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Conjugate Convex Functions without Infinity

Fumioki Wada
Hokkaido Sapporo Kita Senior High School, Sapporo, Hokkaido 001-0025, Japan
21013351@alumni.tus.ac.jp

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Let $B_r(E)$ be the closed ball of radius r around the origin in a real Banach space E and $\mathcal{F}_r(E)$ be the set of all r-Lipschitz continuous convex functions defined on $B_r(E)$. Suppose f is a real-valued and bounded below function on $B_r(E)$. We define the I-conjugate function f^I of f to improve the Fenchel inequality and investigate the properties of f^I. In particular, $(f^I)^I$ coincides with f on $B_r(E)$ if and only if f is in $\mathcal{F}_r(E)$. Excluding the value $+\infty$, the transformation from f to f^I enlarges the potentiality of the contribution to numerical computation for convex analysis.

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1. Introduction

In this paper, we define conjugate convex functions, such as Legendre-Fenchel transforms and demonstrate that these convex functions improve the Fenchel inequality.

The Legendre-Fenchel transform f^* of an *extended* real-valued function f in a Banach space E can be defined as follows [3]:

$$f^*(x^*) := \sup_{x \in E}\{\langle x^*, x\rangle - f(x)\} \quad (x^* \in E^*).$$

It immediately follows from the definition of f^* that

$$\langle x^*, x\rangle \le f^*(x^*) + f(x)$$

for all $x^* \in E^*$ and all $x \in E$. This inequality, which is known as the Fenchel inequality, can be regarded as a generalization of Young's inequality [5]. However, f^* may have the value of $+\infty$ in a large subset of E^* even if f is continuous convex bounded below or Lipschitz continuous convex. For example, if $f(x) = e^x$, then

$$f^*(x^*) = \begin{cases} x^* \log x^* - x^* & (x^* > 0) \\ 0 & (x^* = 0) \\ +\infty & (x^* < 0) \end{cases}.$$

Further, if $f(x) = ax + b$, then

$$f^*(x^*) = \begin{cases} -b & (x^* = a) \\ +\infty & (x^* \ne a) \end{cases}.$$

In the latter case, the Fenchel inequality is trivial and not useful. To avoid this peculiar aspect of the Fenchel inequality, we can adopt the following method.

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The letter x on the right-hand side of the definition of f^* runs through the domain E. If the domain through which x runs is restricted from the whole space E to the bounded closed convex set, the new transform will never take the value $+\infty$; it is a *usual* real-valued continuous convex function. The transform f_r^I of a bounded below function f is defined as follows:

$$f_r^I(x^*) := \sup_{x \in B_r(E)} \{\langle x^*, x\rangle - f(x)\},$$

where $B_r(E)$ is the closed ball of radius r around the origin in E. Then, we obtain

$$\langle x^*, x\rangle \leq f_{\|x\|}^I(x^*) + f(x)$$

for all $x^* \in E^*$ and all $x \in E$. This inequality is an improved version of the Fenchel inequality, as $f_{\|x\|}^I \leq f^*$ holds for E^*.

Throughout this paper, E denotes the real Banach space and is considered to be a subspace of E^{**}. Let $\mathcal{F}_r(E)$ be the set of all r-Lipschitz continuous convex functions defined on $B_r(E)$.

Section 2 presents the necessary definitions and introduces some of the results of the convex analysis. The $r \in \mathbb{R}^+$ of f_r^I is arbitrarily fixed. It is thus omitted and we commonly write the simplified form of f^I for f_r^I (see Definition 2.1).

The Legendre-Fenchel transforms have been developed in the context of locally convex spaces in the monograph of Moreau and the book by Zalinescu, and they have also been investigated in several studies [1, 2, 5, 7, 8, 9, 11, 12, 13]. For any lower semi-continuous convex function f on $\mathbb{R}^n$, $(f^*)^*$ coincides with f [12]. What is the necessary and sufficient condition for $(f^I)^I = f$?

In Section 3, we prove that $(f^I)^I|_{B_r(E)} = f$ holds if and only if $f \in \mathcal{F}_r(E)$ (see Theorem 3.1). We call the transform f^I the *I-conjugate function* of f if $f \in \mathcal{F}_r(E)$. The Fenchel-Rockafellar theorem is useful [3, Theorem 12], and we obtain an analogous result for the I-conjugate function:

$$\inf_{x \in B_r(E)} \{f(x) + g(x)\} = \max_{x^* \in B_r(E^*)} \{-f^I(x^*) - g^I(-x^*)\},$$

which holds if $f, g \in \mathcal{F}_r(E)$ (see Theorem 3.4). This result is a generalization of Theorem 3.1 (see Remark 3.6).

In Section 4, we address the I-conjugate functions in a reflexive space. We investigate the set of all points where the supremum, $\sup_{x \in B_r(E)}\{\langle x^*, x\rangle - f(x)\}$, is attained.

In Section 5, we introduce the notion of a *U-conjugate function* in a Hilbert space E, where U is a unitary operator. The notion of this U-conjugate function is a generalization of the I-conjugate function (see Definition 5.1). We present a few properties of these functions (see Theorem 5.2). For $x^* \in B_r(E)$, $f \in \mathcal{F}_r(E)$ and a unitary operator U, we define the subset $M(x^*, f, U)$ of $B_r(E)$ by

$$M(x^*, f, U) := \{x \in B_r(E) \mid f^U(x^*) = \langle Ux^*, x\rangle - f(x)\}.$$

We prove that the set $M(x^*, f, U)$ is a nonempty convex and weak-compact subset of E, and that the equality

$$\bigcup_{x^* \in B_r(E)} M(x^*, f, U) = B_r(E)$$

holds for any $f \in \mathcal{F}_r(E)$ and any unitary operator U (see Proposition 5.8). Moreover, we show that for any $f \in \mathcal{F}_r(E)$ and any unitary operator U, V in E, there exists some x_0 in $B_r(E)$ such that $\langle Ux_0, Vx_0 \rangle = f^I(Ux_0) + f(Vx_0)$ if E is a finite-dimensional space (see Example 5.9).

Finally, in Section 6, in a Hilbert space, we characterize f^U by three conditions, where U is a self-adjoint unitary operator (see Theorem 6.2). This result represents simple conditions for the transformation $f \mapsto f^U$.

2. Preliminaries

In this section, we provide some essential definitions and basic results of the convex analysis.

Definition 2.1. Let $B_r(E)$ be the closed ball of radius r around the origin in E and $\mathcal{F}_r(E)$ be the set of all r-Lipschitz continuous convex functions defined in $B_r(E)$. Suppose that a function $f\colon B_r(E) \to \mathbb{R}$ is bounded below. The *transform* f^I of f can be defined as follows:

$$f^I(x^*) := \sup_{x \in B_r(E)} \{\langle x^*, x\rangle - f(x)\} \quad (x^* \in B_r(E^*)),$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical duality product between E^* and E. We call f^I the *I-conjugate function* of f if $f \in \mathcal{F}_r(E)$, and the transformation $f \mapsto f^I$ the *I-transformation* from $\mathcal{F}_r(E)$ to $\mathcal{F}_r(E^*)$.

Remark 2.2. We see that $f^I \in \mathcal{F}_r(E^*)$ and $(f^I)^I|_{B_r(E)} \in \mathcal{F}_r(E)$. We recall that the class of convex functions is closed by taking the supremum of a family of functions. For each fixed $x \in B_r(E)$, the function $x^* \mapsto \langle x^*, x\rangle - f(x)$ is affine (more generally, convex) on $B_r(E^*)$. It follows that the superior envelope f^I of these functions (as x runs through $B_r(E)$) is convex. If $x_1^*, x_2^* \in B_r(E^*)$, we have

$$f^I(x_1^*) - f^I(x_2^*) \leq \sup_{x \in B_r(E)} \{(\langle x_1^*, x\rangle - f(x)) - (\langle x_2^*, x\rangle - f(x))\} \leq r\,\|x_1^* - x_2^*\|\,.$$

Interchanging x_1^*, x_2^* proves that $\left|f^I(x_1^*) - f^I(x_2^*)\right| \leq r\,\|x_1^* - x_2^*\|$ holds for all $x_1^*, x_2^* \in B_r(E^*)$. Consequently, we have $f^I \in \mathcal{F}_r(E^*)$.
Similarly, we have $(f^I)^I|_{B_r(E)} \in \mathcal{F}_r(E)$. □

Throughout this paper, the notion of the subdifferential plays an important role. The subdifferential $\partial f(x_0)$ of f at x_0 is defined as follows.

Definition 2.3. Let C be a nonempty convex subset of E and suppose f is a convex function defined on C. An element $z^* \in E^*$ is called a *subgradient* of f at $x_0 \in C$ if $\langle z^*, x - x_0\rangle \leq f(x) - f(x_0)$ holds for all $x \in C$. The set $\partial f(x_0)$ of all the subgradients of f at x_0 is called the *subdifferential* of f at x_0.

The following proposition [10, Proposition 1.11] is essential in Sections 3 and 4.

Proposition 2.4. [10] *Let C be a nonempty convex subset of E and f be a continuous convex function on C. Then, $\partial f(x_0)$ is a nonempty, convex, and weak*-compact subset of E^* for any $x_0 \in \mathrm{int}C$.*

The following definition is presented as preparation for the introduction of the Kakutani fixed-point theorem [6].

Definition 2.5. Let C be a nonempty subset of E and T be a set-valued map from C into 2^{E^*}. T is called *norm-to-norm* [resp. *norm-to-weak*] *upper semi-continuous* at $x_0 \in C$ if for each norm open set [resp. weak open set] G containing Tx_0, there exists a neighborhood $N(x_0)$ of x_0 in C for the induced topology such that $Tx \subset G$ for any $x \in N(x_0)$. T is called *norm-to-norm* [resp. *norm-to-weak*] *upper semi-continuous* on C if it is norm-to-norm [resp. norm-to-weak] upper semi-continuous at each point of C.

As presented in Section 5, the existence of a solution of the equation

$$\langle Ux, Vx\rangle = f^I(Ux) + f(Vx)$$

is based on the following result proposed by Kakutani [6].

Theorem 2.6. (The Kakutani Fixed-Point Theorem) [6] *Let C be a nonempty convex compact set in $\mathbb{R}^n$ and T be a set-valued map from C into 2^C. If T is (norm-to-norm) upper semi-continuous on C and if the set Tx is nonempty and closed convex for each $x \in C$, then T has a fixed point. That is, there exists a point x_0 in C such that $x_0 \in Tx_0$.*

Fan generalized the whole space in the Kakutani fixed-point theorem, from a finite-dimensional space to a locally convex space [4].

3. Involution theorem and duality theorem

In this section, employing Proposition 2.4, we prove the involution theorem for the I-conjugate functions (see Theorem 3.1). We utilize this important theorem in Sections 4 and 5. In addition, we prove the duality theorem for the I-conjugate functions (see Theorem 3.4).

Theorem 3.1. *Suppose a real-valued function f on $B_r(E)$ is bounded below. Then, $(f^I)^I|_{B_r(E)} = f$ holds if and only if $f \in \mathcal{F}_r(E)$.*

Proof. Suppose $f \in \mathcal{F}_r(E)$. It is evident that $(f^I)^I|_{B_r(E)} \leq f$, as

$$\langle x^*, x\rangle - f^I(x^*) \leq f(x)$$

holds for all $x \in B_r(E)$ and all $x^* \in B_r(E^*)$.

To prove that $(f^I)^I|_{B_r(E)} \geq f$, we use the fundamental fact (in Proposition 2.4) that $\partial f(x_0)$ is nonempty for each $x_0 \in \mathrm{int} B_r(E)$. That is, for any $x_0 \in \mathrm{int} B_r(E)$ there exist $z^* \in E^*$ such that $\langle z^*, x - x_0\rangle \leq f(x) - f(x_0)$ for all $x \in B_r(E)$. As f is r-Lipschitz continuous, we have

$$\langle z^*, x - x_0\rangle \leq f(x) - f(x_0) \leq r\|x - x_0\|$$

for all $x \in B_r(E)$. Therefore, we have $z^* \in B_r(E^*)$. From the inequality

$$\langle z^*, x_0\rangle - f(x_0) \geq \langle z^*, x\rangle - f(x)$$

for all $x \in B_r(E)$, we have $\langle z^*, x_0\rangle - f(x_0) = f^I(z^*)$.

Therefore, $(f^I)^I|_{B_r(E)} \geq f$ holds on $\mathrm{int} B_r(E)$. Because of the continuity of f and $(f^I)^I|_{B_r(E)}$, we find that $(f^I)^I|_{B_r(E)} = f$ on $B_r(E)$.

Conversely, suppose we have $(f^I)^I|_{B_r(E)} = f$. Then it follows from Remark 2.2 that $f = (f^I)^I|_{B_r(E)} \in \mathcal{F}_r(E)$. This completes the proof. □

Remark 3.2. As shown in the proof above, for $f \in \mathcal{F}_r(E)$

$$f(x) = (f^I)^I|_{B_r(E)}(x) = \sup_{x^* \in B_r(E^*)} \{\langle x^*, x\rangle - f^I(x^*)\},$$

and for each $x \in \mathrm{int}B_r(E)$, this supremum is attained at some point $x^* \in \partial f(x)$. (In Remark 3.6, we see that this supremum is attained for each $x \in B_r(E)$.) Furthermore, if E is a reflexive Banach space, for each $x^* \in \mathrm{int}B_r(E^*)$, the supremum of the function

$$f^I(x^*) = \sup_{x \in B_r(E)} \{\langle x^*, x\rangle - f(x)\}$$

is attained at some point $x \in \partial f^I(x^*)$. Indeed, from Theorem 3.1, we have

$$f^I(x^*) = \sup_{x \in B_r(E)} \{\langle x^*, x\rangle - (f^I)^I(x)\}.$$

It follows that there exists $z \in \partial f^I(x^*)$ such that

$$f^I(x^*) = \langle x^*, z\rangle - (f^I)^I(z) = \langle x^*, z\rangle - f(z). \qquad \square$$

Suppose $f \in \mathcal{F}_r(E)$. Then, an r-Lipschitz continuous convex function $\widetilde{f}$ defined in the whole space E is said to be an *extension* of the function f if $\widetilde{f}(x) = f(x)$ holds for all $x \in B_r(E)$.

The following lemma prepares the proof of Theorem 3.4.

Lemma 3.3. *If $f \in \mathcal{F}_r(E)$, there exists an r-Lipschitz continuous convex function $\widetilde{f}$ defined in the whole space E such that $\widetilde{f}(x) = f(x)$ for all $x \in B_r(E)$.*

Proof. For $x \in E$, let $\widetilde{f}(x) := \sup_{x^* \in B_r(E^*)}\{\langle x^*, x\rangle - f^I(x^*)\}$. It immediately follows from Remark 2.2 that $\widetilde{f}$ is an extension of $f \in \mathcal{F}_r(E)$. $\square$

Another well-known r-Lipschitz continuous convex extension is given by (see Remark 3.6):

$$\widetilde{f}(x) := \inf_{y \in B_r(E)} \{r\,\|x - y\| + f(y)\} \quad (x \in E).$$

Theorem 3.4. *If $f, g \in \mathcal{F}_r(E)$, then*

$$\inf_{x \in B_r(E)} \{f(x) + g(x)\} = \max_{x^* \in B_r(E^*)} \{-f^I(x^*) - g^I(-x^*)\}.$$

Proof. Let $p := \inf_{x \in B_r(E)}\{f(x) + g(x)\}$ and $q := \sup_{x^* \in B_r(E^*)}\{-f^I(x^*) - g^I(-x^*)\}$. It is clear from the definition of f^I and g^I that

$$q \leq p. \tag{1}$$

We next show that $p \leq q$. Applying Lemma 3.3 to $f \in \mathcal{F}_r(E)$, we see that there exists an r-Lipschitz continuous convex function $\widetilde{f}$, defined in the whole space E, such that $\widetilde{f} = f$ on $B_r(E)$. For $\widetilde{f}$ and g, let

$$C := \{(x, y) \in E \times \mathbb{R} \mid \widetilde{f}(x) \leq y\}$$

and
$$D := \{(x, y) \in E \times \mathbb{R} \mid x \in B_r(E) \text{ and } y \leq p - g(x)\}.$$

Then, both int$C$ and D are nonempty convex sets. Moreover, it immediately follows that int$C \cap D = \emptyset$. Therefore, we can apply the Hahn-Banach theorem [3, Theorem 1.6] to int$C$ and D. Hence, there exist $(w^*, \alpha) \in E^* \times \mathbb{R}$ and $c \in \mathbb{R}$ such that

$$(w^*, \alpha) \neq (0,0) \qquad \text{and} \qquad \langle w^*, x\rangle + \alpha y \leq c \leq \langle w^*, \widetilde{x}\rangle + \alpha \widetilde{y}$$

for all $(\widetilde{x}, \widetilde{y}) \in \text{int}C$ and all $(x, y) \in D$. As $C = \overline{\text{int}C}$, we have

$$\langle w^*, x\rangle + \alpha y \leq c \leq \langle w^*, \widetilde{x}\rangle + \alpha \widetilde{y}$$

for all $(\widetilde{x}, \widetilde{y}) \in C$ and all $(x, y) \in D$. Therefore, we have

$$\langle w^*, x\rangle + \alpha y \geq c, \text{ for all } (x, y) \in C, \tag{2}$$

and

$$\langle w^*, x\rangle + \alpha y \leq c, \text{ for all } (x, y) \in D. \tag{3}$$

Letting $y \to +\infty$ in (2), we see that $\alpha \geq 0$. We can show further that $\alpha > 0$. First, assume that $\alpha = 0$. From (2) and (3), we have $\langle w^*, x\rangle = c$ for all $x \in B_r(E)$. Therefore, we obtain $w^* = 0$, which is a contradiction. It follows that $\alpha > 0$. Setting $z^* := -w^*/\alpha$ in (2) and (3), we see that

$$\langle z^*, x\rangle - \widetilde{f}(x) \leq -c/\alpha \text{ for all } x \in E, \tag{4}$$

and

$$\langle -z^*, u\rangle - g(u) \leq -p + c/\alpha \text{ for all } u \in B_r(E). \tag{5}$$

Further, for any $\varepsilon > 0$ there exists $x_0 \in B_r(E)$ such that

$$p + \varepsilon > f(x_0) + g(x_0) = \widetilde{f}(x_0) + g(x_0). \tag{6}$$

Using (4), (5), and (6), we have

$$\langle z^*, x - u\rangle < \widetilde{f}(x) - \widetilde{f}(x_0) + g(u) - g(x_0) + \varepsilon$$

for all $x \in E$ and all $u \in B_r(E)$. Setting $u = x_0$, we have

$$\langle z^*, x - x_0\rangle < \widetilde{f}(x) - \widetilde{f}(x_0) + \varepsilon \leq r\,\|x - x_0\| + \varepsilon$$

for all $x \in E$. In particular, if $\|x - x_0\| = 1$, then $\langle z^*, x - x_0\rangle \leq r + \varepsilon$.

It follows that $\|z^*\| \leq r + \varepsilon$. As $\varepsilon > 0$ is arbitrary, it follows that $z^* \in B_r(E^*)$. Using (4), we have

$$f^I(z^*) \leq -c/\alpha. \tag{7}$$

Then, using (5), we have

$$g^I(-z^*) \leq -p + c/\alpha. \tag{8}$$

From (7) and (8), we obtain

$$p \leq -f^I(z^*) - g^I(-z^*) \leq q. \tag{9}$$

Finally, it follows from (1) and (9) that

$$\begin{aligned}\inf_{x \in B_r(E)} \{f(x) + g(x)\} &= \sup_{x^* \in B_r(E^*)} \{-f^I(x^*) - g^I(-x^*)\} \\ &= \max_{x^* \in B_r(E^*)} \{-f^I(x^*) - g^I(-x^*)\}.\end{aligned}$$

Thus, we have proved the theorem. □

Remark 3.5. It immediately follows from the preceding theorem that

$$\sup_{x \in B_r(E)} \{-f(x) - g(x)\} = \min_{x^* \in B_r(E^*)} \{f^I(x^*) + g^I(-x^*)\}$$

holds for any f and $g \in \mathcal{F}_r(E)$. □

Remark 3.6. Artstein-Avidan and Milman characterized a class of transformations containing the Legendre-Fenchel transformation using the notions of both involution and inf-convolution [1, Theorem 11]. The I-transformation does not belong to this class. In terms of the inf-convolution of $n\,\|\cdot\|$ and f, we can show that

$$\lim_{n \to \infty} \inf_{y \in D} \{n\,\|x - y\| + f(y)\} = f(x) \quad (x \in D)$$

holds for any continuous convex function f defined in the open convex set $D \subset E$ from the proof of [10, Lemma 2.31].

By contrast, if $f \in \mathcal{F}_r(E)$, the inf-convolution of $r\,\|\cdot\|$ and f coincides with f:

$$\min_{y \in B_r(E)} \{r\,\|x - y\| + f(y)\} = f(x) \quad (x \in B_r(E)).$$

Let $g(x) := r\,\|u - x\|$, where $u \in B_r(E)$ is arbitrarily fixed. It is clear that we have $g^I(x^*) = \langle x^*, u\rangle$. It follows from Theorem 3.4 that

$$f(u) = \inf_{x \in B_r(E)} \{r\,\|u - x\| + f(x)\} = \max_{x^* \in B_r(E^*)} \{\langle x^*, u\rangle - f^I(x^*)\} = (f^I)^I(u).$$

Thus, we see that Theorem 3.4 is a generalization of Theorem 3.1. □

4. I-conjugate functions in reflexive spaces

Throughout this section, E denotes a reflexive Banach space. We investigate the set $M(x^*, f)$ defined as follows.

Definition 4.1. For $f \in \mathcal{F}_r(E)$ and $x^* \in B_r(E^*)$, let

$$M(x^*, f) := \{x \in B_r(E) \mid f^I(x^*) = \langle x^*, x\rangle - f(x)\}.$$

It follows from Remark 3.6 that $M(x^*, f)$ is nonempty for each $x^* \in B_r(E^*)$.

Since $f(\cdot) - \langle x^*, \cdot\rangle$ is weakly lower semi-continuous and $B_r(E)$ is weakly compact and convex, we have the following lemma.

Lemma 4.2. *If $f \in \mathcal{F}_r(E)$ and $x^* \in B_r(E^*)$, then $M(x^*, f)$ is nonempty, convex, and weak-compact.*

We also have the following.

Lemma 4.3. *If $f \in \mathcal{F}_r(E)$ and $x \in B_r(E)$, then $M(x, f^I)$ is nonempty, convex, and weak-compact.*

Remark 4.4. In consequence of Lemma 4.2, we have $M(x^*, f) = \partial f^I(x^*)$ for each $x^* \in \mathrm{int} B_r(E^*)$. In general, we have $M(x^*, f) = \partial f^I(x^*) \cap B_r(E)$ for each $x^* \in B_r(E^*)$. Furthermore, we have $M(x, f^I) = \partial f(x)$ for each $x \in \mathrm{int} B_r(E)$ and $M(x, f^I) = \partial f(x) \cap B_r(E^*)$ for each $x \in B_r(E)$. □

The following remark, which is evident from the definitions of $M(x^*, f)$ and $M(x, f^I)$, is used in the proof of Proposition 4.8.

Remark 4.5. Suppose $x \in B_r(E)$, $x^* \in B_r(E^*)$, and $f \in \mathcal{F}_r(E)$. The following conditions are then equivalent:

(i) $x \in M(x^*, f)$

(ii) $x^* \in M(x, f^I)$ □

From Remark 4.4 and Remark 4.5, we get the following.

Remark 4.6. Suppose $x \in B_r(E)$, $x^* \in B_r(E^*)$, and $f \in \mathcal{F}_r(E)$. The following conditions are thus equivalent:

(i) $x \in \partial f^I(x^*)$

(ii) $x^* \in \partial f(x)$ □

Since E is reflexive, the subdifferential map $x^* \mapsto \partial f^I(x^*)$ is norm-to-weak upper semi-continuous [10, Proposition 2.5].

From the equality $M(x^*, f) = \partial f^I(x^*) \cap B_r(E)$ in Remark 4.4, we obtain the following lemma.

Lemma 4.7. *If $f \in \mathcal{F}_r(E)$, then $M(\cdot, f)$ is a norm-to-weak upper semi-continuous map on $B_r(E^*)$.*

To conclude this section, we show that the set $M(x^*, f)$ has the following property.

Proposition 4.8. *If $f \in \mathcal{F}_r(E)$, then*

$$\bigcup_{x^* \in B_r(E^*)} M(x^*, f) = B_r(E).$$

Proof. It is evident from the definition of the set $M(x^*, f)$ that

$$\bigcup_{x^* \in B_r(E^*)} M(x^*, f) \subset B_r(E).$$

To see that

$$B_r(E) \subset \bigcup_{x^* \in B_r(E^*)} M(x^*, f),$$

we pick any $x_0 \in B_r(E)$. From Lemma 4.3, there exists an element $x^* \in M(x_0, f^I)$. It follows from Remark 4.5 that

$$x_0 \in M(x^*, f) \subset \bigcup_{x^* \in B_r(E^*)} M(x^*, f).$$

As $x_0 \in B_r(E)$ is arbitrary, we obtain

$$B_r(E) \subset \bigcup_{x^* \in B_r(E^*)} M(x^*, f).$$

Thus, we have proved the theorem. □

5. U-conjugate functions in Hilbert spaces

Throughout this section, suppose E is a real Hilbert space. Let $\mathcal{U}(E)$ be the set of all unitary operators on E.

Definition 5.1. For $f \in \mathcal{F}_r(E)$ and $U \in \mathcal{U}(E)$, let f_U be the *U-function* of f, defined as $f_U(x) := f(Ux)$, where $x \in B_r(E)$, and let f^U be the *U-conjugate function* of f, defined as

$$f^U(x) := \sup_{t \in B_r(E)} \{\langle Ux, t\rangle - f(t)\} \quad (x \in B_r(E)),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of E.

It is evident that $f_I = f$, where I is the identity operator.

Theorem 5.2. *If $f \in \mathcal{F}_r(E)$ and $U, V \in \mathcal{U}(E)$, then we have f_U, $f^U \in \mathcal{F}_r(E)$, $(f_U)^V = (f^U)_V = f^{UV}$, and $(f^U)^V = (f_U)_V = f_{UV}$.*

Proof. Suppose $f \in \mathcal{F}_r(E)$ and $U, V \in \mathcal{U}(E)$. It immediately follows from the definitions of f_U and f^U that f_U, $f^U \in \mathcal{F}_r(E)$.

To show (10)–(15), we use Theorem 3.1 and the definitions of f_U and f^U. From the fact that

$$(f^I)_U(x) = f^I(Ux) = \sup_{t \in B_r(E)} \{\langle Ux, t\rangle - f(t)\} = f^U(x),$$

we have

$$(f^I)_U = f^U. \tag{10}$$

From

$$(f_U)_V(x) = (f_U)(Vx) = f(UVx) = f_{UV}(x),$$

we have

$$(f_U)_V = f_{UV}. \tag{11}$$

Using (10) and (11), we have

$$(f^U)_V = ((f^I)_U)_V = (f^I)_{UV} = f^{UV}.$$

That is,

$$(f^U)_V = f^{UV}. \tag{12}$$

As $U, V \in \mathcal{U}(E)$, it follows that

$$\begin{aligned}(f_U)^V(x) &= \sup_{t \in B_r(E)} \{\langle Vx, t\rangle - f_U(t)\} = \sup_{t \in B_r(E)} \{\langle UVx, Ut\rangle - f(Ut)\} \\ &= \sup_{t \in B_r(E)} \{\langle UVx, t\rangle - f(t)\} = (f^{UV})(x).\end{aligned}$$

That is,

$$(f_U)^V = f^{UV}. \tag{13}$$

From the fact that $U, V \in \mathcal{U}(E)$, and using (10), we have

$$\begin{aligned}(f^I)^U(x) &= \sup_{t \in B_r(E)} \{\langle Ux, t\rangle - f^I(t)\} = \sup_{t \in B_r(E)} \{\langle Ux, Ut\rangle - f^I(Ut)\} \\ &= \sup_{t \in B_r(E)} \{\langle x, t\rangle - (f^I)_U(t)\} = \sup_{t \in B_r(E)} \{\langle x, t\rangle - (f^U)(t)\} = (f^U)^I(x).\end{aligned}$$

That is,

$$(f^I)^U = (f^U)^I. \tag{14}$$

Using (13), (14), (13), Theorem 3.1, and (11) in turn, we obtain

$$(f^U)^V = ((f_U)^I)^V = ((f_U)^V)^I = ((f_U)_V)^I)^I = (f_U)_V = f_{UV}.$$

That is,

$$(f^U)^V = f_{UV}. \tag{15}$$

We have completed the proof. $\square$

We present the following corollary to this theorem.

Corollary 5.3. *If $f \in \mathcal{F}_r(E)$ and $U \in \mathcal{U}(E)$, then $(f_U)^I = f^U$, $(f^U)^I = f_U$, and $(f^U)^{U^*} = (f_U)_{U^*} = f$, where U^* is the conjugate operator of U.*

Now, we define a set $M(x^*, f, U)$ as follows.

Definition 5.4. For $x^* \in B_r(E)$, $f \in \mathcal{F}_r(E)$, and $U \in \mathcal{U}(E)$, let

$$M(x^*, f, U) := \{x \in B_r(E) \mid f^U(x^*) = \langle Ux^*, x\rangle - f(x)\}.$$

It is evident that $M(x^*, f, I) = M(x^*, f)$, where I is the identity operator.

From Theorem 5.2, we have

$$\begin{aligned} M(x^*, f, U) &= \{x \in B_r(E) \mid f^U(x^*) = \langle Ux^*, x\rangle - f(x)\} \\ &= \{x \in B_r(E) \mid (f^I)_U(x^*) = \langle Ux^*, x\rangle - f(x)\} \\ &= \{x \in B_r(E) \mid (f^I)(Ux^*) = \langle Ux^*, x\rangle - f(x)\} \\ &= M(Ux^*, f). \end{aligned}$$

Therefore, we obtain the following lemma.

Lemma 5.5. *If $x^* \in B_r(E)$, $f \in \mathcal{F}_r(E)$, and $U \in \mathcal{U}(E)$, then*

$$M(x^*, f, U) = M(Ux^*, f).$$

Remark 5.6. From Theorem 5.2, we also have $M(x^*, f, U) = U[M(x^*, f_U)]$. Indeed, we have

$$\begin{aligned} M(x^*, f, U) &= \{x \in B_r(E) \mid f^U(x^*) = \langle Ux^*, x\rangle - f(x)\} \\ &= \{Ux \in B_r(E) \mid f^U(x^*) = \langle Ux^*, Ux\rangle - f(Ux)\} \\ &= U[\{x \in B_r(E) \mid (f_U)^I(x^*) = \langle x^*, x\rangle - f_U(x)\}] \\ &= U[M(x^*, f_U)]. \end{aligned}$$ □

Lemma 5.7. *If $f \in \mathcal{F}_r(E)$ and $U \in \mathcal{U}(E)$, then $M(\cdot, f, U)$ is a norm-to-weak upper semi-continuous map on $B_r(E)$.*

Proof. Suppose $f \in \mathcal{F}_r(E)$ and $U \in \mathcal{U}(E)$. We define a set-valued map T by

$$Tx^* = M(x^*, f), \quad \text{where } x^* \in B_r(E).$$

It immediately follows from Definition 2.5 and Lemma 4.7 that TU is a norm-to-weak upper semi-continuous map on $B_r(E)$. Moreover, from Lemma 5.5, we have $M(x^*, f, U) = M(Ux^*, f) = TUx^*$, where $x^* \in B_r(E)$. Consequently, $M(\cdot, f, U)$ is a norm-to-weak upper semi-continuous map on $B_r(E)$. □

In addition, we have the following.

Proposition 5.8. *If $x^* \in B_r(E)$, $f \in \mathcal{F}_r(E)$, and $U \in \mathcal{U}(E)$, then $M(x^*, f, U)$ is a nonempty, convex, and weak-compact subset of E, and we have*

$$\bigcup_{x^* \in B_r(E)} M(x^*, f, U) = B_r(E).$$

Proof. Suppose $x^* \in B_r(E)$, $f \in \mathcal{F}_r(E)$, and $U \in \mathcal{U}(E)$. It follows from Lemma 5.5 and Lemma 4.2 that $M(x^*, f, U)$ is a nonempty convex and weak-compact subset of E. Moreover, from Lemma 5.5 and Proposition 4.8, we have

$$\bigcup_{x^* \in B_r(E)} M(x^*, f, U) = \bigcup_{x^* \in B_r(E)} M(Ux^*, f) = \bigcup_{x^* \in B_r(E)} M(x^*, f) = B_r(E).$$

Thus, we have proved the theorem. □

Example 5.9. Suppose E is a finite-dimensional space, and that $f \in \mathcal{F}_r(E)$ and $U, V \in \mathcal{U}(E)$. Then there exists some point $x_0 \in B_r(E)$ such that we have $\langle Ux_0, Vx_0 \rangle = f^I(Ux_0) + f(Vx_0)$.

Proof. We apply Theorem 2.6 to the map $M(\cdot, f, UV^{-1})$. This is possible as a result of Lemma 5.7 and Proposition 5.8. Then, the map $M(\cdot, f, UV^{-1})$ has a fixed point $z_0 \in M(z_0, f, UV^{-1})$, namely

$$f^{UV^{-1}}(z_0) = \langle UV^{-1}z_0, z_0 \rangle - f(z_0).$$

Setting $x_0 := V^{-1}z_0$, we have $\langle Ux_0, Vx_0 \rangle = f^I(Ux_0) + f(Vx_0)$. □

6. Characterization of U-conjugate functions

In this section, suppose E is a real Hilbert space.

Lemma 6.1. *Suppose $\mathcal{T} \colon \mathcal{F}_r(E) \to \mathcal{F}_r(E)$ satisfies*

$$\text{(i)}\ \ \mathcal{T}\mathcal{T}f = f, \qquad \text{(ii)}\ \ f \le g \text{ implies } \mathcal{T}f \ge \mathcal{T}g,$$

and $f_\alpha \in \mathcal{F}_r(E)$ for all α and $\inf_\alpha(f_\alpha) \in \mathcal{F}_r(E)$. Then, $\mathcal{T}(\inf_\alpha(f_\alpha)) = \sup_\alpha(\mathcal{T}f_\alpha)$.

Proof. Set $h := \mathcal{T}(\sup_\alpha(\mathcal{T}f_\alpha))$. It is evident from (i) that $\sup_\alpha(\mathcal{T}f_\alpha) = \mathcal{T}h$. Then, we have $\mathcal{T}f_\alpha \le \mathcal{T}h$ for all α. Using (i) and (ii), we have $f_\alpha \ge h$ for all α, and $\inf_\alpha(f_\alpha) \ge h$. Using (ii) again, we have $\mathcal{T}(\inf_\alpha(f_\alpha)) \le \mathcal{T}h$. That is,

$$\mathcal{T}(\inf_\alpha(f_\alpha)) \le \sup_\alpha(\mathcal{T}f_\alpha). \tag{16}$$

By contrast, from (ii) and $\inf_\alpha(f_\alpha)) \le f_\alpha$ for all α, we have $\mathcal{T}(\inf_\alpha(f_\alpha)) \ge \mathcal{T}f_\alpha$ for all α, and

$$\mathcal{T}(\inf_\alpha(f_\alpha)) \ge \sup_\alpha(\mathcal{T}f_\alpha). \tag{17}$$

It follows from (16) and (17) that $\mathcal{T}(\inf_\alpha(f_\alpha)) = \sup_\alpha(\mathcal{T}f_\alpha)$. □

Finally, we characterize f^U, where U is a self-adjoint unitary operator, by basic conditions.

Theorem 6.2. *Suppose $\mathcal{T} \colon \mathcal{F}_r(E) \to \mathcal{F}_r(E)$ satisfies*

(i) $\mathcal{T}\mathcal{T}f = f$,

(ii) *$f \le g$ implies $\mathcal{T}f \ge \mathcal{T}g$,*

(iii) *There exists $U \in \mathcal{U}(E)$ such that $\mathcal{T}(r\,\|\cdot - y\| + c) = \langle Uy, \cdot \rangle - c$ for all $y \in B_r(E)$ and all $c \in \mathbb{R}$.*

Then, $U^* = U$ and $\mathcal{T}f = f^U$.

Proof. From Remark 3.6 and Lemma 6.1, we have

$$\begin{aligned}(\mathcal{T}f)(x) &= \sup_{y\in B_r(E)} \mathcal{T}(r\,\|x-y\| + f(y)) = \sup_{y\in B_r(E)} \{\langle Uy, x\rangle - f(y)\} \\ &= \sup_{y\in B_r(E)} \{\langle U^*x, y\rangle - f(y)\}.\end{aligned}$$

That is,

$$\mathcal{T}f = f^{U^*}. \tag{18}$$

From Corollary 5.3, we have

$$(\mathcal{T}f)^I = (f^{U^*})^I = f_{U^*}. \tag{19}$$

Using assumption (i), (18), and (19), we have

$$f = \mathcal{T}\mathcal{T}f = (\mathcal{T}f)^{U^*} = ((\mathcal{T}f)^I)_{U^*} = f_{U^*U^*}.$$

As $f \in \mathcal{F}_r(E)$ is arbitrary, it follows that $U^*U^* = I$. That is,

$$U^* = U. \tag{20}$$

From (18) and (20), we obtain $\mathcal{T}f = f^U$. This completes the proof. □

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