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Daugavet- and Delta-Points in Absolute Sums of Banach Spaces*

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A Daugavet-point (resp. Δ-point) of a Banach space is a norm one element x for which every point in the unit ball (resp. element x itself) is in the closed convex hull of unit ball elements that are almost at distance 2 from x. A Banach space has the well-known Daugavet property (resp. diametral local diameter 2 property) if and only if every norm one element is a Daugavet-point (resp. Δ-point). Our results complement the ones of T. A. Abrahamsen, R. Haller, V. Lima and K. Pirk [*Delta- and Daugavet-points in Banach spaces*, Proc. Edinb. Math. Soc. 63/2 (2020) 475–496] concerning the existence of Daugavet- and Δ-points in absolute sums of Banach spaces.

Keywords: Daugavet property, Daugavet-point, delta-point, absolute sum, diameter two property.

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1. Introduction

Let X be a real Banach space. We denote its closed unit ball by B_X, unit sphere by S_X, and dual space by X^*. Following [1] we say that

(1) an x in S_X is a *Daugavet-point* if $B_X = \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ for every $\varepsilon > 0$,

(2) an x in S_X is a Δ*-point* if $x \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ for every $\varepsilon > 0$,

where
$$\Delta_\varepsilon(x) = \{y \in B_X \colon \|x - y\| \geq 2 - \varepsilon\}.$$

Instead of $\Delta_\varepsilon(x)$ one can equivalently use the set $\{y \in S_X \colon \|x - y\| \geq 2 - \varepsilon\}$ in the definitions of Daugavet- and Δ-point.

The concepts of Daugavet-point and Δ-point originate from the observation in [9] and [2]. On the one hand, a Banach space X has the *Daugavet property* (i.e. for every rank-1 bounded linear operator $T \colon X \to X$ we have $\|Id + T\| = 1 + \|T\|$), if

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and only if every $x \in S_X$ is a Daugavet-point (see [9, Corollary, 2.3]). On the other hand, a Banach space X has the *diametral local diameter 2 property* if and only if every $x \in S_X$ is a Δ-point (see [2]).

In this paper we clarify which absolute sums of Banach spaces have Daugavet- and Δ-points. Recall that a norm N on $\mathbb{R}^2$ is called *absolute* if $N(a,b) = N(|a|,|b|)$ for all $(a,b) \in \mathbb{R}^2$, and *normalised* if $N(1,0) = N(0,1) = 1$. Given two Banach spaces X and Y, and an absolute normalised norm N on $\mathbb{R}^2$, we write $X \oplus_N Y$ to denote the direct sum $X \oplus Y$ with the norm $\|(x,y)\|_N = N(\|x\|,\|y\|)$, and we call this Banach space the *absolute sum* of X and Y. Standard examples of absolute normalised norms are the ℓ_p-norms on $\mathbb{R}^2$ for every $p \in [1,\infty]$.

It is known that among all the absolute sums only ℓ_1-sum and ℓ_∞-sum can have Daugavet property (see [3]). In contrast, Daugavet- and Δ-points may exist in various absolute sums. Some preliminary results, regarding this matter, were obtained in [1], let us first recall two of them about Daugavet-points.

Proposition 1.1. (see [1, Propositions 4.3 and 4.6]*Let X and Y be Banach spaces and N an absolute normalised norm on $\mathbb{R}^2$.*

(a) *The absolute sum $X \oplus_N Y$ does not have any Daugavet-points, whenever N has the following property:*

(α) *for all nonnegative reals a and b with $N(a,b) = 1$, there exist $\varepsilon > 0$ and a neighbourhood W of (a,b) with*

$$\sup_{(c,d)\in W} c < 1 \quad or \quad \sup_{(c,d)\in W} d < 1,$$

such that $(c,d) \in W$ for all nonnegative reals c and d with

$$N(c,d) = 1 \quad and \quad N\big((a,b)+(c,d)\big) \geq 2-\varepsilon.$$

(b) *The absolute sum $X \oplus_N Y$ has Daugavet-points, whenever both X and Y have Daugavet-points, and N has the following property:*

(β) *there exist nonnegative reals a and b with $N(a,b) = 1$ such that*

$$N\big((a,b)+(0,1)\big) = 2 \quad and \quad N\big((a,b)+(1,0)\big) = 2.$$

More precisely, if x and y are Daugavet-points in X and Y, respectively, and (a,b) is from (β), then (ax,by) is a Daugavet-point in $X \oplus_N Y$.

It is easy to see that properties (α) and (β) exclude each other. However, there are absolute normalised norms that have neither property (α) nor property (β) (see discussion in [1]). This motivated us, in part, for further research to clarify the existence of Daugavet-points for all absolute sums (see Section 2).

Concerning Δ-points, however, one can easily construct Δ-points in any absolute sum $X \oplus_N Y$ provided both X and Y have Δ-points (see discussion after Lemma 4.1 in [1]). In this paper, our particular aim is to describe the reverse situation, that is the existence of Δ-points in the summands, by assuming that there are Δ-points in the absolute sum.

Our main tools are the following two lemmas that use slices to describe Daugavet- and Δ-points. These lemmas are easily derived from the Hahn–Banach separation theorem. By a *slice of the unit ball* B_X of X we mean a set of the form

$$S(B_X, x^*, \alpha) = \{y \in B_X : x^*(y) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

Lemma 1.2. ([1, Lemma 2.2]) *Let X be a Banach space and $x \in S_X$. Then the following assertions are equivalent:*

(i) *x is a Daugavet-point;*

(ii) *for every slice S of B_X and every $\varepsilon > 0$ there exists $u \in S$ such that $\|x - u\| \geq 2 - \varepsilon$.*

Lemma 1.3. ([1, Lemma 2.1] *Let X be a Banach space and $x \in S_X$. Then the following assertions are equivalent:*

(i) *x is a Δ-point;*

(ii) *for every slice S of B_X, with $x \in S$, and every $\varepsilon > 0$ there exists $u \in S$ such that $\|x - u\| \geq 2 - \varepsilon$.*

In the following, we present our main results regarding both Daugavet- and Δ-points. In Section 2, we generalise part (b) of Proposition 1.1 to all absolute normalised norms that do not have property (α), which completes the research about the existence of Daugavet-points in the absolute sum provided component spaces have Daugavet-points (see Theorem 2.2).

In Section 3, we consider the existence of Daugavet-points in component spaces, assuming that the absolute sum has Daugavet-points. In this case, we prove that at least one component space has Daugavet-points (see Theorems 3.1 and 3.2).

In Section 4, we deal with the existence of Δ-points in component spaces of an absolute sum with Δ-points. For absolute normalised norms that differ from ℓ_∞-norm the results are similar to the case of Daugavet-points, i.e. at least one of component spaces has Δ-points (see Theorem 4.1). However, the point (x, y) with $\|x\| = \|y\| = 1$ in the absolute sum equipped with ℓ_∞-norm can surprisingly be a Δ-point even if neither x nor y is a Δ-point in the respective component space (see Example 4.3 and Proposition 4.4).

To prove Theorems 3.1 and 4.1, we use the following technical result.

Lemma 1.4. *Let N be an absolute normalised norm on $\mathbb{R}^2$. For every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that for every $p, q, r \geq 0$, if*

$$2 - \delta \leq N(p, q) \leq N(r, q) \leq 2 \quad \text{and} \quad q < 2 - \delta,$$

then $|p - r| < \varepsilon$.

Proof. Fix $\varepsilon \in (0, 2)$ and set $c = \max_{N(p,1)=1} p$. To every $q \in [0, 2)$ there corresponds a unique $p_q \geq 0$ such that $N(p_q, q) = 2$. Choose $s \in (2 - \varepsilon, 2)$ such that $p_s - cs < \varepsilon$. It is easy to see that $\delta = 2 - s$ satisfies the conditions of the lemma, since the function $[0,s] \to \mathbb{R}$, $q \mapsto p_q - r_q$, where $r_q \geq 0$ and $N(r_q,q) = s$, is non-decreasing and $p_s - r_s < \varepsilon$ (see Fig. 1.1). $\square$

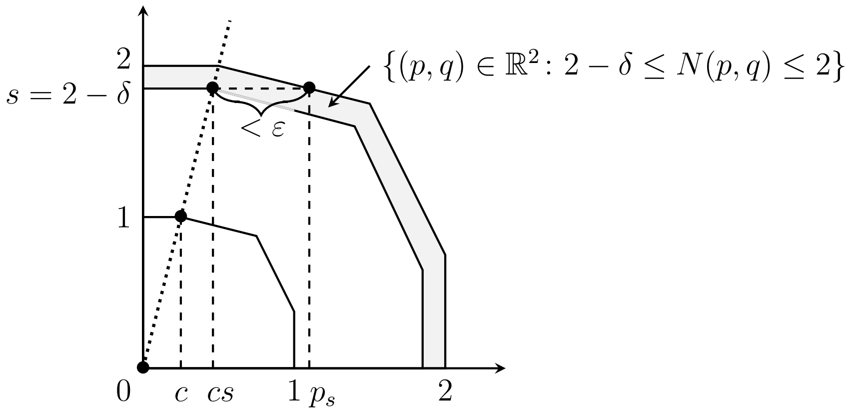

Figure 1.1: The proof of Lemma 1.4

2. Daugavet-points from summands to absolute sums

In this section we focus on absolute sums inheriting Daugavet-points from component spaces. As we pointed out in Introduction, the absolute sum with a norm N satisfying property (α) does not have any Daugavet-points (see part (a) of Proposition 1.1). However, if both component spaces have Daugavet-points, then the absolute sum equipped with a norm N satisfying property (β) also has Daugavet-points (see part (b) of Proposition 1.1). We complete this direction by specifying the situation for all the other absolute normalised norms besides the ones with properties (α) or (β).

Definition 2.1. Let X be a Banach space and $A \subset S_X$. We say that the norm on X is *A-octahedral* (A-OH) if, for every $x_1, \ldots, x_n \in A$ and every $\varepsilon > 0$, there exists $y \in S_X$ such that $\|x_i + y\| \geq 2 - \varepsilon$ for every $i \in \{1, \ldots, n\}$.

It is evident that the octahedrality of a norm in its usual sense (see [4] and [6, Proposition 2.4]) means that the norm is S_X-OH. We use this more general term, A-octahedrality, to describe absolute sums or, more precisely, absolute normalised norms for which the absolute sums possess Daugavet-points. The name, A-octahedrality, is justified by the fact that property (β) has also been called positive octahedrality in [5]. Note that absolute normalised norm N on $\mathbb{R}^2$ with property (β) is exactly $\{(0,1),(1,0)\}$-OH.

Consider an absolute normalised norm N on $\mathbb{R}^2$. We now define a specific set A that is considered from here on until the end of Section 3. Set

$$c = \max_{N(e,1)=1} e \quad \text{and} \quad d = \max_{N(1,f)=1} f, \tag{$*$}$$

and define

$$A = \{(c,1),(1,d)\}.$$

Suppose that the norm N is A-OH. By Definition 2.1 there exists $(a,b) \in \mathbb{R}^2$ with the following property:

$$\begin{aligned} &a, b \geq 0, \quad N(a,b) = 1, \quad \text{and} \\ &N\big((a,b) + (c,1)\big) = 2 \quad \text{and} \quad N\big((a,b) + (1,d)\big) = 2. \end{aligned} \tag{$**$}$$

It is easy to see that, firstly, as mentioned above, the norms with property (β) are A-OH (for the specified A as well), and secondly, A-OH norms do not have property (α). Moreover, part (b) of Proposition 1.1 can be extended for A-OH norms N as well.

Theorem 2.2. *Let X and Y be Banach spaces, $x \in S_X$, $y \in S_Y$, and let N be an A-OH norm with (a,b) as in $(**)$. If x and y are Daugavet-points in X and Y, respectively, then (ax, by) is a Daugavet-point in $X \oplus_N Y$.*

Proof. Assume that x and y are Daugavet-points. Set $Z = X \oplus_N Y$ and fix $f = (x^*, y^*) \in S_{Z^*}$, $\alpha > 0$, and $\varepsilon > 0$. Choose $\delta > 0$ to satisfy $\delta N(1,1) < \varepsilon$. By Lemma 1.2 we obtain $u \in B_X$ and $v \in B_Y$ such that

$$x^*(u) \geq \Big(1 - \frac{\alpha}{2}\Big)\|x^*\| \quad \text{and} \quad y^*(v) \geq \Big(1 - \frac{\alpha}{2}\Big)\|y^*\| \tag{1}$$

and
$$\|x - u\| \geq 2 - \delta \quad \text{and} \quad \|y - v\| \geq 2 - \delta. \tag{2}$$

By the properties of absolute normalised norms (see [7], p 317) and A-OH norms, there exist $k, l \geq 0$ such that

$$N(k,l) = 1, \quad N\big((a,b) + (k,l)\big) = 2, \quad \text{and} \quad k\|x^*\| + l\|y^*\| = 1.$$

Therefore $(ku, lv) \in S(B_Z, f, \alpha)$, because

$$f(ku, lv) = kx^*(u) + ly^*(v) \geq \Big(1 - \frac{\alpha}{2}\Big)(k\|x^*\| + l\|y^*\|) > 1 - \alpha.$$

On the other hand, from $\|x - u\| \geq 2 - \delta$ and $\|y - v\| \geq 2 - \delta$, we get that

$$\big\|ax - ku\big\| \geq a + k - \delta \quad \text{and} \quad \big\|by - lv\big\| \geq b + l - \delta.$$

In conclusion we have that

$$\begin{aligned} &\|(ax, by) - (ku, lv)\|_N = N(\|ax - ku\|, \|by - lv\|) \geq N(a + k - \delta, b + l - \delta) \\ &\geq N(a + k, b + l) - N(\delta, \delta) = N\big((a,b) + (k,l)\big) - \delta N(1,1) > 2 - \varepsilon. \end{aligned}$$

By Lemma 1.2, (ax, by) is a Daugavet-point. □

In order to have Daugavet-points in the absolute sum it is enough, for some A-OH norms, to assume that only one summand has a Daugavet-point. The following two propositions describe these special occasions, we drop the proofs since they are similar to the previous one.

Proposition 2.3. *Let X and Y be Banach spaces, $x \in S_X$ and $y \in S_Y$, and let N be an A-OH norm with (a,b) as in $(**)$.*

(a) *1 If $b = 0$ and x is a Daugavet-point in X, then $(ax, by) = (x, 0)$ is a Daugavet-point in $X \oplus_N Y$.*

(b) *If $a = 0$ and y is a Daugavet-point in Y, then $(ax, by) = (0, y)$ is a Daugavet-point in $X \oplus_N Y$.*

Proposition 2.4. *Let X and Y be Banach spaces, $x \in S_X$ and $y \in S_Y$.*

(a) *If x is a Daugavet-point in X, then (x, by) is a Daugavet-point in $X \oplus_\infty Y$ for every $b \in [0,1]$.*

(b) *If y is a Daugavet-point in Y, then (ax, y) is a Daugavet-point in $X \oplus_\infty Y$ for every $a \in [0,1]$.*

With this we have widened the class of absolute sums inheriting Daugavet-points from the component spaces (see also the discussion after Remark 5.9 in [1]). At this point, it is still not clear, though, whether all absolute normalised norms are covered, since there could be other absolute normalised norms 'in between' the norms with property (α) and A-OH norms. We will prove now that actually this is not the case, all absolute normalised norms are either with property (α) or A-OH.

It is not hard to see that in the definition of property (α), we can relax the condition $\varepsilon > 0$ and let $\varepsilon = 0$. The corresponding reformulation of property (α) is the following:

(α) for all nonnegative reals a and b with $N(a,b) = 1$, there exists a neighbourhood W of (a,b) with

$$\sup_{(c,d)\in W} c < 1 \quad \text{or} \quad \sup_{(c,d)\in W} d < 1,$$

such that $(c,d) \in W$ for all nonnegative reals c and d with

$$N(c,d) = 1 \quad \text{and} \quad N\big((a,b) + (c,d)\big) = 2.$$

Proposition 2.5. *Every absolute normalised norm on $\mathbb{R}^2$ is either with property (α) or A-OH.*

Proof. Let N be an absolute normalised norm that does not have property (α). Then there exist $a, b \geq 0$ with $N(a,b) = 1$ such that for every (a,b)-neighbourhood W which satisfies either $\sup_{(c,d)\in W} c < 1$ or $\sup_{(c,d)\in W} d < 1$, there exist $c, d \geq 0$ with $N(c,d) = 1$ such that $(c,d) \notin W$ and $N\big((a,b) + (c,d)\big) = 2$. To show that N is A-OH, we need to find $c, d \geq 0$ satisfying

$$N(c,1) = N(1,d) = 1 \quad \text{and} \quad N\big((a,b) + (c,1)\big) = N\big((a,b) + (1,d)\big) = 2.$$

If $a = 1$ ($b = 1$, respectively), then take $d = b$ ($c = a$, respectively).

However, if $a \neq 1$, then for every $n \in \mathbb{N}$ large enough ($a < 1 - 1/n$), by taking $W_n = \{(x,y) \colon x \leq 1 - 1/n\}$, we can find $c_n, d_n \geq 0$ with $N(c_n, d_n) = 1$ such that $(c_n, d_n) \notin W_n$ and $N\big((a,b) + (c_n, d_n)\big) = 2$. Passing to a subsequence if necessary, we can assume that $(c_n, d_n) \to (1, d)$ for some $d \geq 0$. Obviously $N(1,d) = 1$ and

$N\big((a,b)+(1,d)\big)=2$. It can be proved similarly that if $b\neq 1$, then there exists $c\geq 0$ with $N(c,1)=1$ and $N\big((a,b)+(c,1)\big)=2$. Combining these facts we have that N is A-OH. $\square$

3. Daugavet-points from absolute sums to summands

In [1] the authors stated a question about the existence of Daugavet-points in Banach spaces X and Y provided the absolute sum $X\oplus_N Y$ has Daugavet-points (see [1, Problem 1]). In this section we give an answer to that question. Note that due to Propositions 1.1 (a) and 2.5 the question is open only for A-OH norms with $A=\{(c,1),(1,d)\}$ where c,d are as in $(*)$.

Theorem 3.1. *Let X and Y be Banach spaces, $x\in B_X$, $y\in B_Y$, and let N be an absolute normalised norm on $\mathbb{R}^2$, different from ℓ_∞-norm. Assume that (x,y) is a Daugavet-point in $X\oplus_N Y$.*

(a) *If $x\neq 0$, then $x/\|x\|$ is a Daugavet-point in X.*

(b) *If $y\neq 0$, then $y/\|y\|$ is a Daugavet-point in Y.*

Proof. We prove only the first statement, because the second can be proved similarly. Suppose that $x\neq 0$. Fix $x^*\in S_{X^*}$, $\alpha>0$, and $\varepsilon>0$. We will find $u\in S(B_X,x^*,\alpha)$ such that $\big\|x/\|x\|-u\big\|\geq 2-\varepsilon$. Set $f=(x^*,0)$ and $Z=X\oplus_N Y$. Then $f\in S_{Z^*}$. Using Lemma 1.4, choose $\delta>0$ such that, for every $p,q,r\geq 0$, if

$$2-\delta\leq N(p,q)\leq N(r,q)\leq 2 \quad\text{and}\quad q<2-\delta,$$

then $|p-r|<\|x\|\varepsilon/2$. There is no loss of generality in assuming that $\delta\leq\varepsilon/2$, $\delta\leq\alpha$, and $(1-\delta)N(1,1)>1$ (here we use the fact that $N(1,1)>1$, i.e. N is not ℓ_∞-norm).

Since (x,y) is a Daugavet-point in Z, there exists $(u,v)\in S(B_Z,f,\delta)$ such that $\|(x,y)-(u,v)\|_N\geq 2-\delta$. Consequently,

$$x^*(u)=f(u,v)>1-\delta\geq 1-\alpha,$$

which gives us that $u\in S(B_X,x^*,\alpha)$ and $\|u\|>1-\delta$. We also conclude that $\|v\|<1-\delta$, because otherwise

$$N(\|u\|,\|v\|)\geq N(1-\delta,1-\delta)=(1-\delta)N(1,1)>1,$$

a contradiction. In addition we have that

$$2-\delta\leq N\big(\|x-u\|,\|y-v\|\big)\leq N\big(\|x\|+\|u\|,\|y-v\|\big)\leq 2$$

and
$$\|y-v\|\leq\|y\|+\|v\|<1+1-\delta=2-\delta.$$

Hence, by the choice of δ, we have that

$$\big|\|x-u\| - (\|x\| + \|u\|)\big| < \|x\|\varepsilon/2.$$

Thus $\|x-u\| > \|x\| + \|u\| - \|x\|\varepsilon/2$, and therefore,

$$\begin{aligned}\Big\|\frac{x}{\|x\|} - u\Big\| &= \Big\|\frac{1}{\|x\|}(x-u) - \Big(u - \frac{1}{\|x\|}u\Big)\Big\| \\ &\geq \frac{1}{\|x\|}\|x-u\| - \Big(\frac{1}{\|x\|} - 1\Big)\|u\| \\ &\geq \frac{1}{\|x\|}\Big(\|x\| + \|u\| - \frac{\|x\|\varepsilon}{2}\Big) - \frac{\|u\|}{\|x\|} + \|u\| \\ &= 1 + \|u\| - \frac{\varepsilon}{2} > 1 + 1 - \delta - \frac{\varepsilon}{2} \geq 2 - \varepsilon.\end{aligned}$$

According to Lemma 1.2, the element $x/\|x\|$ is a Daugavet-point in X. □

Let us now move on to the case of ℓ_∞-norm. Recall that if x is a Daugavet-point in X, then (x, y) is a Daugavet-point in $X \oplus_\infty Y$ for every $y \in B_Y$ (see Proposition 2.4). This means that ℓ_∞-sum of two Banach spaces may have Daugavet-points even if both factors fail to have a Daugavet-point. However, in the following we prove that at least one of the summands has a Daugavet-point.

Theorem 3.2. *Let X and Y be Banach spaces, $x \in B_X$, $y \in B_Y$. Assume that (x, y) is a Daugavet-point in $X \oplus_\infty Y$. Then x is a Daugavet-point in X or y is a Daugavet-point in Y.*

Proof. Firstly, look at the case, where only one of x and y has norm 1, and the other has norm less than 1. Assume that $\|y\| < 1$ and let us prove that x is a Daugavet-point in X. (The statement, if $\|x\| < 1$, then y is a Daugavet-point in Y, can be proved similarly.) Choose $\delta > 0$ such that $\delta \leq \varepsilon$ and $\|y\| < 1 - \delta$.

Fix $x^* \in S_{X^*}$, $\alpha > 0$ and $\varepsilon > 0$. Set $Z = X \oplus_\infty Y$ and $f = (x^*, 0) \in S_{Z^*}$.

Since (x, y) is a Daugavet-point then there exists $(u, v) \in S(B_Z, f, \alpha)$ such that $\|(x, y) - (u, v)\|_\infty \geq 2 - \delta$. Therefore, $x^*(u) = f(u, v) > 1 - \alpha$, i.e. $u \in S(B_X, x^*, \alpha)$ and

$$\|y - v\| \leq \|y\| + \|v\| < 1 - \delta + 1 = 2 - \delta.$$

Combining this with the fact that

$$\|(x, y) - (u, v)\|_\infty = \max\{\|x - u\|, \|y - v\|\} \geq 2 - \delta,$$

we get that $\|x - u\| \geq 2 - \delta \geq 2 - \varepsilon$. Thus, x is a Daugavet-point in X.

Secondly, assume that x and y are norm-one elements which fail to be a Daugavet-point. Then we can fix slices $S(B_X, x^*, \alpha)$ and $S(B_Y, y^*, \alpha)$, and $\varepsilon > 0$ such that $S(B_X, x^*, \alpha) \cap \Delta_\varepsilon(x) = \emptyset$ and $S(B_Y, y^*, \alpha) \cap \Delta_\varepsilon(y) = \emptyset$. There is no loss of generality in assuming that $\alpha < \varepsilon < 1$. Set $f = 1/2(x^*, y^*)$ and $Z = X \oplus_\infty Y$, and consider the slice $S(B_Z, f, \alpha/2)$. Note that

$$S(B_Z, f, \alpha/2) \subset S(B_X, x^*, \alpha) \times S(B_Y, y^*, \alpha).$$

Let $(u,v) \in S(B_Z, f, \alpha/2) \cap S_Z$ be arbitrary. Then

$$\|u\| > 1-\alpha > 0 \quad \text{and} \quad \|v\| > 1-\alpha > 0,$$

and
$$u/\|u\| \in S(B_X, x^*, \alpha) \quad \text{and} \quad v/\|v\| \in S(B_Y, y^*, \beta).$$

Therefore

$$\begin{aligned}\|(u,v)-(x,y)\|_\infty &= \max\{\|u-x\|, \|v-y\|\}\\ &\le \max\left\{\left\|u-\frac{u}{\|u\|}\right\| + \left\|\frac{u}{\|u\|}-x\right\|, \left\|v-\frac{v}{\|v\|}\right\| + \left\|\frac{v}{\|v\|}-y\right\|\right\}\\ &< \alpha + 2 - \varepsilon = 2-(\varepsilon-\alpha).\end{aligned}$$

As a result, $S(B_Z, f, \alpha/2) \cap \Delta_{\varepsilon-\alpha}(x,y) = \emptyset$, which by Lemma 1.2 implies that (x,y) is not a Daugavet-point. □

4. Delta-points from direct sums to summands

As mentioned in Introduction, Δ-points pass from component spaces to the absolute sum for every absolute normalised norm. In fact, this observation let the authors of [1] conclude that Δ-points are indeed different from Daugavet-points. In this section we clarify the existence of Δ-points in the component spaces provided the absolute sum has Δ-points. Surprisingly, the case of Δ-points is different from the case of Daugavet-points even for ℓ_∞-norm (see Proposition 4.4).

From here on, we consider arbitrary absolute normalised norms N. Firstly, we show that, as expected, for most absolute normalised norms N the component spaces have Δ-points whenever the absolute sum has Δ-points.

Theorem 4.1. *Let X and Y be Banach spaces, $x \in S_X$, $y \in S_Y$, N an absolute normalised norm on $\mathbb{R}^2$, and $a, b \ge 0$ such that $N(a,b)=1$. Assume that (ax, by) is a Δ-point in $X \oplus_N Y$.*

(a) *If $b \ne 1$, then x is a Δ-point in X.*

(b) *If $a \ne 1$, then y is a Δ-point in Y.*

Proof. We prove only the first statement, the second can be proved similarly. Assume that $b \ne 1$. Note that then $a \ne 0$. Let $c, d \ge 0$ be such that $N^*(c,d)=1$ and $ac+bd=1$.

Suppose that x is not a Δ-point in X. Then, by Lemma 1.3, there exist $x^* \in S_{X^*}$, $\alpha > 0$, and $\varepsilon > 0$ such that

$$x \in S(B_X, x^*, \alpha) \quad \text{and} \quad S(B_X, x^*, \alpha) \cap \Delta_\varepsilon(x) = \emptyset.$$

Let $y^* \in S_{Y^*}$ be such that $y^*(y)=1$ and let $f = (cx^*, (1-\alpha)dy^*)$. Then

$$f(ax, by) = acx^*(x) + (1-\alpha)bdy^*(y) > (1-\alpha)(ac+bd) = 1-\alpha.$$

Choose $\beta, \gamma > 0$ such that $\beta < a\varepsilon$ and $\beta < \gamma\varepsilon$, and

$$f(ax, by) > 1-(\alpha-\gamma).$$

Now, using Lemma 1.4, choose $\delta > 0$ such that, for every $p, q, r \geq 0$, if

$$2 - \delta \leq N(p,q) \leq N(r,q) \leq 2 \quad \text{and} \quad q < 2 - \delta,$$

then $|p - r| < \beta$. There is no loss of generality in assuming that $b < 1 - \delta$. By the assumption that (ax, by) is a Daugavet-point there exists $(u,v) \in B_Z$, where $Z = X \oplus_N Y$, such that

$$f(u,v) > 1 - (\alpha - \gamma) \quad \text{and} \quad \|(ax,by) - (u,v)\|_N \geq 2 - \delta.$$

Then
$$\begin{aligned} cx^*(u) + (1-\alpha)d\|v\| &\geq cx^*(u) + (1-\alpha)dy^*(v) \\ &= f(u,v) > 1 - (\alpha - \gamma) > 1 - \alpha \\ &\geq (1-\alpha)(c\|u\| + d\|v\|), \end{aligned}$$

which yields
$$cx^*(u) > (1-\alpha)c\|u\|,$$

taht is, $x^*(u/\|u\|) > 1 - \alpha$. Since $S(B_X, x^*, \alpha) \cap \Delta_\varepsilon(x) = \emptyset$, we know now that $\big\|x - u/\|u\|\big\| < 2 - \varepsilon$. We now show that $\|ax - u\| < a + \|u\| - \beta$. We consider two cases. If $\|u\| \geq a$, then

$$\begin{aligned} \|ax - u\| &\leq \left\|ax - a\frac{u}{\|u\|}\right\| + \left\|a\frac{u}{\|u\|} - u\right\| \\ &\leq a(2-\varepsilon) + \big|a - \|u\|\big| = a + \|u\| - a\varepsilon < a + \|u\| - \beta. \end{aligned}$$

On the other hand, if $a \geq \|u\|$, we have

$$\begin{aligned} c\|u\| + (1-\alpha)d\|v\| &\geq cx^*(u) + (1-\alpha)dy^*(v) \\ &= f(u,v) > 1 - \alpha + \gamma \geq (1-\alpha)d\|v\| + \gamma, \end{aligned}$$

from which we conclude $\|u\| \geq c\|u\| > \gamma$. Now we see that

$$\begin{aligned} \|ax - u\| &\leq \big\|ax - \|u\|x\big\| + \big\|\|u\|x - u\big\| \leq a - \|u\| + \|u\|(2-\varepsilon) \\ &= a + \|u\| - \|u\|\varepsilon < a + \|u\| - \gamma\varepsilon < a + \|u\| - \beta. \end{aligned}$$

That gives us the following:

$$\begin{aligned} 2 - \delta \leq \|(ax,by) - (u,v)\|_N &= N(\|ax - u\|, \|by - v\|) \\ &\leq N(a + \|u\| - \beta, b + \|v\|) \end{aligned}$$

and therefore

$$2 - \delta \leq N(a + \|u\| - \beta, b + \|v\|) \leq N(a + \|u\|, b + \|v\|) \leq 2.$$

Since $b + \|v\| < 2 - \delta$, we have by the choice of δ that $\big|(a + \|u\| - \beta) - (a + \|u\|)\big| < \beta$, i.e. $\beta < \beta$, a contradiction. Hence x is a Δ-point in X. $\square$

Theorem 4.1 does not cover the case $a = b = 1$ (for ℓ_∞-norm). In this case, we show that if $x \in S_X$ and $y \in S_Y$, then (x, y) can be a Δ-point even if x and y are not Δ-points. Moreover, we introduce the conditions that $x \in S_X$ and $y \in S_Y$ must satisfy in order for (x, y) to be a Δ-point in $X \oplus_\infty Y$. These results rely heavily on the concept of another type of unit sphere elements similar to Δ-points (compare with Lemma 1.3).

Definition 4.2. Let X be a Banach space, $x \in S_X$, and $k > 1$. We say that x is a Δ_k*-point* in X, if for every $S(B_X, x^*, \alpha)$ with $x \in S(B_X, x^*, \alpha)$ and for every $\varepsilon > 0$ there exists $u \in S(B_X, x^*, k\alpha)$ such that $\|x - u\| \geq 2 - \varepsilon$.

Every Δ-point is obviously a Δ_k-point for every $k > 1$. In contrast, the reverse does not hold, since the upcoming example shows the existence of a Δ_k-point that is not a Δ-point, which proves that the concepts of Δ-point and Δ_k-point do not coincide.

Example 4.3. Let X and Y be Banach spaces, $x \in S_X$ and $y \in S_Y$, and let $k > 1$. Set $Z = X \oplus_1 Y$ and $z = \big((1 - 1/k)x, y/k\big)$. Assume that x is not a Δ-point in X and y is a Δ-point in Y. Then, according to Theorem 4.1, z is not a Δ-point in Z. Fix $f = (x^*, y^*) \in S_{Z^*}$ and $\alpha > 0$, such that $f(z) > 1 - \alpha$, and fix $\varepsilon > 0$. Then

$$1 - \frac{1}{k} + \frac{1}{k} y^*(y) \geq \Big(1 - \frac{1}{k}\Big) x^*(x) + \frac{1}{k} y^*(y) = f(z) > 1 - \alpha.$$

It follows that $y^*(y) > 1 - \alpha k$. Since y is a Δ-point, there exists $v \in B_Y$ such that

$$y^*(v) > 1 - \alpha k \quad \text{and} \quad \|y - v\| \geq 2 - \varepsilon.$$

Then $f(0, v) = y^*(v) > 1 - \alpha k$, i.e. $(0, v) \in S(B_Z, f, \alpha k)$, and

$$\begin{aligned} \Big\| \Big(\Big(1 - \frac{1}{k}\Big) x, \frac{1}{k} y \Big) - (0, v) \Big\|_1 &= \Big(1 - \frac{1}{k}\Big) \|x\| + \Big\| \frac{1}{k} y - v \Big\| \\ &\geq \Big(1 - \frac{1}{k}\Big) + \|y - v\| - \Big(1 - \frac{1}{k}\Big) \|y\| \\ &\geq 2 - \varepsilon. \end{aligned}$$

This proves that z is a Δ_k-point. □

Given two Banach spaces X and Y and $x \in S_X$, $y \in S_Y$, it follows that (x, y) can be a Δ-point of $X \oplus_\infty Y$ even if neither x nor y is a Δ-point in X and Y, respectively. This is immediate from Example 4.3 and following Proposition 4.4.

Proposition 4.4. *Let X and Y be Banach spaces, $x \in S_X$ and $y \in S_Y$. Let $p, q > 1$ satisfy $1/p + 1/q = 1$.*

(a) *If x is a Δ_p-point in X and y is a Δ_q-point in Y, then (x, y) is a Δ-point in $X \oplus_\infty Y$.*

(b) *If x is not a Δ_p-point in X and y is not a Δ_q-point in Y, then (x, y) is not a Δ-point in $X \oplus_\infty Y$.*

In fact, Proposition 4.4 gives equivalent condition for (x, y) being a Δ-point in $X \oplus_\infty Y$ where neither x nor y is a Δ-point in X and Y, respectively (see Proposition 4.5 below).

Proof of Proposition 4.4. (a) Assume that x is a Δ_p-point in X and y is Δ_q-point in Y. Set $Z = X \oplus_\infty Y$. Fix $f = (x^*, y^*) \in S_{Z^*}$ and $\alpha > 0$ such that $(x, y) \in S(B_Z, f, \alpha)$, and fix $\varepsilon > 0$. Then

$$x^*(x) + y^*(y) = f(x, y) > 1 - \alpha$$

from what we get

$$x^*(x) > 1 - (\alpha + y^*(y)) = \|x^*\| - \big(\alpha + y^*(y) - \|y^*\|\big).$$

We now show that there exists $(u, v) \in S(B_Z, f, \alpha)$ such that

$$\|(x, y) - (u, v)\|_\infty \geq 2 - \varepsilon.$$

Let us consider two cases. If $\alpha + y^*(y) - \|y^*\| \leq \alpha/p$, then $x^*(x) > \|x^*\| - \alpha/p$ and therefore, since x is a Δ_p-point, there exists $u \in B_X$ such that $x^*(u) > \|x^*\| - \alpha$ and $\|x - u\| \geq 2 - \varepsilon$. Let $v \in B_Y$ be such that

$$f(u, v) = x^*(u) + y^*(v) > \|x^*\| - \alpha + \|y^*\| = 1 - \alpha.$$

Then also $\|(x, y) - (u, v)\|_\infty = \max\{\|x - u\|, \|y - v\|\} \geq 2 - \varepsilon$.

If $\alpha + y^*(y) - \|y^*\| > \alpha/p$, then

$$y^*(y) > \|y^*\| - \alpha + \frac{\alpha}{p} = \|y^*\| - \frac{1}{q}\alpha$$

and analogically, using the fact that y is a Δ_q-point, we can find $(u, v) \in S(B_Z, f, \alpha)$ such that $\|(x, y) - (u, v)\|_\infty \geq 2 - \varepsilon$. Therefore (x, y) is a Δ-point.

(b) Assume that x is not a Δ_p-point in X and y is not a Δ_q-point in Y. By definition there exist $x^* \in S_{X^*}$, $y^* \in S_{Y^*}$, and $\alpha_1, \alpha_2, \varepsilon > 0$, with $x \in S(B_X, x^*, \alpha_1)$ and $y \in S(B_X, y^*, \alpha_2)$ such that we have for every $u \in S(B_X, x^*, p\alpha_1)$ and for every $v \in S(B_Y, y^*, q\alpha_2)$

$$\|x - u\| < 2 - \varepsilon \quad \text{and} \quad \|y - v\| < 2 - \varepsilon.$$

Set $Z = X \oplus_\infty Y$. Let $\lambda \in (0, 1)$ satisfy $(1 - \lambda)/\lambda = (p\alpha_1)/(q\alpha_2)$.

Let $\alpha = \lambda\alpha_1 + (1 - \lambda)\alpha_2$ and let $f = (\lambda x^*, (1 - \lambda)y^*) \in S_{Z^*}$. Then

$$\begin{aligned} f(x, y) &= \lambda x^*(x) + (1 - \lambda)y^*(y) \\ &> \lambda(1 - \alpha_1) + (1 - \lambda)(1 - \alpha_2) = 1 - \alpha. \end{aligned}$$

Fix $(u, v) \in S(B_Z, f, \alpha)$. From

$$1 - \alpha < f(u, v) = \lambda x^*(u) + (1 - \lambda)y^*(v) \leq \lambda x^*(u) + 1 - \lambda$$

we get that

$$x^*(u) > 1 - \frac{\alpha}{\lambda} = 1 - \Big(\alpha_1 + \frac{1-\lambda}{\lambda}\alpha_2\Big) = 1 - p\Big(\frac{\alpha_1}{p} + \frac{\alpha_1}{q}\Big) = 1 - p\alpha_1.$$

Therefore $u \in S(B_X, x^*, p\alpha_1)$ and analogically $v \in S(B_Y, y^*, q\alpha_2)$. It follows that $\|x-u\| < 2-\varepsilon$ and $\|y-v\| < 2-\varepsilon$. Consequently

$$\|(x,y)-(u,v)\|_\infty = \max\{\|x-u\|, \|y-v\|\} < 2-\varepsilon$$

and thus, (x,y) is not a Δ-point. □

Proposition 4.5. *Let X and Y be Banach spaces and $x \in S_X$ and $y \in S_Y$. Assume that neither x nor y is a Δ-point in X and Y, respectively. Then the following statements are equivalent:*

(i) *there exist $p, q > 1$ with $1/p + 1/q = 1$ such that x is Δ_p-point in X and y is Δ_q-point in Y;*

(ii) *for every $p, q > 1$ with $1/p+1/q = 1$ either x is Δ_p-point in X or y is Δ_q-point in Y.*

Proof. (i) $\Rightarrow$ (ii). Assume that (i) holds. Let $p, q > 1$ be such that $1/p + 1/q = 1$. According to (i) x is $\Delta_{p'}$-point in X and y is $\Delta_{q'}$-point in Y for some $p', q' > 1$ with $1/p' + 1/q' = 1$. Then $p' \geq p$ or $q' \geq q$ and therefore x is Δ_p-point in X or y is Δ_q-point in Y, hence (ii) holds.

(ii) $\Rightarrow$ (i). Assume that (ii) holds. Define

$$A = \{k \in [1,\infty) \colon x \text{ is } \Delta_k\text{-point in } X\}$$

and

$$B = \{k \in [1,\infty) \colon y \text{ is } \Delta_k\text{-point in } Y\}.$$

Firstly, let us examine the case where set A is nonempty. Let $a = \inf A$. We show that $a \in A$. Fix $x^* \in S_{X^*}$, $\alpha > 0$ and $\varepsilon > 0$ such that $x \in S(B_X, x^*, \alpha)$. Let $\gamma > 0$ be such that $x^*(x) > 1 - (\alpha - \gamma)$ and let $k = a\alpha/(\alpha - \gamma)$. Then $k > a$ and therefore $k \in A$. Since $x \in S(B_X, x^*, \alpha - \gamma)$, there exists

$$u \in S(B_X, x^*, k(\alpha - \gamma)) = S(B_X, x^*, a\alpha)$$

such that $\|x-u\| \geq 2-\varepsilon$. From that we get $a \in A$. Analogically we can show that if B is nonempty, then $b = \inf B \in B$.

It is not hard to see that neither A nor B can be empty. Indeed, if $A = \emptyset$ (the case $B = \emptyset$ is analogical), then by assumption $(1,\infty) \subset B$. However, according to the previous argumentation we now get that $1 \in B$, i.e. y is a Δ-point, which is a contradiction. Therefore, $A = [a,\infty)$ and $B = [b,\infty)$. From the assumption we can easily see that $1/a + 1/b \geq 1$, hence there exist $p, q > 1$ that satisfy $1/p + 1/q = 1$ such that $p \in A$ and $q \in B$. □

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