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Spherically Convex Sets and Spherically Convex Functions*

Qi Guo, Yanling Peng
Department of Mathematics, Suzhou University of Science and Technology, 215009 Suzhou, Jiangsu, China
guoqi@mail.usts.edu.cn, yanlingpeng35@hotmail.com

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We define first the spherical convexity of sets and functions on general curved surfaces by an analytic approach. Then we study several kinds of properties of spherically convex sets and functions. Several analogies of the results for convex sets and convex functions on Euclidean spaces are established or rediscovered for spherically convex sets and spherically convex functions, such as the Radon-type, Helly-type, Carathéodory-type and Minkowski-type theorems for spherically convex sets, and the Jensen's inequality for spherically convex functions etc. The results obtained here might have applications in some areas, e.g. in the optimization theory on general spherical spaces.

Keywords: Spherical convexity, Helly theorem, Carathéodory theorem, Jensen's inequality.

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1. Introduction

The convexity of sets in linear spaces is a very important concept as well as a very useful tool in the study on properties of convex functions defined on linear spaces. Fruitful results on convexity have been obtained and numerous applications of convexity have been found in different branches of mathematics, in particular in the optimization theory. In the light of the great significance of convexity of sets in linear spaces, It is natural to extend the concepts of convexity of sets and functions from the linear space to the curved surface or more generally in manifolds. Since the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ in Euclidean n-space $\mathbb{R}^n$ is a typical Riemannian manifold, mathematicians started to study the sets in $\mathbb{S}^{n-1}$ first (see [12, 18, 19]).

Since the 1940's several different definitions of convex sets in $\mathbb{S}^{n-1}$ have been proposed, e.g. a set $C \subset \mathbb{S}^{n-1}$ is called *strongly convex* if it doesn't contain antipodal points ($x_1, x_2 \in \mathbb{S}^{n-1}$ are called (a pair of) antipodal points if $x_1 = -x_2$) and it contains, with each pair of its points, the short arc of the great circle determined by them (see [1, 2]); *weakly convex* if it contains, with each pair of its points, the short arc or a semicircular arc of a great circle determined by them (see [1, 2]); *Robinson-convex* if it contains, with each of its non-antipodal points, the short arc of the great circle determined by them ([18]); *Horn-convex* if it contains, with each pair of its non-antipodal points, at least one of the great circle arcs determined by them ([12]). All these convexities and the like were often called *spherical convexity*

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(or *curved-surface convexity*). Several equivalent definitions or necessary and sufficient conditions of these spherical convexities were given later (see [4, 8, 21], etc). Among all these equivalent definitions, the one given by Shao and Guo [21] is of its own feature: it is given in a pure analytic approach.

Clearly, every strongly convex set is weakly convex; weak convexity implies Robinson-convexity and the latter implies Horn-convexity (see [5] and the references therein). Also, simple examples show that all the inclusion relations here are proper.

From the 1940's to the 1980's, the main research topics on this stream were the combinatorial properties of convex sets in $\mathbb{S}^{n-1}$ (with different definitions), in particular, the Helly type theorems (see [1, 2, 5, 7, 12, 14, 17, 18, 22, 23], etc. and the references therein). Then, the studies on other properties and applications of spherical convexity emerged gradually, e.g., Lassak [15, 16] gave the definition of width of a strongly convex body in $\mathbb{S}^{n-1}$ and investigated the properties of the so-called spherically reduced bodies. Ferreira, Iusem and Németh [8] defined and studied the projections onto weakly convex sets on the sphere; Besau and Schuster [3] discussed meaningful and possible compositions between spherically convex sets on $\mathbb{S}^{n-1}$. Gao, Hug and Schneider [10] presented the spherical counterpart of two Euclidean inequalities: the Urysohn inequality connecting mean width and volume, and the Blaschke-Santaló inequality for the volumes of polar convex bodies. Of course, much work was done on applications of spherical convexity, in particular, in optimization theory. E.g., Ferreira, Iusem and Németh [9] introduced the concept of (spherically) convex function on strongly convex sets in $\mathbb{S}^{n-1}$ and, as an application, necessary and sufficient optimality conditions for constrained convex optimization problems on the sphere were derived; the spherical minimax location problem was studied by Das, Chakraborti and Chaudhuri [6] and in principle the problem studied by Iusem and Seeger [13] was an optimization problem for functions defined on spherically convex sets on $\mathbb{S}^{n-1}$.

Some progresses have been made in the study on spherically convex sets and functions on $\mathbb{S}^{n-1}$, however, the attempt to establish a systematic spherical convexity theory is far from being completed. One of the main obstacles in establishing such a systematic theory is that most concepts, definitions we already have are given in geometric forms, which make it hard to formulate/prove useful conclusions. Therefore, the purpose of this paper is to have a try in developing a calculus for spherical convexity similar to the corresponding one for convexity in Euclidean spaces. For generality, we are going to work with general spheres rather than $\mathbb{S}^{n-1}$ (it is simply due to the fact that some conclusions for spherically convex sets/functions on general spheres cannot be derived directly from the corresponding ones for spherically convex sets/functions on $\mathbb{S}^{n-1}$). Concretely, we first formulate the spherical convexity of sets and functions by an analytic approach on general spheres, then we study the basic properties of spherically convex sets and functions. It turns out that such an analytic approach works well, for instance, in these spherical spaces, the Radon-type, Helly-type and Carathéodory-type theorems, and the structural theorem of Minkowski type are established. Also, with the help of these theorems, some analogies of results for convex functions are obtained for spherically convex functions on general spheres.

2. Notation and definitions

As already mentioned above, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the classical inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$, or the n-dimensional affine space (so we don't distinguish points and vectors in $\mathbb{R}^n$ intentionally). $\mathbb{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$. We use $u, v, x, y, \cdots$ to denote vectors or points in $\mathbb{R}^n$, in particular, o stands for the zero vector or the origin. For any non-zero $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, $H_{u,\alpha}$ denotes the hyperplane $\{x \in \mathbb{R}^n \mid \langle u, x\rangle = \alpha\}$ ($\{\langle u,\cdot\rangle = \alpha\}$ for brevity) and $H^+_{u,\alpha} := \{\langle u,\cdot\rangle > \alpha\}$ (resp. $\overline{H}^+_{u,\alpha} := \{\langle u,\cdot\rangle \geqslant \alpha\}$) is called the open (resp. the closed) half-space determined by (u,α). If $\alpha = 0$, we write simply H_u, H^+_u and $\overline{H}^+_u$ instead of $H_{u,0}, H^+_{u,0}$ and $\overline{H}^+_{u,0}$. The open (resp. the closed) half-space $H^-_{u,\alpha}$ (resp. $\overline{H}^-_{u,\alpha}$) is defined in same manners (simply replacing "$>$" (resp. "$\geqslant$") by "$<$" (resp. "$\leqslant$") in the above notation).

For non-empty $A \subset \mathbb{R}^n$, co$A$, cone$A$, $\overline{\text{cone}}A$ and lin$A$ denote the convex hull, the convex conical hull, the closed convex conical hull and the linear hull of A respectively, and cl$A$, bd$A$ denote respectively the closure and the boundary of A. For a convex set A, int$A$, ri$A$ and rbd$A$ denote the interior, relative interior and the relative boundary of A respectively.

In this article, we will work with more general spheres in Euclidean spaces, including the Euclidean unit sphere $\mathbb{S}^{n-1}$ as a particular case. So we have the following definition.

Definition 2.1. (i) A continuous function $\Phi : \mathbb{R}^n \to [0, +\infty)$ is called *sphere-generating* (*s-generating* for brevity) if it satisfies $\Phi(x) = 0$ iff $x = 0$ and $\Phi(tx) = t\Phi(x)$ for every $x \in \mathbb{R}^n$ and $t \geqslant 0$.

(ii) If Φ is an s-generating function, then the set $\mathbb{S} = \mathbb{S}_\Phi := \{x \in \mathbb{R}^n \mid \Phi(x) = 1\}$ is called the Φ-*sphere*. $\mathbb{S}_\Phi$ is called *convex* if Φ is convex, and $\mathbb{S}_\Phi$ is called *symmetric* if Φ is symmetric, i.e. $\Phi(x) = \Phi(-x)$ for all $x \in \mathbb{R}^n$.

Remark 2.2. (i) Clearly, if Φ is a norm on $\mathbb{R}^n$, then $\mathbb{S}_\Phi$ is just the unit sphere determined by the norm, in particular, $\mathbb{S}_{\|\cdot\|} = \mathbb{S}^{n-1}$. One may work with the curved surface $\mathbb{S}^r_\Phi := \{x \in \mathbb{R}^n \mid \Phi(x) = r\}$ $(r > 0)$ as well. However, since $\mathbb{S}^r_\Phi = r\mathbb{S}_\Phi$ for any $r > 0$, it is sufficient to work with $\mathbb{S}_\Phi$.

(ii) Observe that, for each s-generating function Φ, there is a bijection $\overline{\rho} : \mathbb{S}_\Phi \to \mathbb{S}^{n-1}$, defined by $\overline{\rho}(x) = \frac{x}{\|x\|}$, $x \in \mathbb{S}_\Phi$, with $(\overline{\rho})^{-1}$ defined as $(\overline{\rho})^{-1}(x) = \frac{x}{\Phi(x)}$, $x \in \mathbb{S}^{n-1}$.

Given a Φ-sphere $\mathbb{S}$. A subset of the form $H^+_u \cap \mathbb{S}$ (resp. $\overline{H}^+_u \cap \mathbb{S}$) is called an open (resp. a closed) Φ-hemisphere; a set of the form $H_u \cap \mathbb{S}$ is called a Φ-hypercircle. If $V \subset \mathbb{R}^n$ is a 2-dimensional subspace, then $V \cap \mathbb{S}$ is called a Φ-greatcircle. For $x \in \mathbb{S}$, the (unique) intersection x^o of $\{-tx \mid t > 0\}$ and $\mathbb{S}$ is called the antipodal (or opposite) point of x (when Φ is symmetric, $x^\circ = -x$). More general, for a set $S \subset \mathbb{S}$, the set $S^o := \{x^o \mid x \in S\}$ is called the antipodal set of S. If a Φ-greatcircle G passes through a pair of antipodal points x, x^o, then x, x^o partition G into two non-overlapping connected components with x, x^o as ends, each of which is called a Φ-semi-greatcircle. If Φ is convex, a subset of the form $H^+_{u,\alpha} \cap \mathbb{S}$ (resp. $\overline{H}^+_{u,\alpha} \cap \mathbb{S}$), $0 < \alpha \leqslant \sup_{x\in\mathbb{S}}\langle u, x\rangle$, is called an open (resp. a closed) Φ-cap.

To define meaningful compositions on $\mathbb{S}$, we modify slightly the definition of radial projection in the following way: for a given Φ-sphere $\mathbb{S}$, the radial projection $\rho = \rho_\Phi : \mathbb{R}^n \to \mathbb{S} \cup \{o\}$ is defined by

$$\rho(x) := \begin{cases} \frac{x}{\Phi(x)}, & x \neq o \\ o, & x = o. \end{cases}$$

Observe that $\rho \circ \rho = \rho$, $\rho(\lambda x) = \rho(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$, and $\rho(x) = x$ iff $x \in \mathbb{S}$ or $x = o$.

Definition 2.3. Let $\mathbb{S}$ be a Φ-sphere. For $x_1, x_2, ..., x_k \in \mathbb{S}$, $k \geqslant 1$, and $\lambda_1, \lambda_2, ..., \lambda_k \in [0,1]$ with $\sum_{i=1}^k \lambda_i = 1$, we define their Φ-*spherical convex combination* (*s-convex combination* for brevity) by

$$(s_\Phi)\sum_{i=1}^k \lambda_i x_i := \rho_\Phi\big(\sum_{i=1}^k \lambda_i x_i\big).$$

When $k = 2$, we write $\lambda x +_{s_\Phi} (1-\lambda)y$ instead of $(s_\Phi)(\lambda x + (1-\lambda)y)$.

For brevity, when Φ is clear, we always write $(s)\sum_{i=1}^k \lambda_i x_i$, $\lambda x +_s (1-\lambda)y$ instead of $(s_\Phi)\sum_{i=1}^k \lambda_i x_i$, $\lambda x +_{s_\Phi} (1-\lambda)y$ respectively.

Remark 2.4. (i) For $x, y \in \mathbb{S}$, x, y are non-antipodal iff $\lambda x +_s (1-\lambda)y \neq 0$ for any $\lambda \in [0,1]$.
(ii) For non-antipodal (distinct) points $x, y \in \mathbb{S}$, the set $\{\lambda x +_s (1-\lambda)y \mid 0 \leqslant \lambda \leqslant 1\}$ is called the short arc of the Φ-greatcircle determined by x, y (when $\Phi = \|\cdot\|$, this definition coincides with the usual one).

In terms of Definition 2.3, we give the following definition.

Definition 2.5. Let $\mathbb{S}$ be a Φ-sphere. A set $C \subset \mathbb{S}$ is called Φ-*spherically convex* (s_Φ-*convex* or *s-convex* simply for brevity) if $\lambda x +_s (1-\lambda)y \in C$ for any $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$.

Remark 2.6. (i) By (i) in Remark 2.4, an s-convex set contains no antipodal points. By (ii) in Remark 2.4, an s-convex set contains the short arc of the Φ-greatcircle determined by its two non-antipodal points. Thus, the s-convexity goes back to the strong convexity defined on $\mathbb{S}^{n-1}$ if $\Phi(\cdot) = \|\cdot\|$.
(ii) Let $\overline{\rho} : \mathbb{S}_\Phi \to \mathbb{S}^{n-1}$ be the bijection mentioned in ii) of Remark 2.2. Then, for each $C \subset \mathbb{S}_\Phi$, $\operatorname{cone} C = \operatorname{cone} \overline{\rho}(C)$ clearly. So, by the following Proposition 2.7, it is easy to check that $C \subset \mathbb{S}_\Phi$ is (closed) s_Φ-convex iff $\overline{\rho}(C) \subset \mathbb{S}^{n-1}$ is (closed) $s_{\|\cdot\|}$-convex.

Therefore, we mention that most notation here are essentially the same as those known for $\mathbb{S}^{n-1}$ and some of the conclusions obtained here, e.g. the following Theorem 4.3, Corollary 4.6 and 4.7 etc., can be derived from those known for $\mathbb{S}^{n-1}$ directly. However, as $\overline{\rho}\big((s_\Phi)\sum_{i=1}^k \lambda_i x_i\big) = (s_{\|\cdot\|})\sum_{i=1}^k \lambda_i \overline{\rho}(x_i)$ does not hold in general, i.e. $\overline{\rho}$ is not s-convex combination-preserving (readers can find counterexamples easily even in the case $k = 2$), other conclusions, such as Theorem 4.8 and Theorem 6.4 etc., need arguments for general $\mathbb{S}_\Phi$.

It is easy to check that if C_i, $i \in I$, are s-convex, then the nonempty intersection $\cap_{i \in I} C_i$ is s-convex, and that if C is s-convex, then its antipodal set C^o is s-convex.

The following proposition shows that the s-convex sets are closely related to the pointed convex cones, which is known and proved repeatedly in the case $\mathbb{S} = \mathbb{S}^{n-1}$. A convex cone $K \subset \mathbb{R}^n$ is called pointed if $K \cap (-K) = \{o\}$.

Proposition 2.7. *Let $\emptyset \neq C \subset \mathbb{S}_\Phi$. Then, the following statements are equivalent:*
(i) *C is s-convex;*
(ii) *$x_1, x_2 \in K$ imples $x_1 + x_2 \in K$, where $K := \mathbb{P}C$ and $\mathbb{P} := (0, \infty)$;*
(iii) *K is convex;*
(iv) *$\mathbb{R}_+ C$, the cone generated by C, is a pointed convex cone, where $\mathbb{R}_+ := [0, \infty)$.*

Proof. (i)⇒(ii): Suppose C is s-convex. Then for any $x_1 = t_1 x_1'$, $x_2 = t_2 x_2' \in K$, where $x_1', x_2' \in C$ and $t_1, t_2 > 0$, writing $u = \frac{t_1}{t_1 +_s t_2} x_1' + \frac{t_2}{t_1 + t_2} x_2'$, we have by the property of radial projection,

$$\begin{aligned} x_1 + x_2 &= \Phi(x_1 + x_2)\rho(x_1 + x_2) = \Phi(x_1 + x_2)\rho\big((t_1 + t_2)u\big) \\ &= \Phi(x_1 + x_2)\Big(\frac{t_1}{t_1 +_s t_2} x_1' +_s \frac{t_2}{t_1 +_s t_2} x_2'\Big) \in K \end{aligned}$$

since $\frac{t_1}{t_1 +_s t_2} x_1' +_s \frac{t_2}{t_1 +_s t_2} x_2' \in C$ and $\Phi(x_1 + x_2) > 0$ (for $x_1 + x_2 = 0$ would imply $x_1' = -\frac{t_2}{t_1} x_2'$, i.e. $x_1' = (x_2')^\circ$, a contradiction to i) in Remark 2.6).

(ii)⇒(iii): For any $x, y \in K$ and $\lambda \in (0, 1)$, $\lambda x + (1 - \lambda) y \in K$ by (ii) since $\lambda x, (1 - \lambda) y \in K$. So K is convex.

(iii)⇒(iv): Suppose K is convex, then $\mathbb{R}_+ C$ is clearly a convex cone since $\mathbb{R}_+ C = K \cup \{o\}$. If $o \neq z \in \mathbb{R}_+ C \cap (-\mathbb{R}_+ C)$, then $z = t_1 x_1 = -t_2 x_2$ for some $x_1, x_2 \in C$ and $t_1 > 0, t_2 > 0$. Thus $t_1 x_1 + t_2 x_2 = o \notin K$ while $t_1 x_1, t_2 x_2 \in K$, a contradiction to (iii). So, $\mathbb{R}_+ C$ is pointed.

(iv)⇒(i): Observe first that $C = \mathbb{R}_+ C \cap \mathbb{S}_\Phi$ obviously. If $x_1, x_2 \in C$ and $\lambda \in (0, 1)$, then $x_1, x_2 \in \mathbb{R}_+ C$ naturally, and so $\lambda x_1 + (1 - \lambda) x_2 \in \mathbb{R}_+ C$, $\lambda x_1 + (1 - \lambda) x_2 \neq o$ since $\mathbb{R}_+ C$ is a pointed convex cone. Thus, $\lambda x_1 +_s (1 - \lambda) x_2 = \rho_\Phi(\lambda x_1 + (1 - \lambda) x_2) \in \mathbb{R}_+ C \cap \mathbb{S}_\Phi = C$. Therefore C is s-convex. □

One may also check that C is closed s-convex iff $\mathbb{R}_+ C$ is a pointed closed convex cone.

Thanks to Proposition 2.7, we may introduce some more notation for s-convex sets. For an s-convex set C, its boundary bd$C$, relative boundary rbd$C$, interior int$C$ and relative interior ri$C$ are defined respectively as the set $\mathrm{bd}\big(\mathrm{cone}\, C\big) \cap \mathbb{S}$, $\mathrm{rbd}\big(\mathrm{cone}\, C\big) \cap \mathbb{S}$, $\mathrm{int}\big(\mathrm{cone}\, C\big) \cap \mathbb{S}$ and $\mathrm{ri}\big(\mathrm{cone}\, C\big) \cap \mathbb{S}$. It is easy to show that if $C \subset \mathbb{S}$ is s-convex, then so is $\mathrm{ri}(C)$ and that if C is closed and s-convex, then $C = \mathrm{rbd}(C) \cup \mathrm{ri}(C)$.

3. Basic properties of s-convex sets and s-convex hulls

In this section, we study the basic properties of s-convex set and s-convex hull of sets on $\mathbb{S}$, starting with an analogue of that for convex sets in $\mathbb{R}^n$.

Theorem 3.1. *Let $\mathbb{S}$ be a Φ-sphere and $C \subset \mathbb{S}$ be an s-convex set. Then for any $x_1, x_2, \cdots, x_k \in C$ and $\lambda_1, \lambda_2, \cdots, \lambda_k \in [0, 1]$ with $\sum_{i=1}^k \lambda_i = 1$, $k = 1, 2, \cdots$, we have $(s) \sum_{i=1}^k \lambda_i x_i \in C$. In particular, $o \notin \mathrm{co}C$.*

Proof. The conclusion holds for $k = 1, 2$ by the definition. Suppose the conclusion holds for all $1 \leqslant j \leqslant k-1$ (in particular, $o \notin \mathrm{co}\{x_1, x_2, \cdots, x_j\}$ for any $x_1, x_2, \cdots, x_j \in C$ and $\mu_1, \mu_2, \cdots, \mu_j \in (0,1)$ with $\sum_{i=1}^{j} \mu_i = 1$). Then, for $x_1, x_2, \cdots, x_k \in C$ and $\mu_1, \mu_2, \cdots, \mu_k \in (0,1)$ with $\sum_{i=1}^{k} \lambda_i = 1$, denoting $z = \sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k} x_i$, we have $z \neq 0$ and $\rho(z) = \frac{z}{\Phi(z)} \in C$ by the induction hypothesis, which together with the properties of ρ leads to

$$
\begin{aligned}
&(s)\sum_{i=1}^{k} \lambda_i x_i = \rho\big(\sum_{i=1}^{k} \lambda_i x_i\big) = \rho\big(\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k} x_i + \frac{\lambda_k}{1-\lambda_k} x_k\big) = \rho\big(z + \frac{\lambda_k}{1-\lambda_k} x_k\big) \\
&= \rho\big(\frac{z}{\Phi(z)} + \frac{\lambda_k}{\Phi(z)(1-\lambda_k)} x_k\big) = \rho\big(\frac{z(1-\lambda_k)}{\Phi(z)} + \frac{\lambda_k}{\Phi(z)} x_k\big) = \rho\big(\frac{z\Phi(z)(1-\lambda_k)}{\Phi(z)} + \lambda_k x_k\big) \\
&= \rho\big(\frac{\Phi(z)(1-\lambda_k)}{\Phi(z)(1-\lambda_k)+\lambda_k} \frac{z}{\Phi(z)} + \frac{\lambda_k}{\Phi(z)(1-\lambda_k)+\lambda_k} x_k\big) \\
&= \frac{\Phi(z)(1-\lambda_k)}{\Phi(z)(1-\lambda_k)+\lambda_k} \frac{z}{\Phi(z)} +_s \frac{\lambda_k}{\Phi(z)(1-\lambda_k)+\lambda_k} x_k \in C \qquad (1)
\end{aligned}
$$

where we used the fact that $\frac{z}{\Phi(z)}$, $x_k \in C$ and $\frac{\Phi(z)(1-\lambda_k)}{\Phi(z)(1-\lambda_k)+\lambda_k} + \frac{\lambda_k}{\Phi(z)(1-\lambda_k)+\lambda_k} = 1$. The fact that $o \notin \mathrm{co}(C)$ is trivial. $\square$

It is easy to check that an open Φ-hemisphere is an s-convex set while a closed Φ-hemisphere is not. Also Φ-caps are s-convex. For general s-convex sets, we have the following conclusion.

Theorem 3.2. *Let $\mathbb{S}$ be a Φ-sphere and $C \subset \mathbb{S}$ be s-convex. Then*

(i) *there is $u_0 \in \mathbb{S}^{n-1}$ such that $C \subset \overline{H}_{u_0}^{+} \cap \mathbb{S}$, i.e. C is contained in a closed Φ-hemisphere.*

(ii) *if further C is closed, there are $u_0 \in \mathbb{S}^{n-1}$ and $\alpha > 0$ such that $C \subset \overline{H}_{u_0,\alpha}^{+} \cap \mathbb{S}$, i.e. C is contained in a closed Φ-cap. In particular, C is contained in an open Φ-hemisphere.*

In order to prove Theorem 3.2, we need the following lemma, stated as a theorem for its own significance.

Theorem 3.3. *Let $C^* \subset \mathbb{S}^{n-1}$ be a compact set. Then, the following statements are equivalent:*

(i) *For each $u \in \mathbb{S}^{n-1}$, there is $v \in C^*$ such that $\langle v, u\rangle \leqslant 0$;*

(ii) *$o \in \mathrm{ri}(\mathrm{co}\{v_0, v_1, \cdots, v_l\})$, i.e. there are positive α_k such that $\sum_{k=0}^{l} \alpha_k v_k = 0$, for some affinely independent $v_0, v_1, \cdots, v_l \in C^*$, $1 \leqslant l \leqslant n$;*

(iii) $\mathrm{cone}\{v_0, v_1, \ldots, v_l\} = \mathrm{lin}\{v_0, v_1, \ldots, v_l\}$ *for some affinely independent points $v_0, v_1, \ldots, v_l \in C^*$, $1 \leqslant l \leqslant n$.*

Proof. (i)$\Rightarrow$(ii): Consider the linear space $V := \mathrm{cone}\, C^* \cap (-\mathrm{cone}\, C^*)$. If $V = \{o\}$, then by the separation theorem for closed cones ($\mathrm{cone}\, C^*$ is closed since C^* is compact), there is $u \in \mathbb{R}^n$ such that $C^* \subset \{x \mid \langle u, x\rangle > 0\}$, which contradicts (i). So $\dim V \geqslant 1$. Now, it's easy to see that $V = \mathrm{lin}\{v_0', v_1', \cdots, v_s'\} = \mathrm{cone}\{v_0', v_1', \cdots, v_s'\}$ for some $v_0', v_1', \cdots, v_s' \in C^*$ $(1 \leqslant s \leqslant n)$.

Thus, since $-v_0' \in \mathrm{lin}\{v_0', v_1', \cdots, v_s'\} = \mathrm{cone}\{v_0', v_1', \cdots, v_s'\}$, we have $-v_0' = \alpha_0' v_0' + \sum_{k=1}^{s} \alpha_k v_k'$ with $\alpha_0', \alpha_k \geqslant 0$, which leads to $o = \sum_{k=0}^{s} \alpha_k v_k'$, where $\alpha_0 := 1 + \alpha_0' > 0$.

We now let l be the smallest positive integer such that

$$o = \sum_{k=0}^{l} \gamma_k v_k \tag{2}$$

for some $v_0, v_1, \ldots, v_l \in C^*$ with all $\gamma_k > 0$ and then claim that $v_0, v_1, \ldots, v_l$ are affinely independent: Suppose $v_0, v_1, \ldots, v_l$ are affinely dependent, then we have $\sum_{k=0}^{l} \beta_k v_k = o$ for some not all zero $\beta_0, \beta_1, \ldots, \beta_l$ with $\sum_{k=0}^{l} \beta_k = 0$. Let

$$\lambda := \min\{\frac{-\gamma_k}{\beta_k} \mid \beta_k < 0\} = (\text{say})\ \frac{-\gamma_l}{\beta_l}$$

then $o = \sum_{k=0}^{l} \gamma_k v_k + \sum_{k=0}^{l} \lambda \beta_k v_k = \sum_{k=0}^{l} (\gamma_k + \lambda\beta_k) v_k = \sum_{k=0}^{l-1} (\gamma_k + \lambda\beta_k) v_k$, where $\gamma_k + \lambda\beta_k \geqslant 0$ $(0 \leqslant k \leqslant l-1)$ and at least one of them is positive, a contradiction to the choice of l. It follows from (2) that

$$o = \frac{1}{\sum_{j=0}^{l} \gamma_j} \cdot \sum_{k=0}^{l} \gamma_k v_k = \sum_{k=0}^{l} \frac{\gamma_k}{\sum_{j=0}^{l} \gamma_j} v_k \in \mathrm{ri}\big(\mathrm{co}\{v_0, v_1, \cdots, v_l\}\big)$$

since all $\frac{\gamma_k}{\sum_{j=0}^{l} \gamma_j} v_k > 0,\ 1 \leqslant k \leqslant l$. Clearly $l \leqslant n$ since $v_0, v_1, \cdots, v_l$ are affinely independent.

(ii)⇒(iii): Suppose that $v_0, v_1, \cdots, v_l \in C^*$ $(1 \leqslant l \leqslant n)$ are affinely independent and $o \in \mathrm{ri}(\mathrm{co}\{v_0, v_1, \cdots, v_l\})$, i.e., $o = \sum_{k=0}^{l} \alpha_k v_k$ with $\alpha_k > 0$. Then we have for each $0 \leqslant k \leqslant l$ that $-v_k = \sum_{j \neq k} \frac{\alpha_j}{\alpha_k} v_j \in \mathrm{cone}\{v_0, v_1, \cdots, v_l\}$. In consequence we obtain $\mathrm{lin}\{v_0, v_1, \cdots, v_l\} = \mathrm{cone}\{v_0, v_1, \cdots, v_l\}$.

(iii)⇒(i): Suppose that $\mathrm{cone}\{v_0, v_1, \cdots, v_l\} = \mathrm{lin}\{v_0, v_1, \cdots, v_l\}$ for some affinely independent $v_0, v_1, \cdots, v_l \in C^*, 1 \leqslant l \leqslant n$. If there exists $u \in \mathbb{R}^n$ such that $\langle v, u\rangle > 0$ for all $v \in C^*$, in particular $\langle v_j, u\rangle > 0$ for $0 \leqslant j \leqslant l$. Observing, since $\mathrm{lin}\{v_0, v_1, \cdots, v_l\} = \mathrm{cone}\{v_0, v_1, \cdots, v_l\}$, $-v_0 = \sum_{j=0}^{l} \alpha_j v_j$ for some $\alpha_j \geqslant 0$ with at least one $\alpha_{j_0} > 0$, we have $\langle -v_0, u\rangle = -\langle v_0, u\rangle < 0$ and $\langle -v_0, u\rangle = \sum_{j=0}^{l} \alpha_j \langle v_j, u\rangle > 0$, a contradiction. □

Proof of Theorem 3.2 (i) Since $\mathrm{cone}\, C \neq \mathbb{R}^n$ (for C containing no antipodal points), $\overline{\mathrm{cone}}C \neq \mathbb{R}^n$. Hence $\overline{\mathrm{cone}}C$ has at least one support hyperplane H and $o \in H$ by Theorem 1.3.9 in [20]. We may choose a suitable normal $u_0 \in \mathbb{S}^{n-1}$ such that $H = H_{u_0}$, $\overline{\mathrm{cone}}C \subset \overline{H}_{u_0}^{+}$ and in turn $C \subset \overline{H}_{u_0}^{+} \cap \mathbb{S}$.

(ii) Observe first that, as a closed subset of the compact set $\mathbb{S}$, C is compact. Then we claim that there is $u_0 \in \mathbb{S}^{n-1}$ such that $\langle v, u_0\rangle > 0$ for all $v \in C$. Indeed, denoting $C^* := \{\frac{x}{\|x\|} \mid x \in C\}$ $(\subset \mathbb{S}^{n-1})$, we have that $\langle v, u_0\rangle > 0$ for all $v \in C$ iff $\langle v, u_0\rangle > 0$ for all $v \in C^*$. So, we need only to confirm our claim for C^* (observe that C^* is compact as well).

Suppose for each $u \in \mathbb{S}^{n-1}$, there is $v \in C^*$ satisfying $\langle v, u\rangle \leqslant 0$, then by Theorem 3.3, there are affinely independent $v_0, v_1, \cdots, v_l \in C^*$, $1 \leqslant l \leqslant n$, such that $\mathrm{cone}\{v_0, v_1, \cdots, v_l\} = \mathrm{lin}\{v_0, v_1, \cdots, v_l\}$.

So, we have $\rho\big(\mathrm{cone}\{v_0, v_1, \cdots, v_l\} \setminus \{o\}\big) = \rho\big(\mathrm{lin}\{v_0, v_1, \cdots, v_l\} \setminus \{o\}\big)$.
Thus, since $o \notin \mathrm{co}\{v_0, v_1, \cdots, v_l\}$ by Theorem 3.1, we have further on one hand

$$o \notin \rho\big(\mathrm{co}\{v_0, v_1, \cdots, v_l\}\big) = \rho\big(\mathrm{cone}\{v_0, v_1, \cdots, v_l\} \setminus \{o\}\big)$$

$\big($observe $\mathrm{cone}\{v_0, v_1, \cdots, v_l\} \setminus \{o\} = \cup_{t>0} t\mathrm{co}\{v_0, v_1, \cdots, v_l\}\big)$ and on the other hand

$$\begin{aligned} o \in \rho\big(\mathrm{lin}\{v_0, v_1, \cdots, v_l\} \cap \mathbb{S}^{n-1}\big) &= \rho\big(\mathrm{lin}\{v_0, v_1, \cdots, v_l\} \setminus \{o\}\big) \\ &= \rho\big(\mathrm{cone}\{v_0, v_1, \cdots, v_l\} \setminus \{o\}\big) \end{aligned}$$

(observe that there are clearly antipodal points in $\mathrm{lin}\{v_0, v_1, \cdots, v_l\} \cap \mathbb{S}^{n-1}$), a contradiction. The claim is confirmed.

Now, let $\alpha := \inf_{v \in C} \langle v, u_0 \rangle$. Then we have $\alpha > 0$ by the compactness of C and the continuity of inner product, $C \subset \overline{H}^+_{u_0,\alpha}$ and in turn $C \subset \overline{H}^+_{u_0,\alpha} \cap \mathbb{S}$. □

In order to investigate other properties of s-convex sets, we introduce, as counterparts of support hyperplanes and support half-spaces for convex sets in $\mathbb{R}^n$, the concepts of support Φ-hypercircles and support Φ-hemispheres for s-convex sets.

Definition 3.4. Let $\mathbb{S}$ be a Φ-sphere, $C \subset \mathbb{S}$ be s-convex and $x \in C$. A closed Φ-hemisphere $\mathbb{S}^+_u := \overline{H}^+_u \cap \mathbb{S}$ (resp. a Φ-hypercircle $\mathbb{S}_u := H_u \cap \mathbb{S}$) is called a *support Φ-hemisphere* (resp. a *support Φ-hypercircle*) of C at x if $C \subset \mathbb{S}^+_u$ and $x \in \mathbb{S}_u$.

It is worth observing that if there is a supporting Φ-hemisphere of C at x, then $x \in \mathrm{bd}C$. We say simply that $\mathbb{S}^+_u$ (*resp.* $\mathbb{S}_u$) supports C if it supports C at some point. The following result is natural.

Proposition 3.5. *Let $\mathbb{S}$ be a Φ-sphere and $C \subset \mathbb{S}$ be s-convex. Then for each $x \in \mathrm{bd}C \cap C$, there is a closed Φ-hemisphere $\mathbb{S}_u$* (resp. *a Φ-hypercircle $\mathbb{S}_u$*) *supporting C at x.*

Proof. Consider the closed convex cone $\overline{\mathrm{cone}}C$. By the definition of $\mathrm{bd}C$, $x \in \mathrm{bd}(\overline{\mathrm{cone}}C)$. By Theorem 1.3.9 in [20] there is a hyperplane H_u supporting $\overline{\mathrm{cone}}C$ at x with $\overline{\mathrm{cone}}C \subset \overline{H}^+_u$ and $o \in H_u$. So, $\mathbb{S}^+_u$ (*resp.* $\mathbb{S}_u$) supports C at x. □

In fact, we have the following theorem.

Theorem 3.6. *Let C be closed and s-convex. Then $C = \cap\{\mathbb{S}^+_u \mid \mathbb{S}^+_u$ supports $C\}$.*

Proof. Denote the set in the right-handed side of the equality by D. Then $C \subset D$ obviously.

Conversely, let $x \in \mathbb{S}$ and $x \notin C$. We claim that $x \notin D$: first we immediately have $x \notin \mathrm{cone}\, C \cap \mathbb{S} (= \overline{\mathrm{cone}}C \cap \mathbb{S})$; then if $x \in (\mathrm{cone}\, C)^* \cap \mathbb{S}$, where $(\mathrm{cone}\, C)^*$ denotes the polar cone of $\mathrm{cone}\, C$, then $x \notin D$ clearly since $\langle y, z \rangle \leqslant 0$ for all $y \in \mathrm{cone}\, C$, $z \in (\mathrm{cone}\, C)^*$; if $x \notin (\mathrm{cone}\, C)^* \cap \mathbb{S}$, then let nonzero $x^* := P(x)$, where P denotes the project from $\mathbb{R}^n$ to $\mathrm{cone}\, C$, and further $u^* := \frac{x^* - x}{\|x^* - x\|}$. By Proposition 3.2.3 in [11], H_{u^*} supports $\mathrm{cone}\, C$ at x^*, which implies that $\mathbb{S}^+_{u^*} := \overline{H}^+_{u^*} \cap \mathbb{S}$ supports C at $\rho(x^*)$. Since $x \in H^-_{u^*}$, $x \notin \mathbb{S}^+_{u^*}$ and so $x \notin D$. □

Next, we consider the Φ-spherically convex hull of a set in $\mathbb{S}$, i.e. the smallest s-convex set containing a set. Observe first that not every subset has its spherically convex hull: by Theorem 3.1, if $C \subset \mathbb{S}$ is s-convex, then $o \notin \mathrm{co}C$, which implies that $o \notin \mathrm{co}S$ is necessary for $S \subset \mathbb{S}$ to have its spherically convex hull.

We call a nonempty set $S \subset \mathbb{S}$ hull-addible if $o \notin \mathrm{co}S$ and give the following definition.

Definition 3.7. Given a hull-addible $S \subset \mathbb{S}_\Phi$, we define its Φ-*spherically convex hull* (*s-convex hull* for brevity) $\mathrm{sco}S$ by

$$\mathrm{sco}S := \rho(\mathrm{co}S) = \{(s)\sum_{i=1}^{k}\lambda_i x_i \mid x_i \in S,\ \lambda_i \geqslant 0,\ \sum_{i=1}^{k}\lambda_i = 1,\ k = 1, 2, \cdots\}.$$

Theorem 3.8. *Let $S \subset \mathbb{S}$ be hull-addible. Then*

(i) $\mathrm{sco}S$ *is s-convex.*

(ii) $\mathrm{sco}S = \cap\{C \subset \mathbb{S} \mid C \supset S$ *and* C *is s-convex*$\}$, *i.e.* $\mathrm{sco}S$ *is the smallest s-convex set containing* S.

(iii) $\mathrm{sco}S = \mathrm{cone}S \cap \mathbb{S}$.

Proof. (i) Let $x := (s)\sum_{i=1}^{k}\lambda_i x_i,\ y := (s)\sum_{j=1}^{l}\mu_i y_j \in \mathrm{sco}S$ and $0 < \lambda < 1$. Then, writing $a := \Phi\big(\sum_{i=1}^{k}\lambda_i x_i\big),\ b := \Phi\big(\sum_{j=1}^{l}\mu_i y_j\big)$, we have by the properties of ρ

$$\begin{aligned}
\lambda x +_s (1-\lambda)y = \rho\big(\lambda x + (1-\lambda)y\big) &= \rho\left(\frac{\lambda}{a}\sum_{i=1}^{k}\lambda_i x_i + \frac{(1-\lambda)}{b}\sum_{j=1}^{l}\mu_i y_j\right) \\
&= \rho\left(\lambda b\sum_{i=1}^{k}\lambda_i x_i + (1-\lambda)a\sum_{j=1}^{l}\mu_i y_j\right) \\
&= \rho\left(\sum_{i=1}^{k}\frac{\lambda b\lambda_i}{\lambda b + (1-\lambda)a}x_i + \sum_{j=1}^{l}\frac{(1-\lambda)a\mu_i}{\lambda b + (1-\lambda)a}y_j\right) \\
&= (s)\left(\sum_{i=1}^{k}\frac{\lambda b\lambda_i}{\lambda b + (1-\lambda)a}x_i + \sum_{j=1}^{l}\frac{(1-\lambda)a\mu_i}{\lambda b + (1-\lambda)a}y_j\right) \in \mathrm{sco}S
\end{aligned}$$

where we used the fact that $\sum_{i=1}^{k}\frac{\lambda b\lambda_i}{\lambda b+(1-\lambda)a} + \sum_{j=1}^{l}\frac{(1-\lambda)a\mu_i}{\lambda b+(1-\lambda)a} = 1,\ \frac{\lambda b\lambda_i}{\lambda b+(1-\lambda)a} \geqslant 0$, and, finally, $\frac{(1-\lambda)a\mu_i}{\lambda b+(1-\lambda)a} \geqslant 0$.

(ii) Denote $D := \cap\{C \subset \mathbb{S} \mid C \supset S$ and C is s-convex$\}$. Then, $D \subset \mathrm{sco}S$ clearly since the s-convex set $\mathrm{sco}S \supset S$. Conversely, by Theorem 3.1 $\mathrm{sco}S \subset C$ for each s-convex $C \supset S$, and so we have $\mathrm{sco}S \subset D$. Therefore $\mathrm{sco}S = D$.

(iii) If $x \in \mathrm{sco}S$, then $x = (\Phi(\sum_{i=1}^{k}\lambda_i x_i))^{-1}\sum_{i=1}^{k}\lambda_i x_i$ for some $x_1, x_2, \cdots, x_k \in S$ and $\lambda_1, \lambda_2, \cdots, \lambda_k \in [0,1]$ with $\sum_{i=1}^{k}\lambda_i = 1$. So, $x \in \big(\cup_{t\geqslant 0}\, t\mathrm{co}S\big) \cap \mathbb{S} = \mathrm{cone}S \cap \mathbb{S}$. Hence $\mathrm{sco}S \subset \mathrm{cone}S \cap \mathbb{S}$. Conversely, if $x \in \mathrm{cone}S \cap \mathbb{S}$, then $x = \sum_{i=1}^{k}\lambda_i x_i$ with $x_i \in S$, $\lambda_i \geqslant 0$ and $\Phi(x) = 1$ (so $\sum_{i=1}^{k}\lambda_i \neq 0$). Thus, we have

$$x = \rho(x) = \rho\Big(\frac{x}{\sum_{j=1}^{k}\lambda_j}\Big) = (s)\sum_{i=1}^{k}\frac{\lambda_i}{\sum_{j=1}^{k}\lambda_j}x_i \in \mathrm{sco}S.$$

Hence $\mathrm{cone}S \cap \mathbb{S} \subset \mathrm{sco}S$ and in turn $\mathrm{sco}S = \mathrm{cone}S \cap \mathbb{S}$. □

Remark 3.9. From Theorem 3.8, we see that there is a smallest s-convex set containing $S \subset \mathbb{S}$ iff S is hull-addible, and that $S \subset \mathbb{S}$ is s-convex iff $S = \mathrm{sco}S$.

In the following, we consider the closed s-convex hull of a set in $\mathbb{S}$, i.e. the smallest closed s-convex set containing a set. Observing that an open Φ-hemisphere is s-convex but its closure, a closed Φ-hemisphere, is not s-convex, we see that the condition "hull-addible" does not suffice for the existence of closed s-convex hull of a set. In fact, we have the following proposition.

Proposition 3.10. *Let Φ be convex, $\mathbb{S}$ be the Φ-sphere and $S \subset \mathbb{S}$. Then there is a closed s-convex set containing S iff S is contained in a Φ-cap.*

Proof. Suppose there is a closed s-convex set C containing S. Then by Theorem 3.2 there are $u \in \mathbb{S}^{n-1}$ and $\alpha > 0$ such that $C \subset \overline{H}^+_{u,\alpha} \cap \mathbb{S}$ and so S is contained in the Φ-cap $\overline{H}^+_{u,\alpha} \cap \mathbb{S}$.

Conversely, if $S \subset \overline{H}^+_{u,\alpha} \cap \mathbb{S}$ for some $u \in \mathbb{S}^{n-1}$ and $\alpha > 0$, then S is hull-addible since $\mathrm{co}S \subset \overline{H}^+_{u,\alpha}$ clearly and so $\mathrm{sco}S$ exists by Theorem 3.8 and $\mathrm{sco}S \subset \overline{H}^+_{u,\alpha} \cap \mathbb{S}$. Clearly $S \subset \mathrm{cl}(\mathrm{sco}S)$. Now, we show that $\mathrm{cl}(\mathrm{sco}S)$ is s-convex.

Notice that $\mathrm{cl}\,(\mathrm{sco}S) \subset \overline{H}^+_{u,\alpha} \cap \mathbb{S}$ since the later is closed. Let $x, y \in \mathrm{cl}\,(\mathrm{sco}S)$, then $x = \lim x_k, y = \lim y_k$ for some sequences $\{x_k\}, \{y_k\} \subset \mathrm{sco}S$. Observing that for any $\lambda \in [0,1]$, $\lim_{k\to\infty}(\lambda x_k + (1-\lambda)y_k) = \lambda x + (1-\lambda)y \neq o$ by the closednes of $\overline{H}^+_{u,\alpha}$, we have by the continuity of ρ at nonzero points,

$$\begin{aligned}\lambda x +_s (1-\lambda)y = \rho\big(\lambda x + (1-\lambda)y\big) &= \lim \rho\big(\lambda x_k + (1-\lambda)y_k\big)\\ &= \lim \big(\lambda x_k +_s (1-\lambda)y_k\big) \in \mathrm{cl}(\mathrm{sco}S).\end{aligned}$$

Thus we showed that $\mathrm{cl}\,(\mathrm{sco}S)$ is a closed s-convex set containing S. □

Remark 3.11. With a little more complicated argument, one may show that Proposition 3.10 still holds for general s-generating function Φ if "Φ-cap" is replaced by "$\overline{H}^+_{u,\alpha} \cap \mathbb{S}$" for some $u \in \mathbb{S}^{n-1}$ and $\alpha > 0$ (noticing that Φ-cap is defined only when Φ is convex, and $\overline{H}^+_{u,\alpha} \cap \mathbb{S}$ may not be s-convex if Φ is not convex).

From the argument for Proposition 3.10, we see that if Φ is convex and $S \subset \overline{H}^+_{u,\alpha} \cap \mathbb{S}_\Phi$ for some $u \in \mathbb{S}^{n-1}$ and $\alpha > 0$, then $\mathrm{cl}\,(\mathrm{sco}S)$ is a closed and s-convex set. We denote $\overline{\mathrm{sco}}S := \mathrm{cl}\,(\mathrm{sco}S)$, called the closed s-convex hull of S. The following theorem shows that $\overline{\mathrm{sco}}S$ is indeed the smallest closed s-convex set containing S.

Theorem 3.12. *Let Φ be convex and $S \subset \overline{H}^+_{u,\alpha} \cap \mathbb{S}_\Phi$ for some $u \in \mathbb{S}^{n-1}$ and $0 < \alpha \leqslant \sup_{x\in\mathbb{S}}\langle x, u\rangle$. Then*

$$\overline{\mathrm{sco}}S = \cap\{C \subset \mathbb{S}_\Phi \mid C \supset S \text{ and } C \text{ is closed and s-convex}\}.$$

Proof. The argument is similar to that for *ii*) of Theorem 3.8. □

4. Combinatorial properties of S-convex Sets

In this section we discuss the combinatorial properties of s-convex sets, more precisely, we will show that the Radon-type, Helly-type and the Carathéodory-type theorems hold for s-convex sets.

Theorem 4.1. (Radon-type Theorem) *Let* $\mathbb{S}$ *be a* Φ*-sphere and* $S \subset \mathbb{S}$ *with at least* $n{+}1$ *points and* $o \notin \mathrm{co}S$*. Then there are* $S_1, S_2 \subset S$ *such that* $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \emptyset$ *and* $\mathrm{sco}S_1 \cap \mathrm{sco}S_2 \neq \emptyset$.

Proof We point out first that, by Theorem 3.8, for any $S^* \subset S$, $\mathrm{sco}S^*$ exists and $\mathrm{sco}S^* = \mathrm{cone}S^* \cap \mathbb{S} = \mathrm{cone}(\mathrm{co}S^*) \cap \mathbb{S}$.

Then, since there are at least $n{+}2$ points in $S \cup \{o\}$, by the classical Radon's Theorem, there are two disjoint subsets $S_1, S_2 \subset S$ such that $S_1 \cup S_2 = S$, $S_1 \cap (S_2 \cup \{o\}) = \emptyset$ (so naturally $S_1 \cap S_2 = \emptyset$) and $\mathrm{co}S_1 \cap \mathrm{co}\big(S_2 \cup \{o\}\big) \neq \emptyset$ (where $S_2 \neq \emptyset$ since $o \notin \mathrm{co}S_1$).

Finally, observing that $\mathrm{sco}S_1 = \mathrm{cone}\big(\mathrm{co}S_1\big) \cap \mathbb{S}$,

$$\mathrm{sco}S_2 = \mathrm{cone}\big(\mathrm{co}S_2\big) \cap \mathbb{S} = \mathrm{cone}\big(\mathrm{co}(S_2 \cup \{o\})\big) \cap \mathbb{S}$$

and $\mathrm{co}S_1 \cap \mathrm{co}S_2$ contains nonzero element (for $o \notin \mathrm{co}S_1$), we have

$$\begin{aligned} \mathrm{sco}S_1 \cap \mathrm{sco}S_2 &= \mathrm{cone}\big(\mathrm{co}S_1\big) \cap \mathrm{cone}\big(\mathrm{co}(S_2 \cup \{o\})\big) \cap \mathbb{S} \\ &\supset \mathrm{cone}\big(\mathrm{co}S_1 \cap \big(\mathrm{co}(S_2 \cup \{o\})\big) \cap \mathbb{S} \neq \emptyset. \end{aligned}$$

Remark 4.2. One may easily find examples to show that the number $n+1$ in Theorem 4.1 is sharp in general. Also the condition $o \notin \mathrm{co}S$ can not be omitted in general as shown by the example: on $\mathbb{R}^3$, let $\Phi(\cdot) = \|\cdot\|$ and consider the Φ-sphere $\mathbb{S}^2$. It is easy to check that the conclusion of Theorem 4.1 is not valid for the set $S = \{(-1,0,0),(1,0,0),(0,-1,0),(0,1,0)\} \subset \mathbb{S}^2$ (observing $o \in \mathrm{co}S$).

The following theorem is already known for $\mathbb{S}^{n-1}$ (with so many different proofs, see, e.g., [2, 5, 12, 14, 17, 18, 19], etc.) and can be derived from the one for $\mathbb{S}^{n-1}$ (*cf.* ii) in Remark 2.6). However, we prefer to give an argument for the general case $\mathbb{S}_\Phi$ with the help of Theorem 4.1 since the argument here might have some further applications.

Theorem 4.3. (Helly-type theorem) *Let* $\mathbb{S}$ *be a* Φ*-sphere and* $C_1, C_2, \cdots, C_m \subset \mathbb{S}$, $m \geqslant n+1$*, be a family of s-convex sets. If*

$$\begin{aligned} &C_{i_1} \cap C_{i_2} \cap \cdots \cap C_{i_n} \neq \emptyset &&\textit{for any } \{i_1, i_2, \cdots, i_n\} \subset \{1,2,\cdots,m\} \\ &C_{j_1} \cup C_{j_2} \cup \cdots \cup C_{j_{n+1}} \subsetneq \mathbb{S} &&\textit{for any } \{j_1, j_2, \cdots, j_{n+1}\} \subset \{1,2,\cdots,m\} \end{aligned}$$

then $\cap_{i=1}^m C_i \neq \emptyset$.

In order to prove Theorem 4.3, we present first the following lemma.

Lemma 4.4. *Let* $\mathbb{S}$ *be a* Φ*-sphere and* $C_1, C_2, \cdots, C_m \subset \mathbb{S}$, $m \geqslant n{+}1$*, be a family of s-convex sets. If* $C_{i_1} \cap C_{i_2} \cap \cdots \cap C_{i_{n+1}} \neq \emptyset$ *for any* $\{i_1, i_2, \cdots, i_{n+1}\} \subset \{1,2,\cdots,m\}$*, then* $\cap_{i=1}^m C_i \neq \emptyset$.

Proof. We will argue by induction on $m (\geqslant n{+}1)$ with a "standard trick" – applying a Radon-type theorem on a suitable set (see page 4 in [20]):

The conclusion holds for $m = n+1$ trivially. Suppose that the conclusion holds for m $(\geqslant n+1)$. Then for s-convex sets $C_1, C_2, \cdots, C_{m+1} \subset S$ with the desired properties as in the lemma, by the induction hypothesis, we have m nonempty sets

$$C_{(i)} := C_1 \cap \cdots \cap \check{C}_i \cap \cdots \cap C_{m+1}, \ i = 1,2,\cdots,m$$

where $\check{C}_i$ means "omiting C_i in $\cap_{j=1}^{m+1} C_j$". Choose $x_i \in C_{(i)}$, $i = 1, 2, \cdots, m$. Clearly all $x_i \in C_{m+1}$. If $x_i = x_j$ for some $i \neq j$, then it is done. If $x_1, x_2, \cdots, x_m$ are distinct, then by Theorem 4.1 (observing $m \geqslant n+1$ and $o \notin \mathrm{co}\{x_i\}_{i=1}^m$ since $\{x_i\}_{i=1}^m \subset C_{m+1}$), there are two disjoint nonempty sets, say, $S_1 := \{x_1, x_2, \cdots, x_k\}$ and $S_2 := \{x_{k+1}, x_{k+2}, ..., x_m\}$ such that $\mathrm{sco}S_1 \cap \mathrm{sco}S_2 \neq \emptyset$. Choose $x^* \in \mathrm{sco}S_1 \cap \mathrm{sco}S_2$. Then, since $x^* \in \mathrm{sco}S_1$ implies $x^* \in \cap_{i=k+1}^{m+1} C_i$ (for $x_1, \cdots, x_k \in \cap_{i=k+1}^{m+1} C_i$) and $x^* \in \mathrm{sco}S_2$ implies $x^* \in C_{m+1} \cap_{i=1}^k C_i$ (for $x_{k+1}, \cdots, x_m \in C_{m+1} \cap_{i=1}^k C_i$), we have $x^* \in \cap_{i=1}^{m+1} C_i$. □

Proof of Theorem 4.3. By Lemma 4.4, we need only to show that the conclusion holds for $m = n+1$.

For $n+1$ s-convex sets $C_1, C_2, \cdots, C_{n+1}$ with the desired properties as in the theorem, we choose $x_i \in C_{(i)} := C_1 \cap \cdots \cap \check{C}_i \cap \cdots \cap C_{n+1}$, $i = 1, 2, \cdots, n$. Then all x_i are in C_{n+1} and so $o \notin \mathrm{co}\{x_i\}_{i=1}^n$. If $x_i = x_j$ for some $i \neq j$, then it is done. So, we suppose that $x_1, x_2, \cdots, x_n$ are distinct.

Now, let $H := \mathrm{lin}\{x_i\}_{i=1}^n$. If $\dim H < n$, then applying Theorem 4.1 to the set $\{x_i\}_{i=1}^n$ in the Φ-sphere $H \cap \mathbb{S}$, we may find some $x^* \in \cap_{i=1}^{n+1} C_i$ by the "standard trick" used in the proof of Lemma 4.4.

If $\dim H = n$, then $\mathrm{co}\{o, x_1, \cdots, x_n\}$ is an n-dimensional simplex. Now define $A_i = \{x_1, \cdots, \check{x}_i, \cdots, x_n\}$ for each $1 \leqslant i \leqslant n$, where $\check{x}_i$ means "without x_i", then $\mathrm{lin}A_i$ is a hyperplane passing through the origin o. Thus $\mathrm{lin}A_i = H_{u_i}$ for some $u_i \in \mathbb{S}^{n-1}$ where u_i is chosen so that $x_i \in H_{u_i}^+$ (and so $\{x_1, x_2, \cdots, x_n\} \subset \cap_{i=1}^n \overline{H}_{u_i}^+$).

Now, we choose $x_{n+1} \in \cap_{i=1}^n C_i$ and claim that $x_{n+1} \in H_{u_{i_0}}^+$ for some $1 \leqslant i_0 \leqslant n$ as well. Suppose not, then $x_{n+1} \in \cap_{i=1}^n \overline{H}_{u_i}^-$. However, $x_{n+1} \notin H_{u_i}$ for each $1 \leqslant i \leqslant n$ since otherwise we would have $x_{n+1} \in \big(\mathrm{sco}A_i\big)^o \subset C_i^o$, a contradiction.

Therefore we have $x_{n+1} \in \cap_{i=1}^n H_{u_i}^-$ and so $x_{n+1} \in \mathrm{int}\big(\mathrm{sco}\{x_i\}_{i=1}^n\big)^o$ which implies $x_{n+1}^o \in \mathrm{int}\big(\mathrm{sco}\{x_i\}_{i=1}^n\big)$. Thus, for any hemi-greatcircle L joining x_{n+1} and x_{n+1}^o, since L intersects $\mathrm{sco}A_i$ at $y_i \in \mathbb{S}$ for some $1 \leqslant i \leqslant n$, we would have

$$L = \mathrm{sco}\{x_{n+1}^o, y_i\} \cup \mathrm{sco}\{y_i, x_{n+1}\} \subset C_i \cup C_{n+1}$$

since $x_{n+1}^o \in C_{n+1}, x_{n+1} \in C_i$ and $y_i \in C_{n+1} \cap \mathrm{sco}A_i \subset C_{n+1} \cap C_i$, which implies $\mathbb{S} = \cup_{i=1}^{n+1} C_i$ by the obvious fact that for any $x \in \mathbb{S}$ there is a hemi-greatcircle joining x_{n+1} and x_{n+1}^o, and passing through x, a contradiction too. The claim is confirmed.

Again, applying Theorem 4.1 to the set $\{x_i\}_{i=1}^{n+1}$ (observing that $o \notin \mathrm{co}\{x_i\}_{i=1}^{n+1}$ since $o \notin \mathrm{co}\{x_i\}_{i=1}^n \subset \overline{H}_{u_{i_0}}^+$ and $x_{n+1} \in H_{u_{i_0}}^+$), we may find $x^* \in \cap_{i=1}^{n+1} C_i$ by the "standard trick" used in the proof of Lemma 4.4. □

Remark 4.5. One may easily find examples to show that the finiteness of the family of s-convex sets cannot be omitted in general.

As for the condition "$\cup_{k=1}^{n+1} C_{j_k} \subsetneq \mathbb{S}$", it is compulsive: for any $x \in \cap_{i=1}^l C_i$, its antipodal point $x^\circ \notin \cup_{i=1}^l C_i$ since each C_i contains no antipodal points.

As a consequence of Theorem 4.3, we present the following corollary which is equivalent to Molnár's result for $\mathbb{S}^{n-1}$ ([17]) in the light of (ii) of Remark 2.6.

Corollary 4.6. *Let $\mathbb{S}$ be a Φ-sphere and $\{C_i\}_{i\in I}$ be a (not necessarily finite) family of closed s-convex sets in $\mathbb{S}$. If $\cap_{k=1}^n C_{i_k} \neq \emptyset$, $\cup_{k=1}^{n+1} C_{j_k} \subsetneqq \mathbb{S}$, for any $\{i_k\}_{k=1}^n, \{j_k\}_{k=1}^{n+1} \subset I$, then $\cap_{i\in I} C_i \neq \emptyset$.*

Proof. By Theorem 4.3, we have $\cap_{j=1}^m C_{i_j} \neq \emptyset$ for any positive integer m and $i_1, i_2, \cdots, i_m \in I$. Thus, observing that all C_i are compact, we have $\cap_{i\in I} C_i \neq \emptyset$ by the finite-intersection property of families of compact sets. □

The next corollary is obvious since for any l, $\cap_{i=1}^l C_i \neq \emptyset$ implies $\cup_{i=1}^l C_i \subsetneqq \mathbb{S}$ when each C_i contains no antipodal points.

Corollary 4.7. *Let $\mathbb{S}$ be a Φ-sphere and $\{C_i\}_{i\in I}$, where $I = \{1, 2, \cdots, m\}$ and $m \geqslant n+1$, be a family of s-convex sets with the property that $\cap_{k=1}^n C_{i_k} \neq \emptyset$ for any $\{i_k\}_{k=1}^n \subset I$. If $\cup_{k=1}^{n+1} C_{j_k} \subsetneqq \mathbb{S}$ for any $\{j_k\}_{k=1}^{n+1} \subset I$, then $\cup_{i=1}^m C_i \subsetneqq \mathbb{S}$. Moreover, if all C_i are closed, then I can be any index set.*

At the end of this section, we present a Carathéodory-type theorem for s-convex hulls.

Theorem 4.8. (Carathéodory-type Theorem) *Let $\mathbb{S}$ be a Φ-sphere and $S \subset \mathbb{S}$ be hull-addible. Then for each $x \in \mathrm{sco}S$, there exist $x_1, x_2, \cdots, x_n \in S$ and $\lambda_1, \lambda_2, \cdots, \lambda_n \in [0,1]$ with $\sum_{i=1}^n \lambda_i = 1$, such that $x = (s)\sum_{i=1}^n \lambda_i x_i$, i.e. x can be expressed as an s-convex combination of at most n points in S.*

Proof. Let $x \in \mathrm{sco}S$. Then $x = (s)\sum_{i=1}^m \lambda_i' x_i$ for some $x_1, x_2, \cdots, x_m \in S$ and $\lambda_1', \lambda_2', \cdots, \lambda_m' \in [0,1]$ with $\sum_{i=1}^m \lambda_i' = 1$. If $m \leqslant n$, there is nothing to prove. So we assume $m > n$.

Since $\mathrm{co}\{x_i\}_{i=1}^m$ is compact and $\mathrm{cone}\{x_i\}_{i=1}^m$ is closed and pointed, there is a hyperplane $H_{u,\alpha}$ ($\alpha > 0$) such that $\mathrm{co}\{x_i\}_{i=1}^m \subset H_{u,\alpha}^+$. Denote $S^* := \mathrm{cone}\{x_i\}_{i=1}^m \cap H_{u,\alpha}$. Then S^* is closed and convex. Moreover, S^* is a polytope:

$$S^* = \mathrm{co}\{y_i = t_i x_i \mid t_i > 0 \text{ such that } t_i x_i \in H_{u,\alpha}\} =: P.$$

Indeed, $P \subset S^*$ is clear since S^* is convex and all $y_i \in S^*$. Conversely, if $z \in S^*$, then $\langle u, z\rangle = \alpha$ and $z = \sum_{i=1}^m s_i x_i$ $(= \sum_{i=1}^m \frac{s_i}{t_i} y_i)$ for some nonnegative s_i. Thus, since $\langle u, y_i\rangle = \alpha$ for each i, we obtain

$$\alpha = \langle u, z\rangle = \sum_{i=1}^m \frac{s_i}{t_i}\langle u, y_i\rangle = \alpha \sum_{i=1}^m \frac{s_i}{t_i}$$

which leads to $\sum_{i=1}^m \frac{s_i}{t_i} = 1$ and in turn $z = \sum_{i=1}^m \frac{s_i}{t_i} y_i \in P$.

Choose (the unique) $x^* \in S^*$ such that $x = \rho(x^*)$. By the classical Carathéodory's theorem and Minkowski's structure theorem for closed convex sets, $x^* = \sum_{i=1}^n \mu_i y_{j_i}$ for some $\mu_i \in [0,1]$ with $\sum_{i=1}^n \mu_i = 1$ (noticing $\dim H_{u,\alpha} = n-1$). Thus, since $y_{j_i} = t_{j_i} x_{j_i}$ for some $x_{j_i} \in \{x_i\}_{i=1}^m$ and $t_{j_i} > 0$, we have by the homogeneity of ρ

$$x = \rho(x^*) = \rho\Big(\sum_{i=1}^n \mu_i y_{j_i}\Big) = \rho\Big(\sum_{i=1}^n \mu_i t_{j_i} x_{j_i}\Big) = \rho\Big(\sum_{i=1}^n \lambda_i x_{j_i}\Big) = (s)\sum_{i=1}^n \lambda_i x_{j_i}$$

where $\lambda_i := \frac{\mu_i t_{j_i}}{\sum_{k=1}^n \mu_k t_{j_k}} \geqslant 0$ and $\sum_{i=1}^n \frac{\mu_i t_{j_i}}{\sum_{k=1}^n \mu_k t_{j_k}} = 1$. □

5. Structural properties of S-convex sets

In this section, we explore the structure of a s-convex set. More precisely, we establish a theorem of Minkowski type which states that a closed s-convex set can be expressed as the s-convex hull of its s-extreme points (see below for definition). We start with some necessary notation and terms.

For an s-convex $C \subset \mathbb{S}^{n-1}$, we denote $\dim C := \dim(\mathrm{cone}(C))$ (notice that $\mathrm{cone}(C)$ is convex), called the dimension of C.

Definition 5.1. Let $\mathbb{S}$ be a Φ-sphere and $C \subset \mathbb{S}$ be a closed and s-convex set. A point $x \in C$ is called an s-extreme point of C if $x = \frac{1}{2}x_1 +_s \frac{1}{2}x_2$ for some $x_1, x_2 \in C$, then $x_1 = x_2$. The set of s-extreme points of C is denoted by $\mathrm{sext}C$.

Remark 5.2. (i) It is easy to check that if x is an s-extreme point of C, then $x \in \mathrm{rbd}C$.

(ii) In Definition 5.1, the condition "$x = \frac{1}{2}x_1 +_s \frac{1}{2}x_2$" can be replaced by "$x = \lambda x_1 +_s (1-\lambda)x_2$ for some $0 < \lambda < 1$".

Proposition 5.3. *Let $\mathbb{S}$ be a Φ-sphere and $C \subset \mathbb{S}$ be closed and s-convex. Then $x \in C$ is an s-extreme point of C iff $\{x\} = L \cap C$ for some extreme ray L of* $\mathrm{cone}\, C$. *Consequently, if L is an extreme ray of* $\mathrm{cone}\, C$, *then $L = \{tx \mid t \geqslant 0\}$ for some s-extreme point x of C.*

Proof. Observe first that $\mathrm{cone}\, C$ is closed by Proposition 2.1.

Suppose $\{x\} = L \cap C$ for some extreme ray L of $\mathrm{cone}\, C$, then $L = \{tx \mid t \geqslant 0\}$. Now, if $x = \frac{1}{2}x_1 +_s \frac{1}{2}x_2 = \frac{\frac{1}{2}x_1 + \frac{1}{2}x_2}{\Phi(\frac{1}{2}x_1 + \frac{1}{2}x_2)}$ for some $x_1, x_2 \in C$, then $\frac{1}{2}x_1 + \frac{1}{2}x_2 = \Phi(\frac{1}{2}x_1 + \frac{1}{2}x_2)x \in L$. So $x_1, x_2 \in L$ (since L is an extreme ray) and in turn $x_1, x_2 \in L \cap C$ which leads to $x_1 = x_2 = x$ clearly. Thus, x is an s-extreme point.

Conversely, let x be an s-extreme point of C. Denote $L := \{tx \mid t \geqslant 0\}$, which is a ray and $L \cap C = \{x\}$. Now, we show that L is an extreme ray of $\mathrm{cone}\, C$: since $L \ni o = \frac{1}{2}x_1 + \frac{1}{2}x_2$ for some $x_1, x_2 \in \mathrm{cone}\, C$ leads to $x_1 = x_2 = o \in L$ obviously, we consider only nonzero $tx \in L$. If $tx = \frac{1}{2}x_1 + \frac{1}{2}x_2$ for some $x_1, x_2 \in \mathrm{cone}\, C$, then we have (here we assume that both x_1 and x_2 are nonzero for x_1(or x_2) $= o \in L$ implying x_2(or x_1) $= 2tx \in L$).

$$\begin{aligned} x = \rho(tx) = \rho(\frac{1}{2}x_1 + \frac{1}{2}x_2) &= \rho(x_1 + x_2) \\ &= \rho(\frac{\Phi(x_1)}{\Phi(x_1) + \Phi(x_2)}\frac{x_1}{\Phi(x_1)} + \frac{\Phi(x_2)}{\Phi(x_1) + \Phi(x_2)}\frac{x_2}{\Phi(x_2)}) \\ &= \frac{\Phi(x_1)}{\Phi(x_1) + \Phi(x_2)}\frac{x_1}{\Phi(x_1)} +_s \frac{\Phi(x_2)}{\Phi(x_1) + \Phi(x_2)}\frac{x_2}{\Phi(x_2)} \end{aligned}$$

which leads to $\frac{x_1}{\Phi(x_1)} = \frac{x_2}{\Phi(x_2)} = x$ (notice that $\frac{x_1}{\Phi(x_1)}, \frac{x_2}{\Phi(x_2)} \in C$). Thus we have $x_1 = \Phi(x_1)x$, $x_2 = \Phi(x_2)x \in L$. So L is an extreme ray of $\mathrm{cone}\, C$.

Now, if L is an extreme ray of $\mathrm{cone}\, C$, then letting $\{x\} = L \cap C$, we have that x is an s-extreme point of C by what just proved above and $L = \{rx \mid r \geqslant 0\}$. $\square$

Corollary 5.4. *If $C \subset \mathbb{S}_\Phi$ is a closed and s-convex set, then* $\mathrm{sext}C \neq \emptyset$.

Proof. This follows from Proposition 5.3 and the known fact that a pointed closed convex cone has at least one extreme ray (Theorem 1.3.9 in [20]). $\square$

Corollary 5.5. *Let $C \subset \mathbb{S}_\Phi$ be a closed s-convex set and $x \in \mathrm{sext}C$. If we have $x = (s)\sum_{i=1}^k \lambda_i x_i$ for some $x_1, x_2, \cdots, x_k \in C$ and $\lambda_1, \lambda_2, \cdots, \lambda_k \in (0,1)$ with $\sum_{i=1}^k \lambda_i = 1$, then $x_1 = x_2 = \cdots = x_k = x$.*

Proof. By Proposition 5.3, $L := \{tx \mid t \geqslant 0\}$ is an extreme ray of $\mathrm{cone}C$. Observing that $x = (s)\sum_{i=1}^k \lambda_i x_i$ is equivalent to $\sum_{i=1}^k \lambda_i x_i = \Phi(\sum_{i=1}^k \lambda_i x_i)x$, we have $\sum_{i=1}^k \lambda_i x_i \in L$. Therefore $x_1, x_2, \cdots, x_k \in L$ since L is an extreme ray and further $x_1 = x_2 = \cdots = x_k = x$ since $C \cap L$ is the singleton $\{x\}$. $\square$

The following is a theorem of Minkowski type for closed s-convex sets in $\mathbb{S}$.

Theorem 5.6. *Let $C \subset \mathbb{S}_\Phi$ be a closed and s-convex set. Then $C = \mathrm{sco}\,(\mathrm{sext}C)$.*

Proof. Since $\mathrm{cone}\,C$ is closed and line-free by Proposition 2.7, we have $\mathrm{cone}\,C = \mathrm{co}(\mathrm{extr}(\mathrm{cone}\,C))$ by Theorem 1.4.3 in [20], where $\mathrm{extr}(\mathrm{cone}\,C)$ denotes the union of extreme rays of $\mathrm{cone}\,C$. Thus, if $x \in C \subset \mathrm{cone}\,C$, then $x = \sum_{i=1}^m \lambda_i x_i$ with non-zero $x_i \in L_i$ for some extreme rays L_i and $\lambda_i > 0$. Hence, we have,

$$\begin{aligned}
x = \rho(x) = \rho(\sum_{i=1}^m \lambda_i x_i) &= \rho(\sum_{i=1}^m \lambda_i \Phi(x_i) \frac{x_i}{\Phi(x_i)}) \\
&= \rho(\sum_{i=1}^m \frac{\lambda_i \Phi(x_i)}{\sum_{j=1}^m \lambda_j \Phi(x_j)} \frac{x_i}{\Phi(x_i)}) \\
&= (s)(\sum_{i=1}^m \frac{\lambda_i \Phi(x_i)}{\sum_{j=1}^m \lambda_j \Phi(x_j)} \frac{x_i}{\Phi(x_i)}) \in \mathrm{sco}\,(\mathrm{sext}C)
\end{aligned}$$

where we used the fact that $\left\{\frac{x_i}{\Phi(x_i)}\right\} = L_i \cap C$ which leads to $\frac{x_i}{\Phi(x_i)} \in \mathrm{sext}(C)$ by Proposition 5.3, and $\sum_{i=1}^m \frac{\lambda_i \Phi(x_i)}{\sum_{j=1}^m \lambda_j \Phi(x_j)} = 1$. $\square$

6. S-convex Functions on S-convex Sets

As an application of s-convexity of sets in $\mathbb{S}_\Phi$, in this section, we study the Φ-spherically convex functions on s-convex sets.

Definition 6.1. Let $C \subset \mathbb{S}_\Phi$ be an s-convex set. A function $f : C \to \mathbb{R}$ is called Φ-*spherically convex* (*s-convex* for brevity) if $f\big(\lambda x +_s (1-\lambda)y\big) \leqslant \lambda f(x) + (1-\lambda) f(y)$ hold for any $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$. A function $f : C \to \mathbb{R}$ is called Φ-*spherically concave* (*s-concave* for brevity) if $-f$ is s-convex.

Example 6.2. Let Φ be convex and $\mathbb{S}$ be the Φ-sphere. For $u \in \mathbb{S}^{n-1}$, assume $C_1 \subset \overline{H}_u^- \cap \mathbb{S}$ and $C_2 \subset \overline{H}_u^+ \cap \mathbb{S}$ to be s-convex sets. We define a function $f : \mathbb{S} \to \mathbb{R}$ by $f(x) := a\langle u, x\rangle + b,\ x \in \mathbb{S}$, where $a, b \in \mathbb{R}$. Then f is s-convex on C_1 if $a \geqslant 0$ and s-convex on C_2 if $a \leqslant 0$: We only check the case $a \geqslant 0$ since the argument for the case $a \leqslant 0$ is the same. The conclusion is trivial if $a = 0$.

Suppose $a > 0$, then for $x, y \in C_1$ and $0 \leqslant \lambda \leqslant 1$, we have

$$\begin{aligned} f\big(\lambda x +_s (1-\lambda)y\big) &= \frac{a}{\Phi\big(\lambda x + (1-\lambda)y\big)}\langle u, \lambda x + (1-\lambda)y\rangle + b \\ &= \frac{\lambda a\langle u, x\rangle}{\Phi\big(\lambda x + (1-\lambda)y\big)} + \frac{(1-\lambda)a\langle u, y\rangle}{\Phi\big(\lambda x + (1-\lambda)y\big)} + b \\ &\leqslant \lambda a\langle u, x\rangle + (1-\lambda)a\langle u, y\rangle + \lambda b + (1-\lambda)b = \lambda f(x) + (1-\lambda)f(y) \end{aligned}$$

since $\Phi\big(\lambda x + (1-\lambda)y\big) \leqslant 1$ by the convexity of Φ, and $a\langle u, x\rangle \leqslant 0, a\langle u, y\rangle \leqslant 0$. Similarly, the function f is s-concave if $a \geqslant 0$ (resp. $a \leqslant 0$) on C_2 (resp. on C_1).

Also, one may see that the function $\exp\{a\langle u, x\rangle + b\}$ is s-convex on C_1. Generally, one may easily check that the following proposition holds.

Proposition 6.3. *Let $g : \mathbb{R} \to \mathbb{R}$ be a increasing convex (resp. concave) function. If f is s-convex (resp. s-concave) on an s-convex set C, then the function $g \circ f$ is s-convex (resp. s-concave) on C.*

As expected, s-convex functions share many nice properties with convex functions on Euclidean spaces. Here we present few illustrations, starting with a theorem of Jensen's type. From now on, we always assume that Φ is convex.

Theorem 6.4. (Inequality of Jensen) *Let $C \subset \mathbb{S}_\Phi$ be an s-convex set and $f : C \to \mathbb{R}$ be an s-convex function. Then, for any $x_1, x_2, \cdots, x_k \in C$ $(k \geqslant 1)$ and $\lambda_1, \lambda_2, \cdots, \lambda_k \in [0, 1]$ with $\sum_{i=1}^k \lambda_i = 1$, we have*

$$f\big((s)\textstyle\sum_{i=1}^k \lambda_i x_i\big) \leqslant \sum_{i=1}^k \lambda_i f(x_i).$$

To prove Theorem 6.4, the following elementary lemma is needed.

Lemma 6.5. *Let $f(t) = \frac{t}{(1-b)t+b}A + \frac{b}{(1-b)t+b}B$, $0 \leqslant t \leqslant 1$, where $A, B, 0 < b < 1$ are constants. Then $f(t) \leqslant \max\{A + bB, B\}$.*

Proof. An elementary calculation shows that $f'(t) = \frac{b(A+(b-1)B)}{((1-b)t+b)^2}$. So $f(t)$ is increasing if $A + (b-1)B \geqslant 0$ or decreasing if $A + (b-1)B \leqslant 0$, which together with the fact that $f(0) = B$ and $f(1) = A + bB$ leads to the inequality desired. □

Proof of Theorem 6.4. We will argue by induction on k. The inequality holds when $k = 2$ by definition. Suppose it holds for positive integers less than k. Then for $x_1, x_2, \cdots, x_k \in C$ and $\lambda_1, \lambda_2, \cdots, \lambda_k \in (0, 1)$ with $\sum_{i=1}^k \lambda_i = 1$, letting $f(x_k) = \min\{f(x_i)\}_{i=1}^k$ (renumbering if necessary) and denoting $z = \sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k}x_i$ (noticing $z \neq o$ since $\rho(z) = (\mathrm{s})\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k}x_i \in C$), we have by (1) in Section 3 and by the hypothesis of induction (observing $\frac{z}{\Phi(z)} = (\mathrm{s})\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k}x_i$)

$$\begin{aligned} f\big((\mathrm{s})\sum_{i=1}^k \lambda_i x_i\big) &= f\left(\frac{(1-\lambda_k)\Phi(z)}{(1-\lambda_k)\Phi(z)+\lambda_k}\frac{z}{\Phi(z)} +_{\mathrm{s}} \frac{\lambda_k}{(1-\lambda_k)\Phi(z)+\lambda_k}x_k\right) \\ &\leqslant \frac{(1-\lambda_k)\Phi(z)}{(1-\lambda_k)\Phi(z)+\lambda_k} f\left(\frac{z}{\Phi(z)}\right) + \frac{\lambda_k}{(1-\lambda_k)\Phi(z)+\lambda_k} f(x_k) \end{aligned}$$

$$\leqslant \frac{(1-\lambda_k)\Phi(z)}{(1-\lambda_k)\Phi(z)+\lambda_k}\sum_{i=1}^{k-1}\frac{\lambda_i}{1-\lambda_k}f(x_i)+\frac{\lambda_k}{(1-\lambda_k)\Phi(z)+\lambda_k}f(x_k)$$

$$=\frac{\Phi(z)}{(1-\lambda_k)\Phi(z)+\lambda_k}\sum_{i=1}^{k-1}\lambda_i f(x_i)+\frac{\lambda_k}{(1-\lambda_k)\Phi(z)+\lambda_k}f(x_k). \tag{3}$$

Now, applying Lemma 6.5 to (3) with $A = \sum_{i=1}^{k-1}\lambda_i f(x_i), B = f(x_k),\ b = \lambda_k$, $t = \Phi(z)$ (noticing $0 < t \leqslant 1$ by the convexity of Φ), we obtain

$$f\big((\mathrm{s})\textstyle\sum_{i=1}^{k}\lambda_i x_i\big) \leqslant \max\big\{\sum_{i=1}^{k-1}\lambda_i f(x_i)+\lambda_k f(x_k), f(x_k)\big\} = \sum_{i=1}^{k}\lambda_i f(x_i),$$

where we used the fact that $f(x_k) = \sum_{i=1}^{k}\lambda_i f(x_k) \leqslant \sum_{i=1}^{k}\lambda_i f(x_i)$. □

Next, we will show that all s-convex functions are lower bounded. To do this, for a given s-convex function defined on an s-convex set C, we construct first a (unique) convex function defined on co$C$.

Let $C \subset \mathbb{S}_\Phi$ be an s-convex set and $f : C \to \mathbb{R}$ be an s-convex function. Define

$$\widehat{C} := \mathrm{co}\{(x, f(x)) \mid x \in C\},\ \text{a convex set in } \mathbb{R}^n \times \mathbb{R}.$$

Then for each $z \in \mathrm{co}C$, define $l_{\widehat{C}}(z) := \inf\{r \mid (z, r) \in \widehat{C}\}$.

Proposition 6.6. *Let $C, f, l_{\widehat{C}}$ be as above, then*
(i) $l_{\widehat{C}}(z) > -\infty$ *for each* $z \in \mathrm{co}C$*;*
(ii) $l_{\widehat{C}}(x) = f(x)$ *for each* $x \in C$*.*
Therefore $l_{\widehat{C}}$ is a convex function defined on co$C$*.*

Proof. (i) Suppose $l_{\widehat{C}}(z_0) = -\infty$ for some $z_0 \in \mathrm{co}C$. Then by the definition of $l_{\widehat{C}}$ and by the classical Carathéodory Theorem, for each positive integer k, there are $\lambda_1^{(k)}, \lambda_2^{(k)}, \cdots, \lambda_{n+1}^{(k)} \in [0, 1]$, $\sum_{i=1}^{n+1}\lambda_i^{(k)} = 1$ and $x_1^{(k)}, x_2^{(k)}, \cdots, x_{n+1}^{(k)} \in C$, such that

$$z_0 = \textstyle\sum_{i=1}^{n+1}\lambda_i^{(k)}x_i^{(k)} \text{ and } \sum_{i=1}^{n+1}\lambda_i^{(k)}f\big(x_i^{(k)}\big) < -k.$$

Observing that $\rho(z_0) = \rho(\sum_{i=1}^{n+1}\lambda_i^{(k)}x_i^{(k)}) = (s)\sum_{i=1}^{n+1}\lambda_i^{(k)}x_i^{(k)} \in C$ by Theorem 3.1, we have by Theorem 6.4

$$f\big(\rho(z_0)\big) \leqslant \textstyle\sum_{i=1}^{n+1}\lambda_i^{(k)}f\big(x_i^{(k)}\big) < -k \to -\infty \text{ as } k \to \infty,$$

a contradiction.

(ii) Let $x \in C$. Then $l_{\widehat{C}}(x) \leqslant f(x)$ is clear by the definition. Conversely, for $(x, r) \in \widehat{C}$, by the definition of $\widehat{C}$ and the classical Carathéodory theorem, there are $x_1, x_2, \cdots, x_{n+2} \in C$ and $\lambda_1, \lambda_2, \cdots, \lambda_{n+2} \in [0, 1]$ with $\sum_{i=1}^{n+2}\lambda_i = 1$, such that $x = \sum_{i=1}^{n+2}\lambda_i x_i$ and $r = \sum_{i=1}^{n+2}\lambda_i f(x_i)$. Observing $x = \rho(x) = (s)\sum_{i=1}^{n+2}\lambda_i x_i$, we have by Theorem 6.4, $f(x) \leqslant \sum_{i=1}^{n+2}\lambda_i f(x_i) = r$, which leads to $f(x) \leqslant l_{\widehat{C}}(x)$ by the arbitrariness of applicable r.

By (i) above, we see that $l_{\widehat{C}}$ is nothing else but the lower bound function of $\widehat{C}$ (cf. p.85 in [11]) and therefore is convex on co$C$ by Theorem 1.3.1 of [11]. □

Definition 6.7. The convex function $l_{\widehat{C}}$ constructed above is called the *convex expansion* of the s-convex function f, denoted by co$f$.

As a consequence of Proposition 6.6, we have the following corollary, which is the analogy of a result about affine functions for convex functions on $\mathbb{R}^n$..

Corollary 6.8. *Let $C \subset \mathbb{S}_\Phi$ be an s-convex set and $f : C \to \mathbb{R}$ be an s-convex function. Then, for each $x_0 \in \mathrm{ri}\,\mathrm{co}C$ there is $(u, b) \in \mathbb{R}^n \times \mathbb{R}$ such that*

$$f(x) \geqslant \langle u, x - x_0\rangle + b \text{ for all } x \in C.$$

Therefore f is lower bounded, i.e. $\inf_{x\in C} f(x) > -\infty$.

Proof. By Proposition 6.6, the convex expansion co$f$ is convex on co$C$. So, by Proposition 1.2.1 in [11], for any $x_0 \in \mathrm{ri}\,\mathrm{co}C$, there is $u \in \mathbb{R}^n$ such that we have $\mathrm{co}f(x) \geqslant \mathrm{co}f(x_0) + \langle u, x - x_0\rangle$ for all $x \in \mathrm{co}C$, in particular, for any $x \in C$, $f(x) = \mathrm{co}f(x) \geqslant \mathrm{co}f(x_0) + \langle u, x - x_0\rangle$. Letting $b = \mathrm{co}f(x_0)$, we obtain the required inequality.

Now, by the continuity of affine functions and the compactness of $\mathbb{S}$, $\langle u, x - x_0\rangle + b$ is lower bounded on $\mathbb{S}$ and further f is lower bounded on C. □

Remark 6.9. Since an s-convex function f on C is certainly lower bounded, when Φ is convex, one may find an s-convex affine function minorizing f: by Theorem 3.2 there is $v_0 \in \mathbb{R}^n$ ($v_0 = -u_0$ in (i) of Theorem 3.2) such that $C \subset \overline{H}^-_{v_0} \cap \mathbb{S}$. Choosing b such that $b \leqslant \inf_{x\in C} f(x) - \sup_{x\in C}\langle v_0, x\rangle$, we have $f(x) \geqslant \langle v_0, x\rangle + b$ for $x \in C$. By Example 6.2, $\langle v_0, \cdot\rangle + b$ is s-convex on C.

At the end, we give another two analog results to convex functions on $\mathbb{R}^n$.

Theorem 6.10. *Let $C \subset \mathbb{S}_\Phi$ be an s-convex set and function $f : C \to \mathbb{R}$ be s-convex on C. If $f(x_0)$ is a local minimal value of f, i.e. $f(x) \geqslant f(x_0)$ for $x \in C \cap U(x_0)$, where $U(x_0)$ is a neighbourhood of x_0 in $\mathbb{R}^n$, then $f(x_0)$ is the global minimum of f on C.*

Proof. Suppose there is $x_1 \in C$ such that $f(x_1) < f(x_0)$. Then, for $0 < \lambda < 1$, we would have

$$f(\lambda x_1 +_s (1 - \lambda)x_0) \leqslant \lambda f(x_1) + (1 - \lambda)f(x_0) < f(x_0).$$

However, $\lambda x_1 +_s (1 - \lambda)x_0 \in C \cap U(x_0)$ for λ smaller enough, a contradiction. □

Theorem 6.11. *Let $C \subset \mathbb{S}_\Phi$ be a closed s-convex set and $f : C \to \mathbb{R}$ be an s-convex function on C. If $\max_{x\in C} f(x)$ exists, then*

$$\max_{x\in C} f(x) = \max_{x\in \mathrm{sext}C} f(x),$$

i.e. f attains its maximum (if applicable) at some s-extreme points of C.

Proof. Let $f(x_0) = \max_{x\in C} f(x)$ for some $x_0 \in C$. By Theorem 5.6 there are $x_1, x_2, \cdots, x_k \in \mathrm{sext}C$ such that $x_0 = (s)\sum_{i=1}^k \lambda_i x_i$ with $\lambda_i \in (0, 1)$ and $\sum_{i=1}^k \lambda_i = 1$. Thus we have

$$f(x_0) = f\big((s)\sum\nolimits_{i=1}^k \lambda_i x_i\big) \leqslant \sum\nolimits_{i=1}^k \lambda_i f(x_i) \leqslant \sum\nolimits_{i=1}^k \lambda_i f(x_0) = f(x_0)$$

leading directly to $f(x_1) = f(x_2) = \cdots = f(x_k) = f(x_0) = \max_{x\in C} f(x)$. □

Final Remark. In this article, trying to establish a systematic spherical convexity theory for sets and functions with the aim of achieving meaningful results, we define and study the spherical (i.e. strong) convexity of sets and functions on general spheres by an analytic approach. The results show that such an approach does give some meaningful results. However, the study of s-convex functions in this paper is just an approach to the general theory. We mention here that the discussion for continuity and differentiation of s-convex functions already leads to complications, which we leave to another paper.

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