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Strictly Stable Generalized Monotone Maps and Two Versions of Wald's Axiom

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We consider the stable generalization of monotone maps in normed spaces. A dense subclass of s-quasimonotone maps, namely strictly s-quasimonotone maps, is introduced. For a differentiable map, strict s-quasimonotonicity of the gradient is equivalent to strict s-quasiconvexity of the underlying function. Some necessary and sufficient conditions of these maps are presented. Uses of these generalized convexity and generalized monotonicity in economics are considered. In particular, we investigate excess demand maps F in $\mathbb{R}^n$ satisfying a strong version of Wald's axiom and its weaker version on a convex cone (i.e., $-F$ is s-quasimonotone).

Keywords: Excess demand, generalized monotonicity, density in normed spaces, s-quasimonotonicity, stability, strong version of Wald's axiom.

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1. Introduction

It is well-known that the pseudomonotonicity property of excess demand functions guarantees at least a convex equilibrium set (see [13]). John [12] showed that a demand function which yields a pseudomonotone excess demand if it is combined with an arbitrary supply function (in particular, a constant function) is necessarily monotone.

There arises a question: What kind of a demand function is it if the assumption “arbitrary supply function” is replaced by the assumption “supply function with sufficiently small norm”? In this case, this demand function is called *stable*.

It was shown that well-known kinds of generalized convex functions and generalized monotone maps in $\mathbb{R}^n$ are often not stable with respect to the property they have to keep during the generalization. Then the so-called s-quasiconvex functions, s-quasimonotone maps, and strictly s-quasiconvex functions, were introduced in Phu and An [20], An [2], [3] in $\mathbb{R}^n$ respectively. This leads to the following questions: Is the set of strictly generalized convex functions (strictly generalized monotone maps,

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respectively) dense in the set of generalized convex functions (generalized monotone maps, respectively)? Do the properties of these generalized convex functions and generalized monotone maps still hold true in normed spaces?

As an aplication of this type of generalized monotonicity, we introduced in [4] the strong version (SWA) of Wald's Axiom in the theory of general economic equilibrium (see [17]) and showed that an excess demand function Z satisfies (SWA) iff $-Z$ is stable with respect to the pseudomonotonicity property

In this paper, we consider generalized monotone maps in normed spaces and introduce a subclass of s-quasimonotone maps that are stable, namely strictly s-quasimonotone maps. In the case of a differentiable map, strict s-quasimonotonicity of the gradient is equivalent to strict s-quasiconvexity of the underlying function (Theorem 3.5). The density of the subclass of strictly s-quasiconvex functions (strictly s-quasimonotone maps) in the class of s-quasiconvex functions (s-quasimonotone maps) is presented in Theorem 3.7 (Theorem 3.9, respectively). We then investigate excess demand maps F satisfying (SWA) in $\mathbb{R}^n$. Some properties of excess demand maps satisfying (SWA) are shown. In addition, a weaker condition than that of (SWA) is introduced in Sect. 4 (i.e., formula (13)) and we show that an excess demand map Z on a convex cone satisfies this condition iff $-Z$ is s-quasimonotone (Theorem 4.11). Thus, there exists a new stronger version of Wald's axiom being equivalent to strict s-quasimonotonicity, that is dense in our version of Wald's axiom.

Let us start with some terminology. Let $(X, \|\cdot\|)$ be a normed space with dual X^*. Let D be a nonempty convex subset of X. The following concepts are similar to ones in finite dimensions ([14], [15], and [21]). A map $F\colon D \to X^*$ is said to be *monotone* if for all $x_0, x_1 \in D$,

$$\big\langle F(x_1) - F(x_0), x_1 - x_0 \big\rangle \geq 0. \tag{1}$$

F is said to be *strictly monotone* if inequality (1) is strict for every distinct $x_0, x_1 \in D$.

F is called *pseudomonotone* if for every $x_0, x_1 \in D$,

$$\big\langle F(x_0), x_1 - x_0 \big\rangle \geq 0 \quad \Longrightarrow \quad \big\langle F(x_1), x_1 - x_0 \big\rangle \geq 0. \tag{2}$$

F is called *strictly pseudomonotone* if the second inequality in (2) is strict for every distinct $x_0, x_1 \in D$. The function $f\colon D \to \mathbb{R}$ is said to be *convex* if, for all $x_0, x_1 \in D$ and $\lambda \in (0,1)$,

$$f(x_\lambda) \leq (1-\lambda) f(x_0) + \lambda f(x_1), \tag{3}$$

where $x_\lambda := (1-\lambda)x_0 + \lambda x_1$. f is said to be *strictly convex* if (3) is a strict inequality for every distinct $x_0, x_1 \in D$. f is said to be *quasiconvex* if, for all $x_0, x_1 \in D$ and $\lambda \in (0,1)$,

$$f(x_0) \leq f(x_1) \text{ implies } f(x_\lambda) \leq f(x_1). \tag{4}$$

f is said to be *strictly quasiconvex* if the second inequality in (4) is strict, for every distinct $x_0, x_1 \in D, \lambda \in (0,1)$.

A Fréchet differentiable function f is said to be *pseudoconvex* if, for all $x_0, x_1 \in D$,

$$f(x_0) < f(x_1) \text{ implies } \big\langle f'(x_1), x_0 - x_1 \big\rangle < 0, \tag{5}$$

A differentiable function f is said to be *strictly pseudoconvex* if the first inequality in (5) is not strict, for every distinct $x_0, x_1 \in D$.

Note that a Fréchet differentiable function $f\colon D \subset X \to \mathbb{R}$ is convex (respectively, quasiconvex, strictly quasiconvex, pseudoconvex, strictly pseudoconvex) iff f' is monotone (respectively, quasimonotone, strictly quasimonotone, pseudomonotone, strictly pseudomonotone). Proofs of these facts are similar to ones of Theorem 2.3 [5], Theorem 3.1 [14], Propositions 4.1 and 5.2 [15].

We recall the definition of s-quasimonotone maps (s stands for "stable") (see [3]).

Definition 1.1. $F\colon D \to X^*$ is said to be *s-quasimonotone* w.r.t. $\sigma > 0$ if for all $x_0, x_1 \in D$, $x_0 \neq x_1$, and $|\delta| < \sigma$, we have

$$\left\langle F(x_0)\,\frac{x_1 - x_0}{\|x_1 - x_0\|}\right\rangle \geq \delta \quad \Longrightarrow \quad \left\langle F(x_1), \frac{x_1 - x_0}{\|x_1 - x_0\|}\right\rangle \geq \delta. \tag{6}$$

F is said to be *s-quasimonotone* if it is s-quasimonotone w.r.t.some $\sigma > 0$.

Obviously, every monotone map is s-quasimonotone and every s-quasimonotone map is pseudomonotone. In other words, s-quasimonotonicity is an intermediate concept between monotonicity and pseudomonotonicity.

Throughout this paper we denote by $F + \xi$, where $\xi \in X^*$, the map defined by $(F + \xi)(x) = F(x) + \xi$ for all $x \in D$.

The following theorem states some properties of s-quasimonotonicity. Their proofs are similar to ones given in [3].

Theorem 1.2. *Let* $F\colon D \to X^*$.

(a) *F is s-quasimonotone iff there exists $\epsilon > 0$ such that $F + \xi$ is pseudomonotone for each $\xi \in X^*$ satisfying $\|\xi\| < \epsilon$.*

(b) *F is s-quasimonotone iff there exists $\epsilon > 0$ such that $F + \xi$ is s-quasimonotone for each $\xi \in X^*$ satisfying $\|\xi\| < \epsilon$.*

What happens if one (or both) of the inequalities in (6) is strict? This question is answered partly by the following proposition:

Proposition 1.3. *Let $F\colon D \to X^*$. The following statements are equivalent:*

(a) *F is s-quasimonotone.*

(b) *There exists $\sigma > 0$ such that*

$$\left\langle F(x_0), \frac{x_1 - x_0}{\|x_1 - x_0\|}\right\rangle > \delta \quad \Longrightarrow \quad \left\langle F(x_1), \frac{x_1 - x_0}{\|x_1 - x_0\|}\right\rangle > \delta$$

for all $|\delta| < \sigma$, $x_0, x_1 \in D$, $x_0 \neq x_1$.

(c) *There exists $\sigma > 0$ such that*

$$\left\langle F(x_0), \frac{x_1 - x_0}{\|x_1 - x_0\|}\right\rangle > \delta \quad \Longrightarrow \quad \left\langle F(x_1), \frac{x_1 - x_0}{\|x_1 - x_0\|}\right\rangle \geq \delta$$

for all $|\delta| < \sigma$, $x_0, x_1 \in D$, $x_0 \neq x_1$.

Thus Propositions 1.3 shows that in (6), replacing both inequalities by strict inequalities or the first inequality by a strict inequality will not rise to new types of generalized monotonicity. In the next sections, we replace the second inequality by a strict one and obtain a new type of generalized monotonicity.

2. Strictly s-quasimonotone maps

Let D be a nonempty convex set in a normed space X.

Definition 2.1. A map $F\colon D \to X^*$ is said to be *strictly s-quasimonotone w.r.t.* $\sigma > 0$ if for all $x_0, x_1 \in D$, $x_0 \neq x_1$, and $|\delta| < \sigma$,

$$\left\langle F(x_0), \frac{x_1 - x_0}{\|x_1 - x_0\|} \right\rangle \geq \delta \quad \Longrightarrow \quad \left\langle F(x_1), \frac{x_1 - x_0}{\|x_1 - x_0\|} \right\rangle > \delta. \tag{7}$$

F is *strictly s-quasimonotone* if it is strictly s-quasimonotone w.r.t. some $\sigma > 0$.

By setting $\delta = 0$ in Definition 2.1, we see at once that every strictly s-quasimonotone map is strictly pseudomonotone. The following example shows that s-quasimonotonicity does not imply strictly s-quasimonotonicity with the same $\sigma > 0$.

Example 2.2. Let I be an interval in $\mathbb{R}$. Every constant function $F\colon I \to \mathbb{R}$, $F(x) = c$, is s-quasimonotone w.r.t. all $\sigma > 0$. However, F is strictly s-quasimonotone w.r.t. $\sigma > 0$ iff $\sigma \leq |c|$. □

Indeed, since $F(x_1) = F(x_0)$, condition $F(x_0) \cdot \frac{x_1 - x_0}{|x_1 - x_0|} \geq \delta$ implies $F(x_1) \cdot \frac{x_1 - x_0}{|x_1 - x_0|} \geq \delta$. Thus F is s-quasimonotone w.r.t. all $\sigma > 0$.

Suppose now that $0 < \sigma \leq |c|$, $|\delta| < \sigma$, and

$$F(x_0) \cdot \frac{x_1 - x_0}{|x_1 - x_0|} \geq \delta. \tag{8}$$

If $c > 0$, then (8) combined with condition $c > \delta$ implies $x_1 > x_0$ and so we have $F(x_1)\frac{x_1 - x_0}{|x_1 - x_0|} = c > \delta$. If $c < 0$, then (8) gives $x_1 < x_0$ and so $F(x_1)\frac{x_1 - x_0}{|x_1 - x_0|} = |c| > \delta$. Thus F is strictly s-quasimonotone w.r.t. σ.

Next, consider the case $\sigma > |c|$. Set $\delta = c$. Then $|\delta| < \sigma$ and for $x_1 > x_0$, $F(x_0)\frac{x_1 - x_0}{|x_1 - x_0|} \geq \delta$ but $F(x_1)\frac{x_1 - x_0}{|x_1 - x_0|} = \delta$. Thus F is not strictly s-quasimonotone w.r.t. $\sigma > |c|$. Note also that if $c = 0$, i.e., $F(x) \equiv 0$, then F is s-quasimonotone (w.r.t.all $\sigma > 0$) but not strictly s-quasimonotone. It should be noted that every constant function $F\colon I \to \mathbb{R}$, $F(x) \equiv c \neq 0$, is always strictly s-quasimonotone because we can choose $\sigma = |c|$.

We now state a relationship between (strict) s-quasimonotonicity and (strict) monotonicity.

Proposition 2.3. *Let $F\colon D \to X^*$.*

(a) *If F is strictly monotone, then it is strictly s-quasimonotone w.r.t. all $\sigma > 0$. Conversely, if there is $\sigma_0 > 0$ such that F is strictly s-quasimonotone w.r.t. all $\sigma > \sigma_0$, then F is strictly monotone.*

(b) *If F is monotone, then it is s-quasimonotone w.r.t. all $\sigma > 0$. Conversely, if there is $\sigma_0 > 0$ such that F is s-quasimonotone w.r.t. all $\sigma > \sigma_0$, then F is monotone.*

Proof. (a) Suppose that F is strictly monotone. For $x_0, x_1 \in D$, $x_0 \neq x_1$, we have $\big\langle F(x_1) - F(x_0), x_1 - x_0 \big\rangle > 0$, which implies $\big\langle F(x_1), v \big\rangle > \big\langle F(x_0), v \big\rangle$, where $v = (x_1 - x_0)/\|x_1 - x_0\|$.

It follows that $\big\langle F(x_0), v\big\rangle \geq \delta$ implies $\big\langle F(x_1), v\big\rangle > \delta$.

Thus F is strictly s-quasimonotone w.r.t. all $\sigma > 0$.

Conversely, suppose that F is strictly s-quasimonotone w.r.t. all $\sigma > \sigma_0$. Assume $x_1, x_0 \in D$, $x_1 \neq x_0$, and set $v = (x_1 - x_0)/\|x_1 - x_0\|$. Choose $\sigma > \max\{\|F(x_0)\|, \sigma_0\}$ and $\delta := \big\langle F(x_0), v\big\rangle$. Then $|\delta| \leq \|F(x_0)\| < \sigma$. Since F is strictly s-quasimonotone w.r.t. σ and $\big\langle F(x_0), v\big\rangle \geq \delta$, we have $\big\langle F(x_1), v\big\rangle > \delta$. It then follows immediately that $\big\langle F(x_1) - F(x_0), v\big\rangle > 0$ and hence, F is strictly monotone.

Part (b) is proved similarly. □

A version of Theorem 1.2 holds too for strictly s-quasimonotone maps.

Theorem 2.4. *For a map $F\colon D \to X^*$ the following statements are equivalent.*

(a) *F is strictly s-quasimonotone.*

(b) *There exists $\epsilon > 0$ such that $F + \xi$ is strictly s-quasimonotone for each $\xi \in X^*$ with $\|\xi\| < \epsilon$.*

(c) *There exists $\epsilon > 0$ such that $F + \xi$ is strictly pseudomonotone for each $\xi \in X^*$ with $\|\xi\| < \epsilon$.*

Proof. (a)⇒(b): Let F be strictly s-quasimonotone w.r.t. $\sigma > 0$. Set $\epsilon := \sigma/2$. We show that $F + \xi$ is strictly s-pseudomonotone w.r.t. ϵ whenever $\|\xi\| < \epsilon$.

Let $\xi \in X^*$, $\|\xi\| < \epsilon$, $x_0, x_1 \in D$, $x_0 \neq x_1$, and $|\delta| < \epsilon$ satisfy

$$\big\langle F(x_0) + \xi, v\big\rangle \geq \delta \quad \text{where} \quad v := \frac{x_1 - x_0}{\|x_1 - x_0\|}.$$

Then
$$\big\langle F(x_0), v\big\rangle \geq \delta - \big\langle \xi, v\big\rangle =: \delta'.$$

As $|\delta'| \leq |\delta| + \|\xi\| < \epsilon + \epsilon = \sigma$, strict s-quasimonotonicity of F implies $\big\langle F(x_1), v\big\rangle > \delta'$, giving $\big\langle F(x_1) + \xi, v\big\rangle > \delta$. Thus $F + \xi$ is strictly s-pseudomonotone w.r.t. ϵ.

(b)⇒(c): This is obviou since a strictly s-quasimonotone map is always strictly pseudomonotone.

(c)⇒(a): Suppose that there exists $\epsilon > 0$ such that $F + \xi$ is strictly pseudomonotone for each $\xi \in X^*$ satisfying $\|\xi\| < \epsilon$. We show that F is strictly s-quasimonotone w.r.t. ϵ. Assume that

$$\left\langle F(x_0), \frac{x_1 - x_0}{\|x_1 - x_0\|}\right\rangle \geq \delta \quad \text{for some} \quad x_0, x_1 \in D,\ |\delta| < \epsilon. \tag{9}$$

Set $v = (x_1 - x_0)/\|x_1 - x_0\|$. By the Hahn-Banach theorem, there is $\xi \in X^*$ such that (see the Appendix) $\|\xi\| = |\delta|$ and $\big\langle \xi, v\big\rangle = -\delta$.

Then, by (9), we have $\big\langle F(x_0), v\big\rangle \geq -\big\langle \xi, v\big\rangle$, giving $\big\langle F(x_0) + \xi, x_1 - x_0\big\rangle \geq 0$. Since $\|\xi\| = |\delta| < \epsilon$, by assumption, $F + \xi$ is strictly pseudomonotone. Hence $\big\langle F(x_1) + \xi, x_1 - x_0\big\rangle > 0$, which is equivalent to $\big\langle F(x_1), v\big\rangle > -\big\langle \xi, v\big\rangle = \delta$. Therefore F is strictly s-quasimonotone. □

We now present another characterization of s-quasimonotone maps via their derivatives. Let $F\colon D \to X^*$ be differentiable. Consider the following two conditions:

$$\begin{aligned}&\text{There exists } \sigma > 0 \text{ such that } x \in D, v \in X, \|v\| = 1, \text{ and}\\ &|\langle F(x), v\rangle| < \sigma \ \text{ imply } \ F'(x)[v,v] \geq 0\end{aligned} \tag{10}$$

and:

$$\begin{aligned}&\text{There exists } \sigma > 0 \text{ such that } x \in D, v \in X, \|v\| = 1, \text{ and}\\ &|\langle F(x), v\rangle| < \sigma \text{imply } \ F'(x)[v,v] > 0.\end{aligned} \tag{11}$$

Theorem 2.5. *Let $F\colon D \to X^*$ be differentiable on an open convex set D in a normed space X.*

(a) *F is s-quasimonotone iff* (10) *holds.*

(b) *If* (11) *holds, then F is strictly s-quasimonotone.*

Proof. (a) *Necessity.* Suppose that F is s-quasimonotone w.r.t. $\sigma > 0$. Let $x \in D$ and $v \in X$, $\|v\| = 1$ satisfy $|\langle F(x), v\rangle| < \sigma$. Set $\delta := \langle F(x), v\rangle$. As D is open, there is $t^* > 0$ such that $x_t := x + tv \in D$ for all $t \in (0, t^*]$. Then for these t,

$$\left\langle F(x), \frac{x_t - x}{\|x_t - x\|}\right\rangle = \big\langle F(x), v\big\rangle = \delta.$$

Since $|\delta| < \sigma$ we can apply (6) to $x_1 = x_t$ to obtain

$$\big\langle F(x_t), v\big\rangle \geq \delta \quad \text{for all} \quad t \in (0, t^*],$$

which is equivalent to $\big\langle F(x_t) - F(x), v\big\rangle \geq 0$ for all $t \in (0, t^*]$. Dividing both sides of this inequality by t and letting $t \downarrow 0$ gives $F'(x)[v,v] \geq 0$, i.e., (10) holds.

Sufficiency. Let F satisfy (10). Suppose that there exist $x_0, x_1 \in D$, $|\eta| < \sigma$ such that

$$\big\langle F(x_0), v\big\rangle > \eta > \big\langle F(x_1), v\big\rangle \ \text{ where } \ v = \frac{x_1 - x_0}{\|x_1 - x_0\|}. \tag{12}$$

Set $\qquad t_0 = \|x_1 - x_0\|, \quad y_t = x_0 + tv, \quad \text{and} \quad \varphi(t) = \big\langle F(y_t), v\big\rangle.$

Then $y_0 = x_0$, $y_{t_0} = x_1$, and φ is defined on an open interval I containing $[0, t_0]$ since D is open and convex. Moreover, by the change rule,

$$\varphi'(t) = F'(y_t)[v,v] \quad \text{for all} \quad t \in I.$$

Since φ is continuous on $[0, t_0]$ and $\varphi(0) > \eta > \varphi(t_0)$, there is $t \in (0, t_0)$ such that $\varphi(t) = \eta$. Set

$$T = \big\{t \in [0, t_0] : \varphi(t) = \eta\big\} \quad \text{and} \quad \tau = \inf T.$$

Then T is nonempty and it follows from continuity of φ that $\tau \in T$. Since $0 < \tau < t_0$ and $|\varphi(\tau)| = |\eta| < \sigma$, there is $\epsilon > 0$ such that $\epsilon < \tau < \tau + \epsilon < t_0$ and $|\varphi(t)| < \sigma$ for all $t \in (\tau - \epsilon, \tau + \epsilon)$.

As F satisfies (10), $\varphi'(t) = F'(y_t)[v,v] \geq 0$ for all $t \in (\tau - \epsilon, \tau + \epsilon)$.

It follows that φ is nondecreasing on $[\tau - \epsilon, \tau + \epsilon]$. In particular, $\varphi(\tau - \epsilon) \leq \varphi(\tau) = \eta < \varphi(0)$. Again, continuity of φ forces that there is a point $0 < \tau' \leq \tau - \epsilon$ with $\varphi(\tau') = \eta$, i.e., $\tau' \in T$. This contradicts the fact that $\tau = \inf T$. Thus (12) does not hold and hence, F is s-quasimonotone.

(b) Suppose that F satisfies (11) but F is not strictly s-quasimonotone. For $\sigma > 0$ in (11), there exist $x_0, x_1 \in D$, $x_0 \neq x_1$, and $|\delta| < \sigma$ such that

$$\big\langle F(x_0), v\big\rangle \geq \delta \quad \text{and} \quad \big\langle F(x_1), v\big\rangle \leq \delta \quad \text{where} \quad v = \frac{x_1 - x_0}{\|x_1 - x_0\|}.$$

If $\langle F(x_0), v\rangle > \langle F(x_1), v\rangle$, then (12) would hold. By part (a), this is impossible since F satisfies (10). Thus $\langle F(x_0), v\rangle = \langle F(x_1), v\rangle = \delta$. Let t_0, y_t, φ, and I be defined as in part (a). If $\varphi(t) = \delta$ for all $t \in [0, t_0]$, then (11) would imply that

$$\varphi'(t) = F'(y_t)[v, v] > 0 \quad \text{for all} \quad t \in (0, t_0),$$

i.e., φ would be strictly increasing on $[0, t_0]$, a contradiction. Thus there is $t_* \in (0, t_0)$ with $\varphi(t_*) \neq \delta$, say $\varphi(t_*) > \delta$. We now replaced x_0 by y_{t_*} to obtain a contradiction as before. $\square$

Example 2.6. Let $\varphi(x) := \max\{0, \cos x\}$ and set $F(x) = \int_{-\infty}^{x} e^t \varphi(t)dt$ for $x \leq 0$ and $F(x) = \sigma_0 + xe^{-x}$ for $x > 0$, where $\sigma_0 = F(0)$. We show that F is s-quasimonotone but not strictly s-quasimonotone.

Since $e^t\varphi(t)$ is continuous and $0 \leq e^t\varphi(t) \leq e^t$, $0 < F(x) < \int_{-\infty}^{x} e^t dt = e^x$ for all $x \leq 0$. In particular, $0 < \sigma_0 < 1$. We have

$$F'(x) = \begin{cases} e^x\varphi(x) & \text{if } x \leq 0, \\ (1-x)e^{-x} & \text{if } x > 0, \end{cases}$$

so that F is increasing on $(-\infty, 0]$ and $0 < F(x) < F(0) = \sigma_0 < F(y)$ for $x < 0 < y$. If $|F(x)| < \sigma_0$, then $x < 0$, and hence, $F'(x) = e^x\varphi(x) \geq 0$. Thus by Theorem 2.5, F is s-quasimonotone w.r.t. σ_0. Now let $\sigma > 0$ be arbitrary. Since $\lim_{x\to-\infty} F(x) = 0$, there is $M < 0$ such that $|F(x)| < \sigma$ for all $x < M$. Choose a natural number k such that $x_1 := -\frac{\pi}{2} - 2k\pi < M$ and set $x_0 := -\frac{3\pi}{2} - 2k\pi$, $\delta := F(x_0)$. Then $0 < \delta < \sigma$ and as $F'(x) = 0$ on $[x_0, x_1]$, $F(x_0) = F(x_1) = \delta$, i.e., (7) does not holds. Thus F is not strictly s-quasimonotone. Note also that F is decreasing on $[1, \infty)$.

In this paper, the norm $\|\cdot\|$ on $\mathbb{R}^n$ is always assumed to be the Euclidean norm.

Example 2.7. Consider the following map

$$F\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad F(x) = \big[1 - c\sin\big(\|x\|^2\big)\big]x, \qquad c = 0.2.$$

We have $F'(x) = -2c\cos\big(\|x\|^2\big)xx^T + \big[1 - c\sin\big(\|x\|^2\big)\big]I$, $x \in \mathbb{R}^n$, where I is the $n \times n$ identity matrix. We show that F is strictly s-quasimonotone w.r.t. $\sigma = 1$. Suppose $x, v \in \mathbb{R}^n$, $\|v\| = 1$, and

$$\big|\langle F(x), v\rangle\big| = \big[1 - c\sin\big(\|x\|^2\big)\big] \cdot \big|x^Tv\big| < 1.$$

Then
$$\begin{aligned} v^TF'(x)v &= -2c\cos\big(\|x\|^2\big)v^Txx^Tv + \big[1 - c\sin\big(\|x\|^2\big)\big] \\ &\geq -\frac{2c}{\big[1 - c\sin\big(\|x\|^2\big)\big]^2} + \big[1 - c\sin\big(\|x\|^2\big)\big]. \end{aligned}$$

Since $1 - c\sin\big(\|x\|^2\big) \geq \frac{4}{5}$, it follows that $v^TF'(x)v > 0$. Therefore, Theorem 2.5 says that F is strictly s-quasimonotone.

Note that $F'(x)$ is not positive semidefinite in general. If $\cos\big(\|x\|^2\big) = 1$, $\sin\big(\|x\|^2\big) = 0$, then $F'(x) = -2cxx^T + I$, which may be indefinite.

For instance, in $\mathbb{R}^2$, the matrix

$$F'(\sqrt{5\pi}, \sqrt{5\pi}) = -\frac{2}{5}\begin{bmatrix} 5\pi & 5\pi \\ 5\pi & 5\pi \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-2\pi & -2\pi \\ -2\pi & 1-2\pi \end{bmatrix}$$

is indefinite.

Corollary 2.8. *Let $F\colon D \to X^*$ be differentiable on an open convex set D in a normed space X. Then F is s-quasimonotone iff there is some positive ϵ such that $F+\xi$ satisfies* (10) *with $\|\xi\| < \epsilon$.*

Proof. It follows from Theorems 1.2 that F is s-quasimonotone iff there exists $\epsilon > 0$ such that $F+\xi$ is s-quasimonotone for all $\xi \in X^*$ satisfying $\|\xi\| < \epsilon$. By Theorem 2.5(a), for each $\xi \in X^*$ satisfying $\|\xi\| < \epsilon$, the map $F+\xi$ has property (10). $\square$

3. S-quasiconvex and strictly s-quasiconvex functions

In this section, as usual, we assume that D is a nonempty convex set in a normed space X.

We know that a differentiable function $f\colon D \to X^*$ is convex iff f' is monotone (the proof of this conclusion is similar to that of Theorem 2.3 [5]). Corresponding to s-quasimonotone and strictly s-quasimonotone maps are s-quasiconvex and strictly s-quasiconvex functions, respectively.

Definition 3.1. ([20]) A function $f\colon D \to \mathbb{R}$ is said to be *s-quasiconvex* w.r.t. $\sigma > 0$ if

$$\frac{f(x_1)-f(x_0)}{\|x_1-x_0\|} \geq \delta \quad \Longrightarrow \quad \frac{f(x_1)-f(x_\lambda)}{\|x_1-x_\lambda\|} \geq \delta$$

for all $x_0, x_1 \in D$, $x_0 \neq x_1$, $\lambda \in (0,1)$, $x_\lambda = (1-\lambda)x_0 + \lambda x_1$, and $|\delta| < \sigma$. f is said to be *s-quasiconvex* if it is s-quasiconvex w.r.t. some $\sigma > 0$.

Definition 3.2. ([2]) A function $f\colon D \to \mathbb{R}$ is said to be *strictly s-quasiconvex* w.r.t. $\sigma > 0$ if

$$\frac{f(x_1)-f(x_0)}{\|x_1-x_0\|} \geq \delta \quad \Longrightarrow \quad \frac{f(x_1)-f(x_\lambda)}{\|x_1-x_\lambda\|} > \delta$$

for all $x_0, x_1 \in D$, $x_0 \neq x_1$, $\lambda \in (0,1)$, $x_\lambda = (1-\lambda)x_0 + \lambda x_1$, and $|\delta| < \sigma$. f is said to be *strictly s-quasiconvex* if it is strictly s-quasiconvex w.r.t. some $\sigma > 0$.

In [20] and [2] s-quasiconvexity and strict s-quasiconvexity of functions defined on $\mathbb{R}^n$ were introduced. Similar to Theorem 2.1 [2], we have the following property of strictly s-quasiconvex functions in normed spaces.

Theorem 3.3. *A function $f\colon D \to \mathbb{R}$ is strictly s-quasiconvex iff there exists $\epsilon > 0$ such that $f+\xi$ is strictly quasiconvex for each linear functional ξ on X satisfying $\|\xi\| < \epsilon$.*

In normed spaces, the following relationship between s-quasimonotonicity and s-quasiconvexity holds true.

Theorem 3.4. *Suppose that $f\colon D \to \mathbb{R}$ is differentiable on the open and convex set D. Then f is s-quasiconvex iff f' is s-quasimonotone.*

The proof of this theorem is similar to that of Theorem 4.1 [3].

Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^x \cos^2 x + \frac{2}{25} e^x(\sin 2x - 7\cos 2x) & \text{if } x \leq 0, \\ \frac{3}{5}x - \frac{1}{5}\cos x & \text{if } x > 0. \end{cases}$$

Then
$$f'(x) = \begin{cases} e^x \cos^2 x + \frac{1}{5} e^x(\sin 2x - 2\cos 2x) & \text{if } x \leq 0, \\ \frac{3}{5} + \frac{1}{5}\sin x & \text{if } x > 0 \end{cases}$$

and
$$f''(x) = \begin{cases} e^x \cos^2 x & \text{if } x \leq 0, \\ \frac{1}{5}\cos x & \text{if } x > 0. \end{cases}$$

If $|f'(x)| < \frac{2}{5}$, then $x < 0$, so $f''(x) = e^x \cos^2 x \geq 0$. Thus by Theorem 2.5, f' is s-quasimonotone w.r.t. $\sigma = \frac{2}{5}$. According to Theorem 3.4, f is s-quasiconvex.

We now establish a relationship between strict s-quasimonotonicity and strict s-quasiconvexity.

Theorem 3.5. *Suppose that $f\colon D \to \mathbb{R}$ is differentiable on the open and convex set D. Then f is strictly s-quasiconvex iff f' is strictly s-quasimonotone.*

Proof Let f be strictly s-quasiconvex. By Theorem 3.3, there exists $\epsilon > 0$ such that $f + \langle \xi, \cdot \rangle$ is strictly pseudoconvex for each $\xi \in X^*$ satisfying $\|\xi\| < \epsilon$. It follows that $f' + \xi$ is strictly pseudomonotone with $\|\xi\| < \epsilon$. According to Theorem 2.4, f' is strictly s-quasimonotone.

Conversely, suppose that f' is strictly s-quasimonotone. It follows from Theorem 2.4(c) that there exists $\epsilon > 0$ such that $f' + \xi$ is strictly pseudomonotone for all $\xi \in X^*$ satisfying $\|\xi\| < \epsilon$. It follows that $f + \langle \xi, \cdot \rangle$ is strictly s-quasiconvex for all $\xi \in X^*$ satisfying $\|\xi\| < \epsilon$ (for a direct proof, see the Appendix). Hence, f is strictly s-quasiconvex. □

Consider the function $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = 0.1\cos\left(\|x\|^2\right) + 0.5\|x\|^2.$$

Then
$$\nabla f(x) = F(x) \quad \text{and} \quad \nabla^2 f(x) = F'(x),$$

where F and F' were given in Example 2.7. Since $\nabla f(x) = F(x)$ is strictly s-quasimonotone, Theorem 3.5 says that f is strictly s-quasiconvex. However, as its Hessian $\nabla^2 f(x) = F'(x)$ is not positive semidefinite, f is not convex.

The next part of this section is devoted to the problem of approximating an s-quasiconvex function by strictly s-quasiconvex functions and an s-quasimonotone map by strictly s-quasimonotone maps.

Recall that a function $f\colon D \to \mathbb{R}$ is said to be *Lipschitz* provided there is a constant $L > 0$ for which $|f(x) - f(y)| \leq L\|x - y\|$ for all $x, y \in D$.

Lemma 3.6. *Let $f, g\colon D \to \mathbb{R}$. If f is strictly convex and Lipschitz on D and g is s-quasiconvex, then there is a constant $\bar{c} > 0$ such that $cf + g$ is strictly s-quasiconvex for all $0 < c \leq \bar{c}$.*

Proof. Suppose that g is s-quasiconvex w.r.t.$\sigma_0 > 0$ and $|f(x) - f(y)| \leq L\|x - y\|$ for all $x, y \in D$. Set $\bar{c} = \sigma_0/2L$. We show that $cf + g$ is strictly s-quasiconvex w.r.t.$\sigma_0/2$ for all $0 < c \leq \bar{c}$.

Assume that $|\delta| < \sigma_0/2$ and

$$\frac{(cf+g)(x_1)-(cf+g)(x_0)}{\|x_1-x_0\|} \geq \delta, \quad x_0, x_1 \in D.$$

Then
$$\frac{g(x_1)-g(x_0)}{\|x_1-x_0\|} \geq \delta - c \cdot \frac{f(x_1)-f(x_0)}{\|x_1-x_0\|}.$$

Since $|\delta| < \sigma_0/2$ and $0 < c \leq \sigma_0/2L$,

$$\left|\delta - c \cdot \frac{f(x_1)-f(x_0)}{\|x_1-x_0\|}\right| \leq |\delta| + cL < \frac{\sigma_0}{2} + \frac{\sigma_0}{2} = \sigma_0.$$

As g is s-quasiconvex w.r.t. σ_0, we have for all $\lambda \in (0,1)$,

$$\frac{g(x_1)-g(x_\lambda)}{\|x_1-x_\lambda\|} \geq \delta - c \cdot \frac{f(x_1)-f(x_0)}{\|x_1-x_0\|},$$

implying
$$\frac{(cf)(x_1)-(cf)(x_0)}{\|x_1-x_0\|} \geq \delta - \frac{g(x_1)-g(x_\lambda)}{\|x_1-x_\lambda\|}.$$

Since cf is strictly convex, it follows that

$$\frac{(cf)(x_1)-(cf)(x_\lambda)}{\|x_1-x_\lambda\|} > \frac{(cf)(x_1)-(cf)(x_0)}{\|x_1-x_0\|}$$

and hence,
$$\frac{(cf)(x_1)-(cf)(x_\lambda)}{\|x_1-x_\lambda\|} > \delta - \frac{g(x_1)-g(x_\lambda)}{\|x_1-x_\lambda\|}.$$

Therefore
$$\frac{(cf+g)(x_1)-(cf+g)(x_\lambda)}{\|x_1-x_\lambda\|} > \delta \quad \text{for all} \quad \lambda \in (0,1),$$

i.e., $cf+g$ is strictly s-quasiconvex. □

A normed linear space $(X, \|\cdot\|)$ is *strictly convex* if $\|x+y\| = \|x\| + \|y\|$ implies that $x = ty$ for some $t \geq 0$ or else y is the zero vector. Equivalent conditions are given in the Appendix.

Theorem 3.7. *Let $g\colon D \to \mathbb{R}$ be an s-quasiconvex function. If there is a function f on D that is strictly convex, bounded, and Lipschitz, then for every $\epsilon > 0$, there is a strictly s-quasiconvex function h on D such that* $\sup_{x\in D} |h(x) - g(x)| < \epsilon$.

In particular, if D is bounded and there is an equivalent and strictly convex norm $\|\cdot\|_0$ on X, such a function h exists.

Proof. Let $\epsilon > 0$. According to Lemma 3.6, there is $\bar{c} > 0$ such that $cf+g$ is strictly s-quasiconvex for all $0 < c \leq \bar{c}$. As f is bounded on D, we can choose $0 < c \leq \bar{c}$ with $c \cdot \sup_{x\in D} |f(x)| < \epsilon$. Then the function $h = cf + g$ satisfies the requirements. Since any norm is Lipschitz, if the equivalent norm $\|\cdot\|_0$ is strictly convex and D is bounded, we can choose $f(x) = \|x\|_0^2$, see [6]. The proof of boundedness and Lipschitz continuity of f is given in the Appendix. □

In particular, if D is an interval in $\mathbb{R}$ and $D \neq \mathbb{R}$, then any s-quasimonotone function g defined on D can be approximated as closely as desired by a strictly s-quasimonotone function.

Indeed, if $D = (-\infty, a)$ or $(-\infty, a]$, we can choose $f(x) = e^x$, which is strictly convex, bounded, and Lipschitz on D; if $D = [a, \infty)$ or $D = (a, \infty)$, we choose $f(x) = e^{-x}$.

Finally we consider a similar problem for s-quasimonotone maps. The proof of the following lemma is similar to that of Lemma 3.6.

Lemma 3.8. *Given two maps $F, G\colon D \to X^*$. If F is strictly monotone and bounded on D and G is s-quasimonotone, then there is a constant $\bar{c} > 0$ such that $cF + G$ is strictly s-quasimonotone for all $0 < c \leq \bar{c}$.*

A proof of this lemma is given in the Appendix.

Theorem 3.9. *Let $G\colon D \to X^*$ be s-quasimonotone. If there is a strictly monotone and bounded map $F\colon D \to X^*$, then for each $\epsilon > 0$, there is a strictly s-quasimonotone map H on D such that $\sup_{x\in D} \|H(x) - G(x)\| < \epsilon$.*

In particular, if D is bounded and there is an equivalent and strictly convex norm $\|\cdot\|_0$ on X, such a map H exists.

Proof. The first part can be proved in a manner similar to that of Theorem 3.7. Suppose next that $\|\cdot\|_0$ is an equivalent and strictly convex norm on X. The duality map of $(X, \|\cdot\|_0)$ is the set-valued map $J\colon X \to 2^{X^*}$ defined by

$$J(x) = \left\{\xi \in X^* : \langle \xi, x\rangle = \|x\|_0^2 = \|\xi\|_0^2\right\},$$

where $\|\xi\|_0 = \sup\left\{\langle \xi, x\rangle : \|x\|_0 \leq 1\right\}$. By the Hahn-Banach theorem, $J(x) \neq \emptyset$ for all $x \in X$ (see [19], p. 27: $J(x) = \partial f(x)$, where $f(x) = \frac{1}{2}\|x\|_0^2$). For each $x \in D$, take an element $F(x) \in J(x)$. As $\|\cdot\|_0$ is strictly convex, F is strictly monotone. Furthermore, D is bounded and so is F (for more details, see the proof in the Appendix). □

4. Two versions of Wald's Axiom

In this section we use $\mathbb{R}^n_{>0} := \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0 \text{ for all } i = 1, \ldots, n\right\}$.

Let P be a cone in $\mathbb{R}^n$, i.e., $\lambda x \in P$ whenever $x \in P$ and $\lambda > 0$.

In an n-good competitive economy, $F\colon P \subset \mathbb{R}^n_{>0} \to \mathbb{R}^n$ is said to be an *excess demand function* if it is homogeneous in $p \in P$, i.e., $F(p) = F(\lambda p)$, $\lambda > 0$, and satisfies *Walras' Law*, $p^T F(p) = 0$. According to [9] and [10] the following property is called *Wald's Axiom:*

$$p, q \in P,\ p^T F(q) \leq 0 \text{ and } q^T F(p) \leq 0 \text{ imply } F(q) = F(p).$$

Some strong versions of Wald's Axiom were given in [8], [11], [18], etc.

Let D be a nonempty subset of $\mathbb{R}^n$ and $Z\colon D \to \mathbb{R}^n$. A strong version of Wald's Axiom introduced in [4] is as follows.

$$\text{(SWA)}\quad \exists\,\sigma > 0 \text{ such that } \begin{Bmatrix} p, q \in D,\ |\delta| < \sigma,\ Z(q) \neq Z(p) \\ q^T Z(p) - \delta \leq 0 \end{Bmatrix} \Rightarrow p^T Z(q) + \delta > 0.$$

Example 4.1. Any map Z from a subset D of $\mathbb{R}^n_{>0}$ into $\mathbb{R}^n_{>0}$ satisfies (SWA). □

Indeed, for all $p, q \in D$, we have $p, q, Z(p), Z(q) \in \mathbb{R}^n_{>0}$ so $p^T Z(q) > 0$ and $q^T Z(p) > 0$. If $q^T Z(p) - \delta \leq 0$, then $\delta > 0$, implying $p^T Z(q) + \delta > 0$.

Example 4.2. Let $P = \big\{(p_1, \dots, p_n) \in \mathbb{R}^n_{>0} : p_n = \sqrt{p_1^2 + \cdots + p_{n-1}^2}\big\}$.
Consider the map $Z\colon P \to \mathbb{R}^n$ defined by $\mathbb{Z}(p_1, \dots, p_n) = \dfrac{(-p_1, \dots, -p_{n-1}, p_n)}{\|p\|}$.
Then Z is an excess demand function and satisfies (SWA). □

Indeed, P is a cone, Z is homogeneous and $p^T Z(p) = \frac{1}{\|p\|}\big(-p_1^2 - \cdots - p_{n-1}^2 + p_n^2\big) = 0$. Hence Z is an excess demand function. Furthermore, by the Cauchy-Schwarz inequality, we have for all $p, q \in P$

$$p^T Z(q) = \frac{1}{\|q\|}\Big(-p_1q_1 - \cdots - p_{n-1}q_{n-1} + \sqrt{p_1^2 + \cdots + p_{n-1}^2} \cdot \sqrt{q_1^2 + \cdots + q_{n-1}^2}\Big) \geq 0.$$

If there is δ such that $q^T Z(p) - \delta \leq 0$ and $p^T Z(q) + \delta \leq 0$, then adding these inequalities gives $p^T Z(q) + q^T Z(p) \leq 0$. Since $q^T Z(p) \geq 0$ and $p^T Z(q) \geq 0$, we have $p^T Z(q) = q^T F(p) = 0$ and the Cauchy-Schwarz inequality again implies $p = \lambda q$ for some $\lambda > 0$. As Z is homogeneous, $Z(p) = Z(q)$. Therefore Z satisfies (SWA) for arbitrary $\sigma > 0$.

We now introduce a weaker condition than (SWA):

There exists $\sigma > 0$ such that,

$$\text{if } |\delta| < \sigma,\ p, q \in D,\ Z(q) \neq Z(p),\ q^T Z(p) - \delta \leq 0 \ \text{ then } \ p^T Z(q) + \delta \geq 0. \tag{13}$$

Remark 4.3. In fact, (13) is equivalent to each of the following conditions:

There exists $\sigma > 0$ such that,

$$\text{if } \delta \in [0, \sigma),\ p, q \in D,\ Z(q) \neq Z(p),\ q^T Z(p) - \delta \leq 0, \ \text{ then } p^T Z(q) + \delta \geq 0. \tag{14}$$

There exists $\sigma > 0$ such that,

$$\text{if } \delta \in (-\sigma, 0],\ p, q \in D,\ Z(q) \neq Z(p),\ q^T Z(p) - \delta \leq 0, \ \text{ then } p^T Z(q) + \delta \geq 0. \tag{15}$$

Proof. (15) $\Rightarrow$ (13): Indeed, suppose that Z satisfies (15) but not (13). For σ in (15), there are $\delta \in (0, \sigma)$, $p, q \in D$ such that

$$Z(q) \neq Z(p), \quad q^T Z(p) - \delta \leq 0 \quad \text{and} \quad p^T Z(q) + \delta < 0. \tag{16}$$

Set $\delta_1 = -\delta \in (-\sigma, 0)$. Since $p^T Z(q) - \delta_1 < 0$ and Z satisfies (15), $q^T Z(p) + \delta_1 = q^T Z(p) - \delta \geq 0$. This and the second inequality in (16) force $q^T Z(p) = \delta = -\delta_1$. Since $p^T Z(q) - \delta_1 < 0$, we can choose $-\sigma < \delta_2 < \delta_1$ such that $p^T Z(q) - \delta_2 \leq 0$. Again (15) implies $q^T Z(p) + \delta_2 \geq 0$, i.e., $-\delta_1 + \delta_2 \geq 0$, a contradiction. Thus (15) implies (13).

(14) $\Rightarrow$ (13): Indeed, suppose that Z satisfies (14) but not (13). For σ in (14), there are $\delta \in (-\sigma, 0)$, $p, q \in D$ such that

$$Z(q) \neq Z(p), \quad q^T Z(p) - \delta \leq 0 \quad \text{and} \quad p^T Z(q) + \delta < 0. \tag{17}$$

Set $\delta_1 = -\delta \in (0, \sigma)$. Since $p^T Z(q) - \delta_1 < 0$ and Z satisfies (14), $q^T Z(p) + \delta_1 = q^T Z(p) - \delta \geq 0$. This and the second inequality in (17) force $q^T Z(p) = \delta = -\delta_1$. Since $p^T Z(q) - \delta_1 < 0$, we can choose $0 < \delta_2 < \delta_1$ such that $p^T Z(q) - \delta_2 \leq 0$. Again (14) implies $q^T Z(p) + \delta_2 \geq 0$, i.e., $-\delta_1 + \delta_2 \geq 0$, a contradiction. Thus (14) implies (13). □

Clearly (SWA) implies condition (13). The next example shows that the converse is not true.

Example 4.4. Let $n \geq 2$, $1 \leq k < n$,

$$D = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i,\ x_j = 0,\ k+1 \leq j \leq n\},$$

and $$S = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i,\ x_j = 0,\ 1 \leq i \leq k\}.$$

If $Z : D \setminus \{0\} \to S$ satisfies $Z(x) = Z\big(\frac{x}{\|x\|}\big)$, then Z is an excess demand function and satisfies (13) but it does not satisfy Wald's axiom and (SWA). $\square$

It is easily seen that Z is homeogeneous and since $q^T u = 0$ for all $q \in D$ and $u \in S$, it followss that $q^T Z(p) = 0$ for all $p, q \in D$, i.e., Z is an excess demand function and does not satisfy Wald's axiom. Since (SWA) implies Wald's axiom, Z does not satisfy (SWA). Finally if $q^T Z(p) - \delta \leq 0$, then $\delta \geq 0$ so that $p^T Z(q) + \delta \geq 0$, i.e., Z satisfies (13).

Next we present several conditions that are sufficient or necessary for (SWA) or (13).

Proposition 4.5. *Let $Z\colon D \subset \mathbb{R}^n \to \mathbb{R}^n$.*

(a) *If for all $p, q \in D$ with $Z(p) \neq Z(q)$ we have $p^T Z(q) \geq 0$, $q^T Z(p) \geq 0$, and $p^T Z(q) + q^T Z(p) > 0$, then Z satisfies (SWA).*

(b) *If D is a cone, Z is homogeneous and satisfies (SWA), then for all $p, q \in D$ with $Z(p) \neq Z(q)$ we have $p^T Z(q) \geq 0$, $q^T Z(p) \geq 0$, and $p^T Z(q) + q^T Z(p) > 0$.*

Proof. (a) Suppose that $p^T Z(q) \geq 0$, $q^T Z(p) \geq 0$, and $p^T Z(q) + q^T Z(p) > 0$ whenever $p, q \in D$ and $Z(p) \neq Z(q)$. If for $\sigma > 0$, there are $|\delta| < \sigma$ and $p, q \in D$ with $Z(p) \neq Z(q)$ such that $q^T Z(p) - \delta \leq 0$ and $p^T Z(q) + \delta \leq 0$, then adding these inequalities we would have $p^T Z(q) + q^T Z(p) \leq 0$, a contradiction. Thus Z satisfies property (SWA).

(b) Let Z have property (SWA) and let σ be the positive number given in the condition (SWA). Suppose $p, q \in D$, $Z(p) \neq Z(q)$. For any $\delta \in (0, \sigma)$, there is $\lambda > 0$ small enough such that $\lambda q^T Z(p) - \delta \leq 0$. Note that the function $\varphi(\lambda) = \lambda q^T Z(p) - \delta$ is continuous and $\varphi(0) = -\delta < 0$. Hence there is $\lambda_0 > 0$ such that $\varphi(\lambda) \leq 0$ for all $\lambda \in (-\lambda_0, \lambda_0)$.

As $\lambda q \in D$ and Z is homogeneous, $Z(\lambda q) = Z(q) \neq Z(p)$. Since Z satisfies (SWA), we have $p^T Z(\lambda q) + \delta = p^T Z(q) + \delta > 0$. Letting $\delta \to 0$ we get $p^T Z(q) \geq 0$. Likewise, $q^T Z(p) \geq 0$. Furthermore, if $q^T Z(p) = 0$, then taking $\delta = 0$ in the assumption of (SWA) we get $p^T Z(q) > 0$. $\square$

The above results can be easily modified to obtain the following proposition:

Proposition 4.6. *Let P be a cone in $\mathbb{R}^n$ and let $Z\colon P \to \mathbb{R}^n$ be homogeneous. Then Z satisfies* (13) *iff for all $p, q \in P$ with $Z(p) \neq Z(q)$ we have $p^T Z(q) \geq 0$, $q^T Z(p) \geq 0$.*

Proof. *Necessity.* Let Z have property (13) and let σ be the positive number given in the condition (13). Suppose $p, q \in P$, $Z(p) \neq Z(q)$. For any $\delta \in (0, \sigma)$, there is $\lambda > 0$ small enough such that $\lambda q^T Z(p) - \delta \leq 0$. As $\lambda q \in D$ and Z is homogeneous, $Z(\lambda q) = Z(q) \neq Z(p)$. Since Z satisfies (13), we have $p^T Z(\lambda q) + \delta = p^T Z(q) + \delta \geq 0$. Letting $\delta \to 0$ we get $p^T Z(q) \geq 0$. Likewise, $q^T Z(p) \geq 0$.

Sufficiency. Suppose $p^T Z(q) \geq 0$ and $q^T Z(p) \geq 0$ whenever $p, q \in P$, $Z(p) \neq Z(q)$. If for $\sigma > 0$, there are $|\delta| < \sigma$ and $p, q \in D$ with $Z(p) \neq Z(q)$ such that $q^T Z(p) - \delta \leq 0$ and $p^T Z(q) + \delta < 0$, then adding these inequalities we would have $p^T Z(q) + q^T Z(p) < 0$, a contradicting to $p^T Z(q) + q^T Z(p) \geq 0$. Thus Z satisfies property (13). $\square$

Corollary 4.7. *Let $b_1, \ldots, b_k$ be nonzero vectors in $\mathbb{R}^n$ such that*

$$P = \left\{ \textstyle\sum_{i=1}^{k} \lambda_i b_i : \lambda_i > 0,\ i = 1, \ldots, k \right\} \setminus \{0\} \subset \mathbb{R}^n_{>0}.$$

Then a homogeneous function $Z \colon P \to \mathbb{R}^n$ satisfies both Walras' law and (13) *iff $q^T Z(p) = 0$ for all $p, q \in P$.*

Proof. Suppose that Z satisfies Walras' law and equation (13). Fix any $p \in P$, $p = \sum_{i=1}^{k} \lambda_i b_i$ with $\lambda_i > 0$, $i = 1, \ldots, k$. By Walras' law and Proposition 4.6, $0 = p^T Z(p) = \sum_{i=1}^{k} \lambda_i b_i^T Z(p) \geq 0$, it follows that $b_i^T Z(p) = 0$ for all $i = 1, \ldots, k$. Thus $q^T Z(p) = 0$ for all $q \in P$. The converse is clear. $\square$

Example 4.8. Let $\varphi \colon (0, \infty) \to \mathbb{R}$ be an arbitrary function that is always positive or always negative. Let $P \subset \mathbb{R}^2_{>0}$ be a cone and $Z \colon P \to \mathbb{R}^2$ be defined by

$$Z(p_1, p_2) = \left(\varphi\Big(\frac{p_1}{p_2}\Big), -\frac{p_1}{p_2} \varphi\Big(\frac{p_1}{p_2}\Big) \right).$$

Then Z is an excess demand function and satisfies Wald's Axiom. However, Z does not satisfies (13) unless it is constant. $\square$

It is easily seen that Z is homogeneous and for for all $p \in P$ we have

$$p^T Z(p) = p_1 \varphi\Big(\frac{p_1}{p_2}\Big) - p_2 \cdot \frac{p_1}{p_2} \varphi\Big(\frac{p_1}{p_2}\Big) = 0.$$

Thus Z is an excess demand function. Suppose now that

$$p^T Z(q) = p_1 \varphi\Big(\frac{q_1}{q_2}\Big) - p_2 \cdot \frac{q_1}{q_2} \varphi\Big(\frac{q_1}{q_2}\Big) \leq 0 \quad \text{and} \quad q^T Z(p) = q_1 \varphi\Big(\frac{p_1}{p_2}\Big) - q_2 \cdot \frac{p_1}{p_2} \varphi\Big(\frac{p_1}{p_2}\Big) \leq 0.$$

Since φ takes on only one sign, these inequalities yield $p_1 q_2 - p_2 q_1 = 0$, i.e., $p = \lambda q$ for $\lambda = \frac{p_1}{q_1} = \frac{p_2}{q_2} > 0$. Thus $Z(p) = Z(q)$, i.e., Z satisfies Wald's Axiom.

Suppose Z satisfies (13) and is not constant, say, $Z(p) \neq Z(q)$. Since Z is homogeneous, there is no loss of generality in assuming $p = (s, 1)$ and $q = (t, 1)$. According to Proposition 4.6, we have

$$q^T Z(p) = t\varphi(s) - s\varphi(s) = (t - s)\varphi(s) \geq 0,$$

and

$$p^T Z(q) = s\varphi(t) - t\varphi(t) = (s - t)\varphi(t) \geq 0.$$

It follows that $t = s$, and so $p = q$, and hence, $Z(p) = Z(q)$, a contradiction. $\square$

In [4] we considered $\varphi(t) = -t^4 \left(2 + \sin \frac{1}{t}\right) < 0$ for $t > 0$.

Corollary 4.9. *Let $b_1, \ldots, b_k$ be nonzero vectors in $\mathbb{R}^n$ such that*

$$P = \left\{ \textstyle\sum_{i=1}^{k} \lambda_i b_i : \lambda_i > 0,\ i = 1, \ldots, k \right\} \setminus \{0\} \subset \mathbb{R}^n_{>0}.$$

The unique excess demand function defined on P and satisfies (SWA) is the constant map $Z(p) \equiv 0$.

Proof. Suppose Z is an excess demand function on P and satisfies (SWA).

By Corollary 4.7, $q^T Z(p) = 0$ for all $p, q \in P$. If there were $p, q \in P$ with $Z(p) \neq Z(q)$, then for $\delta = 0$ we would get $q^T Z(p) - \delta = 0$ and (SWA) would imply $q^T Z(p) + \delta = 0 > 0$, a contradiction. (An alternative proof is given in the Appendix.) □

Corollary 4.10. *Let Z be a continuous excess demand function defined on a convex cone $P \subset \mathbb{R}^n_{>0}$.*

(a) *If Z satisfies* (13)*, then $p^T Z(q) = 0$ for all $p, q \in P$.*

(b) *If Z satisfies (SWA), then it is a constant map.*

Proof. (a) Assume that Z satisfies (13) and that $p, q \in D$, $Z(p) \neq Z(q)$. Set $x_t = (1-t)p + tq$, $t \in [0,1]$, so $x_0 = p$, $x_1 = q$. According to Proposition 4.6, $p^T Z(x_t) \geq 0$ and $q^T Z(x_t) \geq 0$. Thus $0 = x_t^T Z(x_t) = (1-t)p^T Z(x_t) + tq^T Z(x_t) \geq 0$, whence $p^T Z(x_t) = q^T Z(x_t) = 0$. Letting $t \to 1$ we obtain $p^T Z(q) = 0$. Likewise $q^T Z(p) = 0$.

(b) Suppose that Z is not constant, say $Z(p) \neq Z(q)$. If Z satisfies (SWA), then it satisfies (13) and by part (a), $p^T Z(q) = q^T Z(p) = 0$. This contradicts Proposition 4.5(b). Therefore Z must be constant. □

Finally we state a relationship between the concept s-quasimonotonicity and condition (13).

Theorem 4.11. *Let Z be an excess demand function on a convex cone $P \subset \mathbb{R}^n_{>0}$. Then Z satisfies* (13) *iff $-Z$ is s-quasimonotone.*

Proof. Assume that Z satisfies (13). Let $\sigma > 0$ be given in (13). Suppose that

$$|\delta| < \sigma, \ p, q \in D \quad \text{with} \quad \frac{(q-p)^T}{\|q-p\|}\big(-Z(p)\big) \geq \delta. \tag{18}$$

Then
$$(q-p)^T Z(p) + \delta\|q-p\| \leq 0. \tag{19}$$

Choose $0 < \lambda \leq 1/\|q-p\|$ and let $p' = \lambda p$, $q' = \lambda q$, and $\delta' = -\lambda\delta\|q-p\| = -\delta\|q'-p'\|$. Then $|\delta'| = \lambda|\delta|\|q-p\| \leq |\delta| < \sigma$. Multiplying both sides of (19) by λ gives $(q'-p')^T Z(p') - \delta' = q'^T Z(p') - \delta' \leq 0$. Note that Z is homogeneous and satisfies Walras' Law! As Z satisfies (13), we get $p'^T Z(q') + \delta' \geq 0$. This implies

$$(q'-p')^T Z(q') - \delta' = (q'-p')^T Z(q) - \delta' \leq 0.$$

Thus $(q-p)^T Z(q) + \delta\|q-p\| \leq 0$, i.e., $(q-p)^T/\|q-p\|\big(-Z(q)\big) \geq \delta$. Hence $-Z$ is s-quasimonotone.

Suppose now that $-Z$ is s-quasimonotone w.r.t. $\sigma > 0$, $\delta \in [0, \sigma)$, and $p, q \in P$, $Z(p) \neq Z(q)$ such that
$$q^T Z(p) - \delta \leq 0. \tag{20}$$

Choose $\lambda > 1$ such that $\|q - \lambda p\| \geq 1$. Note that $p \in \mathbb{R}^n_{>0}$, so $\|q-\lambda p\| \geq \lambda\|p\| - \|q\| \to \infty$ as $\lambda \to \infty$. Inequality (20) is equivalent to $q^T Z(p') - \delta \leq 0$ where $p' := \lambda p$. Z satisfies Walras' Law, so we have $(q-p')^T\big(-Z(p')\big) + \delta \geq 0$, which can be rewritten as

$$\frac{(q-p')^T}{\|q-p'\|}\big(-Z(p')\big) + \frac{\delta}{\|q-p'\|} \geq 0.$$

Since $0 \le \frac{\delta}{\|q-p'\|} \le \delta < \sigma$ and $-Z$ is s-quasimonotone w.r.t. $\sigma > 0$,

$$\frac{(q-p')^T}{\|q-p'\|}\big(-Z(q)\big) + \frac{\delta}{\|q-p'\|} \ge 0,$$

or, equivalently, $$p'^T Z(q) + \delta = \lambda p^T Z(q) + \delta \ge 0.$$

It follows that $p^T Z(q) + \delta \ge p^T Z(q) + \delta/\lambda \ge 0$. Finally Remark 4.3 completes the proof. □

A. Appendix

The Hahn-Banach theorem

A1. First formulation: [7], p. 3: *Let X be a normed vector space and denote by X^* the dual space of X. For every $v_0 \in X$ there exists $\xi_0 \in X^*$ such that*

$$\|\xi_0\| = \|v_0\| \quad \textit{and} \quad \langle \xi_0, v_0\rangle = \|v_0\|^2. \tag{21}$$

A2. Second formulation: [1], p. 76: *Let X be a normed vector space and $v_0 \in X$, $v_0 \neq 0$. Then there is an element $\xi_0 \in X^*$ such that $\|\xi_0\| = 1$ and $\langle \xi_0, v_0\rangle = \|v_0\|$.*

A3. Details of the proof of Theorem 2.4: Applying (21) with $v_0 := (x_1 - x_0)/\|x_1 - x_0\|$ we find that there is $\xi_0 \in X^*$ satisfying

$$\|\xi_0\| = 1 \quad \text{and} \quad \langle \xi_0, v_0\rangle = \|v_0\| = 1.$$

Setting $\xi = -\delta\xi_0$ we get $\|\xi\| = |\delta| \cdot \|\xi_0\| = |\delta|$ and $\langle \xi, v_0\rangle = -\delta\langle \xi_0, v_0\rangle = -\delta$. □

The Mean Value Theorem

A4. [1], p. 92: It can also be easily shown that the formula $f(b) - f(a) = f'(c)(b-a)$ remains valid for scalar functions $f(x)$ whose argument belongs to an arbitrary topological linear space.

In this case $[a,b] = \{x : x = a + t(b-a),\ 0 \le t \le 1\}$, and the open interval (a,b) is defined analogously; the differentiability may be understood in Gateaux' sense. Putting $\varphi(t) = f\big(a + t(b-a)\big)$, we reduce the proof to the case of one real variable.

For vector-valued functions the situation is quite different (a counterexample is given in [1]).

A5. (The Mean Value Theorem) *Let X and Y be normed linear spaces, and let an open set $U \subset X$ contain a closed interval $[a,b]$. If $f\colon U \to Y$ is a Gateaux differentiable function at each point $x \in [a,b]$, then*

$$\|f(b) - f(a)\| \le \sup_{c\in[a,b]} \|f'(c)\| \cdot \|b-a\|.$$

A6. Proof of Theorem 3.5. Conversely, suppose that f' is strictly s-quasimonotone. It follows from Theorem 2.4 that there exists $\epsilon > 0$ such that $f' + \xi$ is strictly pseudomonotone for all $\xi \in X^*$ satisfying $\|\xi\| < \epsilon$. Assume that f is not strictly s-quasiconvex w.r.t.$\sigma = \epsilon$. Then there are $x_0, x_1 \in D$, $x_0 \neq x_1$, $\lambda \in (0,1)$, $x_\lambda = (1-\lambda)x_0 + \lambda x_1$, and $|\delta| < \epsilon$ such that

$$\frac{f(x_1) - f(x_0)}{\|x_1 - x_0\|} \ge \delta \quad \text{and} \quad \frac{f(x_1) - f(x_\lambda)}{\|x_1 - x_\lambda\|} \le \delta. \tag{22}$$

Let $x_t := (1-t)x_0 + tx_1$, $\psi(t) = \frac{f(x_t)}{\|x_1 - x_0\|}$ and $v = \frac{x_1 - x_0}{\|x_1 - x_0\|}$. Then (22) is equivalent to

$$\psi(1) - \psi(0) \geq \delta \geq \frac{\psi(1) - \psi(\lambda)}{1 - \lambda}.$$

These inequalities imply $\psi(1) - \psi(0) \geq \lambda\delta + (1-\lambda)\delta \geq \lambda\delta + \psi(1) - \psi(\lambda)$, whence

$$\frac{\psi(\lambda) - \psi(0)}{\lambda} \geq \delta.$$

Applying the Mean Value Theorem to ψ we obtain $\alpha \in (0, \lambda)$ and $\beta \in (\lambda, 1)$ such that $\psi'(\alpha) = \big[\psi(\lambda) - \psi(0)\big]/\lambda$ and $\psi'(\beta) = \big[\psi(1) - \psi(\lambda)\big]/(1-\lambda)$. We then have $\psi'(\alpha) \geq \delta \geq \psi'(\beta)$, i.e.,

$$\big\langle f'(x_\alpha), v \big\rangle \geq \delta \geq \big\langle f'(x_\beta), v \big\rangle. \tag{23}$$

The Hahn-Banach theorem guarantees the existence of $\xi \in X^*$ satisfying

$$\|\xi\| = |\delta| \quad \text{and} \quad \big\langle \xi, v \big\rangle = -\delta.$$

Hence (23) can be rewritten as $\big\langle f'(x_\alpha) + \xi, v \big\rangle \geq 0 \geq \big\langle f'(x_\beta) + \xi, v \big\rangle$, or, equivalently,

$$\big\langle f'(x_\alpha) + \xi, x_\beta - x_\alpha \big\rangle \geq 0 \geq \big\langle f'(x_\beta) + \xi, x_\beta - x_\alpha \big\rangle$$

since $v = (x_\beta - x_\alpha)/\|x_\beta - x_\alpha\|$. This is impossible because condition $\|\xi\| = |\delta| < \epsilon$ implies that $f' + \xi$ is is strictly pseudomonotone. $\square$

A7. Details of the proof of Theorem 3.7: If D is bounded, $\|x\| \leq K$ for all $x \in D$, and $\|x\|_0 \leq \beta\|x\|$ for all $x \in X$, then $f(x) = \|x\|_0^2 \leq \beta^2\|x\|^2 \leq \beta^2 K$, so f is bounded on D. Furthermore, for $x, y \in D$, we have

$$\begin{aligned} \big|f(x) - f(y)\big| &= \big|\|x\|_0^2 - \|y\|_0^2\big| = \big|\|x\|_0 - \|y\|_0\big|(\|x\|_0 + \|y\|_0) \\ &\leq \|x - y\|_0(\|x\|_0 + \|y\|_0) \leq \beta\|x - y\|(\beta\|x\| + \beta\|y\|) \leq 2\beta^2 K\|x - y\|. \end{aligned}$$

Thus f is Lipschitz on D. $\square$

Equivalent conditions for a norm to be strictly convex

A8. [6], p. 35: If a linear normed space enjoys the property "Any nonidentically zero continuous linear functional takes a maximum value on the closed unit ball at most at one point", it is called astrictly convex space. In terms of supporting hyperplanes, this property may be expressed as: *distinct boundary points of the closed unit ball have distinct supporting hyperplanes.*

A linear normed space is strictly convex if and only if one of the following equivalent properties holds:

(a) If $\|x + y\| = \|x\| + \|y\|$ and $x \neq 0$, there is $t \geq 0$ such that $y = tx$.

(b) If $\|x\| = \|y\| = 1$ and $x \neq y$, then $\|\lambda x + (1-\lambda)y\| < 1$ for all $\lambda \in (0, 1)$.

(c) If $\|x\| = \|y\| = 1$ and $x \neq y$, then $\|(x + y)/2\| < 1$.

(d) The function $x \mapsto \|x\|^2$, $x \in X$, is strictly convex.

A9. (Asplund) *Let X be a reflexive Banach space. Then there exists an equivalent norm on X, such that, under this new norm, X and X^* are strictly convex, that is, X and X^* are simultaneously smooth and strictly convex.*

A10. Proof of Lemma 3.8. Suppose that G is s-quasimonotone w.r.t.$\sigma_0 > 0$ and $\|F(x)\| \leq L$ for all $x \in D$. Set $\bar{c} = \sigma_0/2L$. We show that $cF + G$ is strictly s-quasimonotone w.r.t. $\sigma_0/2$ for all $0 < c \leq \bar{c}$. Assume that $x_0, x_1 \in D$, $x_0 \neq x_1$, $|\delta| < \sigma_0/2$ and

$$\big\langle (cF + G)(x_0), v \big\rangle \geq \delta \quad \text{where} \quad v = \frac{x_1 - x_0}{\|x_1 - x_0\|}.$$

Then $\big\langle G(x_0), v \big\rangle \geq \delta - c\big\langle F(x_0), v \big\rangle$. Since $|\delta| < \sigma_0/2$ and $0 < c \leq \sigma_0/2L$,

$$\big|\delta - c\big\langle F(x_0), v \big\rangle\big| \leq |\delta| + cL < \frac{\sigma_0}{2} + \frac{\sigma_0}{2} = \sigma_0.$$

As G is s-quasimonotone w.r.t.σ_0, we have $\big\langle G(x_1), v \big\rangle \geq \delta - c\big\langle F(x_0), v \big\rangle$ implying

$$c\big\langle F(x_0), v \big\rangle \geq \delta - \big\langle G(x_1), v \big\rangle.$$

Since cF is strictly monotone, it follows that $c\big\langle F(x_1), v \big\rangle > c\big\langle F(x_0), v \big\rangle$ and hence, $c\big\langle F(x_1), v \big\rangle > \delta - \big\langle G(x_1), v \big\rangle$. Therefore $\big\langle (cF + G)(x_1), v \big\rangle > \delta$, i.e., $cF + G$ is strictly s-quasimonotone. □

A11. Proof of Theorem 3.9. Suppose $\|F(x)\| \leq L$ for all $x \in D$. Let $\epsilon > 0$. According to Lemma 3.8, there is $\bar{c} > 0$ such that $cF + G$ is strictly s-quasimonotone for all $0 < c \leq \bar{c}$. Choose $0 < c \leq \bar{c}$ with $cL < \epsilon$. Then the function $H = cF + G$ is strictly s-quasimonotone and

$$\|H(x) - G(x)\| = c\|F(x)\| \leq cL < \epsilon \quad \text{for all } x \in D.$$

Suppose now that $F(x) \in J(x)$ for all $x \in D$ and F is not strictly monotone, then there are $x, y \in D$, $x \neq y$, such that

$$\big\langle F(x) - F(y), x - y \big\rangle \leq 0. \tag{24}$$

Set $\xi = F(x)$, $\eta = F(y)$. If $x = 0$, then $\xi = 0$ (since $\|\xi\| = \|x\|_0$) and (24) reduces to $\big\langle \eta, y \big\rangle \leq 0$, which contradicts the fact that $\big\langle \eta, y \big\rangle = \|y\|_0 > 0$ since $y \neq x = 0$. Likewise $y \neq 0$ and $\eta \neq 0$. According to (24),

$$\begin{aligned} 0 &\geq \big\langle \xi - \eta, x - y \big\rangle = \big\langle \xi, x \big\rangle - \big\langle \xi, y \big\rangle - \big\langle \eta, x \big\rangle + \big\langle \eta, y \big\rangle \\ &\geq \|x\|_0^2 - \|\xi\|_0\|y\|_0 - \|\eta\|_0\|x\|_0 + \|y\|_0^2 \\ &= \|x\|_0^2 - 2\|x\|_0\|y\|_0 + \|y\|_0^2 = (\|x\| - \|y\|_0)^2. \end{aligned}$$

It follows that $\|x\|_0 = \|y\|_0$ and $\big\langle \xi, y \big\rangle = \big\langle \eta, x \big\rangle = \|x\|_0\|y\|_0$.

Hence $\|x\|_0 \cdot \|x + y\|_0 \geq \big\langle \xi, x + y \big\rangle = \big\langle \xi, x \big\rangle + \big\langle \xi, y \big\rangle = \|x\|_0^2 + \|x\|_0\|y\|_0$, whence, $\|x + y\|_0 \geq \|x\|_0 + \|y\|_0$ and so $\|x + y\|_0 = \|x\|_0 + \|y\|_0$. Strict convexity of $\|\cdot\|_0$ implies that $x = ty$ for some $t > 0$. This combined with $\|x\|_0 = \|y\|_0$ implies $t = 1$, i.e., $x = y$. This is impossible since $x \neq y$. □

Versions of Wald's Axiom

A12. Alternative proof of Corollary 4.9. Suppose that $Z\colon \mathbb{R}^n_{>0} \to \mathbb{R}^n$ is an excess demand function and satisfies (SWA). Fix an arbitrary $p \in \mathbb{R}^n_{>0}$ and set $Z(p) = (z_1, \dots, z_n)$. If $Z(p)$ has a negative component, say $z_1 < 0$, then for $q_t = (t, 1, \dots, 1) \in \mathbb{R}^n_{>0}$ we have $q_t^T Z(p) = tz_1 + z_2 + \cdots + z_n \to -\infty$ as $t \to \infty$.

By Proposition 4.5, we must have $Z(p) = Z(q_t)$ for all t large enough. But then by Walras' law, $tz_1 + z_2 + \cdots + z_n = q_t^T Z(p) = q_t^T Z(q_t) = 0$ for all t large enough, a contradiction. Thus $z_k \geq 0$ for all $k = 1, \dots, n$. This and Walras' Law $p^T Z(p) = p_1 z_1 + \cdots + p_n z_n = 0$ imply $p_1 z_1 = \cdots = p_n z_n = 0$ so that $z_1 = \cdots = z_n = 0$. Thus Z is the constant map 0 and it is easy to see that this map is an excess demand function and satisfies (SWA). □

A13. Second proof of Corollary 4.10 (b): Let $F\colon P \to \mathbb{R}^n$ be a continuous excess demand function satisfying (SWA). Suppose that F is not constant, say $F(p) \neq F(q)$. By Proposition 4.5 we can assume $p^T F(q) > 0$. Let $x_t = (1-t)p + tq$, $t \in [0, 1]$, so $x_0 = p$, $x_1 = q$. Set

$$\alpha = \sup\big\{t \in [0,1] : F(x_t) = F(p)\big\}.$$

Such an α exists and by continuity of F we have $F(x_\alpha) = F(p)$. Hence $\alpha < 1$. Likewise, set

$$\beta = \inf\big\{t \in [\alpha,1] : F(x_t) = F(q)\big\}.$$

Then $F(x_\beta) = F(q)$ and $\beta > \alpha$. For all $\gamma \in (\alpha, \beta)$ we have $F(x_\gamma) \neq F(p)$ and $F(x_\gamma) \neq F(q)$. So, we can assume, by replacing p with x_α and q with x_β if necessary, that $F(x_t) \neq F(p)$ and $F(x_t) \neq F(q)$ for all $t \in (0, 1)$. Again, according to Proposition 4.5, $p^T F(x_t) \geq 0$ and $q^T F(x_t) \geq 0$. Thus

$$0 = x_t^T F(x_t) = (1-t)p^T F(x_t) + tq^T F(x_t) \geq 0,$$

whence $p^T F(x_t) = q^T F(x_t) = 0$. Letting $t \to 1$ we obtain $p^T F(q) = 0$, contradicting assumption $p^T F(q) > 0$. Therefore F must be constant. □

A14. Example. Let $F\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} \frac{\pi}{2} + \tan^{-1} x & \text{if } x \leq 0, \\ \frac{\pi}{2} + \sin x & \text{if } x > 0. \end{cases}$$

We have
$$F'(x) = \begin{cases} \frac{1}{1+x^2} & \text{if } x \leq 0, \\ \cos x & \text{if } x > 0. \end{cases}$$

It is clear that $|F(x)| < \dfrac{\pi}{2} - 1 \implies x < 0 \implies F'(x) > 0$. Thus F satisfies condition (11) and by Theorem 2.5, F is strictly s-quasimonotone w.r.t. $\sigma = -1 + \frac{\pi}{2}$.

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References

[1] V. M. Alekseev, V. M. Tikhomirov, S. Fomin: *Optimal Control*, Springer, Berlin (1987).

[2] P. T. An: *A new type of stable generalized convex functions*, J. Inequalities Pure Appl. Math. 7 (2006) 10 pp.

[3] P. T. An: *Stability of generalized monotone maps with respect to their characterizations*, Optimization 55 (2006) 289–299.

[4] P. T. An, V. T. T. Binh: *Stability of excess demand functions with respect to a strong version of Wald's Axiom*, Asia-Pacific J. Oper. Res. 26 (2009) 523–532.

[5] M. Avriel, W. E. Diewert, S. Schaible, I. Zang: *Generalized Concavity*, Plenum Publishing Corporation, New York (1988).

[6] V. Barbu, T. Precupanu: *Convexity and Optimization in Banach Spaces*, 4th ed., Springer, Berlin (2012).

[7] H. Brezis: *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, Berlin (2011).

[8] L. Brighi: *A stronger criterion for the Weak Weak Axiom*, J. Math. Economics 40 (2004) 93–103.

[9] G. Debreu: *Theory of Value*, Yale University Press, New Haven (1959).

[10] W. Hildenbrand, M. Jerison: *The demand theory of the weak axioms of revealed preference*, Econ. Letters 29 (1989) 209–213.

[11] H. S. Houthakker: *Revealed preference and the utility function*, Economica 17 (1953) 159–174.

[12] R. John: *Variational inequalities and pseudomonotone functions: some characterizations*, in: *Generalized Convexity, Generalized Monotonicity*, J.-P. Crouzeix et al. (eds.), Kluwer Academic Publishers, Dordrecht (1998) 291–301.

[13] R. John, A. Zaslavski: *Uses of generalized convexity and generalized monotonicity in economics*, in: *Handbook of Generalized Convexity and Generalized Monotonicity*, N. Hadjisavvas, S. Komlósi and S. Schaible (eds.), Springer, Berlin (2005).

[14] S. Karamardian: *Complementary problems over cones with monotone and pseudomonotone maps*, J. Optimization Theory Appl. 18 (1976) 445–454.

[15] S. Karamardian, S. Schaible: *Seven kinds of generalized monotone maps*, J. Optimization Theory Appl. 66 (1990) 37–46.

[16] S. Karamardian, S. Schaible, J. P. Crouzeix: *Characterizations of generalized monotone maps*, J. Optimization Theory Appl. 76 (1993) 399–413.

[17] P. C. Nicola: *Mainstream Mathematical Economics in the 20th Century*, Springer, Berlin (2013).

[18] A. Mossin: *A mean demand function and individual demand functions confronted with the weak and the strong axioms of revealed preference: an empirical test*, Economica 40 (1972) 177–192.

[19] R. R. Phelps: *Convex Functions, Monotone Operators and Differentiability*, 2nd ed., Springer, Berlin (1993).

[20] H. X. Phu, P. T. An: *Stable generalization of convex functions*, Optimization 38 (1996) 309–318.

[21] A. W. Roberts, D. E. Varberg: *Convex Functions*, Academic Press, New York (1973).

[22] P. A. Samuelson: *Foundations of Economic Analysis*, Harvard University Press, Cambridge (1947).