Journal of Convex Analysis
Volume 28 (2021), No. 1, 81–102

A Nonlinear φ_0-Convexity Result for the Bilateral Minimal Time Function

Samara Chamoun
Dept. of Computer Science & Mathematics, Lebanese American University, Byblos, Lebanon
samara.chamoun@lau.edu

Chadi Nour*
Dept. of Computer Science & Mathematics, Lebanese American University, Byblos, Lebanon
cnour@lau.edu.lb

Received: November 4, 2019
Accepted: April 18, 2020

For a nonlinear control system, we derive a φ_0-convexity result for the epigraph of the bilateral minimal time function. This extends the main result of the second author in *Proximal subdifferential of the bilateral minimal time function and some regularity applications* [J. Convex Analysis 20(4) (2013) 1095–1112], where a similar result is obtained for the linear case.

Keywords: Bilateral minimal time function, nonlinear control system, φ_0-convexity, proximal analysis, nonsmooth analysis.

2010 Mathematics Subject Classification: 49N60, 49J52, 34A60.

1. Introduction

Let F be a multifunction mapping points $x \in \mathbb{R}^n$ to subsets of $\mathbb{R}^n$. Associated with F is the differential inclusion

$$\dot{x}(t) \in F(x(t)) \quad \text{a.e. } t \in [0,T], \quad x(0) = x_0 \in \mathbb{R}^n. \tag{1}$$

A *solution* to (1) is an absolutely continuous function $x(\cdot)$ defined on the interval $[0,T]$ with initial value $x(0) = x_0$, in which case we say that $x(\cdot)$ is a *trajectory* of F originating from x_0. The notation $\dot{x}(t)$ refers to the derivative of $x(\cdot)$ at t and is the right derivative if $t = 0$. The *bilateral minimal time function*, denoted by $T(\cdot,\cdot)$, is defined from $\mathbb{R}^n \times \mathbb{R}^n$ to $[0,+\infty]$ as follows: For $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$T(\alpha,\beta) := \begin{cases} \inf T \geq 0 \\ \dot{x}(t) \in F(x(t)) \quad \text{a.e. } t \in [0,T] \\ x(0) = \alpha \text{ and } x(T) = \beta. \end{cases}$$

If we cannot find a trajectory of F that steers α to β then $T(\alpha,\beta) := +\infty$. We set

$$\mathcal{R} := \{(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n : T(\alpha,\beta) < +\infty\},$$

* Corresponding author

ISSN 0944-6532 / $ 2.50

to be the *effective domain* of $T(\cdot,\cdot)$. The function $T(\cdot,\cdot)$ was defined, apparently for the first time, by Clarke and Nour in [14] where the Hamilton-Jacobi equation of the time-optimal control problem was studied in a domain containing the target set. The authors used $T(\cdot,\cdot)$ to construct proximal solutions to this equation, and to study the existence of time-semigeodesic trajectories. Note that for $S \subset \mathbb{R}^n$ a closed set, the function $T(\cdot, S)$ defined by

$$T(\alpha, S) := \inf_{\beta \in S} T(\alpha, \beta),$$

coincides with the well-known *unilateral minimal time function* associated with F and the target set S. This function has a large literature and its regularity is a widely studied topic, see [1, 2, 10, 17, 18, 21, 31, 32] and the references therein.

The regularity of $T(\cdot,\cdot)$ was studied by Nour in [24], where necessary and sufficient conditions were provided for $T(\cdot,\cdot)$ to be continuous and locally Lipschitz in $\mathcal{R}$. Nour also provided in [24] a *semiconvexity* result for the bilateral minimal time function in the linear case. More precisely, Nour proved that for a linear differential inclusion, if $T(\cdot,\cdot)$ is Lipschitz near a given point (α, β) then it is also semiconvex near that point. This generalized to the bilateral case the unilateral semiconvexity result proved by Cannarsa and Sinestrari in [9]. Since a function f is semiconvex if and only if it is locally Lipschitz and its *epigraph* satisfies an *external sphere condition* (this property, for general sets, is often referred to *positive reach* [19], *proximal smoothness* [15], φ_0-*convexity* [16] and *prox-regularity* [28, 29]), Nour provided in [25] a *natural* generalization of the semiconvexity result of [24]. Indeed, Nour proved that if we replace the Lipschitz continuity of $T(\cdot,\cdot)$ near (α, β) by its continuity near (α, β), then the epigraph of $T(\cdot,\cdot)$ becomes φ_0-convex near that point.

In [5], Cannarsa, Marino and Wolenski proved a *semiconcavity* result for the unilateral minimal time function under some regularity assumptions on the *upper Hamiltonian* H associated with F, rather than assuming that F admits a $C^{1,+}$ parameterization as it is done in Cannarsa and Sinestrari [9]. These assumptions given in terms of H, denoted here by hypotheses **(H)** and introduced apparently for the first time in [11], are intrinsic to the trajectories generated by F and are not influenced by any particular parameterization. The proof given in [5], relied heavily on the nonsmooth maximum principle instead of using a priori estimates on trajectories that ensue from a known smooth parameterization for F. Using this approach, several results for smooth parameterized control systems, including the *sensitivity relations* developed in [3], were extended to nonparameterized systems as we can find in [4, 6, 7, 8].

Inspired by the techniques used in [4, 5, 6, 7, 8], Nguyen proved in [22] and under the hypotheses **(H)**, a φ_0-convexity result for the epigraph of the unilateral minimal time function in the *nonlinear case*, a first result of its kind for a nonlinear differential inclusion. To prove this result, Nguyen first developed inclusions for the normal cones to the epigraph and to the level sets of the unilateral minimum time function. As a consequence and using a representation of the proximal horizontal subdifferential and the proximal subdifferential of $T(\cdot, S)$, Nguyen provided sensitivity relations for the unilateral minimal times function similar to whose in [8]. More precisely, Nguyen proved that both the proximal and horizontal subdifferentials of $T(\cdot, S)$ propagate *wholly* along optimal trajectories.

The goal of this paper is to generalize the techniques used in [22] to the bilateral case, that is, for the case in which both initial and end points are variables. This leads us to find a φ_0-convexity result for the epigraph of the bilateral minimal time function associated with a nonlinear differential inclusion which extends, to the nonlinear case, the main result of [25].

The layout of the paper is as follows. Notations, definitions and some preliminaries results from nonsmooth analysis and control theory will be given in the next section. In Section 3, we present our hypotheses and provide some consequences. Finally, Section 4 contains our main result and its proof.

2. Preliminaries

We denote by $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, B and $\bar{B}$, the Euclidean norm, the usual inner product, the open unit ball and the closed unit ball, respectively. For $\rho > 0$ and $x \in \mathbb{R}^n$, we set $B(x;\rho) := x + \rho B$ and $\bar{B}(x;\rho) := x + \rho\bar{B}$. For a set $S \subset \mathbb{R}^n$, S^c, $\operatorname{int} S$, $\operatorname{bdry} S$, $\operatorname{cl} S$ and $\operatorname{conv} S$ are the complement (with respect to $\mathbb{R}^n$), the interior, the boundary the closure and the convex hull of S, respectively. We denote by $\mathcal{D}$ the diagonal set of $\mathbb{R}^{2n}$, that is, $\mathcal{D} = \{(\alpha,\alpha) : \alpha \in \mathbb{R}^n\}$. The distance from a point x to a set S is denoted by $d_S(x)$. We also denote by $\operatorname{proj}_S(x)$ the set of closest points in S to x, that is, the set of points $s \in S$ which satisfy $d_S(x) = \|s - x\|$.

We proceed to provide some definitions from nonsmooth analysis. Our general reference for these constructs is Clarke, Ledyaev, Stern and Wolenski [13]; see also Rockafellar and Wets [30]. Let S be a nonempty closed subset of $\mathbb{R}^n$. For $x \in S$, a vector $\zeta \in \mathbb{R}^n$ is said to be *proximal normal to S at x* provided that there exists $\sigma = \sigma(x,\zeta) \geq 0$ such that

$$\langle \zeta, s - x\rangle \leq \sigma\|s - x\|^2 \quad \forall s \in S. \tag{2}$$

The relation (2) is commonly referred to as the *proximal normal inequality.* This proximal normal inequality can be taken to be a local one, that is, only for s near x in S. We note that no nonzero ζ satisfying (2) exists if $x \in \operatorname{int} S$, but this may also occur for $x \in \operatorname{bdry} S$. For such points, the only proximal normal is $\zeta = 0$. In view of (2), the set of all proximal normals to S at x is a convex cone, and we denote it by $N_S^P(x)$. Now let $x \in \operatorname{bdry} S$, and suppose that $0 \neq \zeta \in \mathbb{R}^n$ and $r > 0$ are such that

$$B\left(x + r\frac{\zeta}{\|\zeta\|}; r\right) \cap S = \emptyset. \tag{3}$$

Then ζ is a proximal normal to S at x and we say that ζ is *realized by an r-sphere.* One can easily prove that ζ being realized by an r-sphere is *equivalent* to the proximal normal inequality holding with $\sigma = \frac{1}{2r}$; that is,

$$\left\langle \frac{\zeta}{\|\zeta\|}, s - x\right\rangle \leq \frac{1}{2r}\|s - x\|^2 \quad \forall s \in S. \tag{4}$$

The *Clarke normal cone* to S at x, denoted by $N_S(x)$, is defined as

$$N_S(x) := \operatorname{conv}\left\{\lim_i \zeta_i : \zeta_i \in N_S^P(x_i) \text{ and } x_i \xrightarrow{S} x\right\}.$$

The notation $x_i \xrightarrow{S} x$ signifies that $x_i \longrightarrow x$ and that $x_i \in S$ for all i.

Now let $f : U \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, let $x \in \operatorname{dom} f$, the *effective domain* of f, and let $\zeta \in \mathbb{R}^n$ be a vector. We say that ζ is *proximal subgradient* of f at x, provided that

$$(\zeta, -1) \in N^P_{\operatorname{epi} f}(x, f(x)),$$

where $\operatorname{epi} f$ denotes the *epigraph* of f. This is equivalent to the existence of two constants $\sigma = \sigma(x, \zeta) \geq 0$ and $\delta > 0$ such that the following *proximal subgradient inequality* holds

$$f(y) - f(x) + \sigma\|y - x\|^2 \geq \langle \zeta, y - x\rangle \;\; \forall y \in B(x; \delta) \cap U.$$

The *proximal subdifferential* of f at x, denoted by $\partial_P f(x)$, is the set of all the proximal subgradients at x. It is an exercise using the proximal normal inequality to show that every proximal normal (ζ, t) to $\operatorname{epi} f$ at $(x, f(x))$ has $t \leq 0$. Similarly, the *Clarke subdifferential* of f at $x \in \operatorname{dom} f$, denoted by $\partial f(x)$, is defined by

$$\partial f(x) := \{\zeta \in \mathbb{R}^n : (\zeta, -1) \in N_{\operatorname{epi} f}(x, f(x))\}.$$

On the other hand, we say that ζ is a *proximal horizontal subgradient* of f at x, provided that

$$(\zeta, 0) \in N^P_{\operatorname{epi} f}(x, f(x)).$$

The set of all proximal horizontal subgradients of f at x, denoted by $\partial^\infty f(x)$, will be called the *horizontal proximal subdifferential* of f at x. Now we introduce a well-known geometric property, the φ_0-*convexity*, see [16]. This property is also known under several names including *positive reach* [19], *proximal smoothness* [15] and *prox-regularity* [28, 29].

Definition 2.1. Let $S \subset \mathbb{R}^n$ be a nonempty and closed set. We say that S is φ_0-*convex* if there exists a constant $\varphi_0 \geq 0$ such that for all $x \in \operatorname{bdry} S$ we have:

$$\left\langle \frac{\zeta}{\|\zeta\|}, s - x \right\rangle \leq \varphi_0 \, \|s - x\|^2 \quad \text{for all } \; s \in S \; \text{ and } \; 0 \neq \zeta \in N^P_S(x).$$

Using the equivalence between (3) and (4), we can easily see that S is φ_0-convex if and only if there exists a constant $\varphi_0 \geq 0$ such that for all $x \in \operatorname{bdry} S$ and nonzero vector $\zeta \in N^P_S(x)$, we have:

- ζ is realized by a $\frac{1}{2\varphi_0}$-sphere, if $\varphi_0 \neq 0$,
- ζ is realized by an r-sphere for all $r > 0$, if $\varphi_0 = 0$.

Remark 2.2. We can define the φ_0-convexity in a *pointwise* way as the following: For $S \subset \mathbb{R}^n$ a nonempty and closed set and $x \in \operatorname{bdry} S$, we say that S is φ_0-convex at x if and only if there exists $\varphi_0 \geq 0$ such that every nonzero proximal normal vector ζ to S at x is realized by a $\frac{1}{2\varphi_0}$-sphere[1]. Clearly S is φ_0-convex if and only if there a constant $\varphi_0 \geq 0$ such that S is φ_0-convex at x for all $x \in \operatorname{bdry} S$.

Example 2.3. The hypograph of the function f defined by, $f(x) = x$ if $x \leq 0$ and $f(x) = x^{3/2}$ if $x \geq 0$, is not φ_0-convex at $(0, 0)$ for any $\varphi_0 \geq 0$. Note that the proximal normal cone is not trivial at $(0, 0)$.

[1]If $\varphi_0 = 0$, then this means that ζ is realized by an r-sphere for all $r > 0$.

For $O \subset \mathbb{R}^n$ an open and convex set, a function $f : O \longrightarrow \mathbb{R}$ is said to be *semiconvex* on O with semiconvexity constant $c \geq 0$ if and only if it is continuous on O and satisfies

$$f(x+h) + f(x-h) + 2f(x) \geq -c\|h\|^2,$$

for all $x \in O$ and $h \in \mathbb{R}^n$ with $x \pm h \in O$. This is equivalent to the convexity of the function $x \mapsto f(x) + \frac{c}{2}\|x\|^2$ on O. Hence, semiconvex functions are essentially a quadratic perturbation of convex functions and therefore inherit several regularity properties from convexity such as local Lipschitzianity and a.e. twice differentiability. For further information about semiconvex functions, we refer the reader to [10]. On the other hand and as we mentioned in the introduction, a function with φ_0-convex epigraph is semiconvex if and only it is locally Lipschitz. This led Colombo and Marigonda to prove in [16] that merely lower semicontinuous functions whose epigraphs are φ_0-convex still enjoy some regularity properties of semiconvex functions, including twice a.e. differentiability, yet not being locally Lipschitz. In [22], Nguyen extended this result to continuous functions but having an epigraph satisfying a weak external sphere condition, called *exterior sphere condition.* For a deep comparison between φ_0-convexity and the exterior sphere condition, we invite the reader to see [26, 27]. Now for $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function and in view of Definition 2.1, the epigraph of f is φ_0-convex if and only if there exits a constant $\varphi_0 \geq 0$ such that for all $x \in \operatorname{dom} f$, we have

$$\left\langle \frac{(\zeta, s)}{\|(\zeta, r)\|}, (y, \rho) - (x, f(x)) \right\rangle \leq \varphi_0 \, \|(y, \rho) - (x, f(x))\|^2, \tag{5}$$

for all $y \in \operatorname{dom} f$, $\rho \geq f(y)$ and $0 \neq (\zeta, s) \in N^P_{\operatorname{epi} f}(x, f(x))$. On the other hand, the epigraph of f is said to be φ_0-convex near a given point $\alpha \in \operatorname{dom} f$, if there exists an open neighborhood O of α such that the epigraph of the restriction of f to O, denoted by $f_{|O}$, is φ_0-convex. In [25], Nour proved the following lemma, see [25, Lemma 2.2], in which he showed that to get the φ_0-convexity of the epigraph of a continuous function, only points in the graph of f, that is, of the form (y, ρ) where $\rho = f(y)$, need to satisfy the inequality (5).

Lemma 2.4. *Let $O \subset \mathbb{R}^n$ be an open and convex set and let $f : O \longrightarrow \mathbb{R}$ be a continuous function. The epigraph of f is φ_0-convex if and only if there exists a constant $\varphi_0 \geq 0$ such that for all $x \in O$, we have:*

$$\left\langle \frac{(\zeta, s)}{\|(\zeta, r)\|}, (y, f(y)) - (x, f(x)) \right\rangle \leq \varphi_0 \, \|(y, \rho) - (x, f(x))\|^2,$$

for all $y \in O$ and $0 \neq (\zeta, s) \in N^P_{\operatorname{epi} f}(x, f(x))$.

We recall that for F a multifunction mapping points $x \in \mathbb{R}^n$ to subsets of $\mathbb{R}^n$, the *bilateral minimal time function* $T(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0, +\infty]$ is defined as follows

$$T(\alpha, \beta) := \inf\{T \geq 0 : \text{some trajectory } x(\cdot) \text{ of } F \text{ has } x(0) = \alpha \text{ and } x(T) = \beta\}.$$

If no trajectory between α and β exists, then $T(\alpha, \beta) = +\infty$. Clearly we have $T(\alpha, \alpha) = 0$ for all $\alpha \in \mathbb{R}^n$. We denote by $\mathcal{R}$ the *effective domain* of $T(\cdot, \cdot)$, that is,

$$\mathcal{R} := \{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n : T(\alpha, \beta) < +\infty\}.$$

We also denote the *level set* of $T(\cdot,\cdot)$ corresponding to $t \geq 0$ by

$$\bar{\mathcal{R}}(t) := \{(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n : T(\alpha,\beta) \leq t\}.$$

We assume throughout this paper that the multifunction F satisfies the following *Standing Hypotheses*:

- For every $x \in \mathbb{R}^n$, $F(x)$ is a nonempty compact convex set.
- The linear growth condition: For some positive constants γ and c, and for all $x \in \mathbb{R}^n$: $\quad v \in F(x) \Longrightarrow \|v\| \leq \gamma\|x\| + c$.
- F is locally Lipschitz; that is, every $x \in \mathbb{R}^n$ admits a neighborhood $U = U(x)$ and a positive constant $K = K(x)$ such that

$$x_1, x_2 \in U \Longrightarrow F(x_2) \subseteq F(x_1) + K\|x_1 - x_2\|\bar{B}.$$

Under these hypotheses, one can easily verify that:

- $T(\cdot,\cdot)$ and $T(\cdot,\beta)$ are lower semicontinuous.
- $T(\alpha,\beta) = 0$ if and only if $(\alpha,\beta) \in \mathcal{D}$.
- If $T(\alpha,\beta) < +\infty$, then the infimum defining $T(\alpha,\beta)$ is attained.
- For all $\alpha,\beta,\gamma \in \mathbb{R}^n$, the following triangle inequality holds

$$T(\alpha,\beta) \leq T(\alpha,\gamma) + T(\gamma,\beta).$$

We associate with F the *lower Hamiltonian* function h and the *upper Hamiltonian* function H defined on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$h(x,p) := \min_{v \in F(x)} \langle p, v\rangle \quad \text{and} \quad H(x,p) := \max_{v \in F(x)} \langle p, v\rangle, \quad \forall (x,p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Finally and for $\beta \in \mathbb{R}^n$, we recall the following two *controllability* definitions:

- F is β-STLC (β-*small-time locally controllable*), if and only if $T(\cdot,\beta)$ is continuous at β, that is, $\forall t > 0\ \exists\, \delta > 0$ such that $T(\cdot,\beta) < t$ on $B(\beta;\delta)$.
- F satisfies the *positive basis condition* (or *Petrov condition*) at β, if and only if $h(\beta,\gamma) < 0$ for any unit vector γ (or equivalently, $0 \in \operatorname{int} F(\beta)$).

For more information about these two properties and their relations with the continuity and the Lipschitz continuity (local and global) of $T(\cdot,\cdot)$, we invite the reader to see [24, Section 4.1].

We terminate this section by the following direct consequence of Gronwall's lemma, for the proof see [7, Lemma 3.1].

Lemma 2.5. *Let G be an upper semicontinuous nonautonomous multifunction mapping points $(t,x) \in \mathbb{R}\times\mathbb{R}^n$ to subsets of $\mathbb{R}^n$. Assume that $G(t,\cdot)$ satisfies the Standing Hypotheses uniformly in $t \in [0,T]$, and is such that, for some $K_0 > 0$,*

$$\|v\| \leq K_0\|p\| \quad \forall v \in G(t,p), \quad \forall (t,p) \in [0,T] \times \mathbb{R}^n.$$

Let $p(\cdot)$ be a solution for the differential inclusion

$$\dot{p}(t) \in G(t,p(t)) \ \text{ a.e. } \ t \in [0,T], \quad p(0) = p_0.$$

Then
$$e^{-K_0 t}\|p(0)\| \leq \|p(t)\| \leq e^{K_0 t}\|p(0)\| \quad \forall t \in [0,T]. \tag{6}$$

Moreover, for all $0 \leq t_1 \leq t_2 \leq T$,

$$e^{-K_0(t_2-t_1)}\|p(t_2)\| \leq \|p(t_1)\| \leq e^{K_0(t_2-t_1)}\|p(t_2)\|$$

and

$$\|p(t_2) - p(t_1)\| \leq K_0 e^{K_0(t_2-t_1)}(t_2 - t_1)\|p(t_2)\|. \tag{7}$$

3. Hypotheses and some consequences

In this section, we present the hypotheses **(H)** needed, besides the Standing Hypotheses, to prove our main result. We also provide some direct consequences of **(H)**. Our general references for these hypotheses and consequences are [7, 11]. We say that the hypotheses **(H)** are satisfied if and only if for all $r > 0$:

(H1) There exists $c \geq 0$ such that for any unit vector $p \in \mathbb{R}^n$, the function $x \mapsto H(x,p)$ is semiconvex with semiconvexity c.

(H2) The function $\nabla_p H(x,p)$ exists and is Lipschitz in x on $B(0;r)$, uniformly for all $p \neq 0$.

In [11], Cannarsa and Wolenski presented several examples that generate multifunctions satisfying **(H)** but do not admit a C^1 parameterization. They also provided some consequences and equivalent statements of the hypotheses **(H)**. The following is an important consequence. For the proof, see [7, 11].

Proposition 3.1. *Assume* **(H)** *and let* $0 \neq p$ *and* x *in* $\mathbb{R}^n$. *Then we have*

$$\partial H(x,p) \subset \partial_x H(x,p) \times \partial_p H(x,p) \tag{8}$$

and

$$\nabla_p H(x,p) \in F(x), \quad \langle \nabla_p H(x,p), p\rangle = H(x,p). \tag{9}$$

The following is a version of the maximum principle adapted for the bilateral minimal time function. Its proof follows directly from [12, Theorem 3.5.4] and inclusion (8).

Theorem 3.2. (Maximum principle) *Assume* **(H)** *and let* $(\alpha,\beta) \in \mathcal{R}\setminus\mathcal{D}$. *Then for* $r := T(\alpha,\beta)$ *and* $x(\cdot)$ *an optimal trajectory from* α *to* β, *there exists an absolutely continuous arc* $p : [0,r] \to \mathbb{R}^n$, *never vanishing, such that*

$$\dot{x}(s) = \nabla_p H(x(s),p(s)), \quad -\dot{p}(s) \in \partial_x H(x(s),p(s)) \ \ \textit{for a.e.} \ \ s \in [0,r]. \tag{10}$$

An absolutely continuous function $p(\cdot)$ satisfying the system (10) is called a *dual arc* associated to the trajectory $x(\cdot)$. From (9), we have

$$H(x(t),p(t)) = \langle \dot{x}(t), p(t)\rangle \quad \text{a.e. } t \in [0,r]. \tag{11}$$

Remark 3.3. We remark that, under our assumptions, if (x,p) solves the Hamiltonian inclusion

$$\dot{x}(s) = \nabla_p H(x(s),p(s)), \quad -\dot{p}(s) \in \partial_x H(x(s),p(s)) \ \text{ for a.e. } \ s \in [0,r]$$

then there are two possible cases: either $p(s) \neq 0$ for all $s \in [0,T]$, or $p(s) = 0$ for all $s \in [0,T]$. Moreover, for $r_0 > 0$ such that $x([0,T]) \subset B(0,r_0)$ and for $K = K(r_0)$ a Lipschitz constant of F on $B(0,r_0)$, we have

$$\|\dot{p}(s)\| \leq K\|p(s)\| \quad \text{a.e.} \ \ s \in [0,T]. \tag{12}$$

See [11] for more details and discussions.

We terminate this section by the following lemma which will be useful in the sequel. Its proof is straightforward due to **(H2)**.

Lemma 3.4. *Assume* **(H)** *and let* $p(\cdot)$ *be an absolutely continuous arc on* $[0, T]$ *with* $p(t) \neq 0$ *for all* $t \in [0, T]$. *Then for each* $x \in \mathbb{R}^n$, *the problem*

$$\dot{x}(t) = \nabla_p H(x(t), p(t)) \ \text{ a.e. } \ t \in [0, T], \quad x(0) = x. \tag{13}$$

has a unique solution.

4. Main result

The goal of this section is to present, and to prove, the main result of this paper, that is, a local φ_0-convexity result for the epigraph of the bilateral minimal time function $T(\cdot, \cdot)$ associated with a nonlinear differential inclusion. We begin by introducing the following new hypothesis. For $(\alpha, \beta) \in \mathcal{R} \setminus \mathcal{D}$, we say that the hypothesis **(R)** is satisfied at (α, β) if and only if

(R) There exists $\varphi_0 \geq 0$ such that $\bar{\mathcal{R}}(\alpha', \beta')$ is φ_0-convex at (α', β') for all (α', β') near (α, β).

Remark 4.1. One can verify, using [25, Lemma 3.2], that for a linear multifunction F, hypothesis **(R)** is satisfied at $(\alpha, \beta) \in \mathcal{R} \setminus \mathcal{D}$ provided that F is small-time locally controllable near α or β. Indeed, in [25, Lemma 3.2] Nour proved the existence of $\delta > 0$ such that $\bar{\mathcal{R}}(\alpha', \beta') \cap B((\alpha', \beta'); \delta)$ is convex for all $(\alpha', \beta') \in B((\alpha, \beta); \delta)$. This gives that $\bar{\mathcal{R}}(\alpha', \beta')$ is $\frac{1}{\delta}$-convex at (α', β') for all $(\alpha', \beta') \in B((\alpha, \beta); \delta)$ and then hypothesis **(R)** is satisfied at (α, β). An example of a nonlinear multifunction F satisfying **(H)** and **(R)** at a point $(\alpha, \beta) \in \mathcal{R} \setminus \mathcal{D}$, will be given at the end of this section.

The following is the main result of this paper.

Theorem 4.2. *Let* $(\alpha, \beta) \in \mathcal{R} \setminus \mathcal{D}$ *and assume that* **(H)** *are satisfied,* **(R)** *is satisfied at* (α, β) *and that* $T(\cdot, \cdot)$ *is continuous near* (α, β). *Then the epigraph of* $T(\cdot, \cdot)$ *is* φ_0*-convex near* (α, β).

Before giving the proof of our main result, we present some direct consequences. We begin by the following corollary that extends to the nonlinear case the main result of [25], Theorem 1.2.

Corollary 4.3. *Let* $(\alpha, \beta) \in \mathcal{R} \setminus \mathcal{D}$ *and assume that* **(H)** *are satisfied,* **(R)** *is satisfied at* (α, β) *and that* F *and* $-F$ *are small-time locally controllable near* α *or* β. *Then the epigraph of* $T(\cdot, \cdot)$ *is* φ_0*-convex near* (α, β).

Proof. Follows from Theorem 4.2 and [24, Proposition 4.2]. □

The following corollary also extends to the nonlinear case the semiconvexity result [25, Corollary 4.8] obtained for a linear multifunction.

Corollary 4.4. *Let* $(\alpha, \beta) \in \mathcal{R} \setminus \mathcal{D}$ *and assume that* **(H)** *are satisfied,* **(R)** *is satisfied at* (α, β) *and that* $0 \in \operatorname{int} F(\alpha)$ *or* $0 \in \operatorname{int} F(\beta)$. *Then* $T(\cdot, \cdot)$ *is semiconvex near* (α, β).

Proof. Follows from Theorem 4.2, [24, Proposition 4.5] and the fact that the semiconvexity of a function f near a point, is equivalent to its Lipschitz continuity and the φ_0-convexity of its epigraph near that point. □

The proof of Theorem 4.2, given at the end of this section, follows using several lemmas. We begin by the following lemma that can be found in [24], and which gives a characterization of the proximal subdifferential of $T(\cdot,\cdot)$ at points outside the diagonal set in terms of the proximal normal cone to the level set and the lower hamiltonian.

Lemma 4.5. *Let $(\alpha,\beta)\in\mathcal{R}\setminus\mathcal{D}$ and let $r:=T(\alpha,\beta)>0$. We have*

$$\partial_P T(\alpha,\beta)=N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)\cap\{(\xi,\theta)\in\mathbb{R}^n\times\mathbb{R}^n : h(\alpha,\xi)=-1\}.$$

The following lemma characterizes the elements of $N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)$ for $(\alpha,\beta)\in\mathcal{R}\setminus\mathcal{D}$ and $r:=T(\alpha,\beta)>0$.

Lemma 4.6. *Let $(\alpha,\beta)\in\mathcal{R}\setminus\mathcal{D}$ and let $r:=T(\alpha,\beta)>0$. Then*

$$(\xi,\theta)\in N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)\Longrightarrow h(\alpha,\xi)\le 0.$$

Proof. Since $(\xi,\theta)\in N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)$, there exists $\sigma>0$ such that

$$\langle(\xi,\theta),(\alpha',\beta')-(\alpha,\beta)\rangle\le\sigma\|(\alpha',\beta')-(\alpha,\beta)\|^2\quad\forall(\alpha',\beta')\in\bar{\mathcal{R}}(r). \tag{14}$$

Let $x(\cdot)$ be an optimal trajectory of F from α to β. By the principle of optimality we have that for all $t\in[0,r]$

$$\begin{aligned} T(x(t),\beta)+t=T(\alpha,\beta) &\;\Rightarrow\; T(x(t),\beta)\le T(x(t),\beta)+t=T(\alpha,\beta)\\ &\;\Rightarrow\; T(x(t),\beta)\le r\;\Rightarrow\;(x(t),\beta)\in\bar{\mathcal{R}}(r).\end{aligned}$$

We define $y(\cdot)$ to be the following measurable function

$$y(t)=\mathrm{proj}_{F(\alpha)}(\dot{x}(t))\in F(\alpha),\qquad\text{a.e. } t\in[0,r].$$

It follows from Gronwall's lemma, see [13, Proposition 4.1.4] and the Lipschitz continuity of F the existence of $M>0$ such that

$$\|\dot{x}(t)-y(t)\|\le K\|x(t)-\alpha\|\le KMt\qquad\text{a.e. } t\in[0,r].$$

Let $t\in(0,r)$, taking in (14) $\alpha':=x(t)$ and $\beta':=\beta$, we obtain

$$\begin{gathered}\langle(\xi,\theta),(x(t),\beta)-(\alpha,\beta)\rangle\le\sigma\|(x(t),\beta)-(\alpha,\beta)\|^2\\ \Rightarrow\;\langle\xi,x(t)-\alpha\rangle\le\sigma\|x(t)-\alpha\|^2\;\Rightarrow\;\langle\xi,x(t)-\alpha\rangle\le\sigma M^2t^2.\end{gathered}$$

Hence,
$$\left\langle\xi,\int_0^t\dot{x}(s)ds\right\rangle\le\sigma M^2t^2.$$

Therefore, for all $t\in(0,r)$, we get

$$\begin{aligned} h(\alpha,\xi)t &\le\int_0^t\langle\xi,y(s)\rangle\,ds=\int_0^t\langle\xi,\dot{x}(s)\rangle\,ds+\int_0^t\langle\xi,y(s)-\dot{x}(s)\rangle\,ds\\ &\le\langle\xi,\int_0^t\dot{x}(s)\rangle\,ds+\int_0^t\|\xi\|\|y(s)-\dot{x}(s)\|\,ds\\ &\le\sigma M^2t^2+\frac{M}{2}K\|\xi\|t^2\le\left(\sigma M^2+\frac{M}{2}K\|\xi\|\right)t^2.\end{aligned}$$

Now if we divide the previous inequality by t and then we take $t \longrightarrow 0$, we obtain that $h(\alpha,\xi) \leq 0$. This terminates the proof of the lemma. □

Similar to Lemma 4.5, we have the following for the proximal horizontal subdifferential of $T(\cdot,\cdot)$

Lemma 4.7. *Let $(\alpha,\beta) \in \mathcal{R} \setminus \mathcal{D}$ and let $r := T(\alpha,\beta) > 0$. We have*

$$\partial^\infty T(\alpha,\beta) = N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta) \cap \{(\xi,\theta) \in \mathbb{R}^n \times \mathbb{R}^n : h(\alpha,\xi) = 0\}.$$

Proof. We begin by the first inclusion and we conisder $(\xi,\theta) \in \partial^\infty T(\alpha,\beta)$. Then there exists $\sigma > 0$ such that for all $(\alpha',\beta') \in \mathcal{R}$ and $\bar{r}' \geq r' := T(\alpha',\beta')$, we have

$$\langle (\xi,\theta), (\alpha',\beta') - (\alpha,\beta) \rangle \leq \sigma\big(\|(\alpha',\beta') - (\alpha,\beta)\|^2 + (\bar{r}' - r)^2\big). \tag{15}$$

Then for $T(\alpha',\beta') \leq r$ and taking $\bar{r}' = r$ in (15) we get

$$\langle (\xi,\theta), (\alpha',\beta') - (\alpha,\beta) \rangle \leq \sigma\|(\alpha',\beta') - (\alpha,\beta)\|^2 \quad \forall (\alpha',\beta') \in \bar{\mathcal{R}}(r).$$

Then $(\xi,\theta) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)$ and, due to Lemma 4.6, we conclude that $h(\alpha,\xi) \leq 0$.

Now, we need to prove that $h(\alpha,\xi) \geq 0$. Let $w \in F(\alpha)$ such that

$$\langle \xi, w \rangle = h(\alpha,\xi) = \min_{v \in F(\alpha)} \langle v, \xi \rangle.$$

By [32, Proposition 2.3], we can find $T > 0$ and a C^1 trajectory $x(\cdot)$ of $-F$ on $[0,T]$ with $\dot{x}(0) = -w$ and $x(0) = \alpha$. By Gronwall's lemma there exists $M > 0$ such that

$$\|x(t) - \alpha\| \leq Mt, \quad \forall t \in [0,T]. \tag{16}$$

Since $\alpha \neq \beta$, there exists $\varepsilon > 0$ such that $x(t) \neq \beta$ for all $t \in [0,\varepsilon]$. We fix $t \in (0,\varepsilon)$ and we consider $y(s) = x(t-s)$ for $s \in [0,t]$. Then $y(\cdot)$ is a trajectory of F such that $y(t) = \alpha$. By the principle of optimality

$$T(x(t),\beta) = T(y(0),\beta) + 0 \leq T(y(t),\beta) + t = r + t.$$

In (15), taking $\alpha' := x(t)$, $\beta' := \beta$ and $\bar{r}' = r + t \geq T(x(t),\beta)$, we get that

$$\langle (\xi,\theta), (x(t),\beta) - (\alpha,\beta) \rangle \leq \sigma\big(\|(x(t),\beta) - (\alpha,\beta)\|^2 + t^2\big). \tag{17}$$

Thus
$$\langle \xi, x(t) - \alpha \rangle \leq \sigma\big(\|x(t) - \alpha\|^2 + t^2\big) \leq \sigma\big(M^2t^2 + t^2\big).$$

This leads to
$$\left\langle \xi, \frac{x(t) - \alpha}{t - 0} \right\rangle \leq \sigma(M^2 + 1)t.$$

Now, taking $t \to 0$ in the previous inequality, we obtain that $\langle \xi, -w \rangle \leq 0$ and then

$$h(\alpha,\xi) \geq 0.$$

We proceed to prove the other inclusion. Let $(\xi,\theta) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)$ such that $h(\alpha,\xi) = 0$. We need to prove that $(\xi,\theta) \in \partial^\infty T(\alpha,\beta)$, that is, the existence of $\sigma > 0$ such that

$$\langle (\xi,\theta), (\alpha',\beta') - (\alpha,\beta) \rangle \leq \sigma\big(\|(\alpha',\beta') - (\alpha,\beta)\|^2 + (\bar{r}' - r)^2\big), \tag{18}$$

for all $(\alpha',\beta')\in\mathcal{R}$ and for all $\bar r'\geq r' := T(\alpha',\beta')$. Let $(\alpha',\beta')\in\mathcal{R}$ and consider the following two cases:

Case 1. $r'\leq r$. Then $(\alpha',\beta')\in\bar{\mathcal{R}}(r)$. Since $(\xi,\theta)\in N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)$, we get the existence of $\sigma_1>0$ such that, for all $\bar r'\geq r'$,

$$\begin{aligned}\langle(\xi,\theta),(\alpha',\beta')-(\alpha,\beta)\rangle &\leq \sigma_1\|(\alpha',\beta')-(\alpha,\beta)\|^2\\ &\leq \sigma_1\big(\|(\alpha',\beta')-(\alpha,\beta)\|^2+(\bar r'-r)^2\big).\end{aligned}$$

Case 2. $r'>r$. Let $y(\cdot)$ be an optimal trajectory from α' to β'. By the principle of optimality we have $T(y(0),\beta')+0=T(y(t),\beta')+t$ for all $t\in[0,r']$.

Set $r_1=r'-r>0$. Then $T(y(r_1),\beta')+r_1=T(\alpha',\beta')+0=r'=r_1+r$.

Hence $T(y(r_1),\beta')=r$ and $(y(r_1),\beta')\in\bar{\mathcal{R}}(r)$. By Case 1 above, we have

$$\langle(\xi,\theta),(y(r_1),\beta')-(\alpha,\beta)\rangle\leq\sigma_1\|(y(r_1),\beta')-(\alpha,\beta)\|^2.$$

Using Gronwall's lemma, we get the existence of $M>0$ such that for all $t\in[0,r']$

$$\|y(t)-\alpha\|\leq\|y(t)-\alpha'\|+\|\alpha'-\alpha\|\leq Mt+\|\alpha'-\alpha\|.$$

In particular for $t=r_1$ we obtain $\|y(r_1)-\alpha\|\leq Mr_1+\|\alpha'-\alpha\|$.

Let $z(\cdot)$ be the measurable function defined on $[0,r']$ as the projection of $\dot y(\cdot)$ on the set $F(\alpha)$, that is, for a.e. $t\in[0,r']$,

$$z(t)=\operatorname{proj}_{F(\alpha)}\dot y(t)\in F(\alpha).$$

By the Lipschitz continuity of F we get

$$\|\dot y(t)-z(t)\|\leq K\|y(t)-\alpha\|\leq K(Mt+\|\alpha'-\alpha\|)\quad\text{for a.e. } t\in[0,r']$$

Therefore we have

$$\begin{aligned}&\langle(\xi,\theta),(\alpha',\beta')-(\alpha,\beta)\rangle=\langle\xi,\alpha'-\alpha\rangle+\langle\theta,\beta'-\beta\rangle\\
&\quad=\langle\xi,\alpha'-y(r_1)\rangle+\langle\xi,y(r_1)-\alpha\rangle+\langle\theta,\beta'-\beta\rangle\\
&\quad=-\left\langle\xi,\int_0^{r_1}\dot y(t)\,dt\right\rangle+\langle\xi,y(r_1)-\alpha\rangle+\langle\theta,\beta'-\beta\rangle\\
&\quad=-\left\langle\xi,\int_0^{r_1}z(t)\,dt\right\rangle+\left\langle\xi,\int_0^{r_1}(z(t)-\dot y(t))\,dt\right\rangle+\langle\xi,y(r_1)-\alpha\rangle+\langle\theta,\beta'-\beta\rangle\\
&\quad=-\int_0^{r_1}\langle\xi,z(t)\rangle\,dt+\left\langle\xi,\int_0^{r_1}(z(t)-\dot y(t))\,dt\right\rangle+\langle\xi,y(r_1)-\alpha\rangle+\langle\theta,\beta'-\beta\rangle\\
&\quad\leq-h(\alpha,\xi)r_1+\|\xi\|K(Mr_1^2+\|\alpha'-\alpha\|r_1)+\sigma_1\|(y(r_1),\beta')-(\alpha,\beta)\|^2\\
&\quad\leq\|\xi\|K(Mr_1^2+\|\alpha'-\alpha\|r_1)+\sigma_1((Mr_1+\|\alpha'-\alpha\|)^2+\|\beta'-\beta\|^2)\\
&\quad\leq\sigma_2\big(r_1^2+\|(\alpha',\beta')-(\alpha,\beta)\|^2\big),\end{aligned}$$

where $\sigma_2>0$ is a suitable constant. Therefore

$$\begin{aligned}\langle(\xi,\theta),(\alpha',\beta')-(\alpha,\beta)\rangle&\leq\sigma_2\big((r'-r)^2+\|(\alpha',\beta')-(\alpha,\beta)\|^2\big)\\&\leq\sigma_2\big((\bar r'-r)^2+\|(\alpha',\beta')-(\alpha,\beta)\|^2\big),\end{aligned}$$

for all $\bar{r}' \geq r'$. Now taking $\sigma := \max\{\sigma_1, \sigma_2\}$ and combining Case 1 and Case 2, we get that (18) holds. □

The following two lemmas give a relation between the proximal normal cone to level set of the bilateral minimal time function and the proximal normal cone to the epigraph of this function.

Lemma 4.8. *Let $(\alpha, \beta) \in \mathcal{R} \setminus \mathcal{D}$ and let $r := T(\alpha, \beta) > 0$. If $(\xi, \theta) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\eta \in \mathbb{R}$ with $((\xi, \theta), \eta) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha, \beta), r)$ then $\eta \leq 0$, $(\xi, \theta) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha, \beta)$ and $h(\alpha, \xi) = \eta$.*

Proof. Let $((\xi, \theta), \eta) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha, \beta), r)$. By the definition of the epigraph we have $\eta \leq 0$. We consider the following two cases:

Case 1. $\eta = 0$. Then $((\xi, \theta), 0) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha, \beta), r)$ which gives that (ξ, θ) belongs to $\partial^\infty T(\alpha, \beta)$. By Lemma 4.7, we get $(\xi, \theta) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha, \beta)$ and $h(\alpha, \xi) = 0 = \eta$.

Case 2. $\eta < 0$. We set $(\xi_1, \theta_1) := \frac{-1}{\eta}(\xi, \theta)$. Therefore, by the definition of a cone, we have

$$((\xi_1, \theta_1), -1) = \frac{-1}{\eta}((\xi, \theta), \eta) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha, \beta), r).$$

Equivalently $(\xi_1, \theta_1) \in \partial_P T(\alpha, \beta)$. By Lemma 4.5

$$(\xi_1, \theta_1) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha, \beta) \text{ and } h(\alpha, \xi_1) = -1.$$

It follows that $(\xi, \theta) = -\eta(\xi_1, \theta_1) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha, \beta)$ and that

$$h(\beta, -\theta) = h(\alpha, \xi) = \min_{v \in F(\alpha)} \langle v, \xi \rangle = \min_{v \in F(\alpha)} \langle v, -\eta\, \xi_1 \rangle = -\eta\, h(\alpha, \xi_1) = \eta.$$

This terminates the proof. □

Lemma 4.9. *Let $(\alpha, \beta) \in \mathcal{R} \setminus \mathcal{D}$ and let $r := T(\alpha, \beta) > 0$. If $(\xi, \theta) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha, \beta)$ then*

$$((\xi, \theta), h(\alpha, \xi)) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha, \beta), r).$$

Proof. Let $(\xi, \theta) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha, \beta)$. By Lemma 4.6 we have that $h(\alpha, \xi) \leq 0$. Two cases follow:

Case 1. $h(\alpha, \xi) = 0$. Then by Lemma 4.7 we get that $(\xi, \theta) \in \partial^\infty T(\alpha, \beta)$. It follows that $((\xi, \theta), 0) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha, \beta), r)$ or equivalently

$$((\xi, \theta), h(\alpha, \xi)) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha, \beta), r).$$

Case 2. $h(\alpha, \xi) < 0$. We set $(\xi_1, \theta_1) = \frac{-1}{h(\alpha, \xi)}(\xi, \theta)$. By the definition of a cone, we have $(\xi_1, \theta_1) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha, \beta)$. But $h(\alpha, \xi_1) = -1$, then by Lemma 4.5 we get that $(\xi_1, \theta_1) \in \partial_P T(\alpha, \beta)$, that is,

$$((\xi_1, \theta_1), -1) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha, \beta), r).$$

Using the definition of a cone we conclude that

$$((\xi, \theta), h(\alpha, \xi)) = -h(\alpha, \xi)((\xi_1, \theta_1), -1) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha, \beta), r).$$

This terminates the proof. □

Before proceeding with the last two lemmas, we note that we can find in [23] similar results to whose of the previous lemmas but for the Fréchet subdifferentials and the Fréchet normal cones. Moreover, a unilateral version of these lemmas can be found in [22]. In the following two lemmas, we prove, under hypotheses **(H)**, important results concerning the propagation of proximal normals to the epigraph and to the level set of $T(\cdot,\cdot)$ wholly along optimal trajectories.

Lemma 4.10. *Assume that hypotheses* **(H)** *hold. Let* $(\alpha,\beta) \in \mathcal{R} \setminus \mathcal{D}$ *and let* $r := T(\alpha,\beta) > 0$. *For* $\bar{x} : [0,r] \longrightarrow \mathbb{R}^n$ *an optimal trajectory from* α *to* β, *suppose that* $\bar{p} : [0,r] \longrightarrow \mathbb{R}^n$ *is an arc such that* $(\bar{x},\bar{p})$ *is a solution of the system*

$$\dot{x}(s) = \nabla_p H(x(s),p(s)), \quad -\dot{p}(s) \in \partial_x H(x(s),p(s)) \ \text{ for a.e. } \ s \in [0,r] \qquad (19)$$

satisfying $\bar{x}(0) = \alpha$ *and* $((-\bar{p}(0),\bar{p}(r)),h(\alpha,-\bar{p}(0))) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha,\beta),r)$. *Then for all* $t \in [0,r)$, *we have*

$$((-\bar{p}(t),\bar{p}(r)),h(\bar{x}(t),-\bar{p}(t))) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\bar{x}(t),\beta),T(\bar{x}(t),\beta)).$$

Moreover, $h(\bar{x}(t),-\bar{p}(t)) = h(\alpha,-\bar{p}(0))$ *for all* $t \in [0,r]$.

Proof. By Remark 3.3, we have the following two cases:

Case 1. $\bar{p}(t) = 0$ for all $t \in [0,r]$, and then nothing to prove.

Case 2. $\bar{p}(t) \neq 0$ for all $t \in [0,r]$. Set $m := h(\alpha,-\bar{p}(0))$. Since

$$((-\bar{p}(0),\bar{p}(r)),m) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha,\beta),r),$$

there exist $\sigma > 0$ and $\eta > 0$ such that

$$\begin{aligned}\langle -\bar{p}(0),\alpha'-\alpha\rangle + \langle \bar{p}(r),\beta'-\beta\rangle + m(\bar{r}'-r)&\\ \le \sigma(\|\alpha'-\alpha\|^2 + \|\beta'-\beta\|^2 + (\bar{r}'-r)^2),&\qquad (20)\end{aligned}$$

for all $(\alpha',\beta') \in B((\alpha,\beta);\eta)$ and $\bar{r}' \ge r' := T(\alpha',\beta')$. Now, let $t \in (0,r)$. Since $\bar{p}(\cdot)$ is an absolutely continuous arc with $\bar{p}(s) \neq 0$ for all $s \in [0,t]$, by Lemma 3.4, we have that $\bar{x}(\cdot)$ is the unique solution of

$$\begin{cases}\dot{x}(s) = \nabla_p H(x(s),\bar{p}(s)) \ \text{ a.e. } \ s \in [0,t],\\ x(t) = \bar{x}(t).\end{cases}$$

Let $v := (v_1,v_2) \in B((0,0);\eta)$ and let $x_{v_1} : [0,t] \to \mathbb{R}^n$ be the solution of

$$\begin{cases}\dot{x}(s) = \nabla_p H(x(s),\bar{p}(s)) \ \text{ a.e. } \ s \in [0,t],\\ x(t) = \bar{x}(t) + v_1.\end{cases}$$

Using the continuous dependence on the initial condition (see [13, Theorem 4.3.11]), there exists a constant $k > 0$, independent of t, such that

$$\|x_{v_1}(s) - \bar{x}(s)\| \le k\|v_1\| \qquad \text{for all } s \in [0,t].$$

We can choose $\eta > 0$ sufficiently small such that $x_{v_1}([0,t]) \neq \beta + v_2$ for all v in $B((0,0);\eta)$.

By the principle of optimality, we have

$$T(\bar{x}(0),\beta) = T(\bar{x}(t),\beta) + t$$

and
$$T(x_{v_1}(0),\beta+v_2) \leq T(x_{v_1}(t),\beta+v_2) + t.$$

For $r_0 \geq T(x_{v_1}(t),\beta+v_2)$, we have

$$T(x_{v_1}(0),\beta+v_2) \leq r_0 + T(\bar{x}(0),\beta) - T(\bar{x}(t),\beta).$$

In (20), we take $\alpha' := x_{v_1}(0)$, $\beta' := \beta + v_2$ and $\bar{r}' := r_0 + r - T(\bar{x}(t),\beta)$. In fact, $\bar{r}' \geq r'$ and $(x_{v_1}(0),\beta+v_2) \in B((\alpha,\beta);\eta)$. This is due to the fact that we can take v close to $(0,0)$ so that

$$\|(x_{v_1}(0),\beta+v_2) - (\alpha,\beta)\| \leq k\|v_1\| + \|v_2\| < \eta.$$

Thus, we obtain

$$\begin{aligned}&\langle -\bar{p}(0), x_{v_1}(0)-\alpha\rangle + \langle \bar{p}(r), v_2\rangle \\ &\qquad \leq -m(r_0 - T(\bar{x}(t),\beta)) + \sigma(\|x_{v_1}(0)-\alpha\|^2 + \|v_2\|^2 + (r_0 - T(\bar{x}(t),\beta))^2).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\langle -\bar{p}(t), x_{v_1}(t)-\bar{x}(t)\rangle + \langle \bar{p}(r), v_2\rangle \\ &= \langle -\bar{p}(0), x_{v_1}(0)-\alpha\rangle + \langle \bar{p}(r), v_2\rangle + \langle -\bar{p}(t), x_{v_1}(t)-\bar{x}(t)\rangle + \langle \bar{p}(0), x_{v_1}(0)-\alpha\rangle.\end{aligned}$$

Then, it follows that

$$\begin{aligned}&\langle -\bar{p}(t), x_{v_1}(t)-\bar{x}(t)\rangle + \langle \bar{p}(r), v_2\rangle \\ &\leq -m(r_0 - T(\bar{x}(t),\beta)) + \sigma(\|x_{v_1}(0)-\alpha\|^2 + \|v_2\|^2 + (r_0 - T(\bar{x}(t),\beta))^2) \\ &\quad + \langle -\bar{p}(t), x_{v_1}(t)-\bar{x}(t)\rangle + \langle \bar{p}(0), x_{v_1}(0)-\alpha\rangle. \qquad (21)\end{aligned}$$

But we have

$$\begin{aligned}\langle -\bar{p}(t), x_{v_1}(t)-\bar{x}(t)\rangle + \langle \bar{p}(0), x_{v_1}(0)-\alpha\rangle &= \int_0^t \frac{d}{ds}\langle -\bar{p}(s), x_{v_1}(s)-\bar{x}(s)\rangle\, ds \\ = \int_0^t \langle -\dot{\bar{p}}(s), x_{v_1}(s)-\bar{x}(s)\rangle + \langle -\bar{p}(s), \dot{x}_{v_1}(s)-\dot{\bar{x}}(s)\rangle\, ds & \\ = \int_0^t \left[\langle -\dot{\bar{p}}(s), x_{v_1}(s)-\bar{x}(s)\rangle - \langle \bar{p}(s), \dot{x}_{v_1}(s)\rangle + \langle \bar{p}(s), \dot{\bar{x}}(s)\rangle\right] ds. &\end{aligned}$$

Going back to (9), we notice that

$$\langle \bar{p}(s), \dot{x}_{v_1}(s)\rangle = H(x_{v_1}(s),\bar{p}(s)) \quad\text{and}\quad \langle \bar{p}(s), \dot{\bar{x}}(s)\rangle = H(\bar{x}(s),\bar{p}(s)).$$

Then
$$\begin{aligned}&\langle -\bar{p}(t), x_{v_1}(t)-\bar{x}(t)\rangle + \langle \bar{p}(0), x_{v_1}(0)-\alpha\rangle \\ &= \int_0^t \left[\langle -\dot{\bar{p}}(s), x_{v_1}(s)-\bar{x}(s)\rangle - H(x_{v_1}(s),\bar{p}(s)) + H(\bar{x}(s),\bar{p}(s))\right] ds.\end{aligned}$$

Since $-\dot{\bar{p}}(s) \in \partial_x H(\bar{x}(s), \bar{p}(s))$ a.e. $s \in [0, r]$, it follows from **(H1)** and [10, Proposition 3.3.1] that

$$H(x_{v_1}(s), \bar{p}(s)) - H(\bar{x}(s), \bar{p}(s)) - \langle -\dot{\bar{p}}(s), x_{v_1}(s) - \bar{x}(s)\rangle \geq -c_1\|x_{v_1}(s) - \bar{x}(s)\|^2\|\bar{p}(s)\|,$$

with c_1 being a suitable constant. Thus, for $s \in [0, r]$, we have

$$\langle -\dot{\bar{p}}(s), x_{v_1}(s) - \bar{x}(s)\rangle - H(x_{v_1}(s), \bar{p}(s)) + H(\bar{x}(s), \bar{p}(s)) \leq c_1\|x_{v_1}(s) - \bar{x}(s)\|^2\|\bar{p}(s)\|.$$

Therefore

$$\begin{aligned}\langle -\bar{p}(t), x_{v_1}(t) - \bar{x}(t)\rangle + \langle \bar{p}(0), x_{v_1}(0) - \alpha\rangle &\leq \int_0^t c_1\|x_{v_1}(s) - \bar{x}(s)\|^2\|\bar{p}(s)\|\, ds \\ &\leq \int_0^t c_1 k^2\|v_1\|^2\|\bar{p}(s)\|\, ds \leq c_2\|v_1\|^2 = c_2\|x_{v_1}(t) - \bar{x}(t)\|^2,\end{aligned}$$

where c_2 is a suitable constant independent of t. Thus,

$$\langle -\bar{p}(t), x_{v_1}(t) - \bar{x}(t)\rangle + \langle \bar{p}(0), x_{v_1}(0) - \alpha\rangle \leq c_2\|x_{v_1}(t) - \bar{x}(t)\|^2. \tag{22}$$

For all $v = (v_1, v_2) \in B((0,0);\eta)$ and $r_0 \geq T(x_{v_1}(t), \beta + v_2)$, we have

$$\begin{aligned}&\langle -\bar{p}(t), x_{v_1}(t) - \bar{x}(t)\rangle + \langle \bar{p}(r), v_2\rangle + m(r_0 - T(\bar{x}(t), \beta)) \\ &\quad\leq \sigma(\|x_{v_1}(0) - \alpha\|^2 + \|v_2\|^2 + (r_0 - T(\bar{x}(t), \beta))^2) + c_1\|x_{v_1}(t) - \bar{x}(t)\|^2 \\ &\quad\leq L(\|x_{v_1}(t) - \bar{x}(t)\|^2 + \|v_2\|^2 + (r_0 - T(\bar{x}(t), \beta))^2),\end{aligned}$$

where $L > 0$ is a suitable constant. Then, for all $v = (v_1, v_2) \in B((0,0);\eta)$ and $r_0 \geq T(x_{v_1}(t), \beta + v_2)$, we have

$$\begin{aligned}&\langle (-\bar{p}(t), \bar{p}(r)), (x_{v_1}(t), \beta + v_2) - (\bar{x}(t), \beta)\rangle + m(r_0 - T(\bar{x}(t), \beta)) \\ &\quad\leq L(\|x_{v_1}(t) - \bar{x}(t)\|^2 + \|v_2\|^2 + (r_0 - T(\bar{x}(t), \beta))^2).\end{aligned}$$

Now, let $(\alpha', \beta') \in B((\bar{x}(t), \beta), \eta)$. Taking $v_1 = \alpha' - \bar{x}(t)$ and $v_2 = \beta' - \beta$, we obtain that $v := (v_1, v_2) \in B((0,0);\eta)$. Taking $x_{v_1}(t)$ the solution as mentioned above, we notice that

$$x_{v_1}(t) = \bar{x}(t) + v_1 = \alpha' \text{ and } \beta + v_2 = \beta'.$$

Thus
$$(x_{v_1}(t), \beta + v_2) = (\alpha', \beta') \in B((\bar{x}(t), \beta);\eta).$$

Therefore, for all $(\alpha', \beta') \in B((\bar{x}(t), \beta);\eta)$ and $r_0 \geq T(\alpha', \beta')$, we have

$$\begin{aligned}&\langle (-\bar{p}(t), \bar{p}(r)), (\alpha', \beta') - (\bar{x}(t), \beta)\rangle + m(r_0 - T(\bar{x}(t), \beta)) \\ &\quad\leq L(\|\alpha' - \bar{x}(t)\|^2 + \|b' - \beta\|^2 + (r_0 - T(\bar{x}(t), \beta))^2).\end{aligned}$$

This implies that $((-\bar{p}(t), \bar{p}(r)), h(\alpha, -\bar{p}(0))) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\bar{x}(t), \beta), T(\bar{x}(t), \beta))$.

By Lemma 4.8, we deduce that $h(\bar{x}(t), -\bar{p}(t))) = h(\alpha, -\bar{p}(0)))$. Since $t \in (0, r)$ is arbitrary, we have

$$((-\bar{p}(t), \bar{p}(r)), h(\bar{x}(t), -\bar{p}(t))) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\bar{x}(t), \beta), T(\bar{x}(t), \beta)), \quad \forall t \in [0, r),$$

and $h(\bar{x}(t), -\bar{p}(t)) = h(\alpha, -\bar{p}(0))$ for all $t \in [0, r)$. Using the continuity of $h(\cdot,\cdot)$ we conclude that $h(\bar{x}(t), -\bar{p}(t)) = h(\alpha, -\bar{p}(0))$ for all $t \in [0, r]$. This concludes the proof. $\square$

Lemma 4.11. *Assume that hypotheses* **(H)** *hold. Let* $(\alpha,\beta) \in \mathcal{R} \setminus \mathcal{D}$ *and let* $r := T(\alpha,\beta) > 0$. *For* $\bar{x} : [0,r] \longrightarrow \mathbb{R}^n$ *an optimal trajectory from* α *to* β, *suppose that* $\bar{p} : [0,r] \longrightarrow \mathbb{R}^n$ *is an arc such that* $(\bar{x},\bar{p})$ *is a solution of the system*

$$\begin{cases} \dot{x}(s) = \nabla_p H(x(s),p(s)) \\ -\dot{p}(s) \in \partial_x H(x(s),p(s)), \end{cases} \quad \text{a.e. } s \in [0,r] \tag{23}$$

satisfying $\bar{x}(0) = \alpha$ *and* $(-\bar{p}(0),\bar{p}(r)) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)$. *Then*

$$(-\bar{p}(t),\bar{p}(r)) \in N^P_{\bar{\mathcal{R}}(T(\bar{x}(t),\beta))}(\bar{x}(t),\beta), \quad \forall t \in [0,r]$$

Proof. Since $(-\bar{p}(0),\bar{p}(r)) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)$ we have, by Lemma 4.9, that

$$((-\bar{p}(0),\bar{p}(r)), h(\alpha,-\bar{p}(0))) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha,\beta),r).$$

This gives using Lemma 4.10 that

$$((-\bar{p}(t),\bar{p}(r)), h(\bar{x}(t),-\bar{p}(t))) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\bar{x}(t),\beta),T(\bar{x}(t),\beta)), \quad \forall t \in [0,r).$$

Hence by Lemma 4.8, we conclude that

$$(-\bar{p}(t),\bar{p}(r)) \in N^P_{\bar{\mathcal{R}}(T(\bar{x}(t),\beta))}(\bar{x}(t),\beta), \quad \forall t \in [0,r).$$

It remains to prove that the previous relation holds for $t = r$, that is,

$$(-\bar{p}(r),\bar{p}(r)) \in N^P_{\mathcal{D}}(\beta,\beta).$$

This is easy to verify. □

Now we are ready to give the proof of our main result, Theorem 4.2.

Proof of Theorem 4.2. Since $T(\cdot,\cdot)$ is continuous near (α,β), there exists $\eta > 0$ such that $T(\cdot,\cdot)$ is continuous at (α',β') for all $(\alpha',\beta') \in B((\alpha,\beta);\eta)$. For (α',β') in $B((\alpha,\beta);1)$ and $r' := T(\alpha',\beta')$, we denote by $\bar{x}(\cdot) : [0,r'] \longrightarrow \mathbb{R}^n$ an optimal trajectory from α' to β'. By Gronwall's lemma, there exists $k_1 = k_1(r)$ such that

$$\|\bar{x}(t) - \alpha'\| \le k_1 t, \quad \forall t \in [0,r+1]. \tag{24}$$

We note that if $r' < r+1$ then we extend the trajectory $\bar{x}$ forward to $+\infty$, see [13, Exercise 4.1.3]. Not to mention that $\bar{x}(\cdot)$ is only optimal from 0 to r'. By hypothesis **(R)**, there exists $\delta > 0$ and $\varphi_0 \ge 0$ such that $\bar{\mathcal{R}}(T(\alpha',\beta'))$ is φ_0-convex at (α',β') for all $(\alpha',\beta') \in B((\alpha,\beta);\delta)$. The continuity of $T(\cdot,\cdot)$ at (α,β) gives, for $\varepsilon_0 < \min\left\{1, \frac{\delta}{2k_1}\right\}$, the existence of $\delta_0 > 0$ such that for all $(\alpha',\beta') \in B((\alpha,\beta);\delta_0)$

$$-\varepsilon_0 \le T(\alpha',\beta') - T(\alpha,\beta) \le \varepsilon_0. \tag{25}$$

Now, let $0 < \mu < \min\left\{1, \delta_0, \eta, \frac{\delta - 2k_1\varepsilon_0}{2}\right\}$ and let $O := B((\alpha,\beta);\mu)$. Clearly we have:

- $O \subset \mathcal{R}$ is an open and convex set.
- $T(\cdot,\cdot) : O \longrightarrow \mathbb{R}$ is continuous.

We claim that $\operatorname{epi} T(\cdot,\cdot)|_O$ is φ_0-convex, that is, using Lemma 2.4 we need to prove that existence $\bar{\varphi}_0 > 0$ such that for all (α',β') and $(\alpha'',\beta'') \in O$, $r' := T(\alpha',\beta')$ and $r'' := T(\alpha'',\beta'')$, and $0 \neq ((\rho,\theta),s) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha',\beta'),r')$, we have

$$\left\langle \frac{((\rho,\theta),s)}{\|((\rho,\theta),s)\|}, ((\alpha'',\beta''),r'') - ((\alpha',\beta'),r') \right\rangle \leq \bar{\varphi}_0 \|((\alpha'',\beta''),r'') - ((\alpha',\beta'),r')\|^2. \tag{26}$$

Let $(\alpha',\beta'), (\alpha'',\beta'') \in O$ and let $0 \neq ((\rho,\theta),s) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha',\beta'),r')$. By (25) we have

$$-2\varepsilon_0 \leq r' - r'' \leq 2\varepsilon_0. \tag{27}$$

Since $((\rho,\theta),s) \in N^P_{\operatorname{epi} T(\cdot,\cdot)}((\alpha',\beta'),r')$, Lemma 4.8 gives that

$$(\rho,\theta) \in N^P_{\bar{\mathcal{R}}(r')}(\alpha',\beta') \text{ and } h(\alpha',\rho) = s \leq 0. \tag{28}$$

Two cases follow:

Case 1. $r'' \geq r'$. Let $\bar{y}(\cdot)$ be an optimal trajectory from α'' to β'' and set $r_1 := r'' - r'$. By the principle of optimality, we can easily verify that $T(\bar{y}(r_1),\beta'') = r'$ and then $(\bar{y}(r_1),\beta'') \in \bar{\mathcal{R}}(r')$. On the other hand, we have

$$\|(\alpha',\beta') - (\alpha,\beta)\| = \|(\alpha'-\alpha, \beta'-\beta)\| \leq \mu < \delta.$$

Hence, we have:

- $(\alpha',\beta') \in B((\alpha,\beta);\delta)$ and $T(\alpha',\beta') = r'$,
- $(\bar{y}(r_1),\beta'') \in \bar{\mathcal{R}}(r')$,
- $(\rho,\theta) \in N^P_{\bar{\mathcal{R}}(r')}(\alpha',\beta')$ (by (28)).

Then by **(R)** we have

$$\begin{aligned}
&\left\langle \rho, \bar{y}(r_1) - \alpha' \right\rangle + \left\langle \theta, \beta'' - \beta' \right\rangle = \left\langle (\rho,\theta), (\bar{y}(r_1),\beta'') - (\alpha',\beta') \right\rangle \\
&\quad \leq \varphi_0 \|(\rho,\theta)\| \, \|(\bar{y}(r_1),\beta'') - (\alpha',\beta')\|^2 \\
&\quad \leq \varphi_0 \|(\rho,\theta)\| \left(\|(\bar{y}(r_1),\beta'') - (\alpha'',\beta'')\| + \|(\alpha'',\beta'') - (\alpha',\beta')\| \right)^2 \\
&\quad \leq \varphi_0 \|(\rho,\theta)\| \left(\|\bar{y}(r_1) - \alpha''\| + \|(\alpha'',\beta'') - (\alpha',\beta')\| \right)^2 \\
&\quad \leq k_2 \|(\rho,\theta)\| \left(r_1^2 + \|(\alpha'',\beta'') - (\alpha',\beta')\|^2 \right) \\
&\quad \leq k_2 \|(\rho,\theta)\| \left((r''-r')^2 + \|(\alpha'',\beta'') - (\alpha',\beta')\|^2 \right),
\end{aligned} \tag{29}$$

where $k_2 = k_2(\varphi_0, r)$ is a suitable constant. Now, we define on $[0,r'']$ the measurable function $z(\cdot)$ by

$$z(t) = \operatorname{proj}_{F(\alpha')} \dot{\bar{y}}(t) \quad \text{for a.e. } t \in [0,r''].$$

Using the Standing Hypotheses and (24), there exists a constant $K := K(r)$ such that

$$\begin{aligned}
\|\dot{\bar{y}}(t) - z(t)\| &\leq K \ \|\bar{y}(t) - \alpha'\| \leq K \ \left(\|\bar{y}(t) - \alpha''\| + \|\alpha'' - \alpha'\| \right) \\
&\leq K \ \left(k_1 r_1 + \|\alpha'' - \alpha'\| \right) \quad \text{a.e. } t \in [0,r_1].
\end{aligned}$$

Thus $$\langle \rho, \alpha'' - \bar{y}(r_1)\rangle = -\int_0^{r_1} \langle \rho, z(s)\rangle\, ds + \int_0^{r_1} \langle \rho, z(s) - \dot{\bar{y}}(s)\rangle\, ds$$
$$\leq -r_1 h(\alpha', \rho) + \|\rho\| \left(Kk_1 r_1 + K\|\alpha'' - \alpha'\|\right) r_1$$
$$\leq -(r''-r')h(\alpha', \rho) + k_3\|(\rho, \theta)\| \left(\|(\alpha'', \beta'') - (\alpha', \beta')\|^2 + (r''-r')^2\right), \quad (30)$$

where $k_3 = k_3(r)$ is a suitable constant. Now using (29) and (30) and the fact that
$$\big\langle(\rho, \theta), (\alpha'', \beta'') - (\alpha', \beta')\big\rangle = \langle \rho, \alpha'' - \alpha'\rangle + \langle \theta, \beta'' - \beta'\rangle$$
$$= \big\langle \rho, \bar{y}(r_1) - \alpha'\big\rangle + \big\langle \theta, \beta'' - \beta'\big\rangle + \big\langle \rho, \alpha'' - \bar{y}(r_1)\big\rangle,$$
we get
$$\langle(\rho, \theta), (\alpha'', \beta'') - (\alpha', \beta')\rangle \leq k_2\|(\rho, \theta)\| \left((r'' - r')^2 + \|(\alpha'', \beta'') - (\alpha', \beta')\|^2\right)$$
$$-(r'' - r')h(\alpha', \rho) + k_3\|(\rho, \theta)\| \left(\|(\alpha'', \beta'') - (\alpha', \beta')\|^2 + (r'' - r')^2\right).$$
Thus $$\big\langle(\rho, \theta), (\alpha'', \beta'') - (\alpha', \beta')\big\rangle + (r'' - r')s$$
$$\leq (k_2 + k_3)\|(\rho, \theta)\| \left((r'' - r')^2 + \|(\alpha'', \beta'') - (\alpha', \beta')\|^2\right).$$

Therefore
$$\Big\langle \frac{((\rho, \theta), s)}{\|((\rho, \theta), s)\|}, ((\alpha'', \beta''), r'') - ((\alpha', \beta'), r')\Big\rangle \leq \tilde{\varphi}_0\|((\alpha'', \beta''), r'') - ((\alpha', \beta'), r')\|^2,$$

where $\tilde{\varphi}_0 = \tilde{\varphi}_0(r, \varphi_0)$ is a suitable constant.

Case 2. $r'' < r'$. Let $\bar{x}(\cdot)$ be an optimal trajectory from α' to β'. We claim the existence of an arc $\bar{p}(\cdot)$ such that $(\bar{x}, \bar{p})$ solves the system

$$\dot{x}(s) = \nabla_p H(x(s), p(s)), \quad -\dot{p}(s) \in \partial_x H(x(s), p(s)) \quad \text{a.e. } s \in [0, r'] \qquad (31)$$

with $\bar{x}(0) = \alpha'$ and $(\bar{p}(0), -\bar{p}(r')) = -(\rho, \theta)$. Indeed, by (28) we have that
$$(\rho, \theta) \in N^P_{\bar{\mathcal{R}}(r')}(\alpha', \beta').$$

We denote by $\mathcal{B}$ an open ball realizing the proximal normal vector (ρ, θ) to $\bar{\mathcal{R}}(r')$ at (α', β'). Clearly we have

$$\bar{\mathcal{B}} \subset \mathcal{R}(r')^c \quad \text{and} \quad N_{\bar{\mathcal{B}}}(\alpha', \beta') = \{-\lambda(\rho, \theta) : \lambda \geq 0\}, \qquad (32)$$

where $\bar{\mathcal{B}} := \operatorname{cl}\mathcal{B}$. We define the multifunction $\tilde{F} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ by

$$\tilde{F}(x, y) := F(x) \times \{0\}, \quad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

We can easily verify that the trajectory $\bar{z} : [0, r'] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ defined by

$$\bar{z}(t) := (\bar{x}(t), \beta'), \quad \forall t \in [0, r'],$$

is a trajectory of $\tilde{F}$ and is optimal from (α', β') to $\mathcal{D}$. This gives that $\bar{z}$ is optimal from $\mathcal{R}(r')^c$ to $\mathcal{D}$ and then also optimal from $\bar{\mathcal{B}}$ to $\mathcal{D}$. Now we obtain the existence of a never vanishing arc $(\bar p,\bar q):\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}^n\times\mathbb{R}^n$ by applying the maximum principle, see [12, Theorem 3.5.4], such that $(\bar z,(\bar p,\bar q))$ solves

$$\begin{cases} \dot z(s)=\nabla_p\tilde H(z(s),(\bar p(s),\bar q(s)), \\ -(\dot{\bar p}(s),\dot{\bar q}(s))\in\partial_x\tilde H(z(s),(\bar p(s),\bar q(s)), \end{cases}\quad \text{a.e. } s\in[0,r'] \tag{33}$$

with $(\bar p(0,\bar q(0))\in N_{\bar{\mathcal{B}}}(\alpha',\beta')$ and $-(\bar p(r'),\bar q(r'))\in N_D(\beta',\beta')=\{(\gamma,-\gamma):\gamma\in\mathbb{R}^n\}$.

Now using the definition of $\tilde F$, the formula of $N_{\bar{\mathcal{B}}}(\alpha',\beta')$ in (32) and the fact that if $(\bar x,p)$ solves (31) then $(\bar x,\lambda p)$ also solves (31) (for $\lambda>0$), we can deduce from (33) and from $(\bar p(0,\bar q(0))\in N_{\bar{\mathcal{B}}}(\alpha',\beta')$ and $-(\bar p(r'),\bar q(r'))\in N_D(\beta',\beta')$, that $(\bar x,\bar p)$ solves (31) with $\bar x(0)=\alpha'$ and $(\bar p(0),-\bar p(r'))=-(\rho,\theta)$. This terminate the proof of the claim. Since $(\rho,\theta)\in N^P_{\bar{\mathcal{R}}(r')}(\alpha',\beta')$ we have by Lemma 4.11 that

$$(-\bar p(t),\bar p(r'))\in N^P_{\bar{\mathcal{R}}(T(\bar x(t),\beta'))}(\bar x(t),\beta') \text{ for all } t\in[0,r']. \tag{34}$$

Set $r_1:=r'-r''>0$. We have

$$\begin{aligned}\big\langle(\rho,\theta),(\alpha'',\beta'')-(\alpha',\beta')\big\rangle &= \underbrace{\big\langle-(-\bar p(r_1),\bar p(r'))-(\bar p(0),-\bar p(r')),(\alpha'',\beta'')-(\alpha',\beta')\big\rangle}_{I_1}\\ &+\underbrace{\big\langle(-\bar p(r_1),\bar p(r')),(\alpha'',\beta'')-(\bar x(r_1),\beta')\big\rangle}_{I_2}+\underbrace{\big\langle(-\bar p(r_1),\bar p(r')),(\bar x(r_1),\beta')-(\alpha',\beta')\big\rangle}_{I_3},\end{aligned} \tag{35}$$

and
$$\begin{aligned} I_1&=\big\langle-(-\bar p(r_1),\bar p(r'))-(\bar p(0),-\bar p(r')),(\alpha'',\beta'')-(\alpha',\beta')\big\rangle\\ &=\big\langle\bar p(r_1)-\bar p(0),\alpha''-\alpha'\big\rangle=\int_0^{r_1}\big\langle\alpha''-\alpha',\dot{\bar p}(s)\big\rangle\,ds\\ &\le k_4\|\rho\|\|\alpha''-\alpha'\|\,r_1 \qquad \text{(from (12) and (6))}\\ &\le k_5\|(\rho,\theta)\|\big(\|(\alpha'',\beta'')-(\alpha',\beta')\|^2+(r'-r'')^2\big),\end{aligned} \tag{36}$$

for some suitable constants $k_4=k_4(r)$ and $k_5=k_5(r)$. Now we move to the term I_2. By the principle of optimality, we can easily verify that $T(\bar x(r_1),\beta')=r''$ and then $(\alpha'',\beta'')\in\bar{\mathcal{R}}((T(\bar x(r_1),\beta'))$. Moreover,

$$\|(\bar x(r_1),\beta')-(\alpha,\beta)\|\le\|\bar x(r_1)-\alpha\|+\|\beta'-\beta\|$$
$$\le\|\bar x(r_1)-\alpha'\|+\|\alpha'-\alpha\|+\|\beta'-\beta\|\le k_1r_1+2\mu\le(2k_1\varepsilon_0+2\mu)<\delta.$$

Hence, we have:

- $(\bar x(r_1),\beta')\in B((\alpha,\beta);\delta)$ and $T(\bar x(r_1),\beta')=r''$,
- $(\alpha'',\beta'')\in\bar{\mathcal{R}}((T(\bar x(r_1),\beta'))$,
- $(-\bar p(r_1),\bar p(r'))\in N^P_{\bar{\mathcal{R}}(T(\bar x(r_1),\beta'))}(\bar x(r_1),\beta')$ (by (34)).

Then by **(R)** and using (6), we have

$$\begin{aligned} I_2&=\big\langle(-\bar p(r_1),\bar p(r')),(\alpha'',\beta'')-(\bar x(r_1),\beta')\big\rangle\\ &\le\varphi_0\|(-\bar p(r_1),\bar p(r'))\|\|(\alpha'',\beta'')-(\bar x(r_1),\beta')\|^2\\ &\le\varphi_0\|(-\bar p(r_1),\bar p(r'))\|\big(\|(\alpha'',\beta'')-(\alpha',\beta')\|+\|(\alpha',\beta')-(\bar x(r_1),\beta')\|\big)^2\\ &\le k_6\|(\rho,\theta)\|\big(\|(\alpha'',\beta'')-(\alpha',\beta')\|^2+(r'-r'')^2\big),\end{aligned} \tag{37}$$

for some suitable constant $k_6 = k_6(r, \varphi_0)$. Proceeding to the third term I_3 we have

$$\begin{aligned} I_3 &= \big\langle(-\bar{p}(r_1), \bar{p}(r')), (\bar{x}(r_1), \beta') - (\alpha', \beta')\big\rangle = \big\langle -\bar{p}(r_1), \bar{x}(r_1) - \alpha'\big\rangle \\ &= \int_0^{r_1} \big\langle \bar{p}(s) - \bar{p}(r_1), \dot{\bar{x}}(s)\big\rangle\, ds + \int_0^{r_1} \big\langle -\bar{p}(s), \dot{\bar{x}}(s)\big\rangle\, ds. \end{aligned} \tag{38}$$

By the Standing Hypotheses we have $\|\dot{\bar{x}}(s)\| \le c_1\|\bar{x}(s)\| + c_2$ for all $s \in [0, r_1]$. On the other hand, for all $s \in [0, r_1]$ we have

$$\|\bar{x}(s)\| \le \|\bar{x}(s) - \alpha'\| + \|\alpha'\| \le k_1 r_1 + \|\alpha'\| \le k_7, \tag{39}$$

where k_7 is a suitable constant. Thus, for all $s \in [0, r_1]$, we have $\|\dot{\bar{x}}(s)\| \le k_8$, where k_8 is a suitable constant. Therefore, for all $s \in [0, r_1]$, we obtain

$$\begin{aligned} \big\langle \bar{p}(s) - \bar{p}(r_1), \dot{\bar{x}}(s)\big\rangle &\le \|\bar{p}(s) - \bar{p}(r_1)\|\|\dot{\bar{x}}(s)\| \le k_9\|r_1 - s\|\|\bar{p}(r_1)\| \quad \text{(from (7))} \\ &\le k_{10}\|\rho\| r_1, \qquad \text{(from (6) and since } \|r_1 - s\| \le 2r_1) \end{aligned}$$

where k_9 and k_{10} are suitable constants. As a consequence we get that

$$\int_0^{r_1} \big\langle \bar{p}(s) - \bar{p}(r_1), \dot{\bar{x}}(s)\big\rangle\, ds \le k_{10}\|(\rho, \theta)\|(r' - r'')^2. \tag{40}$$

By (9), (31), the fact $-H(u, v) = h(u, -v)$ and Lemma 4.10 we get for all $s \in [0, r_1]$

$$\big\langle -\bar{p}(s), \dot{\bar{x}}(s)\big\rangle = -H(\bar{x}(s), \bar{p}(s)) = h(\bar{x}(s), -\bar{p}(s)) = h(\bar{x}(0), -\bar{p}(0)) = h(\alpha', \rho).$$

Therefore $$\int_0^{r_1} \big\langle -\bar{p}(s), \dot{\bar{x}}(s)\big\rangle\, ds = h(\alpha', \rho) r_1 = h(\alpha', \rho)(r' - r'').$$

The equation above together with (40) and (38) give that

$$I_3 \le k_9\|(\rho, \theta)\|(r' - r'')^2 + h(\alpha', \rho)(r' - r''). \tag{41}$$

Now, from the inequalities (36), (37) and (41), and equation (35), we obtain

$$\big\langle(\rho, \theta), (\alpha'', \beta'') - (\alpha', \beta')\big\rangle \le k_5\|(\rho, \theta)\|\big(\|(\alpha'', \beta'') - (\alpha', \beta')\|^2 + (r' - r'')^2\big)$$

$$+k_6\|(\rho, \theta)\|\big(\|(\alpha'', \beta'') - (\alpha', \beta')\|^2 + (r' - r'')^2\big) + k_{10}\|(\rho, \theta)\|(r' - r'')^2 + h(\alpha', \rho)(r' - r'').$$

As a consequence, there exists $\hat{\varphi} = \hat{\varphi}(r, \varphi_0) > 0$ such that

$$\Big\langle \frac{((\rho, \theta), s)}{\|((\rho, \theta), s)\|}, ((\alpha'', \beta''), r'') - ((\alpha', \beta'), r')\Big\rangle \le \hat{\varphi}\|((\alpha'', \beta''), r'') - ((\alpha', \beta'), r')\|^2.$$

Combining Case 1 and Case 2, we conclude that

$$\Big\langle \frac{((\rho, \theta), s)}{\|((\rho, \theta), s)\|}, ((\alpha'', \beta''), r'') - ((\alpha', \beta'), r')\Big\rangle \le \bar{\varphi}_0\|((\alpha'', \beta''), r'') - ((\alpha', \beta'), r')\|^2,$$

where $\bar{\varphi}_0 := \max\{\tilde{\varphi}, \hat{\varphi}\}$. The proof of Theorem 4.2 is now terminated. □

We terminate this paper by providing an example of a nonlinear differential inclusion satisfying all the hypotheses of Theorem 4.2, that is, the Standing Hypotheses, hypotheses **(H)** and hypothesis **(R)** at a given point $(\alpha, \beta) \in \mathcal{R} \setminus \mathcal{D}$.

Example 4.12. For $n = 1$, let $F(x) = \cos x + [-1, 1]$. It easy to verify that the Standing Hypotheses and the hypotheses **(H)** are satisfied. On the other hand, we calculate $T(\cdot,\cdot)$ and $\mathcal{R}$ and we find the following:

- For the effective domain $\mathcal{R}$, we have that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ where:
  - $\mathcal{R}_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : -\pi + 2k\pi < x \leq y < \pi + 2k\pi, k \in \mathbb{Z}\}$,
  - $\mathcal{R}_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 2k\pi < y \leq x < 2\pi + 2k\pi, k \in \mathbb{Z}\}$.
- For the function $T(\cdot,\cdot)$, we have that

$$T(x, y) = \begin{cases} \tan \dfrac{y}{2} - \tan \dfrac{x}{2} & \text{if } (x, y) \in \mathcal{R}_1, \\ \cot \dfrac{y}{2} - \cot \dfrac{x}{2} & \text{if } (x, y) \in \mathcal{R}_2. \end{cases}$$

Now for $(\alpha, \beta) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, one can easily verify that near (α, β) the Hessian matrix of $T(\cdot,\cdot)$ is positive definite, and then there exists $\delta > 0$ such that the set $\bar{\mathcal{R}}(\alpha', \beta') \cap B((\alpha', \beta'); \delta)$ is convex for all $(\alpha', \beta') \in B((\alpha, \beta); \delta)$. This implies that $\bar{\mathcal{R}}(\alpha', \beta')$ is $\frac{1}{\delta}$-convex at (α', β') for all $(\alpha', \beta') \in B((\alpha, \beta); \delta)$ and then hypothesis **(R)** is satisfied at (α, β).

Acknowledgement. The authors would like to thank Jean Takche for fruitful discussions on this work.

References

[1] M. Bardi, I. Capuzzo-Dolcetta: *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, With appendices by M. Falcone and P. Soravia, Birkhäuser, Boston (1997).

[2] P. Cannarsa, H. Frankowska: *Interior sphere property of attainable sets and time optimal control problems*, ESAIM: Control Optim. Calc. Var. 12 (2006) 350–370.

[3] P. Cannarsa, H. Frankowska, C. Sinestrari: *Optimality conditions and synthesis for the minimum time problem*, Set-Valued Analysis 8 (2000) 127–148.

[4] P. Cannarsa, A. Marigonda, K. T. Nguyen: *Optimality conditions and regularity results for time optimal control problems with differential inclusion*, J. Math. Anal. Appl. 427 (2015) 202–228.

[5] P. Cannarsa, F. Marino, P. R. Wolenski: *Semiconcavity of the minimum time function for differential inclusions*, Dyn. Cont. Discrete Impuls. Syst. Ser. B Appl. Algorithms 19(1-2) (2012) 187–206.

[6] P. Cannarsa, F. Marino, P. R. Wolenski: *The dual arc inclusion with differential inclusions*, Nonlinear Analysis 79 (2013) 176–189.

[7] P. Cannarsa, K. T. Nguyen: *Exterior sphere condition and time optimal control for differential inclusions*, SIAM J. Control Optimization 49(6) (2011) 2558–2576.

[8] P. Cannarsa, T. Scarinci: *Conjugate times and regularity of the minimum time function with differential inclusions*, in: *Analysis and Geometry in Control Theory and its Applications*, INdAM Ser. 11, Springer, Cham (2015) 85–110.

[9] P. Cannarsa, C. Sinestrari: *Convexity properties of the minimum time function*, Calc. Var. 3 (1995) 273–298.

[10] P. Cannarsa, C. Sinestrari: *Semiconcave Functions, Hamiton-Jacobi Equations and Optimal Control*, Birkhäuser, Boston (2004).

[11] P. Cannarsa, P. Wolenski: *Semiconcavity of the value function for a class of differential inclusions*, Discrete Cont. Dyn. Syst. 29(2) (2011) 453–466.

[12] F. H. Clarke: *Optimization and Nonsmooth Analysis*, Wiley, New York (1983).

[13] F. H. Clarke, Yu. Ledyaev, R. J. Stern, P. Wolenski: *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics 178, Springer, New York (1998).

[14] F. H. Clarke, C. Nour: *The Hamilton-Jacobi equation of minimal time control*, J. Convex Analysis 11(2) (2004) 413–436.

[15] F. H. Clarke, R. J. Stern, P. Wolenski: *Proximal smoothness and the lower-C^2 property*, J. Convex Analysis 2(1-2) (1995) 117–144.

[16] G. Colombo, A. Marigonda: *Differentiability properties for a class of non-convex functions*, Calc. Var. Partial Differential Equations 25(1) (2006) 1–31.

[17] G. Colombo, A. Marigonda, P. Wolenski: *Some new regularity properties for the minimal time function*, SIAM J. Control Optimization 44(6) (2006) 2285–2299.

[18] G. Colombo, K. T. Nguyen: *On the structure of the minimum time function*, SIAM J. Control Optimization 48(7) (2010) 4776–4814.

[19] H. Federer: *Curvature measures*, Trans. Amer. Math. Soc. 93 (1959) 418–491.

[20] P. L. Lions: *Generalized Solutions of Hamilton-Jacobi Equations*, Pitman, Boston (1982).

[21] K. T. Nguyen: *Hypographs satisfying an external sphere condition and the regularity of the minimum time function*, J. Math. Anal. Appl. 372(2) (2010) 611–628.

[22] L. V. Nguyen: *Variational analysis and regularity of the minimum time function for differential inclusions*, SIAM J. Control Optimization 54(5) (2016) 2235–2258.

[23] L. V. Nguyen: *Variational analysis for the bilateral minimal time function*, J. Convex Analysis 24(3) (2017) 1029–1050.

[24] C. Nour: *The bilateral minimal time function*, J. Convex Analysis 13(1) (2006) 61–80.

[25] C. Nour: *Proximal subdifferential of the bilateral minimal time function and some regularity applications*, J. Convex Analysis 20(4) (2013) 1095–1112.

[26] C. Nour, R. J. Stern, J. Takche: *Proximal smoothness and the exterior sphere condition*, J. Convex Analysis 16(2) (2009) 501–514.

[27] C. Nour, R. J. Stern, J. Takche: *The θ-exterior sphere condition, φ-convexity and local semiconcavity*, Nonlinear Analysis 73(2) (2010) 573–589.

[28] R. A. Poliquin, R. T. Rockafellar: *Prox-regular functions in variational analysis*, Trans. Amer. Math. Soc. 348(5) (1996) 1805–1838.

[29] R. A. Poliquin, R. T. Rockafellar, L. Thibault: *Local differentiability of distance functions*, Trans. Amer. Math. Soc. 352(11) (2000) 5231–5249.

[30] R. T. Rockafellar, R. J.-B. Wets: *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften 317, Springer, New York (1998).

[31] C. Sinestrari: *Semiconcavity of the value function for exit time problems with nonsmooth target*, Commun. Pure Appl. Analysis 3(4) (2004) 757–774.

[32] P. Wolenski, Y. Zhuang: *Proximal analysis and the minimal time function*, SIAM J. Control Optimization 36 (1998) 1048–1072.