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Asymptotic Hyers-Ulam Stability or Superstability by Unilateral Perturbations on the Concavity Side for Generalized Linear Equations

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In a recent paper [*Asymptotic Hyers-Ulam stability or superstability for generalized linear equations by unilateral perturbations*, J. Convex Analysis 26(2) (2019) 543–562] we considered in relation to the famous problem of Ulam "Give conditions in order for a linear mapping near an approximated linear mapping to exist" the stability or superstability of a generalized linear equation

$$||f(x+y)-f(x)-f(y)|| = B[\phi(x)+\phi(y)]$$

by perturbations on the convexity side, named right perturbations. In this paper we continue this research for perturbations on the concavity side with some hypotheses of concavity.

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1. Introduction

Compared to some results of Aoki, Gajda, Gavruta, Hyers, Rassias and others ([1], [2], [3], [6], [8]) we propose three changes:

(1) Firstly when the range of f is ordered and satisfies properties similar to those of the set of real numbers, R, in place of $||f(x+y)-f(x)-f(y)|| \leq H_1(x,y)$, we propose to explore "left" unilateral perturbations of the form

$$-H_1(x,y) \leq \; f(x+y)-f(x)-f(y).$$

Defining $-H_1 = H$ and $H(x,y) = B[\phi(x)+\phi(y)]$, we will write

$$f(x+y)-f(x)-f(y) \geq H(x,y).$$

For reasons that became evident after Theorem 2.9 in [7] we named the right perturbations by "convexity side"; after a similar theorem in this paper, we will name the left ones by "concavity side".

(2) We introduce the following special sense of "closeness": the functions f and g are said to be "near" at infinity if they are asymptotically close, that is, if

$$\lim_{x\to\infty} [f(x)-g(x)] = 0.$$

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Using the above concept, we obtain some stability or superstability results for the generalized linear equation

$$||f(x+y) - f(x) - f(y)|| = B[\phi(x) + \phi(y)]$$

in the presence of some concavity hypothesis.

(3) Finally we remark that the functions in Hyers' theorem [6] and those of Gajda, Gavruta and Rassias ([2], [3], [8]) implicitly satisfy the condition $f(0) = 0$, whereas our theorems are applicable to a broader class of functions that are not forced to vanish at 0.

Our paper is organized as follows: we begin with Theorem 2.1 and Corollary 2.2 that give necessary and sufficient conditions for a concave function to have an asymptote, that is to be asymptotically close to a linear or affine function, or, namely, to be additive.

In the following we will deal more generally with functions satisfying

$$f(x+y) - f(x) - f(y) \geq B[\phi(x) + \phi(y)], \tag{1}$$

but given that we first establish our results for concave functions for which the difference $f(2x) - 2f(x)$ is particularly important, see Lemmas 2.4, 3.1 and 3.2, we will consider this relation only for $x = y$, that is, $f(2x) - 2f(x) \geq 2B\phi(x)$.

Then we generalize the above theorem to functions $f\colon [0; +\infty[\to R$ satisfying (1), where $\phi\colon [0; +\infty[\to R$ satisfies

$$\phi(2x) \leq \beta\phi(x) \text{ or } \ \phi(2x) \geq \beta\phi(x),$$

for $\beta \neq 2$, in order to examine if f can be asymptotically close, or even equal, to any $ax - b - k\phi(x)$, which can be interpreted as Hyers-Ulam stability or superstability. See for this Theorem 3.15, one of our main result.

Some special cases of this Theorem, where $\phi(2x) = \beta\phi(x)$ or $\phi(x) = x^p$ are considered in Theorems 3.17 and Theorem 3.18.

The condition $\lim_{x\to\infty} \phi(x) = +\infty$ that appears in our lemmas and in some parts of our theorems, generalizes Aoki's results [1], that first allowed growth of the form $K(||x^p|| + ||y^p||)$ for the norm of the Cauchy difference $f(x+y) - f(x) - f(y)$ where $0 \leq p < 1$, generalizing so Hyers' idea in [6], where he considered the case of $p = 0$.

These results will be carried over to obtain some asymptotic Hyers-Ulam stability or superstability results for the exponential equation

$$\frac{f(x+y)}{f(x)f(y)} = 1, \quad \text{or its generalized version} \quad \frac{f(x+y)}{f(x)f(y)} = [\phi(x)\phi(y)]^B$$

by means of unilateral perturbations in a forthcomming paper.

2. Stability or superstability by unilateral perturbations on the concavity side for generalized linear mappings

The first part of the following theorems may be regarded as asymptotic stability results, while the second part concerns superstability results of the linear equation in the sense of Hyers-Ulam.

Theorem 2.1. *A concave function* $f\colon [0;+\infty[\ \to R$ *has an asymptote in* $+\infty$ *if and only if,* $b = \inf\{f(x+y) - f(x) - f(y) : x \geq 0, y \geq 0\}$ *is finite. In this case, the equation of the asymptote is* $y = ax - b$ *, where* $a = \lim_{x\to\infty}(f(x)/x)$ *(asymptotic stability). If moreover* $f(0) = -b$*, then* $f(x) = ax - b$ *(superstability result).*

Corollary 2.2. *A concave function* $f\colon [0;+\infty[\ \to R$ *is additive if and only if, for every* $x > 0$ *and every* $y > 0$*,* $f(x+y) - f(x) - f(y) \geq 0$ *and* $f(0) = 0$*.*

Corollary 2.3. *Let* $f\colon [0;+\infty[\ \to R$ *be a concave function satisfying the inequality* $f(x+y) - f(x) - f(y) \geq B$ *for every* $x \geq 0$ *and every* $y \geq 0$*. Then the function* f *is asymptotically close to an affine function* $y = ax - b$*, where*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} \quad \text{and} \quad b = \inf\{f(x+y) - f(x) - f(y)\}.$$

If moreover $f(0) = -B$*, then* $f(x) = ax - B$*.*

We begin the proof of the above theorem and corollaries by proving some auxiliary results stated in the following lemmas:

Lemma 2.4. *For a concave function* $f\colon [0;+\infty[\ \to R$ *we have*

$$\inf\{f(x+y) - f(x) - f(y) : x \geq 0,\ y \geq 0\} = \lim_{x\to\infty}[f(2x) - 2f(x)]$$

and

$$\sup\{f(x+y) - f(x) - f(y) : x \geq 0,\ y \geq 0\} = -f(0).$$

Proof of Lemma 2.4. Define $g(x) = -f(x)$; $g(x)$ is convex and by Lemma 2.4 of [7]

$$\sup\{g(x+y) - g(x) - g(y) : x \geq 0, y \geq 0\} = \lim_{x\to\infty}[g(2x) - 2g(x)]$$

and

$$\inf\{g(x+y) - g(x) - g(y) : x \geq 0, y \geq 0\} = -g(0);$$

as for any numerical set A, $\sup(-A) = -\inf(A)$ and $\inf(-A) = -\sup(A)$, we have

$$\begin{aligned}
&\inf\{f(x+y) - f(x) - f(y) : x \geq 0,\ y \geq 0\}\\
&\qquad = -\sup\{g(x+y) - g(x) - g(y) : x \geq 0, y \geq 0\}\\
&\qquad = -\lim_{x\to\infty}[g(2x) - 2g(x)] = \lim_{x\to\infty}[f(2x) - 2f(x)], \quad \text{and}\\
&\sup\{f(x+y) - f(x) - f(y) : x \geq 0,\ y \geq 0\}\\
&\qquad = -\inf\{g(x+y) - g(x) - g(y) : x \geq 0,\ y \geq 0\} = -[-g(0)] = -f(0).
\end{aligned}$$

This completes the proof. □

Lemma 2.5. *If* $f\colon [0;+\infty[\ \to R$ *and* $g\colon [0;+\infty[\ \to R$ *are two concave functions satisfying* $\lim_{x\to\infty}[f(x) - g(x)] = 0$*, then*

$$\inf\{f(x+y) - f(x) - f(y) : x \geq 0, y \geq 0\} = \inf\{g(x+y) - g(x) - g(y) : x \geq 0, y \geq 0\}$$

In particular, if $g(x) = ax - b$ *is an asymptote for* f*, then*

$$\inf\{f(x+y) - f(x) - f(y) : x \geq 0, y \geq 0\} = b = -g(0).$$

Proof of Lemma 2.5. From

$$g(2x)-2g(x)=[g(2x)-f(2x)]+[2f(x)-2g(x)]+[f(2x)-2f(x)]$$

and $\lim_{x\to\infty}[f(x)-g(x)]=0$, we deduce that

$$\lim_{x\to\infty}[g(2x)-2g(x)]=\lim_{x\to\infty}[f(2x)-2f(x)].$$

By Lemma 2.4 this means that

$$\inf\{f(x+y)-f(x)-f(y):x\geq 0,y\geq 0\}=\inf\{g(x+y)-g(x)-g(y):x\geq 0,y\geq 0\}.$$

If $g(x)=ax-b$ is an asymptote for f, then

$$\inf\{f(x+y)-f(x)-f(y):x\geq 0,y\geq 0\}=\inf\{a(x+y)-b-ax+b-ay+b\}=b.$$

In particular, if $\lim_{x\to\infty}[f(2x)-2f(x)]=\lim_{x\to\infty}[g(2x)-2g(x)]\geq 0$, f and g are super-additive in the sense of Hille-Phillips [5], that is, $f(x+y)\geq f(x)+f(y)$. □

We have now all the ingredients to prove the statement of Theorem 2.1.

Proof of Theorem 2.1.

Necessity: Assume that $f\colon [0,+\infty[\,\to R$ is concave and has $y=ax-b$ as asymptote. By Lemma 2.5, $\inf\{f(x+y)-f(x)-f(y):x\geq 0,\ y\geq 0\}=b$. Hence b is finite.

Sufficiency: Assume that $\inf\{f(x+y)-f(x)-f(y):x\geq 0,y\geq 0\}=b$ is finite. Let $g=-f$. The function g is convex, and

$$\begin{aligned}&\sup\{g(x+y)-g(x)-g(y):x\geq 0,\ y\geq 0\}\\&\qquad=-\inf\{f(x+y)-f(x)-f(y):x\geq 0,y\geq 0\}=-b\end{aligned}$$

is finite; by Theorem 2.1 of [7], the function g has an asymptote of the form $y=a_1x+b$, where $a_1=\lim_{x\to\infty}(g(x)/x)$. But f has an asymptote if and only if $g=-f$ has an asymptote. So f has an asymptote with equation $y=-a_1x-b=ax-b$ where $a=\lim_{x\to\infty}(f(x)/x)$. If $f(0)=-b$, then $g(0)=b=-\sup\{g(x+y)-g(x)-g(y)\}$ and by Theorem 2.1 of [7] again, $g(x)=a_1x+b$, so $f(x)=-a_1x-b=ax-b$.

Remark 2.6. Without the concavity of f, the result of Theorem 2.1 does not hold: indeed, the function $f(x)=-\sqrt{x}$ is not concave and satisfies $f(x+y)-f(x)-f(y)\geq 0$ for all $x\geq 0$ and $y\geq 0$. Therefore $\inf\{f(x+y)-f(x)-f(y):x\geq 0,y\geq 0\}\geq 0$, but f has no asymptote.

Remark 2.7. If $y=ax-b$ is an asymptote in $+\infty$ for a concave function $f\colon]0;+\infty[\,\to R$, we have by Lemma 2.5

$$\inf\{f(x+y)-f(x)-f(y):x\geq 0,y\geq 0\}=b;$$

but if f is not concave, this does not imply that

$$\inf\{f(x+y)-f(x)-f(y):x\geq 0,\ y\geq 0\}=b.$$

This is the case for the next function defined by

$$f(x)=\begin{cases}cx+d & \text{if } 0<x\leq\gamma\\ ax-b & \text{if } x>\gamma\end{cases}$$

with $\gamma > 0$, $a\gamma - b = c\gamma + d$ and $a > c > 0$. It has the asymptote $y = ax - b$, is not concave, but convex and, applying Lemma 2.4 of [7],

$$\inf\{f(x+y) - f(x) - f(y) : x \geq 0, y \geq 0\} = -d.$$

But the previous conditions imply $(a - c)\gamma = b + d$, then $b + d > 0$, that is $b \neq -d$!

Proof of Corollary 2.2. The necessity part of the claim is clear.
Conversely the condition $f(x + y) - f(x) - f(y) \geq 0$ implies that

$$b = \inf\{f(x+y) - f(x) - f(y) : x \geq 0,\ y \geq 0\} \geq 0.$$

If $f(0) = 0$, then

$$b = \inf\{f(x+y) - f(x) - f(y) : x \geq 0,\ y \geq 0\} \leq f(0+0) - f(0) - f(0) = 0,$$

so $b = 0$. As $f(0) = 0 = -b$, by Theorem 2.1, $f(x) = ax$. Hence, it is additive.

Proof of Corollary 2.3. The condition $f(x + y) - f(x) - f(y) \geq B$ implies that $b = \inf\{f(x + y) - f(x) - f(y) : x \geq 0,\ y \geq 0\}$ is finite and $\geq B$.

By Theorem 2.1 f has asymptote $y = ax - b$. If $f(0) = -B$,

$$\begin{aligned} b &= \inf\{f(x+y) - f(x) - f(y) : x \geq 0,\ y \geq 0\} \\ &\leq f(0+0) - f(0) - f(0) \leq (-B + B + B) = B. \end{aligned}$$

Hence $b \leq B$. From $b \geq B$ and $b \leq B$, it follows $b = B$, so $f(0) = -B = -b$, and by Theorem 2.1, $f(x) = ax - B$ for all $x \geq 0$. □

3. Main results

In the following, we will consider the generalized, left perturbed, linear equation

$$f(x+y) - f(x) - f(y) \geq B[\phi(x) + \phi(y)],$$

with ϕ: $[0; +\infty[\to R$, satisfying relations as $\phi(2x) \leq \beta\phi(x)$, or $\phi(2x) \geq \beta\phi(x)$ for $\beta \neq 2$, with a concavity hypothesis, in order to obtain stability or superstability results.

Since the difference $f(2x) - 2f(x)$ is a very important feature for concave functions, see Lemma 2.4, we consider in Lemmas 3.1, 3.5 and 3.6 this relation only for $x = y$, that is, $f(2x) - 2f(x) \geq 2B\phi(x)$.

The simplest functions satisfying these relations are the functions $\phi(x) = x^p$ for $p \neq 1$ and $f(x) = ax + k\phi(x)$ with $\beta = 2^p$ and $2B = k(2^p - 2)$ (in fact $\phi(2x) = \beta\phi(x)$).

It is natural to begin the study with functions of the form $g(x) = f(x) + k\phi(x)$, which have an asymptote. This leads us to Lemmas 3.1, 3.2, 3.5, 3.6 that justify the special role of $k = 2B/(2 - \beta)$ in Theorem 3.15.

Lemma 3.1. *Let f and ϕ be two functions defined on $[0; +\infty[$ and let $(x_n) \to +\infty$, be a sequence satisfying*

$$\left[\lim_{n\to+\infty} \phi(x_n) = +\infty \quad or \quad \lim_{n\to+\infty} \phi(x_n) = -\infty\right] \quad and \quad f(2x_n) - 2f(x_n) \geq 2B\phi(x_n).$$

Suppose that for any real k the function $g(x) = f(x) + k\phi(x)$ has an asymptote in $+\infty$, and that there exists a real β, satisfying $\forall n \geq n_0 \in N$, either one of the following relations:

$$\text{(i) } \big[\phi(2x_n) \leq \beta\phi(x_n) \textit{ and } k < 0\big] \textit{ or}$$

$$\text{(ii) } \big[\phi(2x_n) \geq \beta\phi(x_n) \textit{ and } k > 0\big] \textit{ or } \text{ (iii) } \phi(2x_n) = \beta\phi(x_n).$$

Then the following statements hold true:

(1) *if* $\lim_{n\to+\infty} \phi(x_n) = +\infty$, *then* $2B + k(\beta - 2) \leq 0$, *that is* $k \leq \frac{2B}{2-\beta}$ *if* $\beta > 2$, $k \geq \frac{2B}{2-\beta}$ *if* $\beta < 2$, *and* $B \leq 0$ *if* $\beta = 2$*;*

(2) *if* $\lim_{n\to+\infty} \phi(x_n) = -\infty$, *then* $2B + k(\beta - 2) \geq 0$, *that is* $k \leq \frac{2B}{2-\beta}$ *if* $\beta < 2$, $k \geq \frac{2B}{2-\beta}$ *if* $\beta > 2$, *and* $B \geq 0$ *if* $\beta = 2$.

Proof of Lemma 3.1. By hypothesis, let $y = ax - b$ be the asymptote of g. It follows that $\gamma(x) = g(x) - (ax - b) \to 0$ for $x \to +\infty$, $g(x) = ax - b + \gamma(x)$, $g(2x) - 2g(x) = b + \gamma(2x) - 2\gamma(x)$, and

$$\lim_{n\to+\infty} \frac{g(2x_n) - 2g(x_n)}{\phi(x_n)} = \lim_{n\to+\infty} \frac{b + \gamma(2x_n) - 2\gamma(x_n)}{\phi(x_n)} = 0.$$

On the other hand, from $g(x) = f(x) + k\phi(x)$ it follows

$$\frac{g(2x_n) - 2g(x_n)}{\phi(x_n)} = \frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} + k\Big[\frac{\phi(2x_n)}{\phi(x_n)} - 2\Big].$$

(1) Suppose now that $\lim_{n\to+\infty} \phi(x_n) = +\infty$, so that $\phi(x_n)$ are positive for n large enough and assume that one of the conditions (i),(ii),(iii) is fulfilled. It follows that

$$\frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} \geq 2B, \quad k\Big[\frac{\phi(2x_n)}{\phi(x_n)} - 2\Big] \geq k(\beta - 2)$$

$\big($in the third case, $k\big[\frac{\phi(2x_n)}{\phi(x_n)} - 2\big] = k(\beta - 2)\big)$ and

$$\frac{g(2x_n) - 2g(x_n)}{\phi(x_n)} = \frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} + k\Big[\frac{\phi(2x_n)}{\phi(x_n)} - 2\Big] \geq 2B + k(\beta - 2).$$

Passing to the limit implies $2B + k(\beta - 2) \leq 0$, and the desired statement follows.

(2) Suppose now that $\lim_{n\to+\infty} \phi(x_n) = -\infty$, so that $\phi(x_n)$ are negative for n large enough and assume that one of the conditions (i), (ii), (iii), is fulfilled. It follows that

$$\frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} \leq 2B, \quad k\Big[\frac{\phi(2x_n)}{\phi(x_n)} - 2\Big] \leq k(\beta - 2)$$

$\big($in the third case, $k\big[\frac{\phi(2x_n)}{\phi(x_n)} - 2\big] = k(\beta - 2)\big)$ and

$$\frac{g(2x_n) - 2g(x_n)}{\phi(x_n)} = \frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} + k\Big[\frac{\phi(2x_n)}{\phi(x_n)} - 2\Big] \leq 2B + k(\beta - 2).$$

Passing to the limit implies $2B+k(\beta-2) \geq 0$, and the desired statement follows. $\square$

We could have proved Lemma 3.1 also by defining $F = -f$, $\Phi = -\phi$ and using conclusions of Lemma 3.1 of [7].

Lemma 3.2. *Let ϕ and f be two functions defined on $[0;+\infty[$, and k a real number for which the function $g(x) = f(x) + k\phi(x)$ has an asymptote in $+\infty$. Suppose moreover that there exists a sequence $(x_n) \to +\infty$ for which $\lim_{n\to+\infty} |\phi(x_n)| = +\infty$. Then the following statements hold true:*

(a) *If* $\lim\limits_{n\to+\infty} \dfrac{\phi(2x_n)}{\phi(x_n)} = \beta$ *and* $\lim\limits_{n\to+\infty} \dfrac{f(2x_n) - 2f(x_n)}{\phi(x_n)} = 2B$, *then* $k = \frac{2B}{2-\beta}$ *if* $\beta \neq 2$ *and* $B = 0$ *if* $\beta = 2$;

(b) $\lim\limits_{n\to+\infty} \dfrac{\phi(2x_n)}{\phi(x_n)} = \beta$ *is equivalent to* $\lim\limits_{n\to+\infty} \dfrac{f(2x_n)-2f(x_n)}{\phi(x_n)} = k(2-\beta)$ *for* $k \neq 0$.

Proof of Lemma 3.2. The proof is similar to the one of Lemma 3.2 of [7] and can therefore be omitted. □

Remark 3.3. If the asymptote of g has the form $y = ax$, that is, if $b = 0$, we can replace the condition "there exists a sequence $(x_n) \to +\infty$ for which $\lim_{n\to+\infty} |\phi(x_n)| = +\infty$" by "there exists a sequence $(x_n) \to +\infty$ for which $|\phi(x_n)| \geq c > 0$" and keep the same proof.

Remark 3.4. The simplest examples satisfying the requirements of Lemma 3.2 are the functions $\phi\colon [0;+\infty[\to R$, defined by $\phi(x) = x^p$ for $p > 0$ and $f\colon [0;+\infty[\to R$ defined by $f(x) = cx + d\phi(x)$. In these cases, $\beta = 2^p$, $2B = d(2^p - 2)$ and $k = 2B/(2-\beta) = -d$.

The following Lemma is in some sense the inverse of part (a) of Lemma 3.2; its proof is very similar to that of Lemma 3.5 of [7] and can be omitted here.

Lemma 3.5. *Let f and ϕ be two functions defined on $[0;+\infty[$ and $(x_n) \to +\infty$ a sequence satisfying*

$$\Big[\lim_{n\to+\infty} \phi(x_n) = +\infty, \quad \text{or} \quad \lim_{n\to+\infty} \phi(x_n) = -\infty\Big] \quad \text{and} \quad f(2x_n) - 2f(x_n) \geq 2B\phi(x_n).$$

If there exists $\beta \neq 2$, for which either

$$\Big[\forall n \geq n_0,\ \phi(2x_n) \geq \beta\phi(x_n) \ \ \text{and} \ \ \frac{2B}{2-\beta} > 0\Big], \quad \text{or}$$

$$\Big[\forall n \geq n_0,\ \phi(2x_n) \leq \beta\phi(x_n) \ \ \text{and} \ \ \frac{2B}{2-\beta} < 0\Big], \quad \text{or} \quad \Big[\forall n \geq n_0,\ \phi(2x_n) = \beta\phi(x_n)\Big],$$

and, furthermore, if the function $g(x) = f(x) + k\phi(x)$ has an asymptote at $+\infty$ for $k = 2B/(2-\beta)$, then

$$\lim_{n\to+\infty} \frac{\phi(2x_n)}{\phi(x_n)} = \beta \quad \text{and} \quad \lim_{n\to+\infty} \frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} = 2B.$$

As in [7], very slight modifications of Lemma 3.5 yield the following lemma:

Lemma 3.6. *Let f and ϕ be two functions defined on $[0;+\infty[$, satisfying*

$$\Big[\lim_{x\to+\infty} \phi(x) = +\infty, \ \text{or} \ \lim_{x\to+\infty} \phi(x) = -\infty\Big] \quad \text{and} \quad f(2x) - 2f(x) \geq 2B\phi(x).$$

If there exists $\beta \neq 2$, such that $\forall x \geq x_0$ either one of the following conditions holds

$$\text{(i)} \ \Big[\phi(2x) \geq \beta\phi(x) \ \ \text{and} \ \ 2B/(2-\beta) > 0\Big], \ \text{or}$$

$$\text{(ii)} \ \Big[\phi(2x) \leq \beta\phi(x) \ \ \text{and} \ \ 2B/(2-\beta) < 0\Big], \ \text{or (iii)} \ \phi(2x) = \beta\phi(x),$$

and the function $g(x)=f(x)+k\phi(x)$ has an asymptote at $+\infty$ for $k = 2B/(2-\beta)$, then

$$\lim_{x\to+\infty}\frac{\phi(2x)}{\phi(x)}=\beta \quad \textit{and} \quad \lim_{x\to+\infty}\frac{f(2x)-2f(x)}{\phi(x)}=2B.$$

Remark 3.7. In Lemma 3.5, if the asymptote of g has the equation $y = ax$, i.e., if $b=0$, we can replace the condition $\left[\lim_{n\to+\infty}\phi(x_n)=+\infty \text{ or } \lim_{n\to+\infty}\phi(x_n)=-\infty\right]$ by

$$\left[\forall n \geq n_0,\ \phi(x_n) \geq c > 0 \text{ or } \forall n \geq n_0,\ \phi(x_n) \leq c < 0\right]$$

(and in Lemma 3.6 the condition $\left[\lim_{x\to+\infty}\phi(x) = +\infty \text{ or } \lim_{x\to+\infty}\phi(x) = -\infty\right]$ by $[\,\forall x \geq x_0,\ \phi(x) \geq c > 0 \text{ or } \forall x \geq x_0,\ \phi(x) \leq c < 0]$, and exactly the same proof works.

Remark 3.8. In relation to Lemma 3.6 (similar to Lemma 3.5), we ask if its statements remain true if, without changing the other conditions of Lemma 3.6, we change only the sign of $2B/(2-\beta)$ as below:

(i) $[\forall x \geq x_0,\ \phi(2x) \geq \beta\phi(x)$ and $2B/(2-\beta) < 0]$, or

(ii) $[\forall x \geq x_0,\ \phi(2x) \leq \beta\phi(x)$ and $2B/(2-\beta) > 0]$.

In these cases, there are infinitely many couples (B, β), for which these conditions are satisfied. To prove this, it suffices to change the sense of the inequalities in the proof of Remark 3.8 in [7].

So, if in Lemma 3.6 (and similarly in Lemma 3.5) the conditions (i) or (ii) are fulfilled, all the conditions of these lemmas will be satisfied even if we replace $k = 2B/(2-\beta)$ by $k' = 2B'/(2-\beta')$, where $B' < B$ if $\phi(x) > 0$ and $B' > B$ if $\phi(x) < 0$, and, in both cases, $\beta' = \beta + (2-\beta)(1-\gamma)$ ($\gamma = B'/B$).

This means that, if $\displaystyle\lim_{x\to+\infty}\frac{\phi(2x)}{\phi(x)} = \beta$ and/or $\displaystyle\lim_{x\to+\infty}\frac{f(2x)-2f(x)}{\phi(x)} = 2B$, it is impossible to have

$$\lim_{x\to+\infty}\frac{\phi(2x)}{\phi(x)}=\beta' \quad \text{and / or} \quad \lim_{x\to+\infty}\frac{f(2x)-2f(x)}{\phi(x)}=2B'$$

if $\beta \neq \beta'$ and $B \neq B'$.

This is an indirect argument proving that, without supplementary conditions, if

$$\left[(\phi(2x) \geq \beta\phi(x) \text{ and } \frac{2B}{2-\beta} < 0\right], \text{ or } \left[\phi(2x) \leq \beta\phi(x) \text{ and } \frac{2B}{2-\beta} > 0\right],$$

the conclusions of these lemmas are not true in general (see Example 3.9 below).

However, with supplementaries conditions, as, for example $\phi(2x) = \beta\phi(x)$, which implies that the sign of k is not important, they may be true (see Example 3.10 below).

Example 3.9. Let $\phi(x) = \sqrt{x}$ and $f(x) = x + \sqrt{x}$. For $\beta = 1$ and $2B = -1$ we have $\phi(2x) = \sqrt{2x} \geq \beta\phi(x) = \sqrt{x}$,

$$f(2x) - 2f(x) = (\sqrt{2}-2)\sqrt{x} \geq 2B\sqrt{x} = -\sqrt{x} \text{ and } k = \frac{2B}{2-\beta} = -1 < 0.$$

The function $g(x) = f(x) + k\phi(x) = f(x) - \phi(x) = x$ has clearly an asymptote, but

$$\lim_{x\to+\infty} \frac{\phi(2x)}{\phi(x)} = \lim_{x\to+\infty} \frac{\sqrt{2x}}{\sqrt{x}} = \sqrt{2} \neq 1 = \beta, \quad \text{and}$$

$$\lim_{x\to+\infty} \frac{f(2x) - 2f(x)}{\phi(x)} = \lim_{x\to+\infty} \frac{(\sqrt{2} - 2)\sqrt{x}}{\sqrt{x}} = \sqrt{2} - 2 \neq -1 = 2B.$$

Example 3.10. The functions $\phi(x)$ and $f(x)$ of Example 3.9, with $\beta = \sqrt{2}$ and $2B = \sqrt{2} - 2$, satisfy $\phi(2x) = \sqrt{2x} = \beta\phi(x)$ and $f(2x) - 2f(x) = (\sqrt{2} - 2)\sqrt{x} = 2B\phi(x)$. Therefore, for $k = 2B/(2 - \beta) = -1$, the function $g(x) = f(x) + k\phi(x) = f(x) - \phi(x) = x$ has clearly an asymptote, and

$$\lim_{x\to+\infty} \frac{\phi(2x)}{\phi(x)} = \lim_{x\to+\infty} \frac{\sqrt{2x}}{\sqrt{x}} = \sqrt{2} = \beta, \quad \text{and}$$

$$\lim_{x\to+\infty} \frac{f(2x) - 2f(x)}{\phi(x)} = \lim_{x\to+\infty} \frac{(\sqrt{2} - 2)\sqrt{x}}{\sqrt{x}} = \sqrt{2} - 2 = 2B.$$

This is the case of $\phi(2x) = \beta\phi(x)$, for which the sign of k is not important.

The following example is used to answer several questions posed in Remarks 3.12, 3.13 and 3.14.

Example 3.11. Let ϕ: $[0; +\infty[\to R$ be the function defined by

$$\phi(x) = \begin{cases} \sqrt{x} & \text{if } x \neq 2^n, n \in Z \\ \sqrt[3]{x} & \text{if } x = 2^n, n \in Z \end{cases}$$

and $f(x) = x - \phi(x)$. Since

$$\phi(2x) = \begin{cases} \sqrt{2}\sqrt{x} & \text{if } x \neq 2^n, n \in Z \\ \sqrt[3]{2}\ \sqrt[3]{x} & \text{if } x = 2^n, n \in Z \end{cases}$$

and $\sqrt[3]{2} \leq \sqrt{2}$, we obtain that for every $x \geq 0$, $\sqrt[3]{2}\phi(x) \leq \phi(2x) \leq \sqrt{2}\phi(x)$.

Therefore, if we need to have $\phi(2x) \leq \beta\phi(x)$, we must have $\beta = \sqrt{2}$, while if we need $\phi(2x) \geq \beta\phi(x)$, then we must have $\beta = \sqrt[3]{2}$. On the other hand,

$$f(2x) - 2f(x) = 2\phi(x) - \phi(2x) = \begin{cases} (2 - \sqrt{2})\phi(x) & \text{if } x \neq 2^n, n \in Z \\ (2 - \sqrt[3]{2})\phi(x) & \text{if } x = 2^n, n \in Z. \end{cases}$$

So, $$(2 - \sqrt{2})\phi(x) \leq f(2x) - 2f(x) \leq (2 - \sqrt[3]{2})\phi(x)$$

and if we need to have $f(2x) - 2f(x) \leq 2B\phi(x)$, as in Lemma 3.5 or 3.6 of [7], then $2B = 2 - \sqrt[3]{2}$, but if we need to have $f(2x) - 2f(x) \geq 2B\phi(x)$, as in Lemma 3.5 or 3.6 of this paper, then $2B = 2 - \sqrt{2}$.

Remark 3.12. If there exists at least one k for which $f(x) + k\phi(x)$ has an asymptote, this does not imply that

$$\lim_{x\to+\infty} \phi(2x)/\phi(x) \ \text{ and/or } \ \lim_{x\to+\infty}(f(2x) - 2f(x))/\phi(x)$$

exist: Indeed, in Example 3.11 above for $k = 1$, $f(x) + k\phi(x) = x$ has clearly an asymptote, but none of the limits

$$\lim_{x\to+\infty} \frac{\phi(2x)}{\phi(x)} \quad \text{and/or} \quad \lim_{x\to+\infty} \frac{f(2x) - 2f(x)}{\phi(x)}$$

exists, because for a sequence $(x_n) \to +\infty$, such that $x_n \neq 2^n$ for every $n \in N$, $\phi(2x_n)/\phi(x_n) = \sqrt{2} \to \sqrt{2}$ and $(f(2x_n) - 2f(x_n))/\phi(x_n) = 2 - \sqrt{2} \to 2 - \sqrt{2}$, but for the sequence $(x'_n) \to +\infty$, such that $x'_n = 2^n$ for every $n \in N$, $\phi(2x'_n)/\phi(x'_n) = \sqrt[3]{2} \to \sqrt[3]{2} \neq \sqrt{2}$ and $(f(2x'_n) - 2f(x'_n)/\phi(x'_n) = 2 - \sqrt[3]{2} \to 2 - \sqrt[3]{2} \neq 2 - \sqrt{2}$.

Remark 3.13. Note that if there exists a sequence $(x_n) \to +\infty$ with the property $\lim_{n\to+\infty} \phi(2x_n)/\phi(x_n) = \beta$, this does not imply $\lim_{n\to+\infty}(f(2x_n)-2f(x_n))/\phi(x_n) = 2B$ in general. Indeed, in Example 3.11 above, in which $f(2x) - 2f(x) \geq (2 - \sqrt{2})\phi(x)$, and $\phi(2x) \geq \sqrt[3]{2}\phi(x)$, that is, $\beta = \sqrt[3]{2}$ and $2B = 2 - \sqrt{2}$, for any sequence $(x_n) \to +\infty$, such that $x_n = 2^n$ for every $n \in N$,

$$\frac{\phi(2x_n)}{\phi(x_n)} = \sqrt[3]{2} \quad \text{and} \quad \lim_{n\to+\infty} \frac{\phi(2x_n)}{\phi(x_n)} = \sqrt[3]{2} = \beta,$$

but $\dfrac{f(2x_n)-2f(x_n)}{\phi(x_n)} = 2-\sqrt[3]{2}$ and $\lim\limits_{n\to+\infty} \dfrac{f(2x_n)-2f(x_n)}{\phi(x_n)} = 2-\sqrt[3]{2} \neq 2-\sqrt{2} = 2B$ because for $k = \dfrac{2B}{2-\beta} = \dfrac{2-\sqrt{2}}{2-\sqrt[3]{2}}$ the function $f + k\phi$ has no asymptote.

While, if for $k = 2B/(2 - \beta)$, the function $g(x) = f(x) + k\phi(x)$ would have an asymptote, we have $\lim_{x\to+\infty} \phi(x) = +\infty$ or $-\infty$, and $\lim_{x\to+\infty} \phi(2x)/\phi(x) = \beta$. Then for every sequence $(x_n) \to +\infty$, satisfying $\phi(x_n) \to +\infty$, or $\phi(x_n) \to -\infty$, by part (b) of Lemma 3.2 we would have that

$$\lim_{n\to+\infty} \frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} = k(2 - \beta) = 2B,$$

regardless of the sign of $k = 2B/(2 - \beta)$.

Remark 3.14. If for $k = 2B/(2 - \beta)$ the function $f(x) + k\phi(x)$ has no asymptote, this does not mean that it has no asymptote for another value of k. In Example 3.11, for example, the conditions of Lemma 3.6 are satisfied for $\beta = \sqrt[3]{2}$ and $2B = 2-\sqrt{2}$, the function $f(x) + k\phi(x)$ has clearly no asymptote for

$$k = \frac{2B}{2-\beta} = \frac{2-\sqrt{2}}{2-\sqrt[3]{2}} < 1,$$

but for $k_1 = 1$, $f(x) + k_1\phi(x) = x$ has clearly the asymptote $y = x$.

Lemma 3.2(a) shows, among other things, that if there exists a sequence $(x_n) \to +\infty$, satisfying the conditions $\lim_{n\to+\infty} |\phi(x_n)| = +\infty$, $\lim_{n\to+\infty} \phi(2x_n)/\phi(x_n) = \beta$ and $\lim_{n\to+\infty}(f(2x_n) - 2f(x_n))/\phi(x_n) = 2B$, then the only candidate among the reals k, for which $g(x) = f(x) + k\phi(x)$ can have an asymptote, is $k = 2B/(2 - \beta)$.

This conclusion is fundamental for the formulation of the next Theorem 3.15, which is one of our main results. It is based on the previous lemmas and discussion restricted to the case $k = 2B/(2-\beta)$. However, different versions of this theorem for other values of k, notably when $\phi(2x) = \beta\phi(x)$, may be useful (see, for example, Theorem 3.17 below).

Note that in all the following theorems, the first part of the statement may be regarded as stability result, while the second part of the statement, as superstability result of the affine equation, and in particular of the linear equation, in the sense of Hyers-Ulam.

Theorem 3.15. *Let f and ϕ be two functions defined on $[0;+\infty[$, satisfying*

$$f(x+y) - f(x) - f(y) \geq B[\phi(x) + \phi(y)].$$

Assume that there exists a real number $\beta \neq 2$, for which one of the following conditions is satisfied for all $x \geq 0$:

$$\text{(i) } \left[\phi(2x) \leq \beta\phi(x) \quad \textit{and} \quad \frac{2B}{2-\beta} \leq 0\right], \quad \textit{or}$$

$$\text{(ii) } \left[\phi(2x) \geq \beta\phi(x) \quad \textit{and} \quad \frac{2B}{2-\beta} \geq 0\right], \quad \textit{or} \quad \text{(iii) } \phi(2x) = \beta\phi(x).$$

Let $k = 2B/(2-\beta)$. If the function $g(x) = f(x) + k\phi(x)$ is concave, then f is asymptotically close to $ax - b - k\phi(x)$, where

$$a = \lim_{x\to+\infty} \frac{g(x)}{x} \quad \textit{and} \quad b = \lim_{x\to+\infty}[g(2x) - 2g(x)].$$

If moreover $f(0) + k\phi(0) = -b$, then for all $x \geq 0$, $f(x) = ax - b - k\phi(x)$.
In the particular case $b = 0$, we have additionally to $f(x) = ax - k\phi(x)$, for all $x \geq 0$, $\phi(2x) = \beta\phi(x)$ and $f(2x) - 2f(x) = 2B\phi(x)$.

Proof of Theorem 3.15. The function $g(x) = f(x) + k\phi(x)$ is concave by hypothesis, hence from Theorem 2.1, it has an asymptote if and only if

$$b = \inf\{g(x+y) - g(x) - g(y) : x \geq 0,\ y \geq 0\} = \lim_{x\to\infty}[g(2x) - 2g(x)]$$

is finite. Now setting $x = y$ in (1), gives $f(2x) - 2f(x) \geq 2B\phi(x)$ and in all cases of this theorem $k\phi(2x) \geq k\beta\phi(x)$, hence, since $k = 2B/(2-\beta)$ implies $2B-2k+k\beta = 0$, we obtain

$$\begin{aligned} g(2x) - 2g(x) &= f(2x) - 2f(x) + k[\phi(2x) - 2\phi(x)] \geq (2B - 2k)\phi(x) + k\phi(2x) \\ &\geq (2B - 2k + k\beta)\phi(x) = 0. \end{aligned}$$

It follows that $g(2x) - 2g(x) \geq 0$ and $b \geq 0$ is finite. From Theorem 2.1, g has an asymptote $y = ax - b$, with $a = \lim_{x\to+\infty} g(x)/x$. This means that f is asymptotically close to $ax - b - k\phi(x)$, which is asymptotic Hyers-Ulam stability.

Suppose now that in addition, $g(0) = f(0) + k\phi(0) = -b$. From Theorem 2.1, $g(x) = ax - b$, that is, $f(x) = ax - b - k\phi(x)$ (Hyers-Ulam superstability). In the special case, when $b = 0$ and hence $g(0) = 0$, replacing $f(x) = ax - k\phi(x)$ in the inequality $f(2x) - 2f(x) \geq 2B\phi(x)$, yields $2k\phi(x) - k\phi(2x) \geq 2B\phi(x)$ and $k\phi(2x) \leq (2k - 2B)\phi(x)$, that is $k\phi(2x) \leq k\beta\phi(x)$.

Now, if $k > 0$, and, as required in Theorem 3.15, $\phi(2x) \geq \beta\phi(x)$, the above relation yields $\phi(2x) \leq \beta\phi(x)$, and then $\phi(2x) = \beta\phi(x)$.

Similarly, if $k < 0$, and, as required in Theorem 3.15, $\phi(2x) \leq \beta\phi(x)$, the relation $k\phi(2x) \leq k\beta\phi(x)$ yields $\phi(2x) \geq \beta\phi(x)$, and then, in both cases, $\phi(2x) = \beta\phi(x)$.

Furthermore, in this special case,

$$f(2x) - 2f(x) = 2k\phi(x) - k\phi(2x) = 2k\phi(x) - k\beta\phi(x) = k(2-\beta)\phi(x) = 2B\phi(x).$$

The equalities $f(x) = ax - b - k\phi(x)$, or $f(x) = ax - k\phi(x)$ for all $x \geq 0$, in Theorem 3.15 may be regarded as a superstability result for the linear equation. This completes the proof of the theorem. □

Remark 3.16. If the function $g(x) = f(x) + k\phi(x)$ in Theorem 3.15, is concave and

$$(k > 0 \text{ and } \phi \text{ is convex}) \quad \text{or} \quad (k < 0 \text{ and } \phi \text{ is concave})$$

then the function $f(x) = g(x) - k\phi(x)$ of this theorem is concave.

In the special case when $\phi(2x) = \beta\phi(x)$ (for example if $\phi(x) = x^p$), we can prove the following result where, in contrast to Theorem 3.15, k may be different to $2B/(2-\beta)$:

Theorem 3.17. *Let f and ϕ be two functions defined on $[0; +\infty[$, satisfying*

$$f(x+y) - f(x) - f(y) \geq B[\phi(x) + \phi(y)].$$

Assume that there exists a real number β such that $\phi(2x) = \beta\phi(x)$ is satisfied for all $x \geq 0$. If for any real number k, either

$$(\phi(x) \geq 0 \;\; \textit{and} \;\; k(2-\beta) \leq 2B), \quad \textit{or} \quad (\phi(x) \leq 0 \;\; \textit{and} \;\; k(2-\beta) \geq 2B),$$

and the function $g(x) = f(x) + k\phi(x)$ is concave, then f is asymptotically close to $ax - b - k\phi(x)$, where

$$a = \lim_{x\to+\infty} \frac{g(x)}{x} \qquad \textit{and} \qquad b = \lim_{x\to+\infty} [g(2x) - 2g(x)].$$

If, moreover, $f(0) + k\phi(0) = -b$, then $f(x) = ax - b - k\phi(x)$ for all $x \geq 0$.

In the case $b = 0$, we have moreover $f(2x) - 2f(x) = 2B\phi(x)$ for all $x \geq 0$.

Proof of Theorem 3.17. The function $g(x) = f(x) + k\phi(x)$ is concave by hypothesis, hence from Theorem 2.1, it has an asymptote if and only if

$$b = \inf\{g(x+y) - g(x) - g(y) : x \geq 0, y \geq 0\} = \lim_{x\to\infty}[g(2x) - 2g(x)]$$

is finite. Letting $x = y$ in $f(x+y) - f(x) - f(y) \geq B[\phi(x) + \phi(y)]$, we directly have $f(2x) - 2f(x) \geq 2B\phi(x)$ and $k\phi(2x) = k\beta\phi(x)$, hence

$$g(2x) - 2g(x) = f(2x) - 2f(x) + k[\phi(2x) - 2\phi(x)] \geq (2B - 2k + k\beta)\phi(x) \geq 0,$$

because ($\phi(x) \geq 0$ and $k(2-\beta) \leq 2B$), or ($\phi(x) \leq 0$ and $k(2-\beta) \geq 2B$).

It follows that $g(2x) - 2g(x) \geq 0$ and $b \geq 0$ is finite. Following Theorem 2.1, g has an asymptote $y = ax - b$, with $a = \lim_{x\to+\infty} g(x)/x$. This means that f is asymptotically close to $ax - b - k\phi(x)$ (asymptotic Hyers-Ulam stability).

Suppose now that moreover, $g(0) = f(0) + k\phi(0) = -b$. From Theorem 2.1 we have $g(x) = ax - b$, that is, $f(x) = ax - b - k\phi(x)$ (Hyers-Ulam superstability).

In the special case when, moreover, $b = 0$, we obviously get

$$f(2x) - 2f(x) = 2k\phi(x) - k\phi(2x) = 2k\phi(x) - k\beta\phi(x) = k(2-\beta)\phi(x).$$

Given that ($\phi(x) \geq 0$ and $k(2-\beta) \leq 2B$), or ($\phi(x) \leq 0$ and $k(2-\beta) \geq 2B$), in both cases we have

$$k(2-\beta)\phi(x) \leq 2B\phi(x) \quad \text{and,} \quad \text{consequently,} \quad f(2x) - 2f(x) \leq 2B\phi(x).$$

From $f(2x) - 2f(x) \leq 2B\phi(x)$ and $f(2x) - 2f(x) \geq 2B\phi(x)$, it easily follows that $f(2x) - 2f(x) = 2B\phi(x)$.

Note that the equalities $f(x) = ax - b - k\phi(x)$ and $f(x) = ax - k\phi(x)$ for all $x \geq 0$, in the statement of Theorem 3.17 are indeed superstability results for the linear equation. Hence, Theorem 3.17 is proved. □

The next theorem which is a special case of Theorem 3.15 provides a *stability or superstability* real unilateral version of the theorems by Aoki [1], Rassias [8] (for $0 \leq p < 1$), and by Gajda [2] (for $p > 1$).

Theorem 3.18. *Let $f\colon [0; +\infty[\,\to R$ satisfy $f(x+y) - f(x) - f(y) \geq \theta(x^p + y^p)$, for $0 < p \neq 1$ and $f(0) = 0$. If for $k = 2\theta/(2-2^p)$ the function $g(x) = f(x) + kx^p$ is concave, then f is asymptotically close to $ax - b - kx^p$, where*

$$a = \lim_{x\to+\infty} \frac{f(x) + kx^p}{x} \quad \textit{and} \quad b = \lim_{x\to+\infty} [g(2x) - 2g(x)].$$

If moreover, $b = 0$, then $f(x) = ax - kx^p$, for $x \geq 0$.

Proof of Theorem 3.18. First notice that the function $\phi(x) = x^p$ for $p \neq 1$, satisfies $\phi(2x) = 2^p x^p = 2^p\phi(x)$, whence $\phi(2x) = \beta\phi(x)$ for $\beta = 2^p \neq 2$.

Then Theorem 3.15 implies that f is asymptotically close to $ax - b - kx^p$, where

$$a = \lim_{x\to+\infty} \frac{f(x) + kx^p}{x} \ \text{and} \ b = \lim_{x\to+\infty} [g(2x) - 2g(x)] = \lim_{x\to+\infty} [f(2x) - 2f(x) + 2\theta x^p].$$

If, moreover, $b = 0$, we have $g(0) = f(0) + k\phi(0) = 0$ for $p > 0$. Now it suffices to apply Theorem 3.15 in order to obtain $f(x) = ax - kx^p$, and the proof is complete. □

Remark 3.19. The additional condition $f(0) = 0$, that we added in Theorem 3.18, is also present in both Rassias' and Gajda's theorems, because for $x = y = 0$, the condition $||f(x+y) - f(x) - f(y)|| \leq \theta(||x||^p + ||y||^p)$ in these theorems implies $f(0) = 0$.

Remark 3.20. It is important to note that in the above real unilateral version of Rassias', or Gajda's theorems the additional assumption of g-concavity is rewarded by using only the left, also called "concavity side" of the real Rassias' or Gajda's condition, namely

$$-\theta(||x||^p + ||y||^p) \leq f(x+y) - f(x) - f(y) \leq \theta(||x||^p + ||y||^p).$$

One gets in exchange an asymptotic Hyers-Ulam stability result, and in certain cases, the equality $f(x) = ax - kx^p$, that is, a superstability result of the generalized linear equation.

Remark 3.21. We also note that, as the function $\phi(x) = x^p$ is concave for $0 \leq p < 1$ and convex for $p > 1$, the functions $f(x)$ satisfying the conditions of this theorem, are concave if ($0 \leq p < 1$ and $k < 0$) or if ($p > 1$ and $k > 0$).

Remark 3.22. Gajda [2] showed that the theorem is false for $p = 1$, by constructing a counterexample.

The theorem below is a version of Theorem 3.17 adapted to the case $\phi(x) = x^p$:

Theorem 3.23. *Let $f\colon [0;+\infty[\, \to R$ satisfy $f(x+y) - f(x) - f(y) \geq \theta(x^p + y^p)$, $0 \leq p \neq 1$ and $f(0) = 0$. If for any k satisfying $k(2-2^p) \leq 2\theta$, that is,*

$$k \leq \frac{2\theta}{2-2^p} \ \text{if } p < 1, \ \text{and } k \geq \frac{2\theta}{2-2^p} \ \text{if } p > 1,$$

the function $g(x) = f(x) + kx^p$ is concave, then f is asymptotically close to the function $ax - b - kx^p$, where

$$a = \lim_{x\to+\infty} \frac{f(x) + kx^p}{x} \quad \text{and} \quad b = \lim_{x\to+\infty} g(2x) - 2g(x).$$

If moreover $b = 0$, then $f(x) = ax - kx^p$ for all $x \geq 0$.

Remark 3.24. It is important to recognize that in the assumptions of Theorem 3.23, if there exists a k satisfying $k(2-2^p) \leq 2\theta$ for which the function $f(x) + kx^p$ is concave, and $b = 0$, it must be unique, because $f(x) = a_1x - k_1x^p = a_2x - k_2x^p$ implies $(k_1 - k_2)x^p = (a_1 - a_2)x$ and the last equality is possible only if $k_1 = k_2$ and $a_1 = a_2$.

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