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Permutation-Invariance in Komlós' Theorem for Hilbert-Space Valued Random Variables

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The Komlós theorem states that we can extract a subsequence from every $L^1_{\mathbb{R}}$-bounded sequence of random variables, so that every further subsequence converges Cesàro a.e. to the same limit. The purpose of this paper is to prove that if $\mathbb{H}$ is a Hilbert space, we can extract a subsequence from every $L^1_{\mathbb{H}}$-bounded sequence, so that every permuted subsequence converges Cesàro a.e. in $\mathbb{H}$ to the same limit.

Keywords: Bounded sequences in $L^1_{\mathbb{H}}$, Cesàro-convergence, Hilbert space, Komlós, permutation.

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1. Introduction

Let μ be a probability measure. In 1967, Komlós [6] showed that for every bounded sequence $(f_n)_{n\geq 1}$ in $L^1_{\mathbb{R}}$, there exists a subsequence $(g_n)_{n\geq 1}$ of $(f_n)_{n\geq 1}$ and $f \in L^1_{\mathbb{R}}$ such that for every further subsequence $(h_n)_{n\geq 1}$ of $(g_n)_{n\geq 1}$, the Cesàro sum $\frac{1}{n}\sum_{k=1}^{n} h_k$ converges μ-a.e. to f. By the subsequence principle in [3], the Komlós theorem corresponds to the Kolmogorov strong law of large numbers. And since rearranging the terms of an i.i.d. sequence does not change its asymptotic properties, it is natural to expect that the Komlós theorem remains valid after any permutation of the terms of the subsequence. However, the proof of the general theorem in [6] is not permutation-invariant. In fact, the last property has been proved only in special cases or under additional assumptions. For instance, under the stronger assumption $\sup_n \|f_n\|_2 < \infty$, the result of [7] guarantees the permutational invariance in the Komlós theorem (for another proof see Aldous [1]). Later Berkes [2] proved that for every bounded sequence $(f_n)_{n\geq 1}$ in $L^1_{\mathbb{R}}$ there exists a subsequence $(g_n)_{n\geq 1}$ of $(f_n)_{n\geq 1}$ and $f \in L^1_{\mathbb{R}}$ such that every permutation of $(g_n)_{n\geq 1}$ is almost everywhere Cesàro-convergent to f. On the other hand, in Komlós theorem the almost surely Cesàro-convergence holds true for any further subsequence of $(g_n)_{n\geq 1}$ as well. In 1986, Meyerowitz and Schwartz [8] showed that the subsequences in the Komlós theorem could be permuted for certain "dominated" permutations. Hence, the question of whether a Komlós-type theorem stays valid for arbitrary permutations remained open. The present paper gives an affirmative answer. Inspired by Etemadi [4] and Schwartz [9], we establish the following generalization of Komlós theorem for Hilbert space valued random variables:

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Theorem 1.1. *Let $(f_n)_{n\geq 1}$ be a bounded sequence in $L^1_{\mathbb{H}}(\Omega,\Sigma,\mu)$, $\mathbb{H}$ a Hilbert space. Then, there exists a subsequence $(g_n)_{n\geq 1}$ of $(f_n)_{n\geq 1}$ and f in $L^1_{\mathbb{H}}(\Omega,\Sigma,\mu)$ such that:*

$$\frac{1}{n}\sum_{k=1}^{n} g_{j_{\pi(k)}} \underset{n\to+\infty}{\longrightarrow} f \ \mu\text{-a.e.} \tag{1}$$

for any subsequence $(g_{j_n})_{n\geq 1}$ of $(g_n)_{n\geq 1}$ and for every permutation $(\pi(1), \pi(2), ...)$ of $(1,2,...)$.

2. Notations and preliminaries

Throughout this paper $(\Omega,\ \Sigma,\ \mu)$ is a probability space and $\mathbb{H}$ is a real Hilbert space. If $1\leq p<\infty$, $L^p_{\mathbb{H}} = L^p_{\mathbb{H}}(\Omega,\ \Sigma,\ \mu)$ denotes the Banach space of (equivalence classes of) all strongly Σ-measurable functions $f\colon\Omega\to\mathbb{H}$, such that $\|f(.)\|^p$ is integrable. The inner product in the Hilbert spaces $\mathbb{H}$ and $L^2_{\mathbb{H}}$ is denoted by $\langle .,.\rangle$. For any $a\geq 0$ and $f\colon\Omega\to\mathbb{H}$, we set

$$F_a(f)(w) = \begin{cases} f(w) & \text{if} \quad \|f(w)\| < a, \\ 0 & \text{if not.} \end{cases}$$

We first establish the following lemma,

Lemma 2.1. *Let $(f_n)_{n\geq 1}$ be a bounded sequence in $L^1_{\mathbb{H}}$. Then there exists a subsequence $(g_n)_{n\geq 1}$ of $(f_n)_{n\geq 1}$ such that for any subsequence $(h_n)_{n\geq 1}$ of $(g_n)_{n\geq 1}$ and for every permutation $(\pi(1), \pi(2), ...)$ of $(1,2,...)$, we have the following:*

(a) $\sum\limits_{n\geq 1} \frac{1}{n^2}\left\|F_{n\wedge\pi(n)}\left(h_{\pi(n)}\right)\right\|_2^2 < +\infty,$

(b) $\sum\limits_{n\geq 1} \mu\left(\left\{\left\|h_{\pi(n)}\right\| \geq n\wedge\pi(n)\right\}\right) < +\infty.$

Proof. Put $M = \sup_n \|f_n\|_1$. The real sequence $(\mu(\{k-1\leq\|f_n\|<k\}))_{n\geq 1}$ is bounded for each $k\geq 1$. Then there exists a subsequence $(f^k_n)_{n\geq 1}$ of $(f_n)_{n\geq 1}$ and p_k in $[0,1]$, where $(f^{k+1}_n)_{n\geq 1}$ is a subsequence of $(f^k_n)_{n\geq 1}$, such that

$$\lim_n \mu\left(\left\{k-1\leq\left\|f^k_n\right\|<k\right\}\right) = p_k$$

and $$\forall k\geq 1, \forall n\geq 1,\ \mu\left(\left\{k-1\leq\left\|f^k_n\right\|<k\right\}\right) < p_k + \frac{1}{k^3}.$$

Put $g_n = f^{n^2}_n (n\geq 1)$. Then for any subsequence $(h_n)_{n\geq 1}$ of $(g_n)_{n\geq 1}$:

(i) $\forall k\geq 1\ \lim\limits_n \mu(\{k-1\leq\|h_n\|<k\}) = p_k,$

(ii) $\forall n\in\mathbb{N}^*,\ \forall k\in\{1,...,n^2\}:\quad \mu(\{k-1\leq\|h_n\|<k\}) < p_k + \frac{1}{k^3}.$

One has

$$\sum_{k=1}^{N} p_k = \lim_n \sum_{k=1}^{N} \mu(\{k-1\leq\|g_n\|<k\}) = \lim_n \mu\left(\bigcup_{k=1}^{N}\{k-1\leq\|g_n\|<k\}\right) \leq \mu(\Omega),$$

and
$$\sum_{k=1}^{N}(k-1)\,p_k=\lim_n\sum_{k=1}^{N}(k-1)\,\mu\left(\{k-1\le\|g_n\|<k\}\right)$$
$$\le\overline{\lim_n}\int_{\bigcup_{k=1}^{N}\{k-1\le\|g_n\|<k\}}\|g_n\|\,d\mu\le M.$$

Consequently
$$\sum_{k=1}^{\infty}p_k\le 1\quad\text{and}\quad\sum_{k=1}^{\infty}(k-1)p_k\le M.$$

(a) Let $(\pi(1),\pi(2),...)$ be a permutation of $(1,2,...)$, one has

$$\left\|F_{n\wedge\pi(n)}\left(h_{\pi(n)}\right)\right\|_2^2=\int_{\left\{\left\|h_{\pi(n)}\right\|<n\wedge\pi(n)\right\}}\left\|h_{\pi(n)}\right\|^2d\mu$$
$$\le\sum_{k=1}^{n\wedge\pi(n)}k^2\,\mu\left(\left\{k-1\le\left\|h_{\pi(n)}\right\|<k\right\}\right)<\sum_{k=1}^{n\wedge\pi(n)}k^2\left(p_k+\frac{1}{k^3}\right).$$

Hence

$$\sum_{n\ge1}\frac{1}{n^2}\left\|F_{n\wedge\pi(n)}\left(h_{\pi(n)}\right)\right\|_2^2<\sum_{n\ge1}\frac{1}{n^2}\sum_{k=1}^{n\wedge\pi(n)}k^2\left(p_k+\frac{1}{k^3}\right)\le\sum_{n\ge1}\frac{1}{n^2}\sum_{k=1}^{n}k^2\left(p_k+\frac{1}{k^3}\right)$$
$$\le\sum_{k\ge1}k^2\left(p_k+\frac{1}{k^3}\right)\sum_{n\ge k}\frac{1}{n^2}\le2\sum_{k\ge1}\left(kp_k+\frac{1}{k^2}\right)<\infty.$$

(b) Let $n\ge1$. If $n<\pi(n)^2$, one has

$$\mu\left(\left\|h_{\pi(n)}\right\|\ge n\right)=\sum_{k=n+1}^{(\pi(n))^2}\mu\left(k-1\le\left\|h_{\pi(n)}\right\|<k\right)+\mu\left(\left\|h_{\pi(n)}\right\|\ge(\pi(n))^2\right)$$
$$\le\sum_{k=n+1}^{(\pi(n))^2}\left(p_k+\frac{1}{k^3}\right)+\frac{1}{(\pi(n))^2}\left\|h_{\pi(n)}\right\|_1\le\sum_{k=n+1}^{\infty}p_k+\frac{1}{2n^2}+\frac{M}{(\pi(n))^2}$$

and if $n\ge(\pi(n))^2$, one has

$$\mu\left(\left\|h_{\pi(n)}\right\|\ge n\right)\le\mu\left(\left\|h_{\pi(n)}\right\|\ge(\pi(n))^2\right)\le\frac{M}{(\pi(n))^2}.$$

Consequently

$$\begin{aligned}\sum_{n\ge1}\mu\left(\left\|h_{\pi(n)}\right\|\ge n\right)&\le\sum_{n\ge1}\sum_{k\ge n+1}p_k+\sum_{n\ge1}\frac{1}{2n^2}+M\sum_{n\ge1}\frac{1}{(\pi(n))^2}\\&\le\sum_{k\ge2}(k-1)\,p_k+\frac{1+2M}{2}\sum_{n\ge1}\frac{1}{n^2}<\infty.\end{aligned}$$

If π is the identity this gives $\sum_{n\ge1}\mu\left(\|h_n\|\ge n\right)<\infty$ and therefore, for every permutation $(\pi(1),\pi(2),...)$ of $(1,2,...)$, one has

$$\sum_{n\ge1}\mu\left(\left\|h_{\pi(n)}\right\|\ge\pi(n)\right)<\infty. \tag{2}$$

The following inequality allows to conclude

$$\mu\left(\left\|h_{\pi(n)}\right\| \geq n \wedge \pi(n)\right) \leq \mu\left(\left\|h_{\pi(n)}\right\| \geq n\right) + \mu\left(\left\|h_{\pi(n)}\right\| \geq \pi(n)\right). \qquad \square$$

3. Proof of Theorem 1.1

Proof. Step 1: By applying the preceding Lemma and extracting a subsequence if necessary, we may suppose that for any subsequence $(h_n)_{n\geq 1}$ of $(f_n)_{n\geq 1}$ and for any permutation $(\pi(1), \pi(2), ...)$ of $(1, 2, ...)$, we have the following:

(a) $\sum_{n\geq 1} \frac{1}{n^2} \left\|F_{n\wedge\pi(n)}\left(h_{\pi(n)}\right)\right\|_2^2 < +\infty$,

(b) $\sum_{n\geq 1} \mu\left(\left\{\|h_{\pi(n)}\| \geq n \wedge \pi(n)\right\}\right) < +\infty$.

For each $k \geq 1$, the sequence $(F_k(f_n))_{n\geq 1}$ is weakly relatively compact in $L^2_{\mathbb{H}}$, then there exists, by the diagonal method, a subsequence $(f'_n)_{n\geq 1}$ of $(f_n)_{n\geq 1}$ and a sequence $(v_n)_{n\geq 1}$ in $L^2_{\mathbb{H}}$ such that for all $k \geq 1$ the sequence $(F_k(f'_n))_{n\geq 1}$ converges weakly in $L^2_{\mathbb{H}}$ to v_k, and

$$\|v_k\|_2 \leq 2\left\|F_k\left(f'_n\right)\right\|_2 \quad \forall n \geq k \geq 1.$$

(for more details see Lemma 2.1 in [5]). Consequently, for any subsequence $(h_n)_{n\geq 1}$ of $(f'_n)_{n\geq 1}$ and for any permutation $(\pi(1), \pi(2), ...)$ of $(1, 2, ...)$

$$\sum_{n\geq 1} \frac{1}{n^2} \left\|F_{n\wedge\pi(n)}\left(h_{\pi(n)}\right) - v_{n\wedge\pi(n)}\right\|_2^2 \leq 9 \sum_{n\geq 1} \frac{1}{n^2} \left\|F_{n\wedge\pi(n)}\left(h_{\pi(n)}\right)\right\|_2^2 < +\infty.$$

Also, taking $n_1 = 1$ and proceeding by induction on p, there exist integers $n_1 < n_2 < ... < n_p < n_{p+1} < ...$ such that for all $p \geq 2$, $1 \leq q < p$ and $1 \leq l, k \leq p$

$$\left|\left\langle\left(F_l\left(f'_{n_q}\right) - v_l\right), \left(F_k\left(f'_{n_p}\right) - v_k\right)\right\rangle\right| \leq \frac{1}{2^{p+q}}. \tag{3}$$

Put $g_q = f'_{n_q}$. Then for any subsequence $(h_n)_{n\geq 1}$ of $(g_n)_{n\geq 1}$ and permutation $(\pi(1), \pi(2), ...)$ of $(1, 2, ...)$, and for any $i \neq j$ and $1 \leq l,\ k \leq \max(i, j)$

$$\left|\left\langle\left(F_l(h_i) - v_l\right), \left(F_k(h_j) - v_k\right)\right\rangle\right| \leq \frac{1}{2^{i+j}}.$$

Consequently, for any $i \neq j$

$$\left|\left\langle F_{i\wedge\pi(i)}\left(h_{\pi(i)}\right) - v_{i\wedge\pi(i)}, F_{j\wedge\pi(j)}\left(h_{\pi(j)}\right) - v_{j\wedge\pi(j)}\right\rangle\right| \leq \frac{1}{2^{\pi(i)+\pi(j)}}. \tag{4}$$

Step 2: Let $(h_n)_{n\geq 1}$ be a subsequence of $(g_n)_{n\geq 1}$ and $(\pi(1), \pi(2), ...)$ a permutation of $(1, 2, ...)$. Put:

$$S^*_k = \sum_{i=1}^{k}\left(F_{i\wedge\pi(i)}\left(h_{\pi(i)}\right) - v_{i\wedge\pi(i)}\right), \quad S_k = \sum_{i=1}^{k} h_{\pi(i)} \quad \text{and} \quad \widetilde{S}_k = \sum_{i=1}^{k}\left\|h_{\pi(i)}\right\|.$$

Let $\varepsilon > 0$, $\alpha > 1$ and $k_n = [\alpha^n]$. For any $n \geq 1$ we have

$$\mu\left(\|S^*_{k_n}\| \geq \varepsilon k_n\right) \leq c\frac{1}{k_n^2}\left\|S^*_{k_n}\right\|_2^2 = c\frac{1}{k_n^2}\sum_{i=1}^{k_n}\left\|F_{i\wedge\pi(i)}\left(h_{\pi(i)}\right) - v_{i\wedge\pi(i)}\right\|_2^2 +$$

$$+2c\ \frac{1}{k_n^2}\sum_{1\leq i<j\leq k_n}\left|\left\langle F_{i\wedge\pi(i)}\left(h_{\pi(i)}\right) - v_{i\wedge\pi(i)}, F_{j\wedge\pi(j)}\left(h_{\pi(j)}\right) - v_{j\wedge\pi(j)}\right\rangle\right|.$$

Then by (4)

$$\begin{aligned}\sum_{n=1}^{\infty}\mu\left(\|S^*_{k_n}\| \geq \varepsilon k_n\right) &\leq c\sum_{i=1}^{\infty}\left[\left\|F_{i\wedge\pi(i)}\left(h_{\pi(i)}\right) - v_{i\wedge\pi(i)}\right\|_2^2\sum_{k_n\geq i}\frac{1}{k_n^2}\right] + c'\sum_{n=1}^{\infty}\frac{1}{k_n^2}\\ &\leq c\sum_{i=1}^{\infty}\frac{1}{i^2}\left\|F_{i\wedge\pi(i)}\left(h_{\pi(i)}\right) - v_{i\wedge\pi(i)}\right\|_2^2 + c'\sum_{n=1}^{\infty}\frac{1}{k_n^2}.\end{aligned}$$

So $\sum\limits_{n=1}^{\infty}\mu\left(\|S^*_{k_n}\| \geq \varepsilon k_n\right) < \infty$ and therefore $\frac{1}{k_n}S^*_{k_n}$ converges μ-a.e. to 0.

We have $$\frac{1}{k_n}S_{k_n} = \frac{1}{k_n}\sum_{i=1}^{k_n}\left(h_{\pi(i)} - F_{i\wedge\pi(i)}\left(h_{\pi(i)}\right)\right) + \frac{1}{k_n}S^*_{k_n} + \frac{1}{k_n}\sum_{i=1}^{k_n}v_{i\wedge\pi(i)}.$$

According to Lemma 2.2 in [5], the sequence $(v_n)_{n\geq 1}$ converges μ-a.e. to some limit $f \in L^1_{\mathbb{H}}$. We deduce from **(b)** that for all $\alpha > 1$

$$\frac{1}{k_n}S_{k_n} \text{ converges } \mu\text{-a.e. to } f. \tag{5}$$

Similarly, we may assume without loss of generality that

$$\frac{1}{k_n}\widetilde{S}_{k_n} \text{ converges } \mu\text{-}a.e. \text{ to some limit } S, \text{ for all } \alpha > 1. \tag{6}$$

Step 3: Let us observe that for any $n \in \mathbb{N}^*$, there exists $\varphi(n) \epsilon \mathbb{N}$ such that

$$k_{\varphi(n)-1} < n \leq k_{\varphi(n)}.$$

We shall use the following inequalities

$$1 \leq \frac{k_{\varphi(n)}}{n} \leq \frac{\alpha^{\varphi(n)}}{n} \leq \alpha\frac{k_{\varphi(n)-1}+1}{n} \leq \alpha, \tag{7}$$

and $$\frac{1}{\alpha} - \frac{1}{n} \leq \frac{k_{\varphi(n)-1}}{n} < \frac{k_{\varphi(n)}}{n} \leq \alpha. \tag{8}$$

One has

$$\left\|\frac{1}{n}S_n - \frac{1}{k_{\varphi(n)}}S_{k_{\varphi(n)}}\right\| = \left\|\frac{1}{k_{\varphi(n)}}\left(\frac{k_{\varphi(n)}}{n} - 1\right)S_{k_{\varphi(n)}} - \frac{1}{n}\left(S_{k_{\varphi(n)}} - S_n\right)\right\|$$

$$\leq \left(\frac{k_{\varphi(n)}}{n} - 1\right)\frac{1}{k_{\varphi(n)}}\widetilde{S}_{k_{\varphi(n)}} + \frac{1}{n}\left(\widetilde{S}_{k_{\varphi(n)}} - \widetilde{S}_n\right) = \left(2\frac{k_{\varphi(n)}}{n} - 1\right)\frac{1}{k_{\varphi(n)}}\widetilde{S}_{k_{\varphi(n)}} - \frac{1}{n}\widetilde{S}_n$$

$$\leq (2\alpha - 1)\frac{1}{k_{\varphi(n)}}\widetilde{S}_{k_{\varphi(n)}} - \frac{1}{n}\widetilde{S}_{k_{\varphi(n)-1}} = (2\alpha - 1)\frac{1}{k_{\varphi(n)}}\widetilde{S}_{k_{\varphi(n)}} - \frac{k_{\varphi(n)-1}}{n}\frac{1}{k_{\varphi(n)-1}}\widetilde{S}_{k_{\varphi(n)-1}}$$

$$\leq (2\alpha - 1)\frac{1}{k_{\varphi(n)}}\widetilde{S}_{k_{\varphi(n)}} - \left(\frac{1}{\alpha} - \frac{1}{n}\right)\frac{1}{k_{\varphi(n)-1}}\widetilde{S}_{k_{\varphi(n)-1}}.$$

By the triangular inequality, (5) and (6) we get

$$\overline{\lim_n}\left\|\frac{1}{n}S_n - f\right\| \leq (2\alpha - 1)\,S - \frac{1}{\alpha}S.$$

Since $\alpha > 1$ is arbitrary, then $\frac{1}{n}S_n$ converges μ-a.e. to f. □

Remark 3.1. As mentioned in the introduction, Berkes [2] proved that if $(X_n)_n$ is a bounded sequence in $L^1_{\mathbb{R}}$, then there exists a subsequence $(X_{n_k})_k$ of $(X_n)_n$ so that every permutation of $(X_{n_k})_k$ is a.e. Cesàro-convergent to the same limit $f \in L^1_{\mathbb{R}}$. On the other hand, in this paper, we prove that there exists a subsequence $(X_{n_k})_k$ of $(X_n)_n$ so that every permutation of any subsequence $(Y_k)_k$ of $(X_{n_k})_k$ is a.e. Cesàro-convergent to the same limit $f \in L^1_{\mathbb{R}}$. It was pointed out to the authors by an anonymous referee that a trivial modification of the proof in [2] yields this stronger result. Indeed, as it was mentioned by the referee, in the inductive procedure yielding the sequence $(X_{n_k})_k$ in the proof of lemma 2 in [2], if one chooses n_k in such a way that the property guaranteed in the induction step holds not only for n_k, but for all $n > n_k$, thus every subsequence $(Y_k)_k$ of $(X_{n_k})_k$ satisfied the property (13) in [2] which implies that every permutation of $(Y_k)_k$ is a.e. Cesàro convergent to 0.

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