Journal of Convex Analysis
Volume 28 (2021), No. 1, 19–30

A Family of Volterra Cubic Stochastic Operators

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Received: June 20, 2019
Accepted: February 22, 2020

We consider a convex combination of Volterra cubic stochastic operators defined on a two-dimensional simplex depending on the parameter θ and study their trajectory behaviours. We show that at the values $\theta = 1/2$, the trajectories change their orientation. Moreover, for $\theta < 1/2$ any Volterra cubic stochastic operator has the property being regular and it is non-ergodic while $\theta > 1/2$.

Keywords: Quadratic stochastic operator, Volterra operator, cubic stochastic operator.

2010 Mathematics Subject Classification: Primary 37N25; secondary 92D10.

1. Introduction

The evolution of a population can be studied by a dynamical system of a quadratic stochastic operator [14]. Such evolution operators frequently arise in many models of mathematical genetics, namely in the theory of heredity (see e.g. [1, 3, 4, 5, 6, 7, 9, 13, 14, 17, 18, 26]).

Let $E = \{1, \dots, m\}$ be a finite set. The set of all probability distributions on E is then given by

$$S^{m-1} = \{\mathbf{x} = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : \; x_i \geq 0, \text{ for any } i \text{ and } \sum_{i=1}^{m} x_i = 1\},$$

the $(m-1)$-dimensional simplex. We consider a continuous map $\Psi\colon S^{m-1} \to S^{m-1}$. The trajectory $\{\mathbf{x}^{(n)}\}$, $n = 0, 1, 2, \dots$ of Ψ for an initial value $\mathbf{x}^{(0)} \in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = \Psi\big(\mathbf{x}^{(n)}\big) = \Psi^{n+1}\big(\mathbf{x}^{(0)}\big), \quad n = 0, 1, 2, \dots$$

Denote by $\omega_\Psi\big(\mathbf{x}^{(0)}\big)$ the set of limit points of the trajectory $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$.

A mapping Ψ is called *regular* if the limit $\lim\limits_{n\to\infty} \Psi^n(\mathbf{x})$ exists for any initial value $\mathbf{x} \in S^{m-1}$. A mapping Ψ is said to be *ergodic* if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Psi^k(\mathbf{x})$$

exists for any $\mathbf{x} \in S^{m-1}$.

ISSN 0944-6532 / \$ 2.50

A *quadratic stochastic operator* (QSO) is a mapping $V : S^{m-1} \to S^{m-1}$ of the simplex into itself, of the form $V(\mathbf{x}) = \mathbf{x}' \in S^{m-1}$, where

$$V : x'_k = \sum_{i,j\in E} p_{ij,k} x_i x_j, \quad k \in E,$$

and the coefficients $p_{ij,k}$ satisfy

$$p_{ij,k} = p_{ji,k} \geq 0, \quad \sum_{k=1}^{m} p_{ij,k} = 1, \quad i,j,k \in E.$$

The main problem in mathematical biology consists in the study of the asymptotical behaviour of the trajectories for a given QSO. In other words, the main task is the description of the set $\omega_V\big(\mathbf{x}^{(0)}\big)$ for any initial point $\mathbf{x}^{(0)} \in S^{m-1}$ for a given QSO V. This problem is an open problem even in the two-dimensional case.

A QSO is called *Volterra operator* if $p_{ij,k} = 0$ for any $k \notin \{i,j\}$, $i,j,k = 1,\dots,m$. The asymptotic behaviour of trajectories Volterra QSOs was analysed in [7, 8] using the theory of Lyapunov functions and tournaments.

On the basis of numerical calculations, Ulam conjectured that any QSO is ergodic [20]. But in 1977 Zakharevich [25] considered the following QSO on S^2

$$x'_1 = x_1^2 + 2x_1x_2, \quad x'_2 = x_2^2 + 2x_2x_3, \quad x'_3 = x_3^2 + 2x_1x_3 \tag{1}$$

and showed that it is a non-ergodic transformation, proving that Ulam's conjecture is false in general. Later, Ganikhodjaev and Zanin [5] established a sufficient condition for a QSO defined on S^2 to be a non-ergodic transformation, that is, Zakharevich's result was generalized to a class of Volterra QSOs defined on S^2. In [3], we have shown the correlation between non-ergodicity of Volterra QSOs and rock-paper-scissors games, so that the QSO (1) can be reinterpreted in terms game theory as a rock-paper-scissors game. In [12] the random dynamics of Volterra QSOs is studied.

For a recent review on the theory of quadratic stochastic operators see [9].

A mapping $V\colon S^{m-1} \to S^{m-1}$ is called a *Lotka-Volterra operator* (in short, an LV-operator) if every face of the simplex is invariant under V, i.e., we obtain $V(\Gamma_I) \subset \Gamma_I$ for any $I \subset E$, where $\Gamma_I = conv\{\mathbf{e}_k\}_{k\in I}$ and $\{\mathbf{e}_k\}_{k=1}^m$ is the standard basis of $\mathbb{R}^m$. An LV-operator is a discrete analog of a generalized predator-prey model (see [19]). In [19] the results of [5], [25] were generalized to a class of LV-operators.

In the present paper we consider a certain family of discrete-time dynamical systems generated by a cubic stochastic operators (CSO), i.e. another class of nonlinear evolution operators which are different from quadratic operators. In particular, CSOs depend on a parameter θ. For more motivations of such investigations see [11, 15, 16] and references therein. In [15], a notion of cubic stochastic operator is introduced and investigated.

In Section 2 we recall the definitions and known results. In Section 3 we prove that for $\theta < 1/2$ (resp. $\theta > 1/2$) any Volterra CSO of this family is regular (resp. non-ergodic) operator.

2. Preliminaries and known results

Let $S := S^2 = \{\mathbf{x} \in \mathbb{R}^3 : x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_1 + x_2 + x_3 = 1\}$.

Vallander [21] studied a family of quadratic stochastic operators $V_\theta : S \to S$ defined by $V_\theta = \theta V_1 + (1-\theta)V_0,\ \ \theta \in [0,1]$, where

$$V_0(\mathbf{x}) = \big(x_1^2 + 2x_2x_3,\ x_2^2 + 2x_1x_3,\ x_3^2 + 2x_1x_2\big),$$

$$V_1(\mathbf{x}) = \big(x_1^2 + 2x_1x_2,\ x_2^2 + 2x_1x_3,\ x_3^2 + 2x_2x_3\big).$$

It is proven that the vertices $\mathbf{e}_1 = (1,0,0)$; $\mathbf{e}_2 = (0,1,0)$; $\mathbf{e}_3 = (0,0,1)$ of the simplex and the center $\mathbf{c} = (1/3, 1/3, 1/3)$ are fixed points of the operator V_θ, that is, they are solutions of the equation $V_\theta(\mathbf{x}) = \mathbf{x}$. The set $S \cap \{\mathbf{x} \in S : x_1 = x_3\}$ is an invariant set respect to V_θ. There exists a critic value $\theta_{cr} = 3/4$ and there is a line consisting from the fixed points of V_θ whenever $\theta = \theta_{cr}$.

Theorem 2.1. ([22]) *For the operator V_θ the following statements are true:*

(i) *if $0 \leq \theta < \theta_{cr}$ then $\omega_{V_\theta}\big(\mathbf{x}^{(0)}\big) = \{\mathbf{c}\}$ for any $\mathbf{x}^{(0)} \in S$ except the vertices;*

(ii) *if $\theta = \theta_{cr}$ then $S = \bigcup_{c\in[-1,1]} \{\mathbf{x} \in S : x_1 - x_3 = c\}$ and*

$$\omega_{V_\theta}\big(\mathbf{x}^{(0)}\big) = \Big\{\Big(\frac{1+3c+\sqrt{1+3c^2}}{6}, \frac{2-\sqrt{1+3c^2}}{3}, \frac{1-3c+\sqrt{1+3c^2}}{6}\Big)\Big\}$$

for any initial $\mathbf{x}^{(0)} \in \{\mathbf{x} \in S : x_1 - x_3 = c\}$;

(iii) *if $\theta_{cr} < \theta \leq 1$ then*

$$\omega_{V_\theta}\big(\mathbf{x}^{(0)}\big) = \begin{cases} \{\mathbf{e}_1\}, & \textit{if} \quad x_1^{(0)} > x_3^{(0)}, \\ \{\mathbf{c}\}, & \textit{if} \quad x_1^{(0)} = x_3^{(0)} > 0, \\ \{\mathbf{e}_3\}, & \textit{if} \quad x_1^{(0)} < x_3^{(0)}. \end{cases}$$

Ganikhodjaev [6] considered a family quadratic operators $V_\lambda : S \to S$ defined by $V_\lambda = \lambda V_2 + (1-\lambda)V_3,\ \ \lambda \in [0,1]$, where

$$V_2(\mathbf{x}) = \big(x_1^2 + 2x_1x_2,\ x_2^2 + 2x_2x_3,\ x_3^2 + 2x_1x_3\big),$$

$$V_3(\mathbf{x}) = \big(x_1^2 + 2x_2x_3,\ x_2^2 + 2x_1x_3,\ x_3^2 + 2x_1x_2\big).$$

Note that the operator V_2 is known as Zakharevich's QSO (1). It is also called Stein-Ulam Spiral Map in some references.

Let $DV(\mathbf{x}^*) = \big(\partial V_i/\partial x_j(\mathbf{x}^*)\big)_{i,j=1}^{m}$ be the Jacobi matrix of the operator V at the point $\mathbf{x}^*$.

A fixed point $\mathbf{x}^*$ is called *hyperbolic* if its Jacobi matrix $DV(\mathbf{x}^*)$ has no eigenvalues 1 in absolute value. A hyperbolic fixed point $\mathbf{x}^*$ is called *attracting* (resp. *repelling*) if all the eigenvalues of the Jacobi matrix $DV(\mathbf{x}^*)$ are less (resp. greater) than 1 in absolute value; it is called a saddle if some of eigenvalues of $DV(\mathbf{x}^*)$ are less than 1 in absolute value and other eigenvalues are greater than 1 in absolute value (see [2]).

Define $\operatorname{int} S = \{\mathbf{x} \in S : x_1x_2x_3 > 0\}$ and $\partial S = S \setminus \operatorname{int} S$.

Theorem 2.2. [6] *For the operator V_λ the following statements are true:*

(i) *the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and the center $\mathbf{c}$ are fixed points;*

(ii) *if $\lambda > 1/2$ then the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are repelling points and they are saddles when $\lambda < 1/2$;*

(iii) *if $\lambda > 1 - \sqrt{3}/2$ then the center $\mathbf{c}$ is an attracting point and it is a repelling point when $\lambda < 1 - \sqrt{3}/2$.*

(iv) *if $\lambda > 1/2$ then $\omega_{V_\lambda}\big(\mathbf{x}^{(0)}\big) = \{\mathbf{c}\}$ for any $\mathbf{x}^{(0)} \in S$ except the vertices;*

(v) *if $0 < \lambda < 1 - \sqrt{3}/2$ then $\omega_{V_\lambda}\big(\mathbf{x}^{(0)}\big) \subset int S$ is an infinite compact set for any $\mathbf{x}^{(0)} \in int S$ except the center $\mathbf{c}$.*

Vallander [24] studied a family of quadratic stochastic operators on S defined by $V_\lambda = \lambda V_2 + (1-\lambda)V_4, \quad \lambda \in [0,1]$, where

$$V_4(\mathbf{x}) = \big(x_1^2 + 2x_1x_3,\ x_2^2 + 2x_1x_2,\ x_3^2 + 2x_2x_3\big).$$

It is evident that V_λ is the identity map whenever $\lambda = 1/2$. The trajectories under consideration are (discrete) *spirals* with infinitely many coils approaching ∂S [24].

A continuous function $\varphi : S \to \mathbb{R}$ is called a Lyapunov function for an operator V if $\varphi(V(\mathbf{x})) \le \varphi(\mathbf{x})$ for all $\mathbf{x}$ (or $\varphi(V(\mathbf{x})) \ge \varphi(\mathbf{x})$ for all $\mathbf{x}$).

Theorem 2.3. [24] *For the operator V_λ the following statements are true:*

(i) *if $\lambda \ne 1/2$ then the function $\varphi(\mathbf{x}) = x_1x_2x_3$ is a Lyapunov function;*

(ii) *if $\lambda < 1/2$ (resp. $\lambda > 1/2$) then the trajectories of V_λ are spirals similar to the trajectories of V_2 (resp. V_4).*

It is easy to understand that, as λ approaches $1/2$, the "speed of rotation" of these spirals unboundedly decreases and at the point $\lambda = 1/2$, the trajectories change their orientation.

Ganikhodjaev et al. [4] studied two families of quadratic stochastic operators on S defined by $V_\alpha = (1-\alpha)V_2 + \alpha V_5, \quad \alpha \in [0,1]$ and $V_\beta = (1-\beta)V_2 + \beta V_6, \quad \beta \in [0,1]$, where

$$V_5(\mathbf{x}) = \big(x_2^2 + 2x_1x_2,\ x_3^2 + 2x_2x_3,\ x_1^2 + 2x_1x_3\big),$$

$$V_6(\mathbf{x}) = \big(x_3^2 + 2x_1x_2,\ x_1^2 + 2x_2x_3,\ x_2^2 + 2x_1x_3\big).$$

Theorem 2.4. [4] *For the operator V_α the following statements are true:*

(i) *if $\alpha = 1/2$ then the function $\varphi(\mathbf{x}) = |x_1 - x_2||x_2 - x_3||x_3 - x_1|$ is a Lyapunov function and $\omega_{V_\alpha}\big(\mathbf{x}^{(0)}\big) = \{\mathbf{c}\}$ for any $\mathbf{x}^{(0)} \in S$;*

(ii) *if $\alpha \ne 1/2$ then $\omega_{V_\alpha}\big(\mathbf{x}^{(0)}\big) \subset int S$ is an infinite compact set for any $\mathbf{x}^{(0)} \in int S$ except the center $\mathbf{c}$.*

Theorem 2.5. [4] *For the operator V_β the following statements are true:*

(i) *if $0 < \beta < 1 - \sqrt{3}/2$ then $\omega_{V_\alpha}\big(\mathbf{x}^{(0)}\big) \subset int S$ is an infinite compact set for any $\mathbf{x}^{(0)} \in int S$ except the center $\mathbf{c}$.*

(ii) *if $1 - \sqrt{3}/2 \le \beta < 1$ then $\omega_{V_\beta}\big(\mathbf{x}^{(0)}\big) = \{\mathbf{c}\}$ for any $\mathbf{x}^{(0)} \in S$;*

In [10] a family of quadratic stochastic operators on S was studied defined by $V_\theta = \theta V_7 + (1-\theta)V_8, \quad \theta \in [0,1]$, where

$$V_7(\mathbf{x}) = \big(x_3^2 + 2x_2x_3,\ x_1^2 + 2x_1x_3,\ x_2^2 + 2x_1x_2\big),$$

$$V_8(\mathbf{x}) = \big(x_2^2 + 2x_2x_3,\ x_3^2 + 2x_1x_3,\ x_1^2 + 2x_1x_2\big).$$

Theorem 2.6. [10] *For the operator V_θ the following statements are true:*

(i) *if $\theta = 1/2$ then the function $\varphi(\mathbf{x}) = |x_1 - x_2||x_2 - x_3||x_3 - x_1|$ is a Lyapunov function and $\omega_{V_\theta}\big(\mathbf{x}^{(0)}\big) = \{\mathbf{c}\}$ for any $\mathbf{x}^{(0)} \in S$;*

(ii) *if $\theta \neq 1/2$ then $\omega_{V_\theta}\big(\mathbf{x}^{(0)}\big) \subset int\,S$ is an infinite compact set for any $\mathbf{x}^{(0)} \in int\,S$ except the center $\mathbf{c}$.*

3. Main results

A *cubic stochastic operator* (CSO), mapping of the simplex S^{m-1} into itself, is of the form

$$W : x'_k = \sum_{i,j,l=1}^{m} p_{ijl,k} x_i x_j x_l, \quad k = 1, \dots, m \tag{2}$$

where,
$$p_{ijl,k} \geq 0, \quad \sum_{k=1}^{m} p_{ijl,k} = 1, \quad i,j,k,l = 1,\dots,m. \tag{3}$$

A CSO is called a *Volterra* CSO if $p_{ijl,k} = 0$ for any $k \notin \{i,j,l\},\ i,j,l,k = 1,\dots,m.$

We consider the following family of CSOs $W_\theta = \theta W_1 + (1-\theta)W_2,\ \theta \in [0,1]$ defined on S, where

$$W_1(\mathbf{x}) = \big(x_1(1 + x_1x_2 - x_3^2),\ x_2(1 + x_2x_3 - x_1^2),\ x_3(1 + x_1x_3 - x_2^2)\big),$$

$$W_2(\mathbf{x}) = \big(x_1(1 + x_3^2 - x_1x_2),\ x_2(1 + x_1^2 - x_2x_3),\ x_3(1 + x_2^2 - x_1x_3)\big).$$

Note that for any $\theta \in [0,1]$ the operator W_θ is a Volterra CSO and it can be seen as a discrete version of Kolmogorov equations for three interacting species. Since the operator W_θ is the convex combination of the operators W_1, W_2 from the results of Theorems 2.1-2.6 it is natural expect that also there is a critic value of θ and depending to this value the trajectories are different. Note that if $\theta = 1/2$ the W_θ is identity map.

The faces $\Gamma_{ij} = \{\mathbf{x} \in S : x_k = 0; k \notin \{i,j\}\},\quad i \neq j \in \{1,2,3\}$ are invariant sets with respect to W_θ. The vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and the center $\mathbf{c}$ are the fixed points of the operator W_θ.

Proposition 3.1. *For the operator W_θ the following statements are true:*

(i) *The vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are non-hyperbolic fixed points;*

(ii) *If $\theta < 1/2$ then $\mathbf{c}$ is an attractor point and it is a repelling point whenever $\theta > 1/2$.*

Proof. We rewrite the operator W_θ in the following form

$$\begin{aligned} x_1' &= x_1\Big(2(1-\theta)+(1-2\theta)x_1^2+(1-2\theta)x_2^2+(1-2\theta)x_1x_2+2(2\theta-1)x_1 \\ &\qquad\qquad + 2(2\theta-1)x_2\Big), \\ x_2' &= x_2\Big(1+(1-2\theta)x_1^2+(1-2\theta)x_2^2+(1-2\theta)x_1x_2+(2\theta-1)x_2\Big), \end{aligned} \tag{4}$$

where $(x_1,x_2)\in\tilde{S}=\{(x_1,x_2): x_1,x_2\geq 0, x_1+x_2\in[0;1]\}$ and x_1,x_2 are the first two coordinates of a point lying in the simplex S.

(i) It is easy to verify that the Jacobi matrix of the operator (4) at the vertex $\mathbf{e}_1$ has the eigenvalues $\mu_1=1,\mu_2=2(1-\theta)$. So the vertex $\mathbf{e}_1$ is a non-hyperbolic point. Other vertices can be considered in a similar manner.

(ii) After simple algebra one has that the Jacobi matrix of the operator (4) at the center $\mathbf{c}$ has the eigenvalues

$$\mu_{1,2}=\frac{2\theta+5\pm(2\theta-1)\sqrt{3}i}{6} \quad \text{and} \quad |\mu_{1,2}|=\frac{\sqrt{4\theta^2+2\theta+7}}{3}.$$

The analysis of polynomial $4\theta^2+2\theta+7,\quad \theta\in[0,1]$ gives that if $0\leq\theta<1/2$ then $|\mu_{1,2}|<1$ and $|\mu_{1,2}|>1$ whenever $1/2<\theta\leq 1$. □

Proposition 3.2. *The function $\varphi(\mathbf{x})=x_1x_2x_3$ is a Lyapunov function for the operator W_θ.*

Proof. It is easy to see that the function $\varphi(\mathbf{x})=x_1x_2x_3$ is continuous on S. One has $\varphi(W_\theta(\mathbf{x}))=\varphi(\mathbf{x})\psi(\mathbf{x})$, where

$$\psi(\mathbf{x})=\big(1+(2\theta-1)(x_1x_2-x_3^2)\big)\big(1+(2\theta-1)(x_2x_3-x_1^2)\big)\big(1+(2\theta-1)(x_1x_3-x_2^2)\big).$$

The case $\theta=1/2$ is obvious.

Let $\theta>1/2$ then using the AM-GM relation one has

$$\begin{aligned} \psi(\mathbf{x}) &\leq \left(1+\frac{(2\theta-1)(x_1x_2+x_2x_3+x_1x_3-x_1^2-x_2^2-x_3^2)}{3}\right)^3 \\ &= \left(1-\frac{(2\theta-1)\big((x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2\big)}{6}\right)^3 \leq 1. \end{aligned}$$

Thus $\psi(\mathbf{x})\leq 1$ and $\varphi(W_\theta(\mathbf{x}))\leq\varphi(\mathbf{x})$ for all $\mathbf{x}\in S$.

Let $\theta<1/2$. Then, using the equality $x_3=1-x_1-x_2$, the function $\psi(\mathbf{x})$ takes on $\tilde{S}$ the following form

$$\begin{aligned} \psi(\mathbf{x}) =& \big(1+(2\theta-1)(x_1x_2-(1-x_1-x_2)^2)\big)\big(1+(2\theta-1)(x_2(1-x_1-x_2)-x_1^2)\big) \\ &\cdot\big(1+(2\theta-1)(x_1(1-x_1-x_2)-x_2^2)\big). \end{aligned}$$

After some algebra it follows that the unique solution of the system

$$\frac{\partial\psi(\mathbf{x})}{\partial x_1}=0,\quad \frac{\partial\psi(\mathbf{x})}{\partial x_2}=0,$$

namely $(1/3,1/3)\in\tilde{S}$, is the minimum of $\psi(\mathbf{x})$, that is, $\min\limits_{\mathbf{x}\in S}\psi(\mathbf{x})=1$ iff $\mathbf{x}=\mathbf{c}$.

Therefore $\psi(\mathbf{x})\geq 1$ and $\varphi(W_\theta(\mathbf{x}))\geq\varphi(\mathbf{x})$ for all $\mathbf{x}\in S$.

Thus the function $\varphi(\mathbf{x})=x_1x_2x_3$ is a Lyapunov function for the operator W_θ. It is a decreasing (resp. increasing) function when $\theta>1/2$ (resp. $\theta<1/2$). $\square$

Theorem 3.3. *For the operator W_θ the following statements are true:*

(i) *if $\theta<1/2$ then $\omega_{W_\theta}\big(\mathbf{x}^{(0)}\big)=\{\mathbf{e}_j\}$ for any $\mathbf{x}^{(0)}\in\Gamma_{ij}$, $\{ij\}\in\big\{\{12\},\{23\},\{31\}\big\}$;*

(ii) *if $\theta>1/2$ then $\omega_{W_\theta}\big(\mathbf{x}^{(0)}\big)=\{\mathbf{e}_i\}$ for any $\mathbf{x}^{(0)}\in\Gamma_{ij}$, $\{ij\}\in\big\{\{12\},\{23\},\{31\}\big\}$.*

Proof. Let $\theta<1/2$ and $\mathbf{x}^{(0)}\in\Gamma_{12}$. Using $x_2=1-x_1$ we can rewrite the operator W_θ in the form

$$W_\theta(x_1)=x_1\big(1+(2\theta-1)x_1(1-x_1)\big).$$

It is easy to verify that the function $W_\theta(x_1)$ is monotone decreasing in $(0,1)$ and it has $x_1=1$ repelling and $x_1=0$ non-hyperbolic fixed points and one easily has that for a $x_1\in(0,1)$ the trajectory $W_\theta^n(x_1)\to 0$ when $n\to\infty$. Similarly can be shown that if $\theta>1/2$ and $\mathbf{x}^{(0)}\in\Gamma_{12}$ for any $x_1\in(0,1)$ the trajectory $W_\theta^n(x_1)\to 1$ when $n\to\infty$.

Other faces Γ_{ij}, $\{ij\}\in\big\{\{23\},\{31\}\big\}$ can be considered in a similar manner. $\square$

Theorem 3.4. *For the operator W_θ the following statements are true:*

(i) *if $\theta<1/2$ then $\omega_{W_\theta}\big(\mathbf{x}^{(0)}\big)=\{\mathbf{c}\}$ for any $\mathbf{x}^{(0)}\in \operatorname{int}S$;*

(ii) *if $\theta>1/2$ then $\omega_{W_\theta}\big(\mathbf{x}^{(0)}\big)\subset\partial S$ is an infinite compact set for any $\mathbf{x}^{(0)}\in \operatorname{int}S\setminus\{\mathbf{c}\}$.*

Proof. It is evident that $\varphi(\mathbf{x})=x_1x_2x_3$ iff $\mathbf{x}\in\partial S$ and $\max\limits_{\mathbf{x}\in S}\varphi(\mathbf{x})=1/27$ iff $\mathbf{x}=\mathbf{c}$.

(i) Let $\theta<1/2$. Then, due to $\min\limits_{\mathbf{x}\in S}\psi(\mathbf{x})=1$ if and only if $\mathbf{x}=\mathbf{c}$, we have $\psi(\mathbf{x})>1$ for any $\mathbf{x}\in\operatorname{int}S\setminus\{\mathbf{c}\}$. Consequently it follows that $\varphi(W_\theta(\mathbf{x}))>\varphi(\mathbf{x})$ whenever $\mathbf{x}\neq\mathbf{c}$.

So $\{\varphi(\mathbf{x}^{(n)})\}$, $n=0,1,\dots$ is strict increasing bounded sequence. Hence there exits $\lim\limits_{n\to\infty}\varphi(\mathbf{x}^{(n)})=1/27$. Since $\max\limits_{\mathbf{x}\in S}\varphi(\mathbf{x})=1/27$ iff $\mathbf{x}=\mathbf{c}$ we obtain $\lim\limits_{n\to\infty}\mathbf{x}^{(n)})=\mathbf{c}$.

(ii) Let $\theta>1/2$. Then, using $\psi(\mathbf{x})\leq 1$ and $\varphi(W_\theta(\mathbf{x}))\leq\varphi(\mathbf{x})$, one obtains that for any $\mathbf{x}\in\operatorname{int}S\setminus\{\mathbf{c}\}$ the sequence $\{\varphi(\mathbf{x}^{(n)})\}$, $n=0,1,\dots$ is decreasing and bounded. Therefore there exits $\lim\limits_{n\to\infty}\varphi(\mathbf{x}^{(n)})=\tau$.

We claim that $\tau=0$. Suppose on the contrary that $\tau>0$ then we get

$$1=\lim_{n\to\infty}\frac{\varphi(\mathbf{x}^{(n+1)})}{\varphi(\mathbf{x}^{(n)})}=\lim_{n\to\infty}\frac{\varphi(\mathbf{x}^{(n)})\psi(\mathbf{x}^{(n)})}{\varphi(\mathbf{x}^{(n)})}=\lim_{n\to\infty}\psi(\mathbf{x}^{(n)}).$$

Therefore it follows that $\psi(\mathbf{x}^{(n)})$ converges to its upper bound. So the differences

$$x_1^{(n)}x_2^{(n)} - \left(x_3^{(n)}\right)^2, \quad x_2^{(n)}x_3^{(n)} - \left(x_1^{(n)}\right)^2, \quad x_1^{(n)}x_3^{(n)} - \left(x_2^{(n)}\right)^2$$

converge to zero, consequently their sum

$$x_1^{(n)}x_2^{(n)} + x_2^{(n)}x_3^{(n)} + x_1^{(n)}x_3^{(n)} - \left(x_1^{(n)}\right)^2 - \left(x_2^{(n)}\right)^2 - \left(x_3^{(n)}\right)^2$$

also converge to zero, that is, we obtain

$$\lim_{n\to\infty} x_1^{(n)} = \lim_{n\to\infty} x_2^{(n)} = \lim_{n\to\infty} x_3^{(n)} = \frac{1}{3} \Rightarrow \lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{c}.$$

But this is impossible, because it contradict $\varphi(\mathbf{x}^{(0)}) < \varphi(\mathbf{c}) = 1/27$. Thus $\tau = 0$, that is $\omega_{W_\theta}\left(\mathbf{x}^{(0)}\right) \subset \partial S$.

Since $W_\theta\left(\omega_{W_\theta}\left(\mathbf{x}^{(0)}\right)\right) = \omega_{W_\theta}\left(\mathbf{x}^{(0)}\right)$ it follows that $\{\mathbf{y}^{(n)}\}_{n=0}^{\infty} \subset \omega_{W_\theta}\left(\mathbf{x}^{(0)}\right)$ for any $\mathbf{y}^{(0)} \in \omega_{W_\theta}\left(\mathbf{x}^{(0)}\right) \setminus \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. This trajectory is an infinite set, e.g. the restriction of W_θ to the invariant face Γ_{12} has the form $x_1' = x_1(1 + (2\theta - 1)x_1(1 - x_1))$ which implies if $0 < x_1 < 1$ then $x_1 < x_1' < 1$. Thus $\omega_{W_\theta}\left(\mathbf{x}^{(0)}\right)$ is an infinite set and the proof is complete. □

Remark. Some parts of Theorem 3.4 were announced by Vallander [23] in an abstract of the conference.

Consider the partition of the simplex S by their medians $M_{12} = \{\mathbf{x} \in S : x_1 = x_2\}$, $M_{13} = \{\mathbf{x} \in S : x_1 = x_3\}$, $M_{23} = \{\mathbf{x} \in S : x_2 = x_3\}$ into six parts and their closures:

$$\begin{aligned} &G_1 = \{\mathbf{x} \in S : x_1 \geq x_2 \geq x_3\},\ G_2 = \{\mathbf{x} \in S : x_1 \geq x_3 \geq x_2\}, \\ &G_3 = \{\mathbf{x} \in S : x_3 \geq x_1 \geq x_2\},\ G_4 = \{\mathbf{x} \in S : x_3 \geq x_2 \geq x_1\}, \\ &G_5 = \{\mathbf{x} \in S : x_2 \geq x_3 \geq x_1\},\ G_6 = \{\mathbf{x} \in S : x_2 \geq x_1 \geq x_3\}. \end{aligned}$$

Proposition 3.5. *If $\theta > 1/2$ then in a long run time, the trajectory $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ goes around the periodic path $G_1 \to G_2 \to G_3 \to G_4 \to G_5 \to G_6 \to G_1$.*

Proof. Let $\mathbf{x} \in \operatorname{int} S \setminus \{\mathbf{c}\}$ be an initial point. Suppose $\mathbf{x} \in G_1$ then one easily has that $x_1' \geq x_1$, $x_2' \leq x_2$ and also $x_1' \geq x_3'$, $x_1' \geq x_2'$. Therefore either $x_1' \geq x_2' \geq x_3'$ or $x_1' \geq x_3' \geq x_2'$, that is either $\mathbf{x}' \in G_1$ or $\mathbf{x}' \in G_2$. Similarly , if $\mathbf{x} \in G_2$, then one has either $\mathbf{x}' \in G_2$ or $\mathbf{x}' \in G_3$, etc., proving the proposition. □

Denote by U_0 a neighborhood the center $\mathbf{c}$ and the neighborhoods U_1, U_2, U_3 of the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ respectively, and we define them as

$$U_1 = (G_1 \cup G_2) \setminus U_0, \quad U_2 = (G_3 \cup G_4) \setminus U_0, \quad U_3 = (G_5 \cup G_6) \setminus U_0.$$

Let us choose a neighborhood U_0 such that $U_0 \subset \operatorname{int} S$ and the neighborhoods U_1, U_2, U_3 are convex and they satisfy $\overline{U_1} \cap \overline{U_2} \cap \overline{U_3} = \emptyset$. By Proposition 3.5 the trajectory moves from the neighborhood of a vertex to the neighborhood of another vertex, staying for some time $T(n)$ in this neighborhood and then moving to the neighborhood of a third vertex, etc. We would like to estimate the staying time of the trajectory in a neighborhood of a vertex. Let U be one of neighborhoods U_1, U_2, U_3.

Proposition 3.6. *If $\theta > 1/2$, $\mathbf{x} \notin U$, $\mathbf{x}^{(s)} \in U$ for any $s = 1, \dots, n$ and $\mathbf{x}^{(n+1)} \notin U$, then there is $n \in \mathbb{N}$ such that*

$$n > A \log(B/\varphi(\mathbf{x}')),$$

where A, B some constants and $\mathbf{x}' = W_\theta(\mathbf{x})$.

Proof. Without loss of generality we can assume that $\mathbf{x} \in G_6$, $\mathbf{x}^{(s)} \in U_1$ for any $s = 1, \dots, n$ and $\mathbf{x}^{(n+1)} \in G_3$. Then for $\mathbf{x} \in G_6$ we obtain

$$\frac{x_3^{(n+1)}}{x_3'} = \prod_{k=1}^{n} \Big(1 + (2\theta - 1)\big(x_3^{(k)} x_1^{(k)} - \big(x_2^{(k)}\big)^2\big) \Big) < 2^n.$$

For $\mathbf{x} \in G_6$, $\mathbf{x}' \in U_1$ and $\mathbf{x}^{(n+1)} \in G_3$ one has

$$2^n > \frac{x_3^{(n+1)}}{x_3'} \geq \frac{1}{3x_3'} = \frac{x_1' x_2'}{3\varphi(\mathbf{x}')}.$$

From the form of W_θ it follows that

$$\begin{aligned} x_2' &= x_2\big(x_2 + x_3 + (2\theta-1)x_2x_3 + x_1 - (2\theta-1)x_1^2\big) \geq x_2^2 \geq x_2^3 = \frac{x_2(x_2^2 + 9x_2^2)}{10} \\ &\geq \frac{x_1\big(x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3 + (2\theta-1)x_1x_3\big)}{10} \geq \frac{x_1'}{10}, \end{aligned}$$

and using $\varphi(\mathbf{x}')) \leq 1$ we obtain

$$2^n > \frac{x_1' x_2'}{3\varphi(\mathbf{x}')} \geq \frac{(x_1')^2}{30\varphi(\mathbf{x}')} \geq \frac{1}{270\varphi(\mathbf{x}')}.$$

Thus
$$n > \log_2(1/(270\varphi(\mathbf{x}'))) > A\log(B/\varphi(\mathbf{x}')).$$

Moreover for the point $\mathbf{x}^{(n)}$ we obtain from the last inequalities

$$2^{T(n)}\varphi\big(\mathbf{x}^{(n+1)}\big) > \frac{1}{270}. \qquad \square \tag{5}$$

Proposition 3.7. *Let U be one of the neighborhoods U_i, $i = 1, 2, 3$ and assume $\mathbf{x} \in int\, S \setminus \{\mathbf{c}\}$. Let $\{u_i\}_{i=1}^\infty$, $\{v_i\}_{i=1}^\infty$ be the sequences of natural numbers such that $\mathbf{x}^{(u_i)} \notin U$, $\mathbf{x}^{(u_i+t)} \in U$ for $t = 1, 2, \dots, v_i$, and $\mathbf{x}^{(u_i+v_i+1)} \notin U$. Then there is a positive constant K such that $u_i > Kv_i$.*

Proof. It follows from Propositions 3.2 and 3.5 that $u_i \to \infty$ and

$$\varphi(\mathbf{x}') < \tau\varphi(\mathbf{x}), \quad \text{for any } \ \mathbf{x} \in S \setminus U_0,$$

where $\tau = \max\limits_{\mathbf{x} \in S\setminus U_0} \varphi(\mathbf{x}) < 1$. Using Proposition 3.6 one has

$$u_i > A \log\left(\frac{B}{\varphi\big(\mathbf{x}^{(u_i)}\big)}\right) > A' \log\left(\frac{B}{\tau^{u_i}\varphi\big(\mathbf{x}^{(0)}\big)}\right) > Kv_i. \qquad \square$$

From Propositions 3.5, 3.6, 3.7, it follows that the time spent in any neighborhood of any vertex tends to infinity and the average time spent in the exterior of the union of sufficiently small neighborhoods of all vertices is equal to 0.

Denote by U a neighborhood of the vertex set.

Theorem 3.8. *If $\theta > 1/2$ then for all $\mathbf{x} \in int S \setminus \{\mathbf{c}\}$ and the operator W_θ the limit of averages $\lim\limits_{n\to\infty} \Big(\mathbf{x} + W_\theta(\mathbf{x}) + \cdots + W_\theta^{n-1}(\mathbf{x})\Big)/n$ doesn't exist.*

Proof. Suppose on the contrary, let $\mathbf{x} \in \operatorname{int} S \setminus \{\mathbf{c}\}$ be an initial point and the limit of averages exits.

Due to Proposition 3.2 it follows $\omega_{W_\theta}\big(\mathbf{x}^{(0)}\big) \subset \partial S$ and from Proposition 3.6 one has that most of the time the trajectory belong to U and only finite points of the trajectory belong to ∂S, otherwise it converges to one of the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Therefore if denote by n_1 the time spent of $\mathbf{x}^{(n)}$ in U and n_2 is remaining time, then we obtain $\lim\limits_{n\to\infty} n_1/n = 1$, where $n = n_1 + n_2$. Hence for any $\tau > 0$ there exists a neighborhood U of a vertex set such that $\varphi(\mathbf{x}') \leq \tau\varphi(\mathbf{x})$ for any $\mathbf{x} \in U$. Consequently $\varphi\big(\mathbf{x}^{(n)}\big) \leq \tau^{n_1}\varphi(\mathbf{x})$ and using $\lim\limits_{n\to\infty} n_1/n = 1$, $\limsup\limits_{n\to\infty} \sqrt[n]{\varphi\big(\mathbf{x}^{(n)}\big)} \leq \tau$ and due to the fact that τ is arbitrary we obtain

$$\lim_{n\to\infty} \sqrt[n]{\varphi\big(\mathbf{x}^{(n)}\big)} = 0.$$

Using (5), one has
$$2^{T(n_1)}\varphi\big(\mathbf{x}^{(n_1+1)}\big) > \frac{1}{270}$$

and $T(n_1)/n \to \infty$ as $n \to \infty$. However the average $\big(\mathbf{x} + W_\theta(\mathbf{x}) + \cdots + W_\theta^{n-1}(\mathbf{x})\big)/n$ is arbitrarily close to the closure $\overline{U_1}$. By assumption the limit of averages exists and it lies in $\overline{U_1}$ and by the same argument it lies in $\overline{U_2}$ and $\overline{U_3}$ which contradicts to $\overline{U_1} \cap \overline{U_2} \cap \overline{U_3} = \emptyset$, and the theorem is proved. □

Acknowledgments. We thank the referee for useful comments which helped to improve the presentation.

This work was done at the University of Latvia. The first author (UJ) thanks the University of Latvia for the kind hospitality and for providing all facilities, and the State Education Development Agency of the Republic of Latvia for providing financial support by a fellowship for research. The thanks of the second author (AR) go to the project of the Institute of Mathematics and Computer Science named “Dynamic equations on time scale”.

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