Journal of Convex Analysis
Volume 28 (2021), No. 1, 179–196

Interior Regularity for a Class of Nonlinear Second-Order Elliptic Systems

Josef Daněček
VŠB - Technical University of Ostrava, FEECS, Department of Applied Mathematics, 70833 Ostrava-Poruba, Czech Republic
danecek.j@seznam.cz

Jana Stará
Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, 18600 Praha 8, Czech Republic
stara@karlin.mff.cuni.cz

Eugen Viszus*
Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University, 84248 Bratislava, Slovak Republic
eugen.viszus@fmph.uniba.sk

Received: September 19, 2018
Accepted: May 25, 2020

The interior $C^{1,\gamma}$-regularity is proved for weak solutions to a class of nonlinear second-order elliptic systems. It is typical for the system belonging to the class that the continuity moduli of the gradients of its coefficients become slow growing sufficiently far from zero. This property guarantees the regularity of the gradients of solutions to such system in a case when the ellipticity constant is big enough. Some characteristic features of the obtained result are illustrated by examples at the end of the paper.

Keywords: Nonlinear elliptic systems, weak solutions, regularity, Campanato spaces.

2010 Mathematics Subject Classification: 35J47

1. Introduction

In this paper, we give conditions guaranteeing that a weak solution to a nonlinear elliptic system of second order

$$-\sum_{\alpha=1}^{n} D_\alpha \left(A_i^\alpha(Du)\right) = 0 \qquad \text{in} \quad \Omega, \quad i = 1, \dots, N \tag{1}$$

belongs to $C^{1,\gamma}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ space. For the history and detailed information concerning the problem, see [4], [11], [14], [19], [22] and [24].

By the weak solution to the system (1) we understand $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega \sum_{i,\alpha} A_i^\alpha(Du) D_\alpha \varphi^i \, dx = 0, \quad \forall\, \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N).$$

*Supported by the research projects Slovak Grant Agency No.1/0078/17 and No.1/0358/20.

ISSN 0944-6532 / $ 2.50

Here $\Omega \subset \mathbb{R}^n$ is a bounded open set, $n \geq 3$ and continuously differentiable coefficients $(A_i^\alpha)_{i=1,\dots,N,\alpha=1,\dots,n}$ have the linear controlled growth and satisfy the strong uniform ellipticity condition. More precisely, denoting

$$A_{ij}^{\alpha\beta}(p) = \frac{\partial A_i^\alpha}{\partial p_j^\beta}(p)$$

and supposing $A_i^\alpha(0) = 0$, we require the following assumptions:

(i) There exists a constant $M > 0$ such that for every $p \in \mathbb{R}^{nN}$

$$\sum_{i,\alpha} |A_i^\alpha(p)| \leq M\,(1 + |p|)\,.$$

(ii)
$$\sum_{i,j,\alpha,\beta} \left| A_{ij}^{\alpha\beta}(p) \right| \leq M.$$

(iii) The strong ellipticity: there exists $\nu > 0$ such that for every $p,\ \xi \in \mathbb{R}^{nN}$

$$\sum_{i,j,\alpha,\beta} A_{ij}^{\alpha\beta}(p)\xi_\alpha^i \xi_\beta^j \geq \nu|\xi|^2\,.$$

(iv) There exists a real function ω continuous on $[0,\infty)$, which is bounded, nondecreasing, increasing on a neighbourhood of zero, $\omega(0) = 0$ and such that for all $p,\ q \in \mathbb{R}^{nN}$

$$\sum_{i,j,\alpha,\beta} \left| A_{ij}^{\alpha\beta}(p) - A_{ij}^{\alpha\beta}(q) \right| \leq \omega\left(|p-q|\right).$$

We set $\omega_\infty = \lim_{t\to\infty} \omega(t) \leq 2M$.

It is worth to point out (see [11], p. 169) that for uniformly continuous coefficients $A_{ij}^{\alpha\beta}$ there exists a real function ω satisfying the assumption (iv) and, vice versa, (iv) implies the uniform continuity of coefficients and absolute continuity of ω on $[0,\infty)$.

Here we will consider the continuous functions ω given by the formula

$$\omega(t) = \begin{cases} \omega_0(t) & \text{for} \quad 0 \leq t < t_o,\ t_o \geq 0 \\ \omega_1(t) \leq \omega_\infty, & \text{for} \quad t_o \leq t < \infty \end{cases} \tag{2}$$

where ω_0 is arbitrary continuous, concave, nondecreasing function, increasing on a neighbourhood of zero such that $\omega_0(0) = 0$ and point t_0 and the function ω_1 are chosen in such a way that ω preserves its continuity and concavity on $[0,\infty)$.

It is well known that with the above assumptions the Dirichlet problem

$$-\sum_\alpha D_\alpha\left(A_i^\alpha(Du)\right) = 0 \text{ in } \Omega,\ i = 1,\dots,N, \quad u - g \in W_0^{1,2}(\Omega,\mathbb{R}^N) \tag{3}$$

has for any function $g \in W^{1,2}(\Omega,\mathbb{R}^N)$ the unique solution u in the same space. Moreover, we have

$$\int_\Omega |Du - (Du)_\Omega|^2\, dx \leq \left(\frac{M}{\nu}\right)^2 \int_\Omega |Dg - (Dg)_\Omega|^2\, dx, \tag{4}$$

where $(v)_\Omega$ means the integral mean value of a function v over Ω. The foregoing estimate can be proved by a standard technique (see [8], Appendix).

Much effort has been devoted to understanding of the regularity of solutions to nonlinear elliptic systems. If $n \geq 3$, it was shown in [21] (n large, $N = n^2$), [12] ($n \geq 5$, $N = n^2$), [23] ($n \geq 3$, $N = n^2$), [24] ($n = 3$, $N = 5$) that, even in the case of analytic coefficients A_i^α, weak solutions to (1) may be not of class C^1. The weak solutions constructed in the mentioned papers are Lipschitz continuous and so there was still the possibility to establish the lower order $C^{0,\alpha}$-regularity for weak solutions to (1) (under the assumptions (i), (ii) and (iii) only). Later on the authors of [25] showed that there exist Non-Lipschitz solutions to (1) for $n \geq 3$, $N = n(n+1)/2 - 1$ and for $n = 4$, $N = 3$ that can be even locally unbounded when $n \geq 5$. So, in general, neither the standard $C^{1,\alpha}$-everywhere regularity nor (for $n \geq 5$) the lower order $C^{0,\alpha}$-everywhere regularity of weak solutions to (1) are possible without some additional structural assumptions about (1). There are a lot of papers dealing with the so-called partial regularity (i.e. the regularity properties are preserved at least on large subsets) of weak solutions to (1) under the assumptions (i), (ii) and (iii). Because we study the interior everywhere $C^{1,\alpha}$-regularity and not the partial regularity of weak solutions to (1), we refer here the interested reader to the excellent survey papers [19], [20] and references therein.

An important role for our nonlinear elliptic systems play higher integrability and differentiability. It is proved in the paper [15] that the weak solutions to the system (1) belong to $W^{2,2+\epsilon}_{loc}(\Omega, \mathbb{R}^N)$ under the strong monotonicity and the Lipschitz continuity of the coefficients A_i^α. The constant $\epsilon > 0$ depends only on an absolute constant $\delta_0 > 1/50$ and on the ratio of the strong monotonicity and the Lipschitz continuity constants. Moreover, one can immediately see, using the embedding theorem, that for $n = 3, 4$ the weak solutions to the system (1) possess the $C^{0,\alpha}$-regularity. Regarding the lower order everywhere interior regularity for general n, we can refer the reader to recent papers [2] and [3] where the authors show the interior everywhere Hőlder continuity of weak solutions to some class of the systems (1) under (besides others) a splitting condition on the coefficients. As it is mentioned above, the everywhere interior $C^{1,\alpha}$-regularity of weak solutions to the system (1) is impossible without any additional structural assumptions. So, the everywhere interior $C^{1,\alpha}$-regularity is known for the diagonal systems (see [13], [1]), for the systems where the coefficients $A_i^\alpha(Du) = A_i^\alpha(|Du|)$ (see [27]), for systems with a small dispersion of eigenvalues of matrices $A_{ij}^{\alpha\beta}$ (see [14], [17]) and for the systems with sufficiently small ratio ω_∞/ν where the parameter ω_∞ is defined in (iv) above (see [8] or, for the reader's convenience, Proposition 2.6 below).

Here we also mention the papers [6] (see Th.2.2 there) and [10] where the authors study a class of second order nonlinear elliptic systems, with coefficients split into linear part and the nonlinear one which grows sub-linearly, whose weak solutions have the gradients belonging to the space $\mathcal{L}^{2,n}_{loc}(\Omega, \mathbb{R}^N)$ (known as BMO-space, see Definition 3.1 below). The papers [7], [9] and the present one offer conditions guaranteeing the interior everywhere $C^{1,\alpha}$-regularity expressed in terms of the continuity modulus ω of the first derivatives of the coefficients of (1). More precisely, it is proved in [7] that if the modulus of continuity is of the type square root of logarithm, then there exists $\nu_0 > 0$ such that for every ellipticity constant $\nu \geq \nu_0$ with the ratio $M/\nu \leq P$, where $P > 1$ is a given constant, the gradients of weak solutions to (1) are locally Hölder continuous in Ω (see [7], Th.9).

The paper [9] deals with the modulus of continuity that is of the logarithmic type as well, but only for big arguments. Near zero it can be almost arbitrary. The main result of [9] is, comparing it to that one from [7], that for the mentioned type of modulus of continuity the interior regularity of solutions to (1) is reached for any ellipticity constant, but in a case of small ellipticity constant the growth of continuity modulus should be adequately slow for big argument. Moreover, the bigger is the ellipticity constant, the faster can be the growth of continuity modulus. In most partial regularity results for the system (1) the regular points $x \in \Omega$ of a solution u are characterized in such a way that for some $r_x > 0$ the quantity $U_{r_x}(x)$ (for its definition see beginning of the next paragraph) has to be sufficiently small. In the paper [9] the condition of regularity of solutions to (1) on an open subset $\Omega_o \subset\subset \Omega$ allows $U_r(x)$ not to be necessarily small in almost all points of Ω_o. Our condition of regularity (5) is formally similar to that from [9] (see [9], Th.2.1) however, while the condition for regularity in [9] possesses local character, the condition (5) possesses the global one.

The crucial role in the regularity condition there is played by a constant $\mathcal{M}$ (see (28) below), estimate of which is a bit complicated in a case of general ω. In this paper we focus ourselves to the modulus of continuity described by (8) and (10). This restriction enables us estimate the constant $\mathcal{M}$ (see Appendix below) and so we get the regularity condition (5) below. The main advantage of this condition is that the relationship between the regularity of solutions to (1) and the parameters of the continuity modulus is clearly visible (see Th.2.1 below). For example, it is easily seen how the relation between t_o and t_1 influences the smallness of the constant $\mathcal{M}$. On the other hand, it is visible from (9), (11) and (7) below that the smaller the ratio M/ν and ω_∞/ν, the smaller the parameter $\mathcal{M}$ and the better the condition (5). Here it is useful to remind that the condition of regularity (5) is especially effective in the case of the Dirichlet problem (3) as it is obvious from (4) and the global condition (5) is very significant in the case when the ratio of the measure of domain Ω and the measure of the ball, whose radius equals to the distance of subdomain $\Omega_o \Subset \Omega$ from boundary $\partial\Omega$, is small (see right hand side of (7)). It is good visible in the case when Ω and Ω_o are concentrated balls (see Example 5.1).

In Example 5.2 below we construct a class of systems (1) with the property that the gradients of weak solutions $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ are locally Hölder continuous in Ω and for better understanding we provide some numerical results.

The method for proving the main result of this paper follows the standard procedures used in the direct proofs of the partial regularity. The novelty is an employment of special complementary Young functions and a modification of the Natanson's lemma which allows us to get some key estimates. The essentially improved regularity criterium (5) (comparing it to the main result of the paper [9]) was achieved thanks to a peculiar choice (by far not elementary) of parameters in (30) and (31) below.

2. Main results

By $\Omega_o \Subset \Omega$ we will understand any bounded subdomain Ω_o with smooth boundary which is compactly embedded into Ω (i.e. $\overline{\Omega}_o \subset \Omega$). We set $d_o = \operatorname{dist}(\Omega_o, \partial\Omega)/2$ and $d_x = \operatorname{dist}(x, \partial\Omega)/2$ for $x \in \Omega$ (here $\operatorname{dist}(A, B) = \inf_{x\in A, y\in B}|x - y|$, $A, B \subset \mathbb{R}^n$).

For $x \in \Omega$, $r > 0$ such that $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\} \subset \Omega$ we set

$$U_r(x) = r^{-n}\phi(x,r), \qquad \phi(x,r) = \int\limits_{B_r(x)} |Du(y) - (Du)_{x,r}|^2 \, dy$$

where $(Du)_{x,r}$ means the integral mean value of Du over $B_r(x)$. By κ_n we denote the n-dimensional Lebesgue measure of the unit ball.

We formulate our results in several steps. The main result is given in Theorem 2.1 below. The further results are consequences of this theorem.

Theorem 2.1. *Let $\Omega_o \subset\subset \Omega$ and $n \leq \vartheta < n+2$ be given. Then there exists a constant $\mathcal{M}$ such that if $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a weak solution to the system* (1) *satisfying the condition*

$$⨍\limits_{\Omega} |Du - (Du)_\Omega|^2 \, dy < \frac{1}{\mathcal{M}^2}\,, \tag{5}$$

then $Du \in C^{0,(\vartheta-n)/2}(\Omega_o, \mathbb{R}^{nN})$ in the case $\vartheta > n$ and $Du \in BMO(\Omega_o, \mathbb{R}^{nN})$ for $\vartheta = n$. Here

$$\mathcal{M} = \sup_{t_o < t < \infty} \frac{\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) - \widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t - t_o} < \infty, \qquad \widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right) \leq 5\,. \tag{6}$$

Remark 2.2. In the foregoing formula the function $\widetilde{\Psi}(u) = u\mathrm{e}^{(u/2\sqrt{\mu})^{2/(2\mu-1)}}$. For further properties of $\widetilde{\Psi}$ see (14). Moreover, we assume that $t_o \geq 0$ (t_o is from the definition ω, see (2)), $\varepsilon = \omega_\infty^2 / C_\mu^\alpha$, $C_\mu = ((n-2)\mu/2\mathrm{e})^\mu$ and the constants $\mu \geq 6$, $\alpha > 1 - 2/n$ have to be such that

$$C_\mu^{(\alpha-1)n+2} \geq K L^{\frac{n\vartheta}{n+2-\vartheta}} \left(5 C_S \frac{M\omega_\infty}{\nu^2}\right)^{2n} \frac{|\Omega|}{(2d_o)^n} \tag{7}$$

where $K = 2^{2+[5+(n+5)\vartheta/(n+2-\vartheta)]n}$, L is the constant from Lemma 3.11 below, C_S is the Sobolev embedding constant defined in Lemma 3.4 below and the symbol $|\cdot|$ stands for the n-dimensional Lebesgue measure. $\square$

The regularity theorem we formulated above can be illustrated by two examples of the function ω, defined by (2), for which we give estimates of the parameter $\mathcal{M}$. Broadly speaking, if coefficients of a nonlinear system satisfy (iv) with some ω given below and (5) is fulfilled, we have the regularity.

Example 2.3. For

$$\omega(t) = \begin{cases} \omega_0(t) & \text{for} \quad 0 \leq t < t_o, \\ \omega_\infty \sqrt{\ln\left(1 + \frac{\mathrm{e}^{\varepsilon/\omega_\infty^2} - 1}{t_o} t^\gamma\right)}, & \text{for} \quad t_o \leq t \leq t_1,\ 0 < \gamma \leq 1 \\ \omega_\infty & \text{for} \quad t > t_1 \end{cases} \tag{8}$$

where ω_0 is an arbitrary continuous, concave, nondecreasing function such that $\omega_0(0) = 0$ and points t_o, t_1 are selected in such a way that ω is continuous and concave on $[0,\infty)$. The constant occurred in (5), can be chosen as follows (see the Appendix for more information).

$$\frac{1}{\mathcal{M}^2}=\left(\frac{t_o^{2-\gamma}}{10\,C_\mu^{\frac{2\alpha}{2\mu-1}}}\min\left\{1,\frac{3\,C_\mu^{(-1+1/\gamma)\alpha}\,t_o^{\gamma-2+1/\gamma}}{\mathrm{e}^{C_\mu^{\frac{2\alpha}{2\mu-1}}}}\right\}\right)^2, \tag{9}$$

here $t_o\ge 1$, $\mu\ge 6$, $\alpha>1-2/n$, $0<\gamma<0.9$, $C_\mu>1$.

Example 2.4. If

$$\omega(t)=\omega_\infty\sqrt{\frac{2}{\pi}\arctan\frac{t}{C_\mu^\beta}}\,,\qquad 0\le t<\infty \tag{10}$$

then the constant from (5) can be chosen in the form (in this case $t_o=0$, for more information see the Appendix as well)

$$\frac{1}{\mathcal{M}^2}=\left(\frac{C_\mu^{\beta-\alpha}}{\mathrm{e}^{\left(\frac{C_\mu}{2\sqrt{\mu}}\right)^{\frac{2}{2\mu-1}}}}\right)^2. \tag{11}$$

Here $\beta>\alpha>1-2/n$ and $\mu\ge 6$ satisfy (7).

Remark 2.5. We would like to point out that, in the case of the Dirichlet problem (3), the left-hand side of (5) could be substituted with $(M/\nu)^2 ⨍_\Omega |Dg-(Dg)_\Omega|^2\,dy$ (we remind that we employ the estimate (4)). We note that the last mentioned condition involves the data of the problem (3) only (see Example 5.1 below as well). □

It is useful to point out that in the case when the ratio ω_∞/ν is small enough, the regularity of solution to the problem (3) is guaranteed by the following Proposition.

Proposition 2.6. (see [8], Th.2.1) *Let $n\le\vartheta<n+2$ and $\Omega_o\Subset\Omega$ be given. Let u be a weak solution to the Dirichlet problem* (3) *where $g\in W^{1,2}(\Omega,\mathbb{R}^N)$ and the hypotheses* (i), (ii), (iii), (iv) *be satisfied with M, ν and the function ω for which*

$$\frac{\omega_\infty}{\nu}\le\sqrt{\frac{1}{2}\epsilon_0}:=C_{cr} \tag{12}$$

where the constant ϵ_0 is from Theorem 2.1. *Then*

$$\|Du\|_{\mathcal{L}^{2,\vartheta}(\Omega_o,\mathbb{R}^{nN})}\le c d_\vartheta^{-\vartheta}\|Dg\|_{L^2(\Omega,\mathbb{R}^{nN})} \tag{13}$$

for some $0<d_\vartheta\le 2d_o$. The norm $\|Du\|_{\mathcal{L}^{2,\vartheta}(\Omega_o,\mathbb{R}^{nN})}$ is given in Definition 3.1 *below.*

3. Preliminaries

Besides the standard space $C_0^\infty(\Omega,\mathbb{R}^N)$, the Hölder spaces $C^{0,\alpha}(\Omega,\mathbb{R}^N)$ and the Sobolev spaces $W^{k,p}(\Omega,\mathbb{R}^N)$, $W_0^{k,p}(\Omega,\mathbb{R}^N)$ we use the Campanato spaces $\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)$ (see Definition 3.1 below). By $X_{loc}(\Omega,\mathbb{R}^N)$ we denote the space of functions which belong to $X(\widetilde{\Omega},\mathbb{R}^N)$ for every smoothly bounded subdomain $\widetilde{\Omega}\Subset\Omega$.

Definition 3.1. (see, e.g., [16]) Let $\lambda \in [0, n+q]$, $q \in [1, \infty)$. The Campanato space $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$ is the subspace of all functions $u \in L^q(\Omega, \mathbb{R}^N)$ for which the seminorm

$$[u]^q_{\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)} = \sup_{r>0, x\in\Omega} \frac{1}{r^\lambda} \int_{\Omega_r(x)} |u(y) - u_{x,r}|^q \, dy < \infty$$

where $\Omega_r(x) = \Omega \cap B_r(x)$ and $u_{x,r} = ⨍_{\Omega_r(x)} u(y)\, dy$. The norm in the space $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$ is defined by $\|u\|_{\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)} = \|u\|_{L^q(\Omega,\mathbb{R}^N)} + [u]_{\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)}$.

Proposition 3.2. (see [4], [11], [16]) *For a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary, for $q \in [1, \infty)$ and $0 < \lambda < \mu < \infty$, we have the relation $\mathcal{L}^{q,\mu}(\Omega, \mathbb{R}^N) \subset \mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$. Moreover, $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$ is isomorphic to $C^{0,(\lambda-n)/q}(\overline{\Omega}, \mathbb{R}^N)$, for $n < \lambda \le n + q$.*

Now let Φ, Ψ be a pair of complementary Young functions

$$\Phi(u) = u \ln_+^\mu(au), \;\; \Psi(u) \le \overline{\Psi}(u) = \frac{1}{a} u e^{\left(\frac{u}{2\sqrt{\mu}}\right)^{2/(2\mu-1)}} = \frac{1}{a}\widetilde{\Psi}(u) \quad \text{for } u \ge 0 \tag{14}$$

where $a > 0$ and $\mu \ge 2$ are constants,

$$\ln_+(au) = \begin{cases} 0 & \text{for} \quad 0 \le u < \frac{1}{a}, \\ \ln(au) & \text{for} \quad u \ge \frac{1}{a}\,. \end{cases} \tag{15}$$

Then the Young inequality for Φ, Ψ reads

$$uv \le \Phi(u) + \Psi(v), \quad u, v \ge 0. \tag{16}$$

Lemma 3.3. ([28], p. 37) *Let $\phi\colon [0, \infty) \to [0, \infty)$ be a nondecreasing function which is absolutely continuous on every closed interval of finite length, $\phi(0) = 0$. If $w \ge 0$ is measurable and $E(t) = \{y \in \mathbb{R}^n : w(y) > t\}$ then*

$$\int_{\mathbb{R}^n} \phi \circ w \, dy = \int_0^\infty |E(t)| \phi'(t)\, dt.$$

Lemma 3.4. (Sobolev embedding, see [16]) *Assume $n \ge 2$, $1 \le r$, $s < \infty$, such that $s^{-1} - r^{-1} \le n^{-1}$, $R > 0$, $x \in \mathbb{R}^n$. Then for $u \in W^{1,s}(B_R(x), \mathbb{R}^N)$ we have*

$$\left(\int_{B_R(x)} |u(y)|^r \, dy \right)^{1/r} \le C_S R^{1+n/r-n/s} \left(R^{-s} \int_{B_R(x)} |u(y)|^s \, dy + \int_{B_R(x)} |Du(y)|^s \, dy \right)^{1/s}$$

where $C_S = C_S(n, N, r, s)$ is a constant independent of x, R and u.

We will employ the next lemma in the proof of Theorem 2.1.

Lemma 3.5. *Let $v \ge 0$, $b > 0$, $\mu > 0$ and $q > 1$ be arbitrary. Then*

$$v \ln_+^\mu(bv) \le \left(\frac{\mu}{(q-1)e} \right)^\mu b^{q-1} v^q. \tag{17}$$

Proof. Hint: calculate $\sup\left\{ \frac{\ln_+^\mu(bv)}{v^{q-1}}; v \in (0, \infty) \right\}$. □

The following Lemma is proved in [8] (see Lemma 3.5 there).

Lemma 3.6. *Let A, $R_0 \le R_1$ be positive numbers, $n \le \vartheta < n+2$, η a nonnegative and nondecreasing function on $(0,\infty)$. Then there exist ϵ_0, c positive so that for any nonnegative, nondecreasing function ϕ defined on $[0, 2R_1]$ and satisfying with $(B_1 + B_2\eta(U_{2R_0})) \in [0, \epsilon_0]$ the inequality*

$$\phi(\sigma) \le \left\{ A\left(\frac{\sigma}{R}\right)^{n+2} + \frac{1}{2}\left(1 + A\left(\frac{\sigma}{R}\right)^{n+2}\right)[B_1 + B_2\eta(U_{2R})]\right\}\phi(2R) \tag{18}$$

for all σ, R such that $0 < \sigma < R \le R_0$, it holds

$$\phi(\sigma) \le c\sigma^{\vartheta}\phi(2R_0), \quad \forall \sigma : 0 < \sigma \le R_0. \tag{19}$$

Remark 3.7. Note that we can take

$$\epsilon_0 = \frac{1}{2(2^{n+3}A)^{\frac{\vartheta}{n+2-\vartheta}}}, \qquad c = \left(\frac{(2^{n+3}A)^{\frac{1}{n+2-\vartheta}}}{2R_0}\right)^{\vartheta}.$$

In the proof of Theorem 2.1 we will use a modification of the Natanson's Lemma (for a proof see [8]). It can be read as follows.

Lemma 3.8. *Let $f\colon [a,\infty) \to \mathbb{R}$ be a nonnegative function which is integrable on $[a,b]$ for all $a < b < \infty$ and*

$$\mathcal{N} = \sup_{0<h<\infty} \frac{1}{h}\int_a^{a+h} f(t)\,dt < \infty$$

is satisfied. Let $g : [a,\infty) \to \mathbb{R}$ be an arbitrary nonnegative, non-increasing and integrable function. Then $\int_a^{\infty} f(t)g(t)\,dt$ exists and $\int_a^{\infty} f(t)g(t)\,dt \le \mathcal{N}\int_a^{\infty} g(t)\,dt$ holds.

Remark 3.9. The foregoing estimate is optimal because if we put $f(t) = 1$, where $t \in [a,\infty)$, then an equality will be achieved. □

The following form of the Cacciopoli's inequality can be derived by the difference quotient method (see [11], p. 43-46). For a weak solution to the system (1) it holds

$$\int_{B_\sigma(x)} |D^2u|^2\,dy \le \frac{16}{(\varrho-\sigma)^2}\left(\frac{M}{\nu}\right)^2 \int_{B_\varrho(x)} |Du - (Du)_{x,\varrho}|^2\,dy \tag{20}$$

where $x \in \Omega$, $0 < \sigma < \varrho \le \operatorname{dist}(x, \partial\Omega)$).

In the proof of Theorem 2.1 we use an inequality which can be read as follows.

Proposition 3.10. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to (1) satisfying* (i), (ii) *and* (iii). *Then for every ball $B_{2R}(x) \subset \Omega$ and arbitrary constants $\mu \ge 2$, $b > 0$, $1 < q \le n/(n-2)$ and $c \in \mathbb{R}^{nN}$ we have*

$$\int_{B_R(x)} |Du-(Du)_{x,R}|^2 \ln_+^\mu \left(b|Du-(Du)_{x,R}|^2\right) dy \tag{21}$$
$$\leq 2^{n(q-1)}\left(5C_S\frac{M}{\nu}\right)^{2q}\left(\frac{\mu}{(q-1)\mathrm{e}}\right)^{\mu}\left(\frac{b}{(2R)^n}\int_{B_{2R}(x)}|Du-c|^2\,dy\right)^{q-1}\int_{B_{2R}(x)}|Du-c|^2\,dy$$

where C_S is the constant from Lemma 3.4.

Proof. Let $x\in\Omega$ and $0\leq R\leq \frac{1}{4}\mathrm{dist}(x,\partial\Omega)$. We denote $B_R=B_R(x)$ for simplicity. From the standard regularity theory it follows that $Du-c\in W^{1,2}_{loc}(\Omega,\mathbb{R}^N)$. Lemma 3.4 implies $W^{1,2}(B_R,\mathbb{R}^N)\hookrightarrow L^{2q}(B_R,\mathbb{R}^N)$ where $B_R\subset\Omega$ is an arbitrary and $1<q\leq n/(n-2)$. Now Lemma 3.5, the fact that

$$\textstyle\int_{B_R}|Du-(Du)_R|^2\,dy=\min_{c\in\mathbb{R}}\int_{B_R}|Du-c|^2\,dy$$

and (20) give the following sequence of estimates.

$$\int_{B_R}|Du-(Du)_R|^2\ln_+^\mu\left(b\,|Du-(Du)_R|^2\right)dy\leq\left(\frac{\mu}{(q-1)\mathrm{e}}\right)^{\mu}b^{q-1}\int_{B_R}|Du-(Du)_R|^{2q}\,dy$$
$$\leq C_S^{2q}\left(\frac{\mu}{(q-1)\mathrm{e}}\right)^{\mu}b^{q-1}R^{[n(-1+1/q)+2]q}\left(\frac{1}{R^2}\int_{B_R}|Du-c|^2\,dy+\int_{B_R}|D^2u|^2\,dy\right)^{q}$$
$$\leq\left(\frac{\mu}{(q-1)\mathrm{e}}\right)^{\mu}\frac{C_S^{2q}\,(1+16(M/\nu)^2)^q b^{q-1}}{R^{n(q-1)}}\left(\int_{B_{2R}}|Du-c|^2\,dy\right)^{q}. \qquad\square$$

The claim of the next lemma is well known (see, e.g., [4], [11], [22]).

Lemma 3.11. *Let $v\in W^{1,2}(\Omega,\mathbb{R}^N)$ be a weak solution to the linear system with constant coefficients of the type* (1) *satisfying* (ii) *and* (iii). *Then there exists a constant $L\geq 1$ such that for every $x\in\Omega$ and $0<\sigma\leq R\leq\mathrm{dist}(x,\partial\Omega)$ the following estimate holds:*

$$\int_{B_\sigma(x)}|Dv(y)-(Dv)_{x,\sigma}|^2\,dy\leq L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B_R(x)}|Dv(y)-(Dv)_{x,R}|^2\,dy.$$

Remark 3.12. The constant L from the previous Lemma can be stated as

$$L=c(n,N)\left(\frac{M}{\nu}\right)^{2(2+[n/2])}$$

and choosing $n=3$, $N=2$ we can compute $L<2.1\cdot 10^8(M/\nu)^6$. $\square$

4. Proof of the result

Proof of Theorem 2.1. We recall that we set $\phi(x,r)=\int_{B_r(x)}|Du-(Du)_{x,r}|^2\,dx$ and $U_r(x)=\phi(x,r)/r^n$ for $B_r(x)\subset\Omega$. Let $x\in\Omega$ and $R<d_x/2$ be such that the assumption (5) is satisfied. Where no confusion can result, we will use the notation B_R, U_R, $\phi(R)$ and $(Du)_R$ instead of $B_R(x)$, $U_R(x)$, $\phi(x,R)$ and $(Du)_{x,R}$.

Denoting $A_{ij,0}^{\alpha\beta} = A_{ij}^{\alpha\beta}\left((Du)_R\right)$,

$$\widetilde{A}_{ij}^{\alpha\beta} = \int_0^1 A_{ij}^{\alpha\beta}\left((Du)_R + t\left(Du - (Du)_R\right)\right)\, dt,$$

we can rewrite system (1), $i = 1, \ldots, N$, as

$$-\sum_\alpha D_\alpha\left(\sum_{j,\beta} A_{ij,0}^{\alpha\beta} D_\beta u^j\right) = -\sum_\alpha D_\alpha\left(\sum_{j,\beta}\left(A_{ij,0}^{\alpha\beta} - \tilde{A}_{ij}^{\alpha\beta}\right)\left(D_\beta u^j - \left(D_\beta u^j\right)_R\right)\right).$$

Split u as $v + w$ where v is the solution of the Dirichlet problem

$$-\sum_\alpha D_\alpha\left(\sum_{j,\beta} A_{ij,0}^{\alpha\beta} D_\beta v^j\right) = 0 \text{ in } B_R, \quad i = 1, \ldots, N, \quad v - u \in W_0^{1,2}(B_R, \mathbb{R}^N),$$

and $w \in W_0^{1,2}(B_R), \mathbb{R}^N)$ is the weak solution of the system

$$-\sum_\alpha D_\alpha\left(\sum_{j,\beta} A_{ij,0}^{\alpha\beta} D_\beta w^j\right) = -\sum_\alpha D_\alpha\left(\sum_{j,\beta}\left(A_{ij,0}^{\alpha\beta} - \widetilde{A}_{ij}^{\alpha\beta}\right)\left(D_\beta u^j - \left(D_\beta u^j\right)_R\right)\right),$$

where $i = 1, \ldots, N$. For every $0 < \sigma \le R$ it follows from Lemma 3.11 that

$$\int_{B_\sigma} |Dv - (Dv)_\sigma|^2\, dy \le L\left(\frac{\sigma}{R}\right)^{n+2} \int_{B_R} |Dv - (Dv)_R|^2\, dy,$$

and so

$$\int_{B_\sigma} |Du - (Du)_\sigma|^2\, dy$$

$$\le 4L\left(\frac{\sigma}{R}\right)^{n+2} \int_{B_R} |Du - (Du)_R|^2\, dy + 2\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right) \int_{B_R} |Dw|^2\, dy. \tag{22}$$

Now $w \in W_0^{1,2}(B_R, \mathbb{R}^N)$ satisfies

$$\int_{B_R} \sum_{i,j,\alpha,\beta} A_{ij,0}^{\alpha\beta} D_\beta w^j D_\alpha \varphi^i\, dy \le \int_{B_R} \sum_{i,j,\alpha,\beta} \left|A_{ij,0}^{\alpha\beta} - \widetilde{A}_{ij}^{\alpha\beta}\right| \left|D_\beta u^j - (D_\beta u^j)_R\right| \left|D_\alpha \varphi^i\right|\, dy$$

$$\le \left(\int_{B_R} \omega^2\left(|Du - (Du)_R|\right) |Du - (Du)_R|^2\, dy\right)^{1/2} \left(\int_{B_R} |D\varphi|^2\, dy\right)^{1/2}$$

for any $\varphi \in W_0^{1,2}(B_R, \mathbb{R}^N)$. Hence, choosing $\varphi = w$, we get

$$\int_{B_R} |Dw|^2\, dy \le \frac{1}{\nu^2} \int_{B_R} \omega^2\left(|Du - (Du)_R|\right) |Du - (Du)_R|^2\, dy.$$

The previous estimate and (22) give us

$$\int_{B_\sigma} |Du-(Du)_\sigma|^2\,dy \le 4L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B_R}|Du-(Du)_R|^2\,dy$$
$$+\frac{2}{\nu^2}\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\int_{B_R}\omega^2\left(|Du-(Du)_R|\right)|Du-(Du)_R|^2\,dy. \tag{23}$$

In order to estimate the last integral in the foregoing inequality we use the Young inequality (16) (here complementary functions are defined through (14)) and for any $0<\varepsilon<\omega_\infty^2$ we obtain

$$\int_{B_R}\omega^2\left(|Du-(Du)_R|\right)|Du-(Du)_R|^2\;dy \tag{24}$$
$$\le\left(\varepsilon\int_{B_R}|Du-(Du)_R|^2\ln_+^\mu\left(a\varepsilon\,|Du-(Du)_R|^2\right)\,dy+\int_{B_R}\overline{\Psi}\left(\frac{\omega_R^2}{\varepsilon}\right)\,dy\right)=:(\varepsilon I_1+I_2)$$

where $\omega_R^2(y)=\omega^2\left(|Du(y)-(Du)_R|\right)$. The term I_1 can be estimated by means of Proposition 3.10 (here $q=n/(n-2)$) and we get

$$I_1\le CC_\mu\left(a\varepsilon U_{2R}\right)^{2/(n-2)}\phi(2R) \tag{25}$$

where $$C_\mu=\left(\frac{n-2}{2\mathrm{e}}\mu\right)^\mu,\qquad C=\left(5C_S\frac{M}{\nu}\right)^{2n/(n-2)}$$

Applying Lemma 3.3 to the second integral I_2 we have

$$I_2=\int_{B_R}\overline{\Psi}\left(\frac{\omega_R^2}{\varepsilon}\right)\,dx=\frac{1}{a}\int_0^\infty\frac{d}{dt}\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right)m_R(t)\,dt:=\frac{1}{a}\widetilde{I}_2 \tag{26}$$

where $m_R(t)=|\{y\in B_R(x):|Du-(Du)_R|>t\}|$. Using $m_R(t)\le\kappa_nR^n$, κ_n is the Lebesgue measure of the unit ball, we have (we use Lemma 3.8)

$$\begin{aligned}
\widetilde{I}_2&\le\int_0^{t_o}\frac{d}{dt}\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right)m_R(t)\,dt+\int_{t_o}^\infty\frac{d}{dt}\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right)m_R(t)\,dt\\
&\le\kappa_nR^n\int_0^{t_o}\frac{d}{dt}\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right)\,dt+\sup_{t_o<t<\infty}\left(\frac{1}{t-t_o}\int_{t_o}^t\frac{d}{ds}\widetilde{\Psi}\left(\frac{\omega^2(s)}{\varepsilon}\right)\,ds\right)\int_{t_o}^\infty m_R(s)\,ds\\
&\le\kappa_n\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)R^n+\sup_{t_o<t<\infty}\left[\frac{\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right)-\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t-t_o}\right]\int_{B_R}|Du-(Du)_R|\;dy\\
&\le\frac{\kappa_n}{2^nU_{2R}}\,\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)\phi(2R)+\frac{\mathcal{M}}{2^{n/2}}\,\kappa_n^{1/2}(2R)^{n/2}\phi^{1/2}(2R)\\
&<\left[\frac{\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{U_{2R}}+\frac{\mathcal{M}}{\sqrt{U_{2R}}}\right]\phi(2R),\quad\text{where}
\end{aligned}\tag{27}$$

$$\mathcal{M} = \sup_{t_o<t<\infty} \frac{\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) - \widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t-t_o}$$
$$= \sup_{t_o<t<\infty} \frac{\frac{\omega^2(t)}{\varepsilon}\,\mathrm{e}^{\left(\frac{1}{2\sqrt{\mu}}\frac{\omega^2(t)}{\varepsilon}\right)^{2/(2\mu-1)}} - \frac{\omega^2(t_o)}{\varepsilon}\,\mathrm{e}^{\left(\frac{1}{2\sqrt{\mu}}\frac{\omega^2(t_o)}{\varepsilon}\right)^{2/(2\mu-1)}}}{t-t_o}. \tag{28}$$

If for some $R>0$ the average $U_R = 0$ then it is clear that x is the regular point. So in the next formula we can suppose that U_R is positive for all $R>0$. Inserting (24)–(27) into (23) yields

$$\phi(\sigma) \le 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) + 2\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\times$$
$$\times\left[\frac{\varepsilon\, CC_\mu}{\nu^2}\left(a\varepsilon U_{2R}\right)^{2/(n-2)} + \frac{1}{a\nu^2}\left(\frac{\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{U_{2R}} + \frac{\mathcal{M}}{\sqrt{U_{2R}}}\right)\right]\phi(2R). \tag{29}$$

In (27) we can choose $\qquad \varepsilon = \dfrac{\omega_\infty^2}{C_\mu^\alpha}, \quad a = \dfrac{12|\Omega|^{1/2}}{\nu^\delta\,(2d_o)^{n/2}\,U_{2R}}$ for $\quad U_{2R} > 0 \qquad (30)$

where $\delta = 2 + \ln\epsilon_0/\ln\nu$, $\epsilon_0 = 1/(4(2^{n+5}L)^{\vartheta/(n+2-\vartheta)})$ and $\alpha,\ \mu \in \mathbb{R}$ are suitable constants. Then we obtain for $U_{2R} > 0$

$$\phi(\sigma) \le 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) + \frac{1}{2}\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\times$$
$$\times\left[\frac{4CP^2}{C_\mu^{\alpha-1}}\left(\frac{12P^\delta\omega_\infty^{2-\delta}|\Omega|^{1/2}}{(2d_o)^{n/2}\,C_\mu^\alpha}\right)^{\frac{2}{n-2}} + \frac{(2d_o)^{n/2}}{12|\Omega|^{1/2}\nu^{2-\delta}}\left(\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right) + \mathcal{M}\sqrt{U_{2R}}\right)\right]\phi(2R)$$
$$\le 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) + \frac{1}{2}\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\times$$
$$\times\left[\left(\frac{2^{2(n+2)}C^{n-2}P^{2(\delta+n-2)}\omega_\infty^{2(2-\delta)}|\Omega|}{(2d_o)^n\,C_\mu^{(\alpha-1)n+2}}\right)^{\frac{1}{n-2}} + \frac{1}{12}\left(5 + \frac{(2d_o)^{n/2}}{|\Omega|^{1/2}}\mathcal{M}\sqrt{U_{2R}}\right)\epsilon_0\right]\phi(2R)$$

where $P = \omega_\infty/\nu$. In the last term of the foregoing inequality we employed the estimate from (5). The constants $\alpha > 1 - 2/n$ and $\mu \ge 2$ can be always chosen in such a way that

$$\left(\frac{2^{2(n+2)}C^{n-2}P^{2(\delta+n-2)}\omega_\infty^{2(2-\delta)}|\Omega|}{(2d_o)^n\,C_\mu^{(\alpha-1)n+2}}\right)^{\frac{1}{n-2}} \le \frac{1}{2}\epsilon_0$$
$$\iff \quad C_\mu^{(\alpha-1)n+2} \ge \frac{2^{3n+2}C^{n-2}P^{2n}|\Omega|}{(2d_o)^n}\,\epsilon_0^{-n} \tag{31}$$

and we get for every $0 < \sigma \le R \le d_o$

$$\phi(\sigma) \le 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) + \frac{1}{2}\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\left[\frac{11}{12} + \frac{1}{12}\frac{(2d_o)^{n/2}}{|\Omega|^{1/2}}\mathcal{M}\sqrt{U_{2R}}\right]\epsilon_0\phi(2R).$$

We put $A = 4L$, $B_1 = \dfrac{11}{12}\epsilon_0$, $B_2 = \dfrac{1}{12}\epsilon_0$. Now for $R = d_o$ we get from (5) that $(2d_o)^{n/2}\mathcal{M}\sqrt{U_{2d_o}(x)}/|\Omega|^{1/2} \le 1$ and using Lemma 3.6 we can conclude that $\phi(\sigma) \le c\sigma^\vartheta\phi(2R), \quad \forall\, 0 < \sigma \le R.$

5. Illustrating examples and comments

Example 5.1. Let us consider the weak solution $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ of the system (1) in $\Omega = B_R(0) \subset \mathbb{R}^n$ (the fact, that the ball B_R is centered at zero, has no importance for the next conclusions). Let us put $\Omega_o = B_r(0)$, $0 < r < R$ and $d_o = (R-r)/2$. Taking into account (5), we get

$$\fint_{B_R(0)} |Du - (Du)_R|^2 \, dy < \frac{1}{\mathcal{M}^2} \tag{32}$$

(for more information about the constant $\mathcal{M}$ see the right-hand side of (9) or (11)). The validity of the foregoing estimate guarantees regularity in $B_r(0)$. If, moreover, we consider the Dirichlet problem (3) where $g \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ and we assume that for $0 < \lambda \le n+2$ and every $\sigma > 0$ the estimate $\sigma^{-\lambda} \int_{B_\sigma(0)} |Dg - (Dg)_{0,\sigma}|^2 \, dy \le c_\lambda$, with $c_\lambda > 0$ holds, then, as a result of Remark 2.5, we get the sufficient condition for validity of (32) in the form

$$\frac{c_\lambda}{\kappa_n} \left(\frac{M}{\nu} \right)^2 R^{\lambda - n} < \frac{1}{\mathcal{M}^2}$$

where κ_n is the Lebesgue measure of the unit ball. We have to remark that for all $R > 0$ and all $0 < r \le R/2$ the constant $|\Omega|/(2d_o)^n$ from right hand side of (7) is less or equal to $2^n \kappa_n$. It follows that neither $\mathcal{M}$ depends on R. If, for instance, $r \le R/2$, the previous inequality takes the following forms.

- For $\lambda = n$ and R arbitrary: $\dfrac{c_\lambda}{\kappa_n} \left(\dfrac{M}{\nu} \right)^2 < \dfrac{1}{\mathcal{M}^2}$,
- for $\lambda > n$ and $R \le 1$: $\dfrac{c_\lambda}{\kappa_n} \left(\dfrac{M}{\nu} \right)^2 R^{\lambda - n} < \dfrac{1}{\mathcal{M}^2}$,
- for $\lambda < n$ and $R > 1$: $\dfrac{c_\lambda}{\kappa_n} \left(\dfrac{M}{\nu} \right)^2 R^{n - \lambda} < \dfrac{1}{\mathcal{M}^2}$.

We only notice that the second inequality is automatically satisfied for sufficiently small R and the same holds for the third one in the case of sufficiently large R.

Example 5.2. Here we give a sample of the system under consideration which can illustrate some advantages of our main result. We construct the elliptic system of the type (1) in the following way. Let us put $n \ge 3$, $N = n$, $A_i^\alpha(p) = a_{ij}^{\alpha\beta} p_j^\beta + g_i^\alpha(p)$ where $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$ is a constant matrix and let us suppose that there exists a constant $\nu_1 > 0$ such that $\sum_{i,j,\alpha,\beta} a_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \ge \nu_1 |\xi|^2$ for all $\xi \in \mathbb{R}^{n^2}$, $g_i^\alpha \in C^1(\mathbb{R}^{n^2})$, there exist constants $c_1 > 0$, $c_2 > 0$ such that $|g_i^\alpha(p) - g_i^\alpha(q)| \le c_1 + c_2 |p - q|$ for all $p, q \in \mathbb{R}^{n^2}$, there exists a constant $\nu_2 > 0$ such that

$$\sum_{i,j,\alpha,\beta} \left| \frac{\partial g_i^\alpha}{\partial p_j^\beta}(p) \right| \le \nu_2 \quad \text{and} \quad 0 \le \sum_{i,j,\alpha,\beta} \frac{\partial g_i^\alpha}{\partial p_j^\beta}(p) \xi_\alpha^i \xi_\beta^j, \qquad \forall\, p, \xi \in \mathbb{R}^{n^2}. \tag{33}$$

We set
$$A_{ij}^{\alpha\beta}(p) = \frac{\partial A_i^\alpha}{\partial p_j^\beta}(p) = a_{ij}^{\alpha\beta} + \frac{\partial g_i^\alpha}{\partial p_j^\beta}(p) \tag{34}$$

and moreover we suppose that $\sum_{i,j,\alpha,\beta} \left| A_{ij}^{\alpha\beta}(p) - A_{ij}^{\alpha\beta}(q) \right| \le \omega(|p-q|)$ (ω is defined above by the expression (8) or (10)).

It is easy to construct constant matrices $(d^{\alpha\beta})$ and (b_{ij}) such that $a_{ij}^{\alpha\beta} = d^{\alpha\beta} b_{ij}$, $d^{\alpha\beta} = d^{\beta\alpha}$, $b_{ij} = b_i^{'} b_j^{'}$, $b_i^{'} \neq 0$, $i = 1, ..., N$, $\det(b_{ij}) \neq 0$ and there exist constants $\nu_3, \nu_4 > 0$ for which

$$\nu_3 |\eta|^2 \le \sum_{\alpha,\beta} d^{\alpha\beta} \eta_\alpha \eta_\beta \le \nu_4 |\eta|^2, \qquad \forall\, \eta \in \mathbb{R}^n$$

holds. For instance, if the matrix $(d^{\alpha\beta})$ is positively definite with smallest eigenvalue greater than or equal to $\nu_3 > 0$ and the matrix (b_{ij}) has positive diagonal components, then the matrix $(a_{ij}^{\alpha\beta})$ has the required properties. Now we can write

$$A_{ij}^{\alpha\beta}(p) = \widetilde{d}^{\alpha\beta}(p) b_{ij}, \qquad \widetilde{d}^{\alpha\beta}(p) = d^{\alpha\beta} + \frac{1}{b_{ij}} \frac{\partial g_i^\alpha}{\partial p_j^\beta}(p).$$

We have $$\sum_{i,j,\alpha,\beta} A_{ij}^{\alpha\beta}(p) \xi_\alpha^i \xi_\beta^j = \sum_{i,j,\alpha,\beta} \left(a_{ij}^{\alpha\beta} + \frac{\partial g_i^\alpha}{\partial p_j^\beta}(p) \right) \xi_\alpha^i \xi_\beta^j \ge \nu_1 |\xi|^2,$$

$$\begin{aligned}\sum_{\alpha,\beta} \widetilde{d}^{\alpha\beta}(p) \eta_\alpha \eta_\beta &= \sum_{\alpha,\beta} \left(d^{\alpha\beta} + \sum_{i,j} \frac{1}{b_{ij}} \frac{\partial g_i^\alpha}{\partial p_j^\beta}(p) \right) \eta_\alpha \eta_\beta \\ &= \sum_{\alpha,\beta} \left(d^{\alpha\beta} + \sum_{i,j} \frac{1}{b_i^{'} b_j^{'}} \frac{\partial g_i^\alpha}{\partial p_j^\beta}(p) \right) \eta_\alpha \eta_\beta \ge \nu_3 |\eta|^2\end{aligned}$$

and $$\sum_{\alpha,\beta} \widetilde{d}^{\alpha\beta}(p) \eta_\alpha \eta_\beta = \sum_{\alpha,\beta} \left(d^{\alpha\beta} + \sum_{i,j} \frac{1}{b_i^{'} b_j^{'}} \frac{\partial g_i^\alpha}{\partial p_j^\beta}(p) \right) \eta_\alpha \eta_\beta \le \left(\nu_4 + \frac{n\nu_2}{(b^{'})^2} \right) |\eta|^2$$

where $b^{'} = \min_i |b_i^{'}|$. Now, putting $H = \sum_{i,j,\alpha,\beta} |a_{ij}^{\alpha\beta}|$ and taking into account (33) the constant L, mentioned in Lemma 3.11, can be exactly computed as

$$L = 2^{3n+6} \left(\frac{\nu_4 + \frac{n\nu_2}{(b^{'})^2}}{\nu_3} \right)^n \left(\frac{H + \nu_2}{\nu_1} \right)^2. \tag{35}$$

The foregoing computation was allowed due to the arguments from [18]. In this case the constant ϵ_0 defined in Theorem 2.1 has the form

$$\epsilon_0 = \frac{1}{2 \left(2^{4n+11} \left(\frac{\nu_4 + \frac{n\nu_2}{(b^{'})^2}}{\nu_3} \right)^n \left(\frac{H+\nu_2}{\nu_1} \right)^2 \right)^{\vartheta/(n+2-\vartheta)}}, \qquad n \le \vartheta < n+2. \tag{36}$$

We would like to point out that the constant L (and consequently the constant ϵ_0) can be calculated in general case as well but, its computation is a bit more complicated and the obtained estimate is bigger than the one in (35), as it is visible in Remark 3.12.

Now, because of simplicity, we will consider the above system in $\Omega = B_R(0)$ and put $\Omega_o = B_{R/2}(0)$ (we use the same denotation as in the previous example). Choosing $n=3$, $\omega_\infty = \nu_1$, $((\nu_4 + 3\nu_2/(b')^2)/\nu_3)^3 = 10^2$, $((H+\nu_2)/\nu_1)^2 = 10^4$, $P = \omega_\infty/\nu_1 = 1$, $C_S = 10$ and $\epsilon_0 = 10^{-20}$ (the choice of the value ϵ_0 seems to be realistic with respect to (36), and the same we can say about the constant C_S according to the paper [26]) we can get as follows.

(a) The case when the function ω is defined by Example 2.3:

ω_∞	$=$	10^{20}	10^{30}	10^{40}	10^{50}	10^{60}
t_o	$=$	10^3	10^5	10^7	10^8	10^9
$\omega(t_o)$	$\approx$	10^4	10^{14}	10^{26}	10^{36}	10^{44}
$\omega(\omega_\infty)$	$\approx$	10^{11}	10^{24}	10^{37}	10^{50}	10^{60}
t_1	$\approx$	10^{44}	10^{46}	10^{48}	10^{49}	10^{51}
real value $\frac{1}{\mathcal{M}^2}$	$\approx$	10^5	10^{10}	10^{14}	10^{14}	10^{18}
estimate $\frac{1}{\mathcal{M}^2}$ by means of (9)	$\approx$	1	10^3	10^{11}	10^{12}	10^{12}
α	$=$	2.2	2.19	2.25	2.34	2.35
γ	$=$	0.77	0.77	0.76	0.76	0.79
μ	$=$	22.9	23	22.5	21.8	21.8

(b) The case when the function ω is defined by Example 2.4:

ω_∞	$=$	10^{20}	10^{30}	10^{40}	10^{50}	10^{60}
$\tilde{t}$	$\approx$	10^{43}	10^{46}	10^{47}	10^{49}	10^{50}
$\omega(\omega_\infty)$	$\approx$	10^8	10^{22}	10^{37}	10^{50}	10^{60}
real value $\frac{1}{\mathcal{M}^2}$	$\approx$	10^7	10^{12}	10^{13}	10^{18}	10^{21}
estimate $\frac{1}{\mathcal{M}^2}$ by means of (11)	$\approx$	10^6	10^{11}	10^{12}	10^{17}	10^{20}
α	$=$	1.44	1.45	1.49	1.49	1.49
β	$=$	1.78	1.9	2	2.1	2.17
μ	$=$	31.5	31.3	30.6	30.6	30.6

Here $\tilde{t}$ is the point for which $\omega(\tilde{t}) = 0.95 \cdot \omega_\infty$. It is necessary to remember that the condition (7) from the main Theorem is satisfied for the above-mentioned parameters.

6. Appendix

Here we give an estimate of the constant $\mathcal{M}$ defined by (6). We remind that $\mu \geq 6$, $\alpha > 1 - 2/n$ and $C_\mu \geq (0.18 \cdot (n-2)\mu)^\mu > 1$.

(a) Estimate $\mathcal{M}$ related to the function ω given by (8), $t_o < \xi < t \leq t_1$:

$$\frac{\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) - \widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t - t_o} = \left(\frac{d}{dt}\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right)\right)|_{t=\xi}$$

$$= \frac{2}{\varepsilon}\omega(\xi)\omega'(\xi)\left[1 + \frac{2}{2\mu-1}\left(\frac{1}{2\sqrt{\mu}}\frac{\omega^2(\xi)}{\varepsilon}\right)^{\frac{2}{2\mu-1}}\right] \mathrm{e}^{\left(\frac{1}{2\sqrt{\mu}}\frac{\omega^2(\xi)}{\varepsilon}\right)^{\frac{2}{2\mu-1}}}.$$

Here we consider $t_o \geq 1$ and $0 < \gamma < 0.9$

$$
\begin{aligned}
\mathcal{M} &= \sup_{t_o<t<t_1} \frac{\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) - \widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t - t_o} \\
&= \sup_{t_o<t<t_1} \left(\frac{2}{\varepsilon}\omega(t)\omega'(t)\right) \mathrm{e}^{\left(\frac{1}{2\sqrt{\mu}}\frac{\omega^2(t)}{\varepsilon}\right)^{\frac{2}{2\mu-1}}} \left[1 + \frac{2}{2\mu-1}\left(\frac{1}{2\sqrt{\mu}}\frac{\omega^2(t)}{\varepsilon}\right)^{\frac{2}{2\mu-1}}\right] \\
&= \sup_{t_o<t<t_1} \left(\gamma C_\mu^\alpha \frac{\mathrm{e}^{1/C_\mu^\alpha} - 1}{\left[t_o + \left(\mathrm{e}^{1/C_\mu^\alpha} - 1\right) t^\gamma\right]} t^{\gamma-1}\right) \mathrm{e}^{\left(\frac{C_\mu^\alpha}{2\sqrt{\mu}} \ln\left(1 + \frac{\mathrm{e}^{1/C_\mu^\alpha} - 1}{t_o} t^\gamma\right)\right)^{\frac{2}{2\mu-1}}} \times \\
&\qquad \times \left[1 + \frac{2}{2\mu-1}\left(\frac{C_\mu^\alpha}{2\sqrt{\mu}} \ln\left(1 + \frac{\mathrm{e}^{1/C_\mu^\alpha} - 1}{t_o} t^\gamma\right)\right)^{\frac{2}{2\mu-1}}\right] \\
&\leq \sup_{t_o<t<t_1} \left(\gamma C_\mu^\alpha \frac{\mathrm{e}^{1/C_\mu^\alpha} - 1}{\left[t_o + \left(\mathrm{e}^{1/C_\mu^\alpha} - 1\right) t^\gamma\right]}\right) \sup_{t_o<t<t_1} \left(t^{\gamma-1} \mathrm{e}^{\left(\frac{C_\mu^\alpha}{2\sqrt{\mu}} \ln\left(1 + \frac{\mathrm{e}^{1/C_\mu^\alpha} - 1}{t_o} t^\gamma\right)\right)^{\frac{2}{2\mu-1}}}\right) \times \\
&\qquad \times \left[1 + \frac{2}{2\mu-1} \sup_{t_o<t<t_1} \left(\frac{C_\mu^\alpha}{2\sqrt{\mu}} \ln\left(1 + \frac{\mathrm{e}^{1/C_\mu^\alpha} - 1}{t_o} t^\gamma\right)\right)^{\frac{2}{2\mu-1}}\right] \\
&= S_1 S_2 \left(1 + \frac{2}{2\mu-1} S_3\right).
\end{aligned} \tag{37}
$$

The estimates of S_1–S_3 are as follows:

$$
S_1 \leq \gamma\, C_\mu^\alpha \frac{\mathrm{e}^{1/C_\mu^\alpha} - 1}{t_o} \leq \frac{\gamma(\mathrm{e}-1)}{t_o} \quad \forall\, t_o \leq t \leq t_1, \quad S_2 \leq \sup_{t_o<t<t_1} \frac{\mathrm{e}^{\left(\frac{1}{\sqrt{\mu} t_o} t^\gamma\right)^{\frac{2}{2\mu-1}}}}{t^{1-\gamma}}.
$$

If we denote $$f(t) = \frac{\mathrm{e}^{\left(\frac{1}{\sqrt{\mu} t_o} t^\gamma\right)^{\frac{2}{2\mu-1}}}}{t^{1-\gamma}}, \quad t \in (t_o, \infty),$$

the standard method of differential calculus gives us the estimate

$$
\begin{aligned}
S_2 &\leq \max\{f(t_o), f(t_1)\} \leq \max\left\{\frac{\mathrm{e}^{\left(\frac{1}{\sqrt{\mu}\, t_o^{1-\gamma}}\right)^{\frac{2}{2\mu-1}}}}{t_o^{1-\gamma}}, \frac{\mathrm{e}^{\left(\frac{2C_\mu^\alpha}{\sqrt{\mu}}\right)^{\frac{2}{2\mu-1}}}}{(C_\mu^\alpha t_o)^{\frac{1-\gamma}{\gamma}}}\right\} \\
&\leq \frac{1}{t_o^{1-\gamma}} \max\left\{3, \frac{\mathrm{e}^{\left(\frac{2C_\mu^\alpha}{\sqrt{\mu}}\right)^{\frac{2}{2\mu-1}}}}{C_\mu^{(-1+1/\gamma)\alpha}\, t_o^{\gamma-2+1/\gamma}}\right\}.
\end{aligned}
$$

Finally, $$S_3 \leq \left(\frac{C_\mu^\alpha}{2\sqrt{\mu}}\right)^{\frac{2}{2\mu-1}} \qquad \forall\, t_o \leq t \leq t_1.$$

Inserting the above estimates into (37), we get

$$\mathcal{M} \leq \frac{\gamma(\mathrm{e}-1)}{t_o^{2-\gamma}}\left(1+\frac{2}{2\mu-1}\left(\frac{C_\mu^\alpha}{2\sqrt{\mu}}\right)^{\frac{2}{2\mu-1}}\right)\max\left\{3, \frac{\mathrm{e}^{\left(\frac{2C_\mu^\alpha}{\sqrt{\mu}}\right)^{\frac{2}{2\mu-1}}}}{C_\mu^{(-1+1/\gamma)\alpha}\, t_o^{\gamma-2+1/\gamma}}\right\}$$

$$\leq \frac{10\, C_\mu^{\frac{2\alpha}{2\mu-1}}}{t_o^{2-\gamma}}\max\left\{1, \frac{\mathrm{e}^{C_\mu^{\frac{2\alpha}{2\mu-1}}}}{3\, C_\mu^{(-1+1/\gamma)\alpha}\, t_o^{\gamma-2+1/\gamma}}\right\}. \tag{38}$$

The term $\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)$ from the definition of $\mathcal{M}$ can be estimated as

$$\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right) = \frac{\omega^2(t_o)}{\varepsilon}\mathrm{e}^{\left(\frac{\omega^2(t_o)}{2\sqrt{\mu}\,\varepsilon}\right)^{2/(2\mu-1)}} = C_\mu^\alpha \ln\left(1+\frac{\mathrm{e}^{1/C_\mu^\alpha}-1}{t_o^{1-\gamma}}\right)\mathrm{e}^{\left(\frac{C_\mu^\alpha}{2\sqrt{\mu}}\ln\left(1+\frac{\mathrm{e}^{1/C_\mu^\alpha}-1}{t_o^{1-\gamma}}\right)\right)^{\frac{2}{2\mu-1}}}$$

$$\leq C_\mu^\alpha \frac{\mathrm{e}^{1/C_\mu^\alpha}-1}{t_o^{1-\gamma}}\mathrm{e}^{\left(C_\mu^\alpha\frac{\mathrm{e}^{1/C_\mu^\alpha}-1}{2\sqrt{\mu}\, t_o^{1-\gamma}}\right)^{\frac{2}{2\mu-1}}} \leq \frac{\mathrm{e}-1}{t_o^{1-\gamma}}\mathrm{e}^{\left(\frac{1}{\sqrt{\mu}\, t_o^{1-\gamma}}\right)^{\frac{2}{2\mu-1}}} \leq 5, \qquad \forall\, t_o \geq 1\,.$$

(b) The estimate $\mathcal{M}$ for ω is given by (10):

$$\mathcal{M} \leq \frac{\mathrm{e}^{\left(\frac{C_\mu^\alpha}{2\sqrt{\mu}}\right)^{\frac{2}{2\mu-1}}}}{C_\mu^{\beta-\alpha}}, \quad \beta > \alpha > 1-\frac{2}{n}, \quad \text{and} \quad \widetilde{\Psi}(\omega^2(t_o)/\varepsilon) = 0. \tag{39}$$

References

[1] L. Beck, M. Bulíček, J. Frehse: *Old and new results in regularity theory for diagonal elliptic systems via blowup techniques*, J. Differential Equations 259 (2015) 6528–6572.

[2] M. Bulíček, J. Frehse: *C^α-regularity for a class of non-diagonal elliptic systems with p-growth*, Calc. Var. Partial Diff. Equations 43 (2012) 441–462.

[3] M. Bulíček, J. Frehse, M. Steinhauer: *On Hölder continuity of solutions for a class of nonlinear elliptic systems with p-growth via weighted integral techniques*, Ann. Mat. Pura Appl. 194 (2015) 1025–1069.

[4] S. Campanato: *Sistemi Ellittici in Forma Divergenza. Regolarità all'Interno*, Quaderni Scuola Norm. Sup. Pisa, Pisa (1980).

[5] S. Campanato: *A maximum principle for non linear elliptic systems: Boundary fundamental estimates*, Advances Math. 66 (1987) 293–317.

[6] J. Daněček: *Regularity for nonlinear elliptic systems of second order*, Comment. Math. Univ. Carolinae 27(4) (1986) 755–764.

[7] J. Daněček, O. John, J. Stará: *Structural conditions guaranteeing $C^{1,\gamma}$-regularity of weak solutions to nonlinear second order elliptic systems*, Nonlinear Analysis T.M.A. 66 (2007) 288–300.

[8] J. Daněček, E. Viszus: *On the interior regularity for the gradient of weak solutions to nonlinear second order elliptic systems*, Electron. J. Diff. Equation 2013(121) (2013), 17 pp.

[9] J. Daněček, E. Viszus: *Regularity on the interior for some class of nonlinear second-order elliptic systems*, Nonlinear Analysis T.M.A. 142 (2016) 97–111.

[10] J. Daněček, E. Viszus: *On Morrey and BMO regularity for gradients of weak solutions to nonlinear elliptic systems with non-differentiable coefficients*, Electron. J. Qual. Theory Diff. Equations 11 (2017) 1–14.

[11] M. Giaquinta: *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals Math. Studies 105, Princeton University Press, Princeton (1983).

[12] W. Hao, S. Leonardi, J. Nečas: *An example of irregular solutions to a nonlinear Euler-Lagrange system with real analytic coefficients*, Ann. Scuola Norm. Sup. Pisa 23 (1996) 57–67.

[13] S. Hildebrandt, K. O. Widman: *Some regularity results for quasilinear elliptic systems of second order*, Math. Zeitschrift 142 (1975) 67–86.

[14] A. I. Koshelev: *Regularity Problem for Quasilinear Elliptic and Parabolic Systems*, Lecture Notes in Mathematics 1614, Springer, Berlin (1995).

[15] J. Kristensen, Ch. Melcher: *Regularity in oscillatory nonlinear elliptic systems*, Math. Zeitschrift 260(4) (2008) 813–847.

[16] A. Kufner, O. John, S. Fučík: *Function Spaces*, Academia, Prague (1977).

[17] S. Leonardi: *Remarks on the regularity of solutions of elliptic systems*, in: *Applied Nonlinear Analysis*, A. Sequeira, H. B. da Veiga and J. H. Videman (eds.), Kluwer Academic / Plenum Publishers, New York (1999) 325–344.

[18] F. Leonetti: *An integral estimate for weak solutions to some quasilinear elliptic systems*, Comment. Math. Univ. Carolinae 32(1) (1991) 39–43.

[19] G. Mingione: *Regularity of minima: An invitation to the dark side of the calculus of variations*, Appl. Math. 51(4) (2006) 355–426.

[20] G. Mingione: *Singularities of minima: a walk on the wild side of the Calculus of Variations*, J. Global Optimization 40 (2008) 209–223.

[21] J. Nečas: *Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity*, in: *Theory of Nonlinear Operators*, Proc. Fourth International Summer School, Academy of Science, Berlin 1975, Akademie-Verlag, Berlin (1977) 197–206.

[22] J. Nečas: *Introduction to the Theory of Nonlinear Elliptic Equations*, Texte zur Mathematik 52, Teubner, Leipzig (1983).

[23] J. Nečas, O. John, J. Stará: *Counterexample to the regularity of weak solutions of elliptic systems*, Comment. Math. Univ. Carolinae 21 (1980) 145–154.

[24] V. Šverák, X. Yan: *A singular minimizer of a smooth strongly convex functional in three dimensions*, Calc. Var. Partial Diff. Equations 10 (2000) 213–221.

[25] V. Šverák, X. Yan: *Non-Lipschitz minimizers of smooth uniformly convex functionals*, Proc. Nat. Acad. Sci, USA 99, No.24 (2002) 15269–15276.

[26] G. Talenti: *Best constant in Sobolev inequality*, Ann. Math. Pura Appl. 110 (1976) 353–372.

[27] K. Uhlenbeck: *Regularity for a class of non-linear elliptic systems*, Acta Math. 138 (1977) 219–240.

[28] W. P. Ziemer: *Weakly Differentiable Functions*, Springer, Berlin (1989).