frac{\langle Tu, u\rangle}{\|u\|} = +\infty.$$

Thus, by Minty-Browder's Theorem [9, Teorema 5.16], there exists a unique function $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ that satisfies $Tu = a(x)$. $\square$

Lemma 2.2. *If* Ω *is a bounded domain and if* $u, v \in W_0^{1,\overrightarrow{p}}(\Omega)$ *satisfy*

$$\begin{cases} -\Big[\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{pi-2}\frac{\partial u}{\partial x_i}\Big)\Big] \leq -\Big[\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial v}{\partial x_i}\Big|^{pi-2}\frac{\partial v}{\partial x_i}\Big)\Big] \text{ in } \Omega, \\ u \leq v \text{ on } \partial\Omega, \end{cases}$$

then $u \leq v$ *a.e in* Ω.

Proof: Taking $0 \leq \phi = max\{u-v,0\} \in W_0^{1,\overrightarrow{p}}(\Omega)$ as a test function, we obtain

$$\int_{\Omega \bigcap [u>v]} \sum_{i=1}^{N} \left\langle \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}, \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right\rangle dx \leq 0.$$

From inequality (5), we conclude that $\|(u-v)^+\| \leq 0$, which implies $u \leq v$ a.e in Ω. □

Before proving the L^∞-regularity we enunciate an iteration lemma by Stampacchia that we will use.

Lemma 2.3. (See [23]) *Assume that ϕ: $[0,\infty) \to [0,\infty)$ is a nonincreasing function such that if $h > k > k_0$, for some $\alpha > 0, \beta > 1$, $\phi(h) \leq C(\phi(k))^\beta/(h-k)^\alpha$. Then $\phi(k_0+d) = 0$, where $d^\alpha = C2^{\frac{\alpha\beta}{\beta-1}}\phi(k_0)^{\beta-1}$ and C is a positive constant.*

Lemma 2.4. *Let $v \in W_0^{1,\overrightarrow{p}}(\Omega)$ be a solution to problem*

$$\begin{cases} -\Big[\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial v}{\partial x_i}\Big|^{pi-2}\frac{\partial v}{\partial x_i}\Big)\Big] = f \ \text{in } \Omega, \\ v = 0 \ \text{on } \partial\Omega, \end{cases}$$

where $f \in L^r(\Omega)$ with $r > p^*/(p^*-p_1)$. Then $v \in L^\infty(\Omega)$. In particular, if $\|f\|_r$ is small, then also $\|v\|_\infty$ is small.

Proof: Consider $v_k = sign(u)(|u|-k)^+$, then $v_k \in W_0^{1,\overrightarrow{p}}(\Omega)$ and $\frac{\partial v}{\partial x_i} = \frac{\partial v_k}{\partial x_i}$ in $A(k) = \{x \in \Omega : |u(x)| > k\}$. Let $|A(k)|$ be the Lebesgue measure of $A(k)$. Using v_k as test function and Holder inequality, we have

$$\sum_{i=1}^{N}\int_{A(k)}\left|\frac{\partial v_k}{\partial x_i}\right|^{p_i} dx = \int_\Omega f v_k dx \leq \left(\int_\Omega |v_k|^{p^*}dx\right)^{\frac{1}{p^*}}\left(\int_\Omega |f|^r dx\right)^{\frac{1}{r}}|A(k)|^{1-\left(\frac{1}{p^*}+\frac{1}{r}\right)}.$$

Let $$0 < S = \inf_{u \in D^{1,\overrightarrow{p}}(\mathbb{R}^N), \|u\|_{p^*}=1}\left\{\sum_{i=1}^{N}\frac{1}{p_i}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}^{p_i}\right\}, \text{ see [13].}$$

Once that $p_i \geq p_1 > 1$, we have

$$S\left(\int_\Omega |u|^{p^*}dx\right)^{\frac{p_1}{p^*}} \leq \sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx, \text{ for all } u \in W_0^{1,\overrightarrow{p}}(\Omega).$$

This implies $$S\left(\int_{A(k)} |v_k|^{p^*}dx\right)^{\frac{p_1-1}{p^*}} \leq \left(\int_\Omega |f|^r dx\right)^{\frac{1}{r}}|A(k)|^{1-\left(\frac{1}{p^*}+\frac{1}{r}\right)}.$$

Note that if $0 < k < h$, $A(h) \subset A(k)$ and

$$|A(h)|^{\frac{1}{p^*}}(h-k) = \left(\int_{A(h)}(h-k)^{p^*}dx\right)^{\frac{1}{p^*}} \leq \left(\int_{A(k)}|v_k|^{p^*}dx\right)^{\frac{1}{p^*}},$$

then $$|A(h)| \leq \frac{1}{(h-k)^{p^*}} \frac{1}{S^{\frac{p^*}{p_1-1}}} \|f\|_r^{\frac{p^*}{p_1-1}} |A(k)|^{\frac{p^*}{p_1-1}\left[1-\left(\frac{1}{p^*}+\frac{1}{r}\right)\right]}.$$

Since $r > \frac{p^*}{p^*-p_1}$, we have $\beta := \frac{p^*}{p_1-1}\left[1-\left(\frac{1}{p^*}+\frac{1}{r}\right)\right] > 1$. Therefore, if we define

$$\phi(h) = |A(h)|, \alpha = p^*, \beta = \frac{p^*}{p_1-1}\left[1-\left(\frac{1}{p^*}+\frac{1}{r}\right)\right], k_0 = 0,$$

we have that ϕ is a nonincreasing function and

$$\phi(h) \leq \frac{C}{(h-k)^{\alpha}} \phi(k)^{\beta}, \text{ for all } h > k > 0.$$

By Lemma 2.3, we have $\phi(d) = 0$ for $d = C\|f\|_r^{\frac{1}{p_1-1}} |\Omega|^{\frac{\beta-1}{\alpha}} / S^{\frac{1}{p_1-1}}$, then

$$\|u\|_\infty \leq \frac{C\|f\|_r^{\frac{1}{p_1-1}} |\Omega|^{\frac{\beta-1}{\alpha}}}{S^{\frac{1}{p_1-1}}}. \qquad \square$$

Lemma 2.5. *Assume that* (a_1), (h_1) *and* (h_2) *hold. If* $\|a\|_\infty$ *is small, then there exist* $\underline{v}, \overline{v} \in W_0^{1,\overrightarrow{p}}(\Omega) \bigcap L^\infty(\Omega)$ *such that*

(i) $\|\underline{v}\|_\infty \leq \delta$, *where* δ *is the constant that appeared in the hypothesis* (h_1).

(ii) $0 < \underline{v}(x) \leq \overline{v}(x)$ *a.e in* Ω.

(iii) $\underline{v}$ *is a subsolution and* $\overline{v}$ *is a supersolution of* (1).

Proof: Let $\underline{v} \in W_0^{1,\overrightarrow{p}}(\Omega)$ be the unique solution of problem (4). By Lemma 2.4, $\underline{v} \in L^\infty(\Omega)$ and there exits $C^* > 0$ such that $\|\underline{v}\|_\infty \leq C^*\|a\|_\infty$. Now we fix $\|a\|_\infty$ such that $\|\underline{v}\|_\infty \leq \delta$.

In order to prove (ii), we consider $\overline{v} \in W_0^{1,\overrightarrow{p}}(\Omega) \bigcap L^\infty(\Omega)$, the unique solution of problem

$$\begin{cases} -\Big[\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{v}}{\partial x_i}\Big|^{pi-2} \frac{\partial \overline{v}}{\partial x_i}\Big)\Big] = 1 + a(x) \text{ in } \Omega, \\ \overline{v} = 0 \text{ on } \partial\Omega. \end{cases} \tag{6}$$

Note that, for all $0 \leq \varphi \in W_0^{1,\overrightarrow{p}}(\Omega)$, we have

$$\begin{aligned} \sum_{i=1}^N \int_\Omega \Big|\frac{\partial \overline{v}}{\partial x_i}\Big|^{p_i-2} \frac{\partial \overline{v}}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\, dx &= \int_\Omega [a(x)+1]\varphi\, dx \geq \int_\Omega a(x)\varphi\, dx \\ &= \sum_{i=1}^N \int_\Omega \Big|\frac{\partial \underline{v}}{\partial x_i}\Big|^{p_i-2} \frac{\partial \underline{v}}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\, dx. \end{aligned}$$

Then, from Lemma 2.2 we conclude that $\underline{v}(x) \leq \overline{v}(x)$ a.e in Ω, which proves the condition (ii).

First we use the maximum principle of [12, Corollary 4.4] and conclude that $\underline{v} > 0$.

Now using the definition of $\underline{v}$ and (h_1), we obtain, for each $\varphi \geq 0$,

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial \underline{v}}{\partial x_i}\right|^{p_i-2}\frac{\partial \underline{v}}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\,dx-\int_{\Omega}a(x)\,\underline{v}\,\varphi\,dx-\int_{\Omega}h(x,\underline{v})\,\varphi\,dx\\ &\leq\int_{\Omega}a(x)\,\varphi\,dx-\int_{\Omega}a(x)\,\underline{v}\,\varphi\,dx-\int_{\Omega}a(x)\,\varphi\,dx+\int_{\Omega}a(x)\,\underline{v}\,\varphi\,dx=0.\end{aligned}$$

Since $\overline{v}$ is the solution of problem (6) and (h_2), we have for $\|a\|_\infty$ sufficiently small

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial \overline{v}}{\partial x_i}\right|^{p_i-2}\frac{\partial \overline{v}}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\,dx-\int_{\Omega}a(x)\overline{v}\varphi\,dx-\int_{\Omega}h(x,\overline{v})\varphi\,dx\\ &\geq\Big(1-\|a\|_\infty\|\overline{v}\|_\infty-\|a\|_\infty-\|a\|_\infty\|\overline{v}\|_\infty^{r-1}\Big)\int_{\Omega}\varphi\,dx>0.\end{aligned}$$

Then $\overline{v}$ is a supersolution of (1). □

3. Proof of Theorem 1.1

Consider the function

$$g(x,t)=\begin{cases}a(x)\overline{v}(x)+h(x,\overline{v}(x)), & t>\overline{v}(x)\\ a(x)t+h(x,t), & \underline{v}(x)\leq t\leq\overline{v}(x)\\ a(x)\underline{v}(x)+h(x,\underline{v}(x)), & t<\underline{v}(x),\end{cases} \tag{7}$$

and the auxiliary problem

$$\begin{cases}-\Big[\displaystyle\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{pi-2}\frac{\partial u}{\partial x_i}\Big)\Big] \;=\; g(x,u)\text{ in }\Omega,\\ u>0\text{ in }\Omega,\quad u\in W_0^{1,\overrightarrow{p}}(\Omega).\end{cases} \tag{8}$$

We define the functional $\Phi\colon W_0^{1,\overrightarrow{p}}(\Omega)\to\mathbb{R}$ by

$$\Phi(v)=\sum_{i=1}^{N}\int_{\Omega}\frac{1}{p_i}\left|\frac{\partial v}{\partial x_i}\right|^{p_i}dx-\int_{\Omega}G(x,v)dx, \tag{9}$$

where $G(x,t)=\int_0^t g(x,s)ds$. We have $\Phi\in C^1\big(W_0^{1,\overrightarrow{p}}(\Omega),\mathbb{R}\big)$ with

$$\Phi'(v)\varphi=\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\,dx-\int_{\Omega}g(x,v)\varphi dx,$$

for all $v,\varphi\in W_0^{1,\overrightarrow{p}}(\Omega)$.

From (h_2) and the definition of g, we have that

$$|g(x,t)| \leq K \quad \text{for some} \quad K > 0, \quad \text{a.e in } \Omega. \tag{10}$$

From (10), we have that Φ is coercive. Then, we can obtain that (v_n) is a bounded sequence in $W_0^{1,\overrightarrow{p}}(\Omega)$ such that

$$\Phi(v_n) \to c = \inf_{\mathcal{M}} \Phi,$$

where
$$\mathcal{M} = \left\{ v \in W_0^{1,\overrightarrow{p}}(\Omega) : \underline{v} \leq v \leq \overline{v} \text{ a. e. in } \Omega \right\}.$$

Hence, up to subsequence, we have

$$\begin{cases} v_n \rightharpoonup v \text{ in } W_0^{1,\overrightarrow{p}}(\Omega), \\ v_n \to v \text{ in } L^s(\Omega), \ 1 \leq s < p^*, \\ v_n(x) \to v(x) \text{ a.e in } \Omega. \end{cases} \tag{11}$$

Now, note that $\mathcal{M}$ is closed and convex in $W_0^{1,\overrightarrow{p}}(\Omega)$. By [24, Therem 1.2], the restriction $\Phi\big|_{\mathcal{M}}$ attains its infimum at a point v in $\mathcal{M}$. Using the same argument as in the proof of [24, Therem 2.4], we see that v weakly solves (8). Since $g(x,t) = a(x)t + h(x,t)$ for $t \in [\underline{v}, \overline{v}]$, then v is a positive weak solution of (1). □

4. Proof of Theorem 1.2

Consider h and $\underline{v} \in W_0^{1,\overrightarrow{p}}(\Omega) \bigcap L^\infty(\Omega)$ the subsolution of Problem (1). The next result shows that the functional Φ satisfies the geometric hypotheses of the Mountain Pass Theorem [3]. Consider the function

$$\widehat{g}(x,t) = \begin{cases} a(x)\, t + h(x,t), & t > \underline{v}(x) \\ a(x)\, \underline{v}(x) + h(x, \underline{v}(x)), & t \leq \underline{v}(x) \end{cases} \tag{12}$$

and define the functional $\widehat{\Phi} \colon W_0^{1,\overrightarrow{p}}(\Omega) \to \mathbb{R}$ by

$$\widehat{\Phi}(u) = \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx - \int_\Omega \widehat{G}(x,u) dx, \tag{13}$$

where $\widehat{G}(x,t) = \int_0^t \widehat{g}(x,s)ds$. Note that by (h_2) and (12), we have

$$\widehat{G}(x,t) \leq C|t| + a(x)|t|^r, \text{ for all } t \geq 0, \tag{14}$$

for some constant $C > 0$.

Lemma 4.1. *The functional $\widehat{\Phi}$ satisfies the* (PS)*-condition for every $c \in \mathbb{R}$.*

Proof: Let $(v_n) \subset W_0^{1,\overrightarrow{p}}(\Omega)$ be a sequence such that

$$\widehat{\Phi}(v_n) \to c \quad \text{and} \quad \widehat{\Phi}'(v_n) \to 0. \tag{15}$$

Using (h_3) and (14), we get that (v_n) is a bounded sequence in $W_0^{1,\overrightarrow{p}}(\Omega)$ and hence, up to subsequence, we have

$$\begin{cases} v_n \rightharpoonup v \text{ in } W_0^{1,\overrightarrow{p}}(\Omega), \\ v_n \to v \text{ in } L^s(\Omega),\ 1 \leq s < p^*, \\ v_n(x) \to v(x) \text{ a.e in } \Omega. \end{cases} \tag{16}$$

Using (15), (16), (5), the Lebesgue dominated convergence theorem and standard arguments, up to subsequence, we obtain

$$\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial v_n}{\partial x_i} - \frac{\partial v}{\partial x_i} \right|^{p_i} \leq o_n(1),$$

which implies $v_n \to v$ in $W_0^{1,\overrightarrow{p}}(\Omega)$. $\square$

Lemma 4.2. *Assume that (a_1), $(h_1) - (h_3)$ hold. Then for $\|a\|_{L^\infty}$ small, $\widehat{\Phi}$ satisfies:*

(i) *There are $R > \|\underline{v}\|$ and $\beta > 0$, such that*

$$\widehat{\Phi}(\underline{v}) < 0 < \beta \leq \inf_{\partial B_R(0)} \widehat{\Phi}(u).$$

(ii) *There are $e \in W_0^{1,\overrightarrow{p}}(\Omega) \setminus B_{2R}(0)$ such that $\widehat{\Phi}(e) < \beta$.*

Proof: Since $\underline{v}$ is a subsolution of (1), $\widehat{G}(x, \underline{v}) = \big(a(x)\underline{v} + h(x, \underline{v})\big)\underline{v}$ and $p_i > 1$, for $i = 1, 2, ...N$, we have

$$\widehat{\Phi}(\underline{v}) < \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial \underline{v}}{\partial x_i} \right|^{p_i} dx - \int_\Omega (a(x)\, \underline{v} + h(x, \underline{v}))\underline{v}\ dx \leq 0.$$

Now, let $\|u\| = R > 1$, without loss of generality, we can assume that

$$\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \geq 1, \text{ for all } i = 1, 2, ..., N.$$

Hence, using this inequality and (14) with the Sobolev Embedding Theorem, we find positive constants, $c_1, c_2, c_3 > 0$ such that

$$\widehat{\Phi}(u) \geq c_1 \|u\|^{p_N} - c_2 \|u\| - c_3 |a|_{L^\infty} \|u\|^r, \text{ for all } \|u\| = R.$$

We fix R such that $c_1 R^{p_N} - c_2 R \geq c_1 R^{p_N}/2$ and $R > \|\underline{v}\|$. Choose $|a|_{L^\infty}$ small such that $c_1 R^{p_N}/2 > c_3 |a|_{L^\infty} R^r$, we have (ii) for $\beta = c_1 R^{p_N}/2 - c_3 |a|_{L^\infty} R^r$.

Now, using (h_3), we get

$$H(x,t) \geq H(x,t_0)t_0^{-\theta}t^{\theta} - C, \quad \text{for all } t \geq 0.$$

Then, we obtain for each $t > 1$

$$\widehat{\Phi}(t\underline{v}) \leq t^{p_N} \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i}\left|\frac{\partial \underline{v}}{\partial x_i}\right|^{p_i} dx - t^{\theta} \int_\Omega H(x,t_0)t_0^{-\theta}\underline{v}^{\theta}\, dx + C,$$

for some constant $C > 0$. Since $H(x,t_0)t_0 > 0$ and $\theta > p_N$ we obtain (ii). □

Remark 4.3. Note that $\underline{v}$ and e satisfy the geometric hypotheses of the Mountain Pass Theorem [3]

$$\|e - \underline{v}\| > R > 0 \quad \text{and} \quad \max\left\{\widehat{\Phi}(\underline{v}), \widehat{\Phi}(e)\right\} < \inf\left\{\widehat{\Phi}(u) : \|u - \underline{v}\| < R\right\}.$$

Proof of the Theorem 1.2: Let $\underline{v}, \overline{v}$ be the subsolution and the supersolution of (1) given in Lemma (2.5) and v_1 the solution of (1) obtain in Theorem 1.1.

Using Lemma 4.2, we conclude, with the Mountain Pass Theorem (see [3]), that

$$\widehat{c} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \widehat{\Phi}(\gamma(t)) \text{ where } \Gamma = \{\gamma \in C\big([0,1], W_0^{1,\overrightarrow{p}}(\Omega)\big) \colon \gamma(0) = \underline{v},\ \gamma(1) = e\},$$

is a critical value of $\widehat{\Phi}$.

By (7) and (12), $g(x,t) = \widehat{g}(x,t)$ for $t \in [0,\overline{v}]$, thus $\Phi(v) = \widehat{\Phi}(v)$ for $u \in [0,\overline{v}]$, where Φ and $\widehat{\Phi}$ are given in (9) and (13), respectively. Then,

$$\widehat{\Phi}(v_1) = \inf_{\mathcal{M}} \Phi(u),$$

where $\mathcal{M} = [\underline{v}, \overline{v}]$ was given in proof of Theorem 1.

Therefore, the problem (1) has two weak solutions $v_1, v_2 \in W_0^{1,\overrightarrow{p}}(\Omega)$, such that

$$\widehat{\Phi}(v_1) \leq \widehat{\Phi}(\underline{v}) < 0 < \beta \leq \widehat{c} = \widehat{\Phi}(v_2).$$

Recall that $v_1 \in [\underline{v}, \overline{v}]$, thus $v_1 > 0$. Now, we will show that $v_2 > 0$.

Taking $(\underline{v} - v_2)^+$, as test function and defining $\{v_2 < \underline{v}\} := \{x \in \Omega : v_2(x) < \underline{v}(x)\}$, we have

$$\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial v_2}{\partial x_i}\right|^{p_i-2} \frac{\partial v_2}{\partial x_i}\frac{\partial}{\partial x_i}(\underline{v} - v_2)^+\, dx = \int_{\{v_2<\underline{v}\}} a(x)\underline{v} + h(x,\underline{v})(\underline{v} - v_2)dx. \quad (17)$$

Since $\underline{v}$ is a subsolution of (1), using (17) we obtain

$$\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial \underline{v}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{v}}{\partial x_i}\frac{\partial}{\partial x_i}(\underline{v} - v_2)^+\, dx - \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial v_2}{\partial x_i}\right|^{p_i-2} \frac{\partial v_2}{\partial x_i}\frac{\partial}{\partial x_i}(\underline{v} - v_2)^+\, dx \leq 0.$$

From inequality (5), we conclude that $\|(\underline{v} - v_2)^+\| \leq 0$, this implies

$$0 < \underline{v}(x) \leq v_2(x) \quad \text{a.e in } \Omega.$$

□

References

[1] A. Alberico, G. di Blasio, F. Feo: *A priori estimates for solutions to anisotropic elliptic problems via symmetrization*, Math. Nachrichten 290(7) (2017) 986–1003.

[2] C. O. Alves, A. El Hamidi: *Existence of solution for a anisotropic equation with critical exponent*, Diff. Int. Equations 21(1) (2008) 25–40.

[3] A. Ambrosetti, P. H. Rabinowitz: *Dual variational methods in critical point theory and applications*, J. Funct. Analysis 14 (1973) 349–381.

[4] S. Antontsev, J. I. Diaz, S. Shmarev: *Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics*, Progress in Nonlinear Differential Equations and their Applications 48, Birkhäuser, Boston (2002).

[5] H. Ayadi, F. Mokhtari: *Nonlinear anisotropic elliptic equations with variable exponents and degenerate coercivity*, Electronic J. Diff. Equations (2018), No. 45, 1–23.

[6] J. Bear: *Dynamics of Fluids in Porous Media*, American Elsevier, New York (1972).

[7] M. Bendahmane, K. H. Karlsen: *Renormalized solutions of an anisotropic reaction-diffusion-advection system with L^1 data*, Commun. Pure Appl. Analysis 5(4) (2006) 733–762.

[8] M. Bendahmane, M. Langlais, M. Saad: *On some anisotropic reaction-diffusion systems with L1-data modeling the propagation of an epidemic disease*, Nonlinear Analysis 54(4) (2003) 617–636.

[9] H. Brézis: *Analyse Fonctionnelle: Théorie et Applications*, Masson, Paris (1983).

[10] A. Di Castro: *Existence and regularity results for anisotropic elliptic problems*, Advanced Nonlinear Studies 9 (2009) 367–393.

[11] A. Di Castro: *Anisotropic elliptic problems with natural growth terms*, Manuscripta Math. 135 (2011) 521–543.

[12] A. Di Castro, E. Montefusco: *Nonlinear eigenvalues for anisotropic quasilinear degenerate elliptic equations*, Nonlinear Analysis 70 (2009) 4093–4105.

[13] A. El Hamidi, J. M. Rakotoson: *Extremal functions for the anisotropic Sobolev inequalities*, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007) 741–756.

[14] B. Ellahyani, A. El Hachimi: *Existence and multiplicity of solutions for anisotropic elliptic problems with variable exponent and nonlinear Robin boundary conditions*, Electronic J. Diff. Equations (2017), No. 188, 1–17.

[15] G. M. Figueiredo, J. R. Santos Junior, A. Suarez: *Multiplicity results for an anisotropic equation with subcritical or critical growth*, Adv. Nonlinear Studies 15(2) (2015) 377–394.

[16] I. Fragala, F. Gazzola, B. Kawohl: *Existence and nonexistence results for anistropic quasilinear elliptic quations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004) 715–734.

[17] S. Khademloo, G. A. Afrouzi, T. Norouzi Ghara: *Infinitely many solutions for anisotropic variable exponent problems*, Complex Var. Elliptic Equations 63 (2018) 1353–1369.

[18] A. R. Leggat, S. E. Miri: *Anisotropic problem with singular nonlinearity*, Complex Var. Elliptic Equations 61 (2016) 484–495.

[19] F. Mokhtari: *Regularity of the solution to nonlinear anisotropic elliptic equations with variable exponents and irregular data*, Mediterr. J. Math. 14 (2017), no. 141.

[20] K. Perera, E. A. B. Silva: *Existence and multiple of positive solutions for singular quasilinear problems*, J. Math. Anal. Appl. 323 (2006) 1238–1252.

[21] P. H. Rabinowitz: *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, Regional Conference Series in Mathematics 65, published for the Conference Board of the Mathematical Sciences, American Mathematical Society, Providence (1986).

[22] V. Radulescu, I. L. Stancu: *Combined concave-convex efects in anisotropic elliptic equations with variable exponent*, Nonlinear Diff. Equations Appl. 22 (2015) 391–410.

[23] G. Stampacchia: *Equations Elliptiques du Seconde Ordre à Coefficients Discontinues*, Les Presses de L'Université de Montreal (1966).

[24] M. Struwe: *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Berlin (1996).

[25] N. Thanh Chung, H. Quoc Toan, *On a class of anisotropic elliptic equations without Ambrosetti-Rabinowitz type conditions*, Nonlinear Analysis, Real World Appl. 16 (2014) 132–145.

[26] J. Vétois: *The blow-up of critical anisotropic equations with critical directions*, Nonlinear Diff. Equations Appl. 18 (2011) 173–197.

[27] J. Vétois: *Strong Maximum Principles for Anisotropic Elliptic and Parabolic Equations*, Advanced Nonlinear Studies 12 (2012) 101–114.

[28] J. Vétois: *Decay estimates and a vanishing phenomenon for the solutions of critical anisotropic equations*, Advances Math. 284 (2015) 122–158.

[29] H. Yang, Y. Zhang: *Bubbling solutions for an anisotropic planar elliptic problem with exponential nonlinearity*, Nonlinear Analysis 174 (2018) 141–168.