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# Nondentable Sets in Banach Spaces

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In his study of the Radon-Nikodým property of Banach spaces, Bourgain showed (among other things) that in any closed, bounded, convex set $A$ that is nondentable, one can find a separated, weakly closed bush. In this note, we prove a generalization of Bourgain's result: in any bounded, nondentable set $A$ (not necessarily closed or convex) one can find a separated, weakly closed approximate bush. Similarly, we obtain as corollaries the existence of $A$-valued quasimartingales with sharply divergent behavior.

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## 1. Introduction

We were motivated by the question of whether using the Kuratowski measure of noncompactness in place of diameter leads to a different notion of dentability of (not necessarily closed or convex) subsets of $X$. Proposition 3.1 shows that they do not. This generalizes results from [1, Chap. 4] where $A$ is assumed to be closed, bounded, and convex. In Section 3, we obtain as corollaries $A$-valued quasimartingales and $\overline{\mathrm{co}}\,(A)$-valued martingales with sharply divergent behavior (Corollaries 3.3 and 3.4) whenever $A$ is non-$\varepsilon$-dentable. In Section 4, we improve the results of Section 3 by showing that the range of the quasimartingale can be made weakly closed. As a further corollary, we show that one can find a countable set $F$ with $\lim_{F\ni f\to\infty} d(f,A)\to 0$ such that $\overline{\mathrm{co}}\,(F)\cap \mathrm{Ext}\,(\overline{co}^{w*}(F))=\emptyset$ (Corol. 4.9).

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## 2. Preliminaries

For any topological vector space $V$ over $\mathbb{R}$ and $E \subseteq V$, let $\mathrm{co}\,(E)$ denote the convex hull of $E$, and $\overline{\mathrm{co}}\,(E)$ the closure of $\mathrm{co}\,(E)$ in $V$. Henceforth, let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{R}$. For $r > 0$ and $x \in X$, $B_r(x)$ denotes the open ball of radius $r$ centered at $x$. $B_X$ denotes the closed unit ball of $X$.

**Definition 2.1.** For any $A \subseteq X$, let $\alpha(A)$ be the infimum over all $\varepsilon > 0$ so that $A$ can be covered by finitely many sets of diameter at most $\varepsilon$. $\alpha(A)$ is called the *Kuratowksi measure of noncompactness* of $A$.

**Definition 2.2.** For any bounded, nonempty $A \subseteq X$, $f \in B_{X^*}$ (where $B_{X^*}$ denotes the unit ball of $X^*$), and $\delta > 0$, we define the *slice* $S(f, A, \delta)$ to be the set $\{a \in A : f(a) > \sup f(A) - \delta\}$. A *slice of* $A$ is a set $S(f, A, \delta)$ for some $f \in B_{X^*}$ and $\delta > 0$.

**Remark 2.3.** Geometrically, a slice of $A$ is a nonempty intersection of $A$ with an open half-plane. Note that if $S(f, \overline{\mathrm{co}}\,(A)\,, \delta)$ is a slice of $\overline{\mathrm{co}}\,(A)$, then

$$S(f, \overline{\mathrm{co}}\,(A)\,, \delta) \cap A = S(f, A, \delta)$$

is a slice of $A$. This is due to the fact that

$$\sup(f(\overline{\mathrm{co}}\,(A))) = \sup\left(\overline{f(\mathrm{co}\,(A))}\right) = \sup(f(\mathrm{co}\,(A))) = \sup(\mathrm{co}\,(f(A))) = \sup(f(A)).$$

**Definition 2.4.** A bounded set $A \subseteq X$ is called *$\varepsilon$-dentable* if there exists a slice of $A$ with $\mathrm{diam}(A) \leq \varepsilon$, and *non-$\varepsilon$-dentable* otherwise. $A$ is *dentable* if it is $\varepsilon$-dentable for every $\varepsilon > 0$, and *nondentable* otherwise.

**Remark 2.5.** By Remark 2.3, if $\overline{\mathrm{co}}\,(A)$ is $\varepsilon$-dentable, $A$ is $\varepsilon$-dentable.

**Definition 2.6.** If $V$ is a topological vector space, $E \subseteq V$ and $e \in E$, $e$ is called a *denting point* of $E$ if $e \notin \overline{\mathrm{co}}\,(E \setminus U)$ for every neighborhood $U$ of $e$. Special cases are when $V$ is a Banach space equipped with the weak topology, or a dual Banach space equipped with the weak* topology, in which case we call $e$ a *weak denting point* or a *weak* denting point*, respectively.

**Definition 2.7.** Let $\mathbb{N}^{<\omega}$ denote the set of finite length sequences of natural numbers. A *tree* is a nonempty set $\mathbb{T}$ such that if $b \in \mathbb{T}$ and $b = (b', i)$ for some $b' \in \mathbb{N}^{<\omega}$ and $i \in \mathbb{N}$, then $b' \in \mathbb{T}$. In this case, $b$ is called a *child* of $b'$. We say that $\mathbb{T}$ is *finitely branching* if each $b \in \mathbb{T}$ has only finitely many children. If $b \in \mathbb{T}$ has $k$ children, we assume that they are $(b, 1), \ldots, (b, k)$. Given a sequence $b \in \mathbb{N}^{<\omega}$, we let $|b|$ denote its *length.* For $n \in \mathbb{N}$, we let

$$\mathbb{T}_{\leq n} = \{b \in \mathbb{T} : |b| \leq n\}, \quad \mathbb{T}_n = \{b \in \mathbb{T} : |b| = n\},$$

and

$$\mathbb{T}_{\geq n} = \{b \in \mathbb{T} : |b| \geq n\}.$$

Given a positive sequence $(\delta_n)_{n\geq 0}$, finitely branching tree $\mathbb{T}$, and subset $(x_b)_{b\in\mathbb{T}} \subseteq X$ indexed by $\mathbb{T}$, we say that $(x_b)_{b\in\mathbb{T}}$ is a $(\delta_n)_{n\geq 0}$-*approximate bush* if for each $n \in \mathbb{N}$ and $b \in \mathbb{T}_n$ with children $(b, 1), \ldots (b, k)$, $b \in \mathrm{co}\left(x_{(b,1)}, \ldots x_{(b,k)}\right) + B_{\delta_n}(0)$. If it always holds that $b \in \mathrm{co}\left(x_{(b,1)}, \ldots x_{(b,k)}\right)$, then $(x_b)_{b\in\mathbb{T}}$ is a *bush*. An approximate bush $(x_b)_{b\in\mathbb{T}}$ is *$\delta$-separated* if for each $n \in \mathbb{N}$, $b \in \mathbb{T}_n$ and child $(b, i)$, $\|x_b - x_{(b,i)}\| > \delta$.

**Definition 2.8.** Given a filtration $(\mathcal{A}_n)_{n\geq 0}$ and a positive sequence $(\delta_n)_{n\geq 0}$, we say that a sequence of $X$-valued, $(\mathcal{A}_n)_{n\geq 0}$-adapted random variables $(M_n)_{n\geq 0}$ is a $(\delta_n)_{n\geq 0}$-*quasimartingale* if

$$\|\mathbb{E}(M_{n+1}|\mathcal{A}_n) - M_n\|_\infty \leq \delta_n$$

for all $n \geq 0$. If $\|\mathbb{E}(M_{n+1}|\mathcal{A}_n) - M_n\|_\infty = 0$ always holds, $(M_n)_{n\geq 0}$ is a *martingale.*

The following proposition can be found in [1, Lemme 4.2]. For the sake of self-containment, we include our own proof here.

**Proposition 2.9.** *Let $\varepsilon > 0$ and $\delta > 0$. Suppose that $C$ and $C_1$ are closed, bounded, convex sets with $C_1$ properly contained in $C$. If $C = \overline{\text{co}}(C_1 \cup C_2)$, where $C_2$ is a convex subset of $C$ and* $\text{diam}(C_2) < \varepsilon$*, then there exists a slice $S$ of $C$ with $S \subseteq C_2 + B_\delta(0)$. In particular, $C$ is $\varepsilon$-dentable.*

**Proof.** We may assume that $\text{diam}(C) \leq 1$. Since $C_1$ is a proper convex subset of $C$, by Hahn-Banach separation there exists $f \in B_{X^*}$ such that

$$\sup f(C_1) < M := \sup f(C).$$

Hence $C_1 \subseteq C \setminus S(f, C, \alpha)$ for some $\alpha > 0$. So

$$C = \overline{\text{co}}((C \setminus S(f, C, \alpha)) \cup C_2).$$

For $\gamma > 0$, let $S_\gamma = S(f, C, \gamma)$. Consider $y \in S_\gamma$. Then there exist $\lambda \in [0, 1]$, $z_1 \in \text{co}(C \setminus S(f, C, \alpha))$, and $z_2 \in C_2$ such that $\|y - \lambda z_1 - (1 - \lambda)z_2\| < \gamma$. Hence

$$\begin{aligned} M - \gamma < f(y) &\leq f(\lambda z_1 + (1 - \lambda)z_2) + \|y - \lambda z_1 - (1 - \lambda)z_2\| \\ &< \lambda f(z_1) + (1 - \lambda)f(z_2) + \gamma \leq \lambda(M - \alpha) + (1 - \lambda)M + \gamma \\ &= M - \lambda\alpha + \gamma. \end{aligned}$$

Hence $\lambda < 2\gamma/\alpha$. So

$$\|y - z_2\| < \lambda\|z_1 - z_2\| + \gamma \leq (2\gamma/\alpha)\,\text{diam}(C) + \gamma \leq \gamma(2/\alpha + 1)$$

So, setting $\gamma := \frac{\delta\alpha}{2+\alpha}$, we get $S := S_\gamma \subseteq C_2 + B_\delta(0)$. Note that

$$\text{diam}(S) \leq \text{diam}(C_2) + 2\delta < \varepsilon$$

for $\delta$ sufficiently small. So $C$ is $\varepsilon$-dentable. □

We now derive a corollary of this proposition that will play a crucial role in the proof of Lemma 4.3.

**Corollary 2.10.** *For any closed, bounded, convex, non-$\varepsilon$-dentable $C \subseteq X$, any closed, convex $C' \subseteq C$, and any $D \subseteq C$ with $\alpha(D) < \varepsilon$, if $C = \overline{\text{co}}\,(C' \cup D)$, then $C = C'$.*

**Proof.** Let $C$, $C'$, and $D$ be as above. Assume $C = \overline{\text{co}}\,(C' \cup D)$. Since $\alpha(D) < \varepsilon$, $D = B_1 \cup B_2 \cup \ldots B_n$ for some $B_i \subseteq D$ with $\text{diam}(B_i) < \varepsilon$. Let $C_i = \overline{\text{co}}\,(B_i)$.

Then $\operatorname{diam}(C_i) = \operatorname{diam}(B_i) < \varepsilon$, and $C = \overline{\text{co}}\,(C' \cup C_1 \cup C_2 \cup \ldots C_n)$. Since $C$ is closed, bounded, convex, and not $\varepsilon$-dentable, and since $C_n \subseteq C$ is closed and convex with $\operatorname{diam}(C_n) < \varepsilon$, Proposition 2.9 (where we take $C_2 = C_n$ and $C_1 = \overline{\text{co}}\,(C' \cup C_1 \cup C_2 \cup \ldots C_{n-1}))$ implies that $C = \overline{\text{co}}\,(C' \cup C_1 \cup C_2 \cup \ldots C_{n-1})$. Since $\operatorname{diam}(C_{n-1}) < \varepsilon$, we may apply Proposition 2.9 again to obtain finally $C = \overline{\text{co}}\,(C' \cup C_1 \cup C_2 \cup \ldots C_{n-2})$. Iterating, we get $C = C'$. □

## 3. $\delta$-Separated martingales and bushes

**Proposition 3.1.** *Let $A \subseteq X$ be bounded, and let $\varepsilon > 0$. The following are equivalent:*

(i) *$\alpha(S) \geq \varepsilon$ for every slice $S \subseteq A$;*

(ii) *diam$(S) \geq \varepsilon$ for every slice $S$ of $A$ ($A$ is non-$\varepsilon$-dentable);*

(iii) *diam$(S) \geq \varepsilon$ for every slice $S$ of $\overline{co}\,(A)$ ($\overline{co}\,(A)$ is non-$\varepsilon$-dentable).*

**Proof.** Let $A$, $\varepsilon$ be as above. (1) $\to$ (2) is clear from definition of $\alpha$. (2) $\to$ (3) follows from the fact that every slice of $\overline{\text{co}}\,(A)$ contains a slice of $A$. We now show (3) $\to$ (1) by contradiction. Let $C = \overline{\text{co}}\,(A)$, assume that $C$ is non-$\varepsilon$-dentable and that there exists a slice $S = S(f, A, \delta)$ of $A$ with $\alpha(S) < \varepsilon$. Set $S_C = S(f, C, \delta)$. Then since $C \setminus S_C$ is a closed convex subset of $C$ and $C = \overline{\text{co}}\,((C \setminus S_C) \cup S)$. Then Corollary 2.10 implies that $C = C \setminus S_C$, a contradiction since $S_C \subseteq C$ and $S_C$ is nonempty. □

As in [1, Chapitre 4], we obtain several corollaries.

**Corollary 3.2.** *For any $A \subseteq X$ bounded and $\varepsilon > 0$, if $A$ is non-$\varepsilon$-dentable, then for all $\delta < \frac{\varepsilon}{2}$ and all $a_1, a_2, \ldots a_n \in A$,*

$$\overline{co}\,(A) = \overline{co}\,(A \setminus (B_\delta(a_1) \cup B_\delta(a_2) \cup \ldots B_\delta(a_n))).$$

**Proof.** Let $A$, $\varepsilon$, $\delta$, and $a_1, a_2, \ldots a_n$ be as above. Suppose that there exists

$$x \in \overline{\text{co}}\,(A) \setminus \overline{\text{co}}\,(A \setminus (B_\delta(a_1) \cup B_\delta(a_2) \cup \ldots B_\delta(a_n))).$$

By Hahn-Banach separation, we can pick a slice $S$ of $\overline{\text{co}}\,(A)$ containing $x$ and disjoint from $\overline{\text{co}}\,(A \setminus (B_\delta(a_1) \cup B_\delta(a_2) \cup \ldots B_\delta(a_n)))$.

Then $S \cap A$ is a slice of $A$ disjoint from $A \setminus (B_\delta(a_1) \cup B_\delta(a_2) \cup \ldots B_\delta(a_n))$, and thus $S \cap A \subseteq B_\delta(a_1) \cup B_\delta(a_2) \cup \ldots B_\delta(a_n)$, which implies $\alpha(S \cap A) \leq 2\delta < \varepsilon$, contradicting Proposition 3.1. □

We can use Corollary 3.2 to construct $A$-valued quasimartingales and $\overline{\text{co}}\,(A)$-valued martingales that diverge in a sharp manner.

**Corollary 3.3.** *For any nonempty, bounded, non-$\varepsilon$-dentable $A \subseteq X$, any $\delta < \frac{\varepsilon}{2}$, and any positive, summable sequence $(\delta_n)_{n\geq 0}$, there exists a filtration of finite $\sigma$-algebras $(\mathcal{A}_n)_{n\geq 0}$ on $[0,1]$, each of whose atoms are intervals, and an $(\mathcal{A}_n)_n$-adapted sequence of random variables $(M_n)_{n\geq 0}$ such that, for all $s, t \in [0,1]$ and $m \neq n \geq 0$,*

(i) *$M_n$ takes values in $A$;*

(ii) *$\|M_n(s) - M_m(t)\| > \delta$;*

(iii) *$(M_n)_{n\geq 0}$ is a $(\delta_n)_{n\geq 0}$-quasimartingale: $\|\mathbb{E}(M_{n+1}|\mathcal{A}_n) - M_n\|_\infty < \delta_n$.*

**Proof.** Let $A \subseteq X$ and $\delta > 0$ be as above. We construct the martingale inductively. Let $x_0$ be any point of $A$, $\mathcal{A}_0$ the trivial $\sigma$-algebra, and $M_0 \equiv x_0$. Suppose that, for some $N \in \mathbb{N}$, $\mathcal{A}_n$ and $M_n$ have been constructed for all $n \leq N$ and satisfy the conclusion of the Corollary 3.3. Let $J$ be an atom of $\mathcal{A}_N$, and let $x_J$ be the value of $M_N$ on $J$. Let $\{a_1, a_2, \dots a_k\} \subseteq A$ be the set of all elements in the image of any one of the $M_n$, $n \leq N$. By Corollary 3.2, we have $x_J \in \overline{\mathrm{co}}\,(A \setminus (B_\delta(a_1) \cup B_\delta(a_2) \cup \dots B_\delta(a_k)))$. Thus, there exists some $z_J \in \mathrm{co}\,(A \setminus (B_\delta(a_1) \cup B_\delta(a_2) \cup \dots B_\delta(a_k)))$ such that $\|x_J - z_J\| < \delta_N$. Since $z_J \in \mathrm{co}\,(A \setminus (B_\delta(a_1) \cup B_\delta(a_2) \cup \dots B_\delta(a_k)))$, $z_J = \lambda_1 z_J^1 + \lambda_2 z_J^2 + \dots \lambda_m z_J^m$ for some $z_J^1, z_J^2, \dots z_J^m \in A$ and $\lambda_1, \lambda_2, \dots \lambda_m \in (0,1)$ with $\lambda_1 + \lambda_2 + \dots \lambda_m = 1$ and $\|z_J^i - a_j\| > \delta$ for all $i \leq m$ and $j \leq k$. Now we subdivide the interval $J$ into $m$ pairwise disjoint subintervals, $J_1, J_2, \dots J_m$, with $|J_i| = \lambda_i |J|$ for each $i$.

Repeating this process for each atom $J \in \mathcal{A}_N$ gives us a collection of pairwise disjoint intervals, and we define $\mathcal{A}_{N+1}$ to be the $\sigma$-algebra that they generate. On each $J_i$, we define $M_{N+1}$ to be $z_J^i$. Then conclusions (1) and (2) hold, and (3) holds since $\|\mathbb{E}(M_{N+1}|\mathcal{A}_N) - M_N\|_\infty = \sup_{J,i} \|z_J - z_J^i\| < \delta_N$. □

**Corollary 3.4.** *For any nonempty, bounded, non-$\varepsilon$-dentable $A \subseteq X$, any $\delta < \frac{\varepsilon}{2}$, and any positive, summable sequence $(\delta_n)_{n\geq 0}$, there exists a filtration of finite $\sigma$-algebras $(\mathcal{A}_n)_{n\geq 0}$ on $[0,1]$, each of whose atoms are intervals, an $(\mathcal{A}_n)_{n\geq 0}$-adapted quasimartingale $(M_n)_{n\geq 0}$, and an $(\mathcal{A}_n)_{n\geq 0}$-adapted martingale $(\overline{M}_n)_{n\geq 0}$ such that, for all $s, t \in [0,1]$ and $m \neq n \geq 0$,*

(i) *$M_n$ takes values in $A$;*

(ii) *$\overline{M}_n$ takes values in $\overline{co}\,(A)$;*

(iii) *$\|M_n - \overline{M}_n\|_\infty < \delta_n$;*

(iv) *$\|M_n(s) - M_m(t)\|, \|\overline{M}_n(s) - \overline{M}_m(t)\| > \delta$.*

**Proof.** Let $A \subseteq X$, and $\delta > 0$ be as above. Choose $\delta' \in (\delta, \frac{\varepsilon}{2})$ and assume $\sum_{n=0}^\infty \delta_n < \delta' - \delta$. Choose a positive sequence $(\gamma_k)_{k\geq 0}$ such that $\sum_{k=n}^\infty \gamma_k < \delta_n$, and note that this implies

$$\sum_{n=0}^\infty \gamma_n < \sum_{n=0}^\infty \delta_n < \delta' - \delta.$$

By Corollary 3.3, there is a filtration $(\mathcal{A}_n)_{n\geq 0}$ and an $A$-valued, $(\mathcal{A}_n)_{n\geq 0}$-adapted quasimartingale $(M_n)_{n\geq 0}$ such that $\|M_n(s) - M_m(t)\| > \delta'$ for all $s, t \in [0,1]$, $m \neq n$, and $\|\mathbb{E}(M_{n+1}|\mathcal{A}_n) - M_n\|_\infty < \gamma_n$. This inequality, together with the fact that $(\delta_n)_{n\geq 0}$ is summable (and thus convergent to 0), implies, for each $n \geq 0$, the sequence $(\mathbb{E}(M_k|\mathcal{A}_n))_{k\geq n}$ is Cauchy in $L^\infty(I;X)$. Indeed, for $k > j \geq n$,

$$\begin{aligned}
&\|\mathbb{E}(M_k - M_j|\mathcal{A}_n)\|_{L^\infty(I;X)} \leq \sum_{r=j}^{k-1} \|\mathbb{E}(M_{r+1} - M_r|\mathcal{A}_n)\|_{L^\infty(I;X)} \\
&\leq \sum_{r=j}^{k-1} \|\mathbb{E}(M_{r+1} - M_r|\mathcal{A}_r)\|_{L^\infty(I;X)} = \sum_{r=j}^{k-1} \|\mathbb{E}(M_{r+1}|\mathcal{A}_r) - M_r\|_{L^\infty(I;X)} \\
&\leq \sum_{r=j}^{k-1} \gamma_r \leq \delta_j
\end{aligned}$$

Thus we may set $\overline{M}_n := \lim_{k\to\infty} \mathbb{E}(M_k|\mathcal{A}_n)$. Clearly, $(\overline{M}_n)_{n\geq 0}$ is adapted to $(\mathcal{A}_n)_{n\geq 0}$ and takes values in $\overline{\mathrm{co}}\,(A)$, showing (2). Let us check the martingale property:

$$\begin{aligned}\mathbb{E}(\overline{M}_{n+1}|\mathcal{A}_n) &= \mathbb{E}(\lim_{k\to\infty} \mathbb{E}(M_k|\mathcal{A}_{n+1})|\mathcal{A}_n) = \lim_{k\to\infty} \mathbb{E}(\mathbb{E}(M_k|\mathcal{A}_{n+1})|\mathcal{A}_n)\\ &= \lim_{k\to\infty} \mathbb{E}(M_k|\mathcal{A}_n) = \overline{M}_n\end{aligned}$$

showing (1). Next,

$$\begin{aligned}\|\overline{M}_n - M_n\|_\infty &\leq \sum_{k=n}^{\infty} \|\mathbb{E}(M_{k+1} - M_k|\mathcal{A}_n)\|_\infty \leq \sum_{k=n}^{\infty} \|\mathbb{E}(M_{k+1} - M_k|\mathcal{A}_k)\|_\infty\\ &= \sum_{k=n}^{\infty} \|\mathbb{E}(M_{k+1}|\mathcal{A}_k) - M_k\|_\infty \leq \sum_{k=n}^{\infty} \gamma_k < \delta_n\end{aligned}$$

showing (3). We then use (3) to show (4):

$$\|\overline{M}_n(s) - \overline{M}_m(t)\| \geq \|M_n(s) - M_m(t)\| - \delta_n - \delta_m > \delta' - (\delta' - \delta) = \delta. \qquad \square$$

**Remark 3.5.** The union over $n$ of the image of $M_n$ forms a $\delta$-separated bush in $\overline{\mathrm{co}}\,(A)$. It is norm closed and lacks extreme points.

## 4. Weakly closed $\delta$-separated martingales and bushes

In this section, we sharpen our results from the previous section by constructing an $A$-valued $\delta$-separated approximate bush that is weakly closed. The argument is more involved than those of the previous section. This again extends results from Bourgain in [1]. $A$ is not assumed to be closed or convex in our case.

**Definition 4.1.** Let $A \subseteq B_X$ and let $C = \overline{\mathrm{co}}\,(A)$. For any $\gamma \in (0,1)$ and slice $S = S(f, C, \delta)$ of $C$, we define $S^\gamma = S\left(f, C, \frac{\gamma\delta}{2}\right)$. $S^\gamma$ is called a *$\gamma$-shallow parallel of* $S$.

**Lemma 4.2.** *For any $C \subseteq B_X$ closed and convex, any $\gamma \in (0,1)$, and any slice $S$ of $C$, we have $S^\gamma \subseteq S$. For any $E \subseteq C$ for which $C = \overline{co}\,((C \setminus S) \cup E)$, we have*

$$S^\gamma \subseteq \overline{co}\,(E) + \overline{B}_\gamma(0) \subseteq co\,(E) + B_{2\gamma}(0).$$

**Proof.** Let $\gamma \in (0,1)$ and $S = S(f, C, \delta)$ a slice of $C$. Since $\gamma \in (0,1)$, $\frac{\gamma\delta}{2} < \delta$ implying $S^\gamma = S\left(f, C, \frac{\gamma\delta}{2}\right) \subseteq S(f, C, \delta) = S$. For the second part, let $E \subseteq C$ such that $C = \overline{\mathrm{co}}\,((C \setminus S) \cup E)$. Let $y \in S^\gamma$, $\varepsilon > 0$, and $M := \sup(f(C))$. Since $y \in C = \overline{\mathrm{co}}\,((C \setminus S) \cup E)$, there exist $\lambda \in [0,1]$, $z_1 \in (C \setminus S)$, $z_2 \in \overline{\mathrm{co}}\,(E)$, and $u \in X$ with $\|u\| < \varepsilon$ such that $y = \lambda z_1 + (1-\lambda) z_2 + u$. Then we have

$$M - \frac{\gamma\delta}{2} < f(y) = \lambda f(z_1) + (1-\lambda) f(z_2) + f(u) < \lambda(M - \delta) + (1-\lambda)M + \varepsilon$$

implying $\lambda < \frac{\gamma}{2} + \frac{\varepsilon}{\delta}$. Hence,

$$\begin{aligned}\|y - z_2\| &\leq \|y - (1-\lambda)z_2\| + \|(1-\lambda)z_2 - z_2\|\\ &= \|\lambda z_1 + u\| + \|\lambda z_2\| \leq 2\lambda + \varepsilon \leq \gamma + \frac{2\varepsilon}{\delta} + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows $y \in B_\gamma(z_2) \subseteq \overline{\mathrm{co}}\,(E) + \overline{B}_\gamma(0)$. The final containment $\overline{\mathrm{co}}\,(E) + \overline{B}_\gamma(0) \subseteq \mathrm{co}\,(E) + B_{2\gamma}(0)$ obviously holds. $\square$

**Lemma 4.3.** *Let $A \subseteq X$ be bounded, nonempty, and non-$\varepsilon$-dentable, and let $C = \overline{co}(A)$ (by Remark 2.5, $C$ is non-$\varepsilon$-dentable). For any slice $S_0$ of $C$, $D \subseteq C$ with $\alpha(D) < \varepsilon$, and $\gamma \in (0,1)$, let $\mathcal{S}(S_0, D)$ be the collection of all slices $S$ of $C$ with $S \subseteq S_0 \setminus D$ and $\mathcal{S}^\gamma(S_0, D) = \{S^\gamma\}_{S \in \mathcal{S}(S_0,D)}$. Let $\Lambda = \Lambda(S_0, D, \gamma) \subseteq C$ denote the union of all sets in $\mathcal{S}^\gamma(S_0, D)$. Then $C = \overline{co}((C \setminus S_0) \cup (\Lambda \cap A))$.*

**Proof.** Let $S_0$, $D$, $\gamma$, and $\Lambda$ be as specified above. By Corollary 2.10 (where we take $C' = \overline{\text{co}}((C \setminus S_0) \cup (\Lambda \cap A))$ and $D = D$), it suffices to prove that $C = \overline{\text{co}}((C \setminus S_0) \cup D \cup (\Lambda \cap A))$. Assume $C \neq \overline{\text{co}}((C \setminus S_0) \cup D \cup (\Lambda \cap A))$. Then by Hahn-Banach separation, there exists a slice $S$ of $C$ such that we have $S \subseteq C \setminus \overline{\text{co}}((C \setminus S_0) \cup D \cup (\Lambda \cap A))$. This implies $S \subseteq S_0$, $S \cap D = \emptyset$, and $S \cap (\Lambda \cap A) = \emptyset$. Then $S \subseteq S_0 \setminus D$. Thus, $S \in \mathcal{S}(S_0, D)$, so $S^\gamma \in \mathcal{S}^\gamma(S_0, D)$, and finally $S^\gamma \subseteq \Lambda$. But since we also have $S^\gamma \subseteq S$ and $S \cap (\Lambda \cap A) = \emptyset$, $(S^\gamma \cap A) = S^\gamma \cap (\Lambda \cap A) = \emptyset$, a contradiction since $S^\gamma \cap A$ is a slice of $A$ (since $S^\gamma$ is a slice of $C = \overline{\text{co}}(A)$) and slices of nonempty sets are nonempty. $\square$

### 4.1. The construction

**Theorem 4.4.** *Let $A \subseteq B_X$ be nonempty and non-$\varepsilon$-dentable (not necessarily closed or convex), and $C = \overline{co}(A)$ so that $C$ is also non-$\varepsilon$-dentable. Fix $\delta < \frac{\varepsilon}{2}$, and assume that $A$ is separable. Then $C$ is separable as well, so $C = \bigcup_{i=0}^\infty B_i$ for some open $B_i$ (relative to $C$) with $diam(B_i) < \varepsilon$. Let $(\delta_n)_{n\geq 0}$ be a sequence of numbers in $(0,1)$. There exist a finitely branching tree $\mathbb{T} \subseteq \mathbb{N}^{<\omega}$, a $(2\delta_n)_{n\geq 0}$-approximate bush $(x_b)_{b\in\mathbb{T}} \subseteq A$, and slices $(S_b)_{b\in\mathbb{T}}$ of $C$ such that, for all $n \in \mathbb{N}$,*

(1) *For all $b \in \mathbb{T}_n$, $x_b \in S_b^{\delta_n} \cap A \subseteq S_b$.*

(2) *If $n \geq 1$, then for all $b \in \mathbb{T}_n$, $S_b \cap B_{n-1} = \emptyset$ and $S_b \cap \left(\bigcup_{|p|\leq n-1} B_\delta(x_p)\right) = \emptyset$.*

(3) *If $n \geq 1$, then for all $b \in \mathbb{T}_{n-1}$, if $(b,1), \ldots (b,q)$ are the children of $b$, then $S_{(b,i)} \subseteq S_b$ and the approximate bush property is satisfied: $x_b \in co(x_{(b,1)}, \ldots x_{(b,q)}) + B_{2\delta_{n-1}}(0)$.*

**Proof.** The proof is by induction on $n$. For the base case, let $S_\emptyset = C$ and let $x_\emptyset$ be any element of $S_\emptyset^{\delta_0}$. For the inductive step, let $n \geq 0$ and assume that $\mathbb{T}_{\leq n}$, $(x_b)_{b\in\mathbb{T}_{\leq n}} \subseteq A$, and $(S_b)_{b\in\mathbb{T}_{\leq n}} \subseteq C$ have been constructed, and satisfy (1)–(3).

Let $b \in \mathbb{T}_n$. Let $D := B_n \cup \bigcup_{|p|\leq n} B_\delta(x_p)$, so that $\alpha(D) < \varepsilon$. As in Lemma 4.3, let $\mathcal{S}(S_b, D)$ be the collection of all slices $S$ of $C$ such that $S \subseteq S_b \setminus D$, $\mathcal{S}^{\delta_{n+1}}(S_b, D) = \{S^{\delta_{n+1}}\}_{S\in\mathcal{S}(S_0,D)}$, and $\Lambda = \bigcup \mathcal{S}^{\delta_{n+1}}(S_b, D)$. By Lemma 4.3 we have $C = \overline{\text{co}}((C \setminus S_b) \cup (\Lambda \cap A))$. Then by Lemma 4.2, $S_b^{\delta_n} \subseteq \text{co}(\Lambda \cap A) + B_{2\delta_n}(0)$.

Then since $x_b \in S_b^{\delta_n}$, there exists $z \in \text{co}(\Lambda \cap A)$ such that $\|x_b - z\| < 2\delta_n$. Let $z_1, \ldots z_q \in \Lambda \cap A$ and $\lambda_1^b, \ldots \lambda_q^b \in [0,1]$ such that $z = \lambda_1^b z_1 + \ldots \lambda_q^b z_q$. For each $i = 1, \ldots q$, since $z_i \in \Lambda$, there are slices $S_{z_i} \in \mathcal{S}(S_b, D)$ of $C$ with $z_i \in S_{z_i}^{\delta_{n+1}}$, by definition of $\Lambda$. We now define the children of $b$ to be $(b,1), \ldots (b,q)$, $x_{(b,i)}$ to be $z_i$, and $S_{(b,i)}$ to be $S_{z_i}$. Repeating this process for each $b \in \mathbb{T}_n$ gives us $\mathbb{T}_{n+1}$, $(x_b)_{b\in\mathbb{T}_{n+1}} \subseteq A$, and $(S_b)_{b\in\mathbb{T}_{n+1}} \subseteq C$.

(1) and (3) hold immediately by construction. It is also clear that (2) holds since $S_{(b,i)} \in \mathcal{S}(S_b, D)$, and then using the definition of $D$ and $\mathcal{S}(S_b, D)$. $\square$

**Remark 4.5.** The assumption that $A$ is separable can be removed (at the penalty of replacing $\varepsilon$ by $\varepsilon/2$) because of the following result: under the hypothesis of Theorem 4.4, $A$ contains a countable subset that is non-$\varepsilon/2$-dentable. This is essentially proved in [3, Lemma 2.2], but we'll include the argument here. Since $\operatorname{diam}(S) > \varepsilon$ for every slice $S$ of $A$, it follows that no slice is contained in a closed ball $B_{\varepsilon/2}(x)$. Hence, if $a \in A$, then $a \in \overline{\operatorname{co}}(A \setminus B_{\varepsilon/2}(a))$. So there exists a countable set $T(a) \subseteq A \setminus B_{\varepsilon/2}(a)$ such that $a \in \overline{\operatorname{co}}(T(a))$. By applying this fact iteratively as in [3, Lemma 2.2], we can construct a countable $A_0 \subseteq A$ such that for every $a \in A_0$, we have $a \in \overline{\operatorname{co}}(A_0 \setminus B_{\varepsilon/2}(a))$. Hence every slice $S$ of $A_0$ satisfied $\operatorname{diam}(S) > \varepsilon/2$. Hence $A_0$ is not $\varepsilon/2$-dentable.

**Corollary 4.6.** *For any separable $A \subseteq B_X$ nonempty and non-$\varepsilon$-dentable, any $\delta < \frac{\varepsilon}{2}$, and any positive $(\delta_n)_{n\geq 0}$, there exists a $\delta$-separated, $(\delta_n)_{n-0}^{\infty}$-approximate bush $(x_b)_{b\in\mathbb{T}}$ in $A$ such that any other set $(y_b)_{b\in\mathbb{T}} \subseteq C = \overline{co}\,(A)$, which fulfills $\sup_{b\in\mathbb{T}_n} \|y_b - x_b\| < \gamma_n$ for some $\gamma_n \to 0$, is weakly closed and discrete. In particular, $(x_b)_{b\in\mathbb{T}}$ is weakly closed and discrete.*

**Proof.** Let $A$, $\delta$, $(\delta_n)_{n\geq 0}$ be as above. Applying the construction of Theorem 4.4, with $(\delta_n/2)_{n\geq 0}$ in place of $(\delta_n)_{n\geq 0}$, yields a bush $(x_b)_{b\in\mathbb{T}}$. By Theorem 4.4(1), $x_b \in A$ for all $b \in \mathbb{T}$. Suppose $b_1, b_2 \in \mathbb{T}$ with $|b_2| > |b_1|$. Then by Theorem 4.4(1), $x_{b_2} \in S_{b_2}$, and by Theorem 4.4(2), $S_{b_2} \cap B_\delta(x_{b_1}) = \emptyset$, so $\|x_{b_2} - x_{b_1}\| > \delta$. This means the bush is $\delta$-separated. By Theorem 4.4(3), if $b \in \mathbb{T}$ and $(b,1), \ldots (b,q)$ are the children of $b$, then $x_b \in \operatorname{co}(x_{(b,1)}, \ldots x_{(b,q)}) + B_{\delta_n}(0)$. This means the bush is $(\delta_n)_{n\geq 0}$-approximate.

Finally, let $(y_b)_{b\in\mathbb{T}} \subseteq C$, with $\sup_{b\in\mathbb{T}_n} \|y_b - x_b\| < \gamma_n$ for some $\gamma_n \to 0$, and let $z$ belong to the weak closure of $(y_b)_{b\in\mathbb{T}}$. Since $C$ is norm closed and convex, it is weakly closed, and thus $z \in C$. Then $z \in B_i$ for some $i$. Consider $S_b$ for $|b| = i + 1$. Then $S_b = S(f_b, C, \alpha_b)$ for some $f_b \in B_{X^*}$ and $\alpha_b > 0$. Hence

$$z \in B_i \subseteq C \setminus S_b = \{x \in C : f_b(x) \leq \sup f(C) - \alpha_b\}$$

Since $B_i$ is open in the norm topology relative in $C$ and $C$ is convex, it follows that $B_i \subseteq \{x \in C : f_b(x) < \sup f_b(C) - \alpha_b\}$. Since $\gamma_n \to 0$, we can find $\gamma > 0$ and $N$ large enough so that $B_i \subseteq \{x \in C : f_b(x) < \sup f_b(C) - \alpha_b - \gamma\}$, $N \geq i + 1$, and $\gamma_n < \gamma$ for all $n \geq N$. Then we set $U_b := \{x \in C : f_b(x) < \sup f_b(C) - \alpha_b - \gamma\}$ and observe that it is a weak neighborhood of $z$ in $C$. Hence $U := \cap_{|b|=i+1} U_b$ is a weak neighborhood of $z$ in $C$. Now we wish to show the set $U \cap (y_b)_{b\in\mathbb{T}}$ is finite, which will imply our desired conclusion that $(y_b)_{b\in\mathbb{T}}$ is weakly closed and discrete. We will show that $U \cap (y_b)_{b\in\mathbb{T}}$ is finite by showing that $U \cap (y_b)_{b\in\mathbb{T}_{\geq N}} = \emptyset$.

Consider $b \in \mathbb{T}$ with $|b| \geq N$. Then $\|y_b - x_b\| < \gamma_{|b|} < \gamma$. Let $b_{i+1} \in \mathbb{T}$ denote the unique predecessor of $b$ with $|b_{i+1}| = i + 1$. Then $x_b \in S_b \subseteq S_{b_{i+1}}$, and hence $f_{b_{i+1}}(x_b) > \sup f_{b_{i+1}}(C) - \alpha_{b_{i+1}}$. Since $f_{b_{i+1}} \in B_{X^*}$ and $\|y_b - x_b\| < \gamma$, this implies $f_{b_{i+1}}(y_b) > \sup f_{b_{i+1}}(C) - \alpha_{b_{i+1}} - \gamma$. Thus, by definition of $U_{b_{i+1}}$, $y_b \notin U_{b_{i+1}}$. By definition of $U$ this proves $U \cap (y_b)_{b\in\mathbb{T}_{\geq N}} = \emptyset$. □

**Corollary 4.7.** *For any $A \subseteq B_X$ nonempty and non-$\varepsilon$-dentable, any $\delta < \frac{\varepsilon}{2}$, and any positive sequence $(\delta_n)_{n\geq 0}$, there exists a filtration of finite $\sigma$-algebras $(\mathcal{A}_n)_{n\geq 0}$, an $A$-valued, $(\mathcal{A}_n)_{n\geq 0}$-adapted $(\delta_n)_{n\geq 0}$-quasimartingale $(M_n)_{n\geq 0}$ with $\|M_n(s) - M_m(t)\| > \delta$ for all $n \geq m \geq 0$ and $s, t \in [0,1]$, and the range of this quasimartingale is weakly closed and discrete.*

**Proof.** Let $A$, $\delta$, $(\delta_n)_{n\geq 0}$ be as above, and apply Corollary 4.6 to obtain a $(\delta_n)_{n\geq 0}$-approximate bush $(x_b)_{b\in\mathbb{T}}$ which is weakly closed and discrete. We define the filtration $(\mathcal{A}_n)_{n\geq 0}$ on $[0,1]$ recursively: Let $\mathcal{A}_0$ be the trivial $\sigma$-algebra. Suppose $\mathcal{A}_n$ has been defined as a finite $\sigma$-algebra whose atoms are intervals, the atoms are in bijection with $\mathbb{T}_n$ via $b \mapsto I_b$, and for any $b \in \mathbb{T}_{n-1}$ and child $(b,i) \in \mathbb{T}_n$, $\mathcal{L}(I_{(b,i)}) = \mathcal{L}(I_b)\lambda_i^b$.

Then for any $b' \in \mathbb{T}_n$ with children $(b',1), \dots (b',q)$, we pick any subdivision of $I_{b'}$ into intervals $I_{(b',1)}, \dots I_{(b',q)}$ so that $\mathcal{L}(I_{(b',i)}) = \mathcal{L}(I_{b'})\lambda_i^{b'}$. Take $\mathcal{A}_n$ to be the $\sigma$-algebra generated by these intervals. Then we define $M_n$ to be $\sum_{|b|=n} x_b \chi_{I_b}$. We then have

$$\|\mathbb{E}(M_{n+1}|\mathcal{A}_n) - M_n\|_{L^\infty} = \sup_{b\in\mathbb{T}_n} \|x_b - \lambda_1 x_{(b,1)} - \dots \lambda_q x_{(b,q)}\| < \delta_n.$$

The range of this quasimartingale is exactly the bush, and thus weakly closed and discrete. $\square$

**Corollary 4.8.** *For any $A \subseteq B_X$ nonempty and non-$\varepsilon$-dentable, $\delta < \frac{\varepsilon}{2}$, and positive, summable sequence $(\delta_n)_{n\geq 0}$, there exist a filtration of finite $\sigma$-algebras $(\mathcal{A}_n)_{n\geq 0}$, an $A$-valued, $(\mathcal{A}_n)_{n\geq 0}$-adapted $(\delta_n)_{n\geq 0}$-quasimartingale $(M_n)_{n\geq 0}$ and $\overline{\mathrm{co}}\,(A)$-valued, $(\mathcal{A}_n)_{n\geq 0}$-adapted martingale $(\overline{M}_n)_{n\geq 0}$ with, for all $n \neq m \geq 0$ and $s,t \in [0,1]$,*

(1) $\|\overline{M}_n(s) - \overline{M}_m(t)\| > \delta$;

(2) $\|M_n - \overline{M}_n\|_\infty < \delta_n$;

(3) *The range of $(\overline{M}_n)_{n\geq 0}$ is weakly closed and discrete.*

**Proof.** Let $A$, $\delta$, $(\delta_n)_{n\geq 0}$ be as above, and apply Corollary 4.7 to obtain the $\sigma$-algebra $(\mathcal{A}_n)_{n\geq 0}$ and $A$-valued, $(\delta_n)_{n\geq 0}$-quasimartingale $(M_n)_{n\geq 0}$ with weakly closed and discrete range. Construct $(\overline{M}_n)_{n\geq 0}$ from $(M_n)_{n\geq 0}$ just as in the proof of Corollary 3.4, so that $(\overline{M_n})_{n\geq 0}$ is $\overline{\mathrm{co}}\,(A)$-valued and (1) and (2) hold. To see (3), again note that the range of $(M_n)_{n\geq 0}$ is exactly $(x_b)_{b\in\mathbb{T}_n}$ from Corollary 4.6. Since $(\overline{M}_n)_{n\geq 0}$ is adapted to the same finite filtration as $(M_n)_{n\geq 0}$, (2) implies that the range of $\overline{M}_n$ equals $(y_b)_{b\in\mathbb{T}_n}$ for some $y_b \in \overline{\mathrm{co}}\,(A)$ and $\sup_{b\in\mathbb{T}_n} \|y_b - x_b\| < \delta_n$. Then Corollary 4.6 implies (3). $\square$

**Corollary 4.9.** *For any $A \subseteq B_X$ nonempty and nondentable, there exists a countable set $F \subseteq \overline{co}\,(A)$ such that*

(1) $\lim_{F\ni f\to\infty} d(f,A) = 0$;

(2) *$F$ is weakly closed and discrete and $Ext\,(F) = \emptyset$;*

(3) *$\overline{co}\,(F)$ has no weak denting point;*

(4) $\overline{co}\,(F) \cap Ext\,(\overline{co}^{w*}(F)) = \emptyset$.

**Proof.** Let $A$ be as above. Let $\varepsilon > 0$ such that $A$ is non-$\varepsilon$-dentable and let $\delta < \frac{\varepsilon}{2}$. Let $\delta_n$ be any positive, summable sequence, and let $(\mathcal{A}_n)_{n\geq 0}$, $(M_n)_{n\geq 0}$, and $(\overline{M}_n)_{n\geq 0}$ be the filtration, $(\delta_n)_{n\geq 0}$-quasimartingale, and martingale afforded to us by Corollary 4.8. Let $F \subseteq \overline{\mathrm{co}}\,(A)$ be the range of the martingale. Since $(M_n)_{n\geq 0}$ is $A$-valued and $\|M_n - \overline{M}_n\|_\infty < \delta_n$, $\lim_{F\ni f\to\infty} d(f,A) = 0$, showing (1).

By Corollary 4.8, $F$ is weakly closed and discrete and clearly has no extreme point since it is a $\delta$-separated bush, showing (2).

Since weak denting points of $\overline{\mathrm{co}}\,(F)$ are extreme points, and since $F$ has no extreme points, the set of weak denting points of $\overline{\mathrm{co}}\,(F)$ is contained in $\overline{\mathrm{co}}\,(F) \setminus F$. But since $\overline{\mathrm{co}}\,(F) \setminus F$ is weakly open in $\overline{\mathrm{co}}\,(F)$, it follows that $\overline{\mathrm{co}}\,(F) \setminus F$ contains no weak denting point. This shows (3).

For (4), we first observe that the converse of the Krein-Milman theorem ([2, Lemma 8.5]) implies that every extreme point of $\overline{co}^{w*}(F)$ is a weak* denting point of $\overline{co}^{w*}(F)$. To see this, let $x$ be an extreme point of $\overline{co}^{w*}(F)$ and assume $x$ is not a weak* denting point. Then there is an open neighborhood $U \subseteq X^{**}$ of $x$ such that $x \in \overline{co}^{w*}\,(\overline{co}^{w*}(F) \setminus U)$. Then since $\overline{co}^{w*}(F) \setminus U$ is weak* compact, the converse to Krein-Milman implies every extreme point of $\overline{co}^{w*}\,(\overline{co}^{w*}(F) \setminus U)$, in particular $x$, is contained in $\overline{co}^{w*}(F) \setminus U$, a contradiction. Then (4) follows from (3) since weak* denting points of $\overline{\mathrm{co}}\,(F) \cap \overline{co}^{w*}(F) \subseteq X^{**}$ are the same as weak denting points of $\overline{\mathrm{co}}\,(F) \cap \overline{co}^{w*}(F) \subseteq X$. □

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