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Existence of a Solution to a Stationary Quasi-Variational Inequality in a Multi-Connected Domain

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We consider a stationary quasi-variational inequality in a multi-connected domain and show the existence of a solution to the inequality. The problem is related to the Bean critical-state model in type II superconductors. Mathematically, we are concerned with a quasi-variational inequality containing a p-curl inequality with a curl constraint by a function of the solution.

Keywords: Quasi-variational inequality, stationary variational problem, existence of solution.

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1. Introduction

In this paper, we are interested in a model of a stationary quasi-variational inequality related to an electromagnetic variational p-curl inequality where the curl of the magnetic field is constrained by a function of the solution.

In the stationary electromagnetic theory, the generalized Maxwell's equations are written by

$$\boldsymbol{j} = \operatorname{curl} \boldsymbol{h},\ \operatorname{curl} \boldsymbol{e} = \boldsymbol{f} \text{ and } \operatorname{div} \boldsymbol{h} = 0 \text{ in } \Omega, \tag{1}$$

where $\boldsymbol{e}$ and $\boldsymbol{h}$ are the electric field and the magnetic field, respectively, $\boldsymbol{j}$ denotes the total current density, and $\boldsymbol{f}$ denotes an internal magnetic current. Here we consider a nonlinear extension of the classical Ohm's law in the form

$$\boldsymbol{e} = \rho \boldsymbol{j}, \tag{2}$$

where the scalar resistivity ρ can be taken as a form

$$\rho = \nu |\operatorname{curl} \boldsymbol{h}|^{p-2} \quad (p > 1),$$

where ν is a given function. Then the equilibrium equations for $\boldsymbol{h}$ may be written as

$$\operatorname{curl} [\nu |\operatorname{curl} \boldsymbol{h}|^{p-2} \operatorname{curl} \boldsymbol{h}] = \boldsymbol{f},\ \operatorname{div} \boldsymbol{h} = 0 \text{ in } \Omega. \tag{3}$$

We impose the natural boundary conditions

$$\boldsymbol{h} \cdot \boldsymbol{n} = 0 \ \text{ and } \ \boldsymbol{e} \times \boldsymbol{n} = \boldsymbol{g} \ \text{ on } \Gamma := \partial \Omega,$$

where $\boldsymbol{n}$ denotes the outer unit normal vector to the boundary Γ of Ω.

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An example of a generalization of the constitutive law (2) arises in type-II superconductors and is known as an extension of the Bean critical-state model. In this model, the current density cannot exceed some given critical value $j > 0$ (cf. Prigozhin [15]). Generally, this critical value j varies as a function of the absolute value $|\boldsymbol{h}|$ of the magnetic field $\boldsymbol{h}$ (cf. Kim et al. [12, 11]), that is to say,

$$\boldsymbol{e} = \begin{cases} \nu|\mathrm{curl}\,\boldsymbol{h}|^{p-2}\mathrm{curl}\,\boldsymbol{h} & \text{if } |\mathrm{curl}\,\boldsymbol{h}| < j(|\boldsymbol{h}|), \\ (\nu j^{p-2} + \lambda)\mathrm{curl}\,\boldsymbol{h} & \text{if } |\mathrm{curl}\,\boldsymbol{h}| = j(|\boldsymbol{h}|), \end{cases} \tag{4}$$

where $\lambda = \lambda(x) \geq 0$ is an unknown Lagrange multiplier whose support lies in the superconductivity region

$$S = \{x \in \Omega; |\mathrm{curl}\,\boldsymbol{h}(x)| = j(|\boldsymbol{h}(x)|)\}.$$

Thus we must consider the following quasi-variational inequality

$$\int_\Omega \nu|\mathrm{curl}\,\boldsymbol{h}|^{p-2}\mathrm{curl}\,\boldsymbol{h}\cdot\mathrm{curl}\,(\boldsymbol{v}-\boldsymbol{h})dx \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{h})dx + \int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{h})dS \tag{5}$$

for all test functions $\boldsymbol{v}$ such that $|\mathrm{curl}\,\boldsymbol{v}| \leq j(|\boldsymbol{h}|)$, where dS denotes the surface area of Γ.

In this paper, we are interested in discussing the existence of a solution to a generalized quasi-variational inequality: to find $\boldsymbol{h}$ in a closed convex set $\mathbb{K}^p_{F(\boldsymbol{h})}$ in an appropriate space defined below such that

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{h}|^2)\mathrm{curl}\,\boldsymbol{h}\cdot\mathrm{curl}\,(\boldsymbol{v}-\boldsymbol{h})dx \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{h})dx + \int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{h})dS$$

for all $\boldsymbol{v} \in \mathbb{K}_{F(\boldsymbol{h})}$ in a multi-connected domain Ω. Here a function $S(x,t)$ satisfies some structure conditions.

When facing a similar problem for the gradient ∇u and u of a function instead of $\mathrm{curl}\,\boldsymbol{h}$ and $\boldsymbol{h}$, respectively, with a gradient constraint instead of a curl constraint, Azevedo et al. [5] succeeded in proving the existence of a solution. Since those authors took $W^{1,p}(\Omega)$ as a basis space, it is necessary to add an extra term in order to guarantee the coerciveness in $W^{1,p}(\Omega)$. However, since Ω is multi-connected, we take an appropriate subspace of $\boldsymbol{W}^{1,p}(\Omega)$ (the Sobolev space of vector valued functions) as a basis space. This makes unnecessary to add any extra term.

The paper is organized as follows. In section 2, we give a precise setting of the problem and give the main theorem (Theorem 2.5). Section 3 is devoted to the continuity of the solution of an associated variational inequality on the data. In section 4, we give the proof of the main theorem (Theorem 2.5) using the Schauder fixed point theorem and the Leray-Schauder fixed point theorem.

2. Preliminaries and the main theorem

In this section we give some preliminaries and the main theorem.

Since we allow that Ω is multi-connected, we assume that Ω has the following conditions (i) and (ii) as in Amrouche and Seloula [2] (cf. Amrouche and Seloula [1], Dautray and Lions [6] and Girault and Raviart [10]). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{1,1}$ with the boundary Γ and Ω locally situated on one side of Γ.

(i) Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_m$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

(ii) There exist n connected open surfaces Σ_j, $(j = 1, \dots, n)$, called cuts, contained in Ω such that
   (a) Σ_j is an open subset of a smooth two dimensional manifold $\mathcal{M}_j$.
   (b) $\partial\Sigma_j \subset \Gamma$ $(j = 1, \dots, n)$, where $\partial\Sigma_j$ denotes the boundary of Σ_j, and Σ_j is non-tangential to Γ.
   (c) $\overline{\Sigma_i} \cap \overline{\Sigma_j} = \emptyset$ $(i \neq j)$.
   (d) The open set $\dot{\Omega} = \Omega \setminus (\cup_{i=1}^n \Sigma_i)$ is simply connected and pseudo $C^{1,1}$ class.

From now on we use the notations $L^p(\Omega), W^{m,p}(\Omega)$ $(m \geq 0, \text{integer} \ \text{ and } p > 1)$, $W^{s,p}(\Gamma)$ $(s \in \mathbb{R})$, and so on, for the standard L^p space and Sobolev spaces of functions. For any Banach space B, we denote $B \times B \times B$ by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard Euclidean inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$ by $\boldsymbol{a} \cdot \boldsymbol{b}$.

For $1 < p < \infty$, we define two spaces

$$\mathbb{K}_N^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \operatorname{div} \boldsymbol{v} = 0,\ \operatorname{curl} \boldsymbol{v} = \boldsymbol{0} \ \text{ in } \Omega,\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\},$$

$$\mathbb{K}_T^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \operatorname{div} \boldsymbol{v} = 0,\ \operatorname{curl} \boldsymbol{v} = \boldsymbol{0} \ \text{ in } \Omega,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

Then it is well known that $\dim \mathbb{K}_T^p(\Omega) = n$ and $\dim \mathbb{K}_N^p(\Omega) = m$ (cf. [2]). Moreover, we define

$$\mathbb{W}_T^p(\Omega) = \left\{ \boldsymbol{v} \in L^p(\Omega);\ \begin{array}{l} \operatorname{curl} \boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \Omega,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma, \\ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0,\ j = 1, 2, \dots, n \end{array} \right\}$$

and

$$\mathbb{W}_N^p\Omega) = \left\{ \boldsymbol{v} \in L^p(\Omega);\ \begin{array}{l} \operatorname{curl} \boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma, \\ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0,\ i = 1, 2, \dots, m \end{array} \right\},$$

$\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j}$ denoting the duality between the spaces $W^{-1/p,p}(\Sigma_j)$ and $W^{1-1/p',p'}(\Sigma_j)$, $\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}$ denoting the duality between the spaces $W^{-1/p,p}(\Gamma_i)$ and $W^{1-1/p',p'}(\Gamma_i)$, and p' is the conjugate exponent of p, i.e., $(1/p) + (1/p') = 1$. We note that $\mathbb{W}_T^p(\Omega)$ and $\mathbb{W}_N^p(\Omega)$ are subspaces of $\boldsymbol{W}^{1,p}(\Omega)$ (cf. [2]).

Then we have the following result (cf. Miranda et al. [13, 14] and Aramaki [4]).

Proposition 2.1. *If $1 < p < \infty$, then $\mathbb{W}_T^p(\Omega)$ and $\mathbb{W}_N^p(\Omega)$ are reflexive, separable Banach spaces with the norm*

$$\|\boldsymbol{v}\|_{\mathbb{W}_N^p(\Omega)} = \|\boldsymbol{v}\|_{\mathbb{W}_T^p(\Omega)} := \|\operatorname{curl} \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)},$$

Furthermore, both norms $\|\boldsymbol{v}\|_{\mathbb{W}_N^p(\Omega)}$ and $\|\boldsymbol{v}\|_{\mathbb{W}_T^p(\Omega)}$ are equivalent to $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$.

For the proof, see Aramaki [3] (cf. [2]).

From now on, we denote $\mathbb{W}^p_T(\Omega)$ or $\mathbb{W}^p_N(\Omega)$ by $\mathbb{V}^p(\Omega)$.

The following lemma is for Sobolev-type inequalities and the trace theorems.

Lemma 2.2. *There exist positive constants C_q and C_r such that for any $\boldsymbol{v} \in \mathbb{V}^p(\Omega)$,*

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^q(\Omega)} \le C_q \|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)} \quad \text{with} \quad \begin{cases} q \le 3p/(3-p) & \text{if } 1 < p < 3, \\ \text{any } q \ge 1 & \text{if } p = 3, \\ q = \infty & \text{if } p > 3. \end{cases} \tag{6}$$

In particular, if $p > 3$, $\mathbb{V}^p(\Omega) \subset \boldsymbol{C}(\overline{\Omega})$ and the inclusion map is compact.

$$\|\boldsymbol{v}\big|_\Gamma\|_{\boldsymbol{L}^r(\Gamma)} \le C_r \|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)} \quad \text{with} \quad \begin{cases} r \le 2p/(3-p) & \text{if } 1 < p < 3, \\ \text{any } r \ge 1 & \text{if } p = 3, \\ r = \infty & \text{if } p > 3 \end{cases} \tag{7}$$

where $\boldsymbol{v}\big|_\Gamma$ denotes the trace to the boundary Γ.

Let $S(x,t)$ be a Carathéodory function in $\Omega \times [0,\infty)$ and for a.e. $x \in \Omega$ let $S(x,\cdot) \in C^2((0,\infty))$ satisfying the following structure conditions. There exist $1 < p < \infty$ and $0 < \lambda \le \Lambda < \infty$ such that for a.e. $x \in \Omega$ and $t > 0$,

$$S(x,0) = 0, \quad \lambda t^{(p-2)/2} \le S_t(x,t) \le \Lambda t^{(p-2)/2}, \tag{8a}$$

$$\lambda t^{(p-2)/2} \le S_t(x,t) + 2t S_{tt}(x,t) \le \Lambda t^{(p-2)/2}, \tag{8b}$$

$$\text{If } \ 1 < p < 2, S_{tt}(x,t) < 0, \text{ and if } p \ge 2, \ S_{tt}(x,t) \ge 0. \tag{8c}$$

Here $S_t(x,t) = \frac{\partial S}{\partial t}(x,t)$ and $S_{tt}(x,t) = \frac{\partial^2 S}{\partial t^2}(x,t)$. We note that from (8a), we have

$$\frac{2}{p}\lambda t^{p/2} \le S(x,t) \le \frac{2}{p}\Lambda t^{p/2} \ \text{ for a.e. } x \in \Omega, \ t \ge 0. \tag{9}$$

Example 2.3. If we define $S(x,t) = \nu(x) t^{p/2}$, where $\nu(x)$ is a measurable function in Ω satisfying $0 < \lambda \le \nu(x) \le \Lambda < \infty$ a.e in Ω, then $S(x,t)$ satisfies (8a)–(8c). $\square$

For the proof, see Aramaki [4] and DiBenedetto [7].

Now we give a monotonicity of S_t in the following sense.

Lemma 2.4. *There exists a constant $c > 0$ such that for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$,*

$$\big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big)\cdot(\boldsymbol{a}-\boldsymbol{b}) \ge \begin{cases} c|\boldsymbol{a}-\boldsymbol{b}|^p & \text{if } p \ge 2, \\ c(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2}|\boldsymbol{a}-\boldsymbol{b}|^2 & \text{if } 1 < p < 2. \end{cases}$$

In particular, $\big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big)\cdot(\boldsymbol{a}-\boldsymbol{b}) > 0$ *if* $\boldsymbol{a} \neq \boldsymbol{b}$.

Proof. Put $J(x,p) = \big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big)\cdot(\boldsymbol{a}-\boldsymbol{b})$. Then

$$J(x,p) = \int_0^1 \frac{d}{ds}\big[S_t(x,|s\boldsymbol{a}+(1-s)\boldsymbol{b}|^2)(s\boldsymbol{a}+(1-s)\boldsymbol{b})\big]ds \cdot (\boldsymbol{a}-\boldsymbol{b})$$

$$= \int_0^1 S_t(x, |s\boldsymbol{a} + (1-s)\boldsymbol{b}|^2) ds |\boldsymbol{a} - \boldsymbol{b}|^2 + \\ + \int_0^1 2S_{tt}(x, |s\boldsymbol{a} + (1-s)\boldsymbol{b}|^2)\big((s\boldsymbol{a} + (1-s)\boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})\big)^2 ds.$$

When $p > 2$, since $S_{tt}(x,t) \geq 0$, we have

$$J(x,p) \geq \int_0^1 S_t(x, |s\boldsymbol{a}+(1-s)\boldsymbol{b}|^2) ds |\boldsymbol{a}-\boldsymbol{b}|^2 \geq \lambda \int_0^1 |s\boldsymbol{a}+(1-s)\boldsymbol{b}|^{p-2} |\boldsymbol{a}-\boldsymbol{b}|^2 ds.$$

From [7, p. 14], we can show that $J(x,p) \geq c|\boldsymbol{a} - \boldsymbol{b}|^p$ for some $c > 0$.

When $1 < p < 2$, since $S_{tt}(x,t) < 0$, we have

$$J(x,p) \geq \int_0^1 S_t(x, |s\boldsymbol{a} + (1-s)\boldsymbol{b}|^2)|\boldsymbol{a} - \boldsymbol{b}|^2 ds \\ + \int_0^1 2S_{tt}(x, |s\boldsymbol{a} + (1-s)\boldsymbol{b}|^2)|s\boldsymbol{a} + (1-s)\boldsymbol{b}|^2 |\boldsymbol{a} - \boldsymbol{b}|^2.$$

From (8b), we have

$$J(x,p) \geq \lambda \int_0^1 |s\boldsymbol{a} + (1-s)\boldsymbol{b}|^{p-2} ds |\boldsymbol{a} - \boldsymbol{b}|^2 \geq c(|\boldsymbol{a}| + |\boldsymbol{b}|)^{p-2} |\boldsymbol{a} - \boldsymbol{b}|^2$$

for some $c > 0$. □

We are now in a position to state the main theorem. Let $F \in C(\mathbb{R}^3; \mathbb{R}^+)$ where $\mathbb{R}^+ = \{s \in \mathbb{R}; s > 0\}$ and $\boldsymbol{f} \in \boldsymbol{L}^{q'}(\Omega), \boldsymbol{g} \in \boldsymbol{L}^{r'}(\Gamma)$, where q', r' are conjugate exponents of q, r as in Lemma 2.2, that is, $(1/q) + (1/q') = 1, (1/r) + (1/r') = 1$. For $\boldsymbol{u} \in \mathbb{V}^p(\Omega)$, define a closed convex subset of $\mathbb{V}^p(\Omega)$:

$$\mathbb{K}^p_{F(\boldsymbol{u})}(\Omega) = \{\boldsymbol{v} \in \mathbb{V}^p(\Omega); |\mathrm{curl}\, \boldsymbol{v}| \leq F(\boldsymbol{u}) \text{ a.e. in } \Omega\}. \tag{10}$$

We consider the following problem for a quasi-variational inequality:

$$\text{Find } \boldsymbol{u} \in \mathbb{K}^p_{F(\boldsymbol{u})}(\Omega) \text{ such that}$$

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2) \mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, (\boldsymbol{v} - \boldsymbol{u}) dx \\ \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u}) dx + \int_\Gamma \boldsymbol{g} \cdot (\boldsymbol{v} - \boldsymbol{u}) dS \text{ for all } \boldsymbol{v} \in \mathbb{K}^p_{F(\boldsymbol{u})}. \tag{11}$$

Theorem 2.5. *Let $1 < p < \infty$ and $\boldsymbol{f} \in \boldsymbol{L}^{q'}(\Omega), \boldsymbol{g} \in \boldsymbol{L}^{r'}(\Gamma)$. Assume that $F \in C(\mathbb{R}^3; \mathbb{R}^+)$ satisfies the condition: if $p \leq 3$, then there exist constants $c_0, c_1 > 0$ such that*

$$F(s) \leq c_0 + c_1|s|^\alpha \text{ for all } s = (s_1, s_2, s_3) \in \mathbb{R}^3,$$

where $\alpha \geq 0$ for $p = 3$, $0 \leq \alpha < 3/(3-p)$ for $p < 3$. Then the quasi-variational inequality (11) has a solution. In particular, the solution can be found in $\boldsymbol{C}^\beta(\overline{\Omega})$ for any $\beta \in (0,1)$.

In order to prove Theorem 2.5, we consider the associated variational inequality. For $\varphi \in L^\infty(\Omega)$ satisfying $\varphi \geq 0$, define a closed convex subset of $\mathbb{V}^p(\Omega)$ by

$$\mathbb{K}^p(\varphi) = \{\boldsymbol{v} \in \mathbb{V}^p(\Omega); |\mathrm{curl}\, \boldsymbol{v}| \leq \varphi \text{ a.e. in } \Omega\}, \tag{12}$$

and consider the following problem of variational inequality: for given $\boldsymbol{f} \in \boldsymbol{L}^{q'}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^{r'}(\Gamma)$, find $\boldsymbol{u} \in \mathbb{K}^p(\varphi)$ such that, for all $\boldsymbol{v} \in \mathbb{K}^p(\varphi)$,

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, (\boldsymbol{v}-\boldsymbol{u})dx \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v}-\boldsymbol{u})dx + \int_\Gamma \boldsymbol{g} \cdot (\boldsymbol{v}-\boldsymbol{u})dS. \tag{13}$$

Proposition 2.6. *Let $1 < p < \infty$ and $\boldsymbol{f} \in \boldsymbol{L}^{q'}(\Omega), \boldsymbol{g} \in \boldsymbol{L}^{r'}(\Gamma)$. Then the above inequality* (13) *has a unique solution $\boldsymbol{u} \in \mathbb{K}^p(\varphi)$, and there exists a constant depending only on C_q, C_r, λ and p such that*

$$\|\boldsymbol{u}\|_{\mathbb{V}^p(\Omega)} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{L}^{q'}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^{r'}(\Gamma)})^{1/(p-1)}. \tag{14}$$

Proof. We prove this theorem directly, using a minimization method of an associated functional. Thus we put

$$I[\boldsymbol{v}] = \int_\Omega S(x, |\mathrm{curl}\, \boldsymbol{v}|^2)dx - 2\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}dx - 2\int_\Gamma \boldsymbol{g} \cdot \boldsymbol{v}dS.$$

We consider the minimization problem: to find $\boldsymbol{u} \in \mathbb{K}^p(\varphi)$ such that

$$I[\boldsymbol{u}] = \inf_{\boldsymbol{v} \in \mathbb{K}^p(\varphi)} I[\boldsymbol{v}].$$

From (9), the Hölder inequality and Lemma 2.2, for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that for any $\boldsymbol{v} \in \mathbb{V}^p(\Omega)$,

$$\begin{aligned} I[\boldsymbol{v}] &\geq \frac{2}{p}\lambda\|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)} - C(\varepsilon)\|\boldsymbol{f}\|^{p'}_{\boldsymbol{L}^{q'}(\Omega)} - \varepsilon\|\boldsymbol{v}\|^p_{\boldsymbol{L}^q(\Omega)} - C(\varepsilon)\|\boldsymbol{g}\|^{p'}_{\boldsymbol{L}^{r'}(\Gamma)} - \varepsilon\|\boldsymbol{v}\|_{\boldsymbol{L}^r(\Gamma)} \\ &\geq \left(\frac{2}{p}\lambda - \varepsilon C_q^p - \varepsilon C_r^p\right)\|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)} - C(\varepsilon)\|\boldsymbol{f}\|^{p'}_{\boldsymbol{L}^{q'}(\Omega)} - C(\varepsilon)\|\boldsymbol{g}\|^{p'}_{\boldsymbol{L}^{r'}(\Omega)}. \end{aligned}$$

Choosing $\varepsilon > 0$ small enough, there exist positive constants c and C such that

$$I[\boldsymbol{v}] \geq c\|\boldsymbol{v}\|^p_{\mathbb{V}^p(\Omega)} - C(\|\boldsymbol{f}\|_{\boldsymbol{L}^{q'}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^{r'}(\Gamma)}). \tag{15}$$

This implies $\alpha := \inf_{\boldsymbol{v} \in \mathbb{K}^p(\varphi)} I[\boldsymbol{v}] > -\infty$ and the coerciveness of I, i.e.,

$$\lim_{\boldsymbol{v} \in \mathbb{K}^p(\varphi), \|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)} \to \infty} I[\boldsymbol{v}] = +\infty.$$

Now if we put $F(x,t) = S(x,t^2)$, then it follows from (8a) and (8b) that for a.e. $x \in \Omega$ and $t > 0$,

$$F_t(x,t) = 2tS_t(x,t^2) \geq 2\lambda t^{p-1} > 0$$

and $$F_{tt}(x,t) = 2\{S_t(x,t^2) + 2t^2 S_{tt}(x,t^2)\} \geq 2\lambda t^{p-2} > 0.$$

Thus I is a proper strictly convex functional on a closed convex subset $\mathbb{K}^p(\varphi)$. In order to prove the lower semi continuity of I, let $\boldsymbol{v}_j \in \mathbb{K}^p(\varphi)$, $\boldsymbol{v}_j \to \boldsymbol{v}$ in $\mathbb{V}^p(\Omega)$. Since $\mathbb{K}^p(\varphi)$ is closed, $\boldsymbol{v} \in \mathbb{K}^p(\varphi)$. Since $\operatorname{curl} \boldsymbol{v}_j \to \operatorname{curl} \boldsymbol{v}$ in $\boldsymbol{L}^p(\Omega)$, there exists a subsequence $\{\boldsymbol{v}_{j_k}\}$ of $\{\boldsymbol{v}_j\}$ such that

$$\lim_{k\to\infty} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}_{j_k}|^2)dx = \liminf_{j\to\infty} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}_j|^2)dx,$$

and $\operatorname{curl} \boldsymbol{v}_{j_k} \to \operatorname{curl} \boldsymbol{v}$ a.e. in Ω. Since $S(x,t)$ is continuous with respect to $t \in [0,\infty)$, we see that $S(x, |\operatorname{curl} \boldsymbol{v}_{j_k}|^2) \to S(x, |\operatorname{curl} \boldsymbol{v}|^2)$ a.e. in Ω. Since $S(x,t) \geq 0$, it follows from the Fatou lemma that

$$\begin{aligned} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}|^2)dx &\leq \liminf_{k\to\infty} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}_{j_k}|^2)dx \\ &= \lim_{k\to\infty} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}_{j_k}|^2)dx \\ &= \liminf_{j\to\infty} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}_j|^2)dx. \end{aligned}$$

Since $\boldsymbol{v}_j \to \boldsymbol{v}$ strongly in $\boldsymbol{L}^q(\Omega)$ and $\boldsymbol{L}^r(\Gamma)$ as $j \to \infty$, we have

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_j dx \to \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx \text{ and } \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{v}_j dS \to \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{v} dS.$$

Therefore we have $I[\boldsymbol{v}] \leq \liminf_{j\to\infty} I[\boldsymbol{v}_j]$, so I has lower semicontinuity. Since I is a strictly convex functional on $\mathbb{K}^p(\varphi)$, a unique minimizer exists (cf. Ekeland and Témam [8, Chapter 2, Proposition 1.2]).

Let $\boldsymbol{u} \in \mathbb{K}^p(\varphi)$ be a minimizer of I. Then for any $\boldsymbol{v} \in \mathbb{K}^p(\varphi)$, since $\boldsymbol{u} + \theta(\boldsymbol{v} - \boldsymbol{u}) \in \mathbb{K}^p(\varphi)$ for $\theta \in [0,1]$, we see that $I[\boldsymbol{u}] \leq I[\boldsymbol{u} + \theta(\boldsymbol{v} - \boldsymbol{u})]$. So we have

$$\begin{aligned} 0 &\leq \left.\frac{d}{d\theta} I[\boldsymbol{u} + \theta(\boldsymbol{v} - \boldsymbol{u})]\right|_{\theta=+0} \\ &= \int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{u}|^2) \operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} (\boldsymbol{v} - \boldsymbol{u})dx \\ &\quad - \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u})dx - \int_\Gamma \boldsymbol{g} \cdot (\boldsymbol{v} - \boldsymbol{u})dS. \end{aligned}$$

Thus we see that $\boldsymbol{u}$ is a solution of (13). Next we prove the uniqueness of the solution for the variational inequality (13). Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{K}^p(\varphi)$ be two solutions of (13). Then we have

$$\begin{aligned} &\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{u}_1|^2) \operatorname{curl} \boldsymbol{u}_1 \cdot \operatorname{curl} (\boldsymbol{u}_2 - \boldsymbol{u}_1)dx \\ &\qquad \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_1)dx + \int_\Gamma \boldsymbol{g} \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_1)dS, \\ &\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{u}_2|^2) \operatorname{curl} \boldsymbol{u}_2 \cdot \operatorname{curl} (\boldsymbol{u}_1 - \boldsymbol{u}_2)dx \\ &\qquad \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)dx + \int_\Gamma \boldsymbol{g} \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)dS. \end{aligned}$$

Hence we have

$$\int_\Omega \big(S_t(x, |\mathrm{curl}\,\boldsymbol{u}_1|^2)\mathrm{curl}\,\boldsymbol{u}_1 - S_t(x, |\mathrm{curl}\,\boldsymbol{u}_2|^2)\mathrm{curl}\,\boldsymbol{u}_2\big) \cdot \mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)dx \le 0.$$

By Lemma 2.4, we have $\mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2) = \mathbf{0}$ in Ω. By Lemma 2.2, $\boldsymbol{u}_1 = \boldsymbol{u}_2$.

Finally we prove the estimate (14). If we choose $\boldsymbol{v} = \mathbf{0}$ as a test function in (13), then we have

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)|\mathrm{curl}\,\boldsymbol{u}|^2 dx \le \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} dx + \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{u} dS.$$

Using (8a) and the Hölder inequality,

$$\begin{aligned} \lambda\|\mathrm{curl}\,\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} &\le \|\boldsymbol{f}\|_{\boldsymbol{L}^{q'}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^{r'}(\Gamma)}\|\boldsymbol{u}\|_{\boldsymbol{L}^r(\Gamma)} \\ &\le (C_q\|\boldsymbol{f}\|_{\boldsymbol{L}^{q'}(\Omega)} + C_r\|\boldsymbol{g}\|_{\boldsymbol{L}^{r'}(\Gamma)})\|\boldsymbol{u}\|_{\mathbb{V}^p(\Omega)}. \end{aligned}$$

Therefore we have $\lambda\|\boldsymbol{u}\|^{p-1}_{\mathbb{V}^p(\Omega)} \le C_q\|\boldsymbol{f}\|_{\boldsymbol{L}^{q'}(\Omega)} + C_r\|\boldsymbol{g}\|_{\boldsymbol{L}^{r'}(\Gamma)}$. □

3. Continuous dependence on the data for the solution of the variational inequality (13)

In this section, we consider the continuous dependence of the solution to the variational inequality (13) on the given data. The continuity will be used in the proof of Theorem 2.5 later.

Proposition 3.1. *For $i = 1, 2$, let $\varphi_i \in L^\infty(\Omega)$ with a common positive lower bound $\eta > 0$, and $\boldsymbol{f}_i \in \boldsymbol{L}^{q'}(\Omega)$ and $\boldsymbol{g}_i \in \boldsymbol{L}^{r'}(\Gamma)$. Let $\boldsymbol{u}_i \in \mathbb{K}^p(\varphi_i)$ be a unique solution of (13) with $\boldsymbol{f} = \boldsymbol{f}_i, \boldsymbol{g} = \boldsymbol{g}_i$ and $\varphi = \varphi_i$. Then there exists a constant $C > 0$ depending only on $p, \|\boldsymbol{f}_i\|_{\boldsymbol{L}^{q'}(\Omega)}, \|\boldsymbol{g}_i\|_{\boldsymbol{L}^{r'}(\Gamma)}$ for $i = 1, 2$ and η such that*

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^{p\vee 2}_{\mathbb{V}^p(\Omega)} \le C(\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{L^{q'}(\Omega)} + \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{\boldsymbol{L}^{r'}(\Gamma)} + \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)}), \quad (16)$$

where $p \vee 2 = \max\{p, 2\}$.

Proof. Throughout the proof we denote a constant depending only on p, $\|\boldsymbol{f}_i\|_{L^{q'}(\Omega)}$, $\|\boldsymbol{g}_i\|_{L^{r'}(\Omega)}$, for $i = 1, 2$, and η, by C, which may vary from line to line. Now put $\mu = \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)}$ and define for $j \ne i$,

$$\widehat{\boldsymbol{u}}_j = \frac{\eta}{\eta + \mu}\boldsymbol{u}_i.$$

Then $$|\mathrm{curl}\,\widehat{\boldsymbol{u}}_j| = \frac{\eta}{\eta+\mu}|\mathrm{curl}\,\boldsymbol{u}_i| \le \frac{\eta}{\eta+\mu}\varphi_i = \frac{\eta}{\eta+\mu}(\varphi_j + (\varphi_i - \varphi_j))$$

$$\le \frac{\eta}{\eta+\mu}\varphi_j + \frac{\varphi_j}{\eta+\mu}\mu \le \varphi_j.$$

In consequence we obtain $\widehat{\boldsymbol{u}}_j \in \mathbb{K}^p(\varphi_j)$ and

$$\begin{aligned}\|\mathrm{curl}\,(\boldsymbol{u}_i - \widehat{\boldsymbol{u}}_j)\|_{L^p(\Omega)} &= \left(1 - \frac{\eta}{\eta+\mu}\right)\|\mathrm{curl}\,\boldsymbol{u}_i\|_{L^p(\Omega)}\\ &\le \frac{\mu}{\eta+\mu}\|\mathrm{curl}\,\boldsymbol{u}_i\|_{L^p(\Omega)} \le C_1\mu,\end{aligned}$$

where $C_1 = \|\mathrm{curl}\,\boldsymbol{u}_i\|_{L^p(\Omega)}/\eta$. However taking Proposition 2.6 into consideration, we can choose the constant C_1 independently of $\|\mathrm{curl}\,\boldsymbol{u}_i\|_{\boldsymbol{L}^p(\Omega)}$. Since $\boldsymbol{u}_i \in \mathbb{K}^p(\varphi_i)$ satisfies

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}_i|^2)\mathrm{curl}\,\boldsymbol{u}_i\cdot\mathrm{curl}\,(\boldsymbol{w}-\boldsymbol{u}_i)dx \ge \int_\Omega \boldsymbol{f}_i\cdot(\boldsymbol{w}-\boldsymbol{u}_i)dx + \int_\Gamma \boldsymbol{g}_i\cdot(\boldsymbol{w}-\boldsymbol{u}_i)dS \quad (17)$$

for all $\boldsymbol{w} \in \mathbb{K}^p(\varphi_i)$, taking $\widehat{\boldsymbol{u}}_i$ as a test function of (17), we have

$$\begin{aligned}&\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}_i|^2)\mathrm{curl}\,\boldsymbol{u}_i\cdot\mathrm{curl}\,(\widehat{\boldsymbol{u}}_i - \boldsymbol{u}_i)dx\\ &\qquad\ge \int_\Omega \boldsymbol{f}_i\cdot(\widehat{\boldsymbol{u}}_i - \boldsymbol{u}_i)dx + \int_\Gamma \boldsymbol{g}_i\cdot(\widehat{\boldsymbol{u}}_i - \boldsymbol{u}_i)dS.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}_1|^2)\mathrm{curl}\,\boldsymbol{u}_1\cdot\mathrm{curl}\,(\boldsymbol{u}_2 - \boldsymbol{u}_1)dx\\ &\quad\ge \int_\Omega \boldsymbol{f}_1\cdot(\boldsymbol{u}_2 - \boldsymbol{u}_1)dx + \int_\Gamma \boldsymbol{g}_1\cdot(\boldsymbol{u}_2 - \boldsymbol{u}_1)dS +\\ &\qquad + \int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}_1|^2)\mathrm{curl}\,\boldsymbol{u}_1\cdot\mathrm{curl}\,(\boldsymbol{u}_2 - \widehat{\boldsymbol{u}}_1)dx +\\ &\qquad + \int_\Omega \boldsymbol{f}_1\cdot(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_2)dx + \int_\Gamma \boldsymbol{g}_1\cdot(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_2)dS\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}_2|^2)\mathrm{curl}\,\boldsymbol{u}_2\cdot\mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)dx\\ &\quad\ge \int_\Omega \boldsymbol{f}_2\cdot(\boldsymbol{u}_1 - \boldsymbol{u}_2)dx + \int_\Gamma \boldsymbol{g}_2\cdot(\boldsymbol{u}_1 - \boldsymbol{u}_2)dS +\\ &\qquad + \int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}_2|^2)\mathrm{curl}\,\boldsymbol{u}_2\cdot\mathrm{curl}\,(\boldsymbol{u}_1 - \widehat{\boldsymbol{u}}_2)dx +\\ &\qquad + \int_\Omega \boldsymbol{f}_2\cdot(\widehat{\boldsymbol{u}}_2 - \boldsymbol{u}_1)dx + \int_\Gamma \boldsymbol{g}_2\cdot(\widehat{\boldsymbol{u}}_2 - \boldsymbol{u}_1)dS.\end{aligned}$$

Thus we have

$$\begin{aligned}&\int_\Omega \big(S_t(x, |\mathrm{curl}\,\boldsymbol{u}_1|^2)\mathrm{curl}\,\boldsymbol{u}_1 - S_t(x, |\mathrm{curl}\,\boldsymbol{u}_2|^2)\mathrm{curl}\,\boldsymbol{u}_2\big)\cdot\mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)dx\\ &\qquad\le \int_\Omega (\boldsymbol{f}_1 - \boldsymbol{f}_2)\cdot(\boldsymbol{u}_1 - \boldsymbol{u}_2)dx + \int_\Gamma (\boldsymbol{g}_1 - \boldsymbol{g}_2)\cdot(\boldsymbol{u}_1 - \boldsymbol{u}_2)dS + \Theta, \quad (18)\end{aligned}$$

where
$$\begin{aligned}\Theta &= \int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}_1|^2)\mathrm{curl}\, \boldsymbol{u}_1 \cdot \mathrm{curl}\,(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_2)dx \; + \\ &\quad + \int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}_2|^2)\mathrm{curl}\, \boldsymbol{u}_2 \cdot \mathrm{curl}\,(\widehat{\boldsymbol{u}}_2 - \boldsymbol{u}_1)dx \; + \\ &\quad + \int_\Omega \boldsymbol{f}_1 \cdot (\boldsymbol{u}_2 - \widehat{\boldsymbol{u}}_1)dx + \int_\Omega \boldsymbol{f}_2 \cdot (\boldsymbol{u}_1 - \widehat{\boldsymbol{u}}_2)dx \; + \\ &\quad + \int_\Gamma \boldsymbol{g}_1 \cdot (\boldsymbol{u}_2 - \widehat{\boldsymbol{u}}_1)dS + \int_\Gamma \boldsymbol{g}_2 \cdot (\boldsymbol{u}_1 - \widehat{\boldsymbol{u}}_2)dS.\end{aligned}$$

Now we estimate Θ. By (8a) and the Hölder inequality, we have

$$\begin{aligned}|\Theta| &\leq \Lambda \int_\Omega |\mathrm{curl}\, \boldsymbol{u}_1|^{p-1}|\mathrm{curl}\,(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_2)|dx \; + \\ &\quad + \Lambda \int_\Omega |\mathrm{curl}\, \boldsymbol{u}_2|^{p-1}|\mathrm{curl}\,(\widehat{\boldsymbol{u}}_2 - \boldsymbol{u}_1)|dx \; + \\ &\quad + \int_\Omega |\boldsymbol{f}_1||\boldsymbol{u}_2 - \widehat{\boldsymbol{u}}_1|dx + \int_\Omega |\boldsymbol{f}_2||\boldsymbol{u}_1 - \widehat{\boldsymbol{u}}_2|dx \; + \\ &\quad + \int_\Gamma |\boldsymbol{g}_1||\boldsymbol{u}_2 - \widehat{\boldsymbol{u}}_1|dS + \int_\Omega |\boldsymbol{g}_2||\boldsymbol{u}_1 - \widehat{\boldsymbol{u}}_2|dS,\end{aligned}$$

hence

$$\begin{aligned}|\Theta| &\leq \Lambda \|\mathrm{curl}\, \boldsymbol{u}_1\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \|\mathrm{curl}\,(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_2)\|_{\boldsymbol{L}^p(\Omega)} \; + \\ &\quad + \Lambda \|\mathrm{curl}\, \boldsymbol{u}_2\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \|\mathrm{curl}\,(\widehat{\boldsymbol{u}}_2 - \boldsymbol{u}_1)\|_{\boldsymbol{L}^p(\Omega)} \; + \\ &\quad + \|\boldsymbol{f}_1\|_{\boldsymbol{L}^{q'}(\Omega)} \|\boldsymbol{u}_2 - \widehat{\boldsymbol{u}}_1\|_{\boldsymbol{L}^q(\Omega)} + \|\boldsymbol{f}_2\|_{\boldsymbol{L}^{q'}(\Omega)} \|\boldsymbol{u}_1 - \widehat{\boldsymbol{u}}_2\|_{\boldsymbol{L}^q(\Omega)} \; + \\ &\quad + \|\boldsymbol{g}_1\|_{\boldsymbol{L}^{r'}(\Gamma)} \|\boldsymbol{u}_2 - \widehat{\boldsymbol{u}}_1\|_{\boldsymbol{L}^r(\Gamma)} + \|\boldsymbol{g}_2\|_{\boldsymbol{L}^{r'}(\Gamma)} \|\boldsymbol{u}_1 - \widehat{\boldsymbol{u}}_2\|_{\boldsymbol{L}^r(\Gamma)}.\end{aligned}$$

Here we note that $\boldsymbol{u}_2 - \widehat{\boldsymbol{u}}_1 = \frac{\mu}{\eta+\mu}\boldsymbol{u}_2 \in \mathbb{V}^p(\Omega)$.

Hence it follows from (14) that

$$\|\mathrm{curl}\,(\boldsymbol{u}_2 - \widehat{\boldsymbol{u}}_1)\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\mu}{\eta}\|\mathrm{curl}\, \boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)} = \frac{\mu}{\eta}\|\boldsymbol{u}_2\|_{\mathbb{V}^p(\Omega)} \leq C\mu.$$

Similarly, $\|\mathrm{curl}\,(\boldsymbol{u}_1 - \widehat{\boldsymbol{u}}_2)\|_{\boldsymbol{L}^p(\Omega)} \leq C\mu$. Moreover, we have

$$\|\boldsymbol{u}_j - \widehat{\boldsymbol{u}}_i\|_{\boldsymbol{L}^q(\Omega)} \leq \frac{\mu}{\eta}\|\boldsymbol{u}_j\|_{\boldsymbol{L}^q(\Omega)} \leq C_q\frac{\mu}{\eta}\|\boldsymbol{u}_j\|_{\mathbb{V}^p(\Omega)} \leq C\mu$$

for $i \neq j$. Summing up and using (14), we see that

$$|\Theta| \leq C\|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)}.$$

On the other hand, using (14), we have

$$\begin{aligned}\left|\int_\Omega (\boldsymbol{f}_1 - \boldsymbol{f}_2) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)dx\right| &\leq \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{\boldsymbol{L}^{q'}(\Omega)}(\|\boldsymbol{u}_1\|_{\boldsymbol{L}^q(\Omega)} + \|\boldsymbol{u}_2\|_{\boldsymbol{L}^q(\Omega)}) \\ &\leq \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{\boldsymbol{L}^{q'}(\Omega)} C_q(\|\boldsymbol{u}_1\|_{\mathbb{V}^p(\Omega)} + \|\boldsymbol{u}_2\|_{\mathbb{V}^p(\Omega)}) \leq C\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{\boldsymbol{L}^{q'}(\Omega)}.\end{aligned}$$

Similarly we have $\left|\int_\Gamma (\boldsymbol{g}_1 - \boldsymbol{g}_2) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) dS\right| \le C\|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{\boldsymbol{L}^{r'}(\Gamma)}$.

Finally we estimate the left hand side of (18) from below. From Lemma 2.4,

$$\int_{\widehat{\Omega}} \big(S_t(x, |\mathrm{curl}\,\boldsymbol{u}_1|^2)\mathrm{curl}\,\boldsymbol{u}_1 - S_t(x, |\mathrm{curl}\,\boldsymbol{u}_2|^2)\mathrm{curl}\,\boldsymbol{u}_2\big) \cdot \mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2) dx$$
$$\ge \begin{cases} c\displaystyle\int_{\widehat{\Omega}} |\mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^p dx & \text{if } p \ge 2, \\ c\displaystyle\int_{\widehat{\Omega}} (|\mathrm{curl}\,\boldsymbol{u}_1| + |\mathrm{curl}\,\boldsymbol{u}_2|)^{p-2} |\mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^2 dx & \text{if } 1 < p < 2, \end{cases}$$

where $\widehat{\Omega} = \{x \in \Omega; |\mathrm{curl}\,\boldsymbol{u}_1(x)| + |\mathrm{curl}\,\boldsymbol{u}_2(x)| \ne 0\}$. When $p \ge 2$, we have (16). When $1<p<2$, we use the following reverse Hölder inequality (cf. Sobolev [16, p. 8]): if $0 < s < 1, s' = s/(s-1)$, $F \in L^s(\Omega), FG \in L^1(\Omega)$ and $\int_\Omega |G(s)|^{s'} dx < \infty$, then

$$\left(\int_\Omega |F(x)|^s dx\right)^{1/s} \le \int_\Omega F(x)G(x)dx \left(\int_\Omega |G(x)|^{s'} dx\right)^{-1/s'}.$$

If we apply this inequality with $s = p/2, s' = p/(p-2)$ and $\Omega = \widehat{\Omega}$, then we have

$$\int_{\widehat{\Omega}} (|\mathrm{curl}\,\boldsymbol{u}_1| + |\mathrm{curl}\,\boldsymbol{u}_2|)^{p-2} |\mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^2 dx$$
$$\ge \left(\int_{\widehat{\Omega}} (|\mathrm{curl}\,\boldsymbol{u}_1| + |\mathrm{curl}\,\boldsymbol{u}_2|)^p dx\right)^{(p-2)/p} \|\mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|^2_{\boldsymbol{L}^p(\Omega)}.$$

Since

$$\left(\int_{\widehat{\Omega}} (|\mathrm{curl}\,\boldsymbol{u}_1| + |\mathrm{curl}\,\boldsymbol{u}_2|)^p dx\right)^{(2-p)/p}$$
$$\le C(\|\mathrm{curl}\,\boldsymbol{u}_1\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{curl}\,\boldsymbol{u}_2\|^p_{\boldsymbol{L}^p(\Omega)})^{(2-p)/p} \le C_1,$$

we get (16). □

4. Proof of Theorem 2.5

In this section, we prove the main theorem (Theorem 2.5). Before beginning the proof, we provide a lemma that is necessary to prove Theorem 2.5.

Lemma 4.1. *Let $1 < p < \infty$, $s > 3$ and $\alpha > 3/2$. Assume that $\alpha < 3/(3-p)$ if $p < 3$. Define a sequence $\{s_n\}_{n=1}^\infty$ by*

$$s_1 = s, \quad s_{n+1} = \frac{3\alpha s_n}{3 + \alpha s_n} \quad (n = 1, 2, \ldots). \tag{19}$$

Then the sequence $\{s_n\}$ is convergent to a limit $s_0 \in (1, p)$. In particular, there exists $n \in \mathbb{N}$ such that $1 < s_n \le p$.

For the proof, see [5, Proposition 3].

Proof of Theorem 2.5.

Assume that $\boldsymbol{f} \in \boldsymbol{L}^{q'}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^{r'}(\Gamma)$. For any $\boldsymbol{\varphi} \in \boldsymbol{C}(\overline{\Omega})$, the variational inequality (13) with $\varphi = F(\boldsymbol{\varphi})$ has a unique solution $\boldsymbol{u}_{\boldsymbol{\varphi}} \in \mathbb{K}^p(F(\boldsymbol{\varphi}))$. Since we have $F \in C(\mathbb{R}^3;\mathbb{R}^+)$, we see that $F(\boldsymbol{\varphi})$ is bounded. Since we have $\boldsymbol{u}_{\boldsymbol{\varphi}} \in \mathbb{V}^p(\Omega)$ and $|\mathrm{curl}\, \boldsymbol{u}_{\boldsymbol{\varphi}}| \le F(\boldsymbol{\varphi})$, we can see that $\boldsymbol{u}_{\boldsymbol{\varphi}} \in \mathbb{V}^s(\Omega)$ for any $s \ge p$. Define a map $T\colon \boldsymbol{C}(\overline{\Omega}) \to \mathbb{V}^s(\Omega)$ by $T(\boldsymbol{\varphi}) = \boldsymbol{u}_{\boldsymbol{\varphi}}$.

First we prove the continuity of the map T. Let $\boldsymbol{\varphi} \in \boldsymbol{C}(\overline{\Omega})$. Then for any $0 < \varepsilon \le 1$, there exists $\delta > 0$ such that if $\|\boldsymbol{\psi} - \boldsymbol{\varphi}\|_{\boldsymbol{C}(\overline{\Omega})} < \delta$, then $\|F(\boldsymbol{\psi}) - F(\boldsymbol{\varphi})\|_{\boldsymbol{C}(\overline{\Omega})} < \varepsilon$. Since $F \in C(\mathbb{R}^3;\mathbb{R}^+)$ and $\boldsymbol{\varphi}$ is bounded in $\overline{\Omega}$, there exists $\eta > 0$ such that $F(\boldsymbol{\varphi}(x)) \ge 2\eta$ for all $x \in \overline{\Omega}$. Therefore, if $\|\boldsymbol{\psi} - \boldsymbol{\varphi}\|_{\boldsymbol{C}(\overline{\Omega})} < \delta$, then we have

$$F(\boldsymbol{\psi}(x)) \ge F(\boldsymbol{\varphi}(x)) - \varepsilon \ge 2\eta - \varepsilon \text{ for all } x \in \overline{\Omega}.$$

Choosing $\varepsilon = \eta$, for any $\boldsymbol{\psi} \in \boldsymbol{C}(\overline{\Omega})$ satisfying $\|\boldsymbol{\psi} - \boldsymbol{\varphi}\|_{\boldsymbol{C}(\overline{\Omega})} < \delta$, $F(\boldsymbol{\psi}(x)) \ge \eta$ for all $x \in \overline{\Omega}$. We note that $\|F(\boldsymbol{\psi})\|_{\boldsymbol{C}(\overline{\Omega})} \le \|F(\boldsymbol{\varphi})\|_{\boldsymbol{C}(\overline{\Omega})} + 1$. Therefore we have

$$\begin{aligned}
\|T(\boldsymbol{\psi}) - T(\boldsymbol{\varphi})\|^s_{\mathbb{V}^s(\Omega)} &= \int_\Omega |\mathrm{curl}\,(\boldsymbol{u}_{\boldsymbol{\psi}} - \boldsymbol{u}_{\boldsymbol{\varphi}})|^s dx \\
&= \int_\Omega |\mathrm{curl}\,(\boldsymbol{u}_{\boldsymbol{\psi}} - \boldsymbol{u}_{\boldsymbol{\varphi}})|^p (|\mathrm{curl}\,\boldsymbol{u}_{\boldsymbol{\psi}}| + |\mathrm{curl}\,\boldsymbol{u}_{\boldsymbol{\varphi}}|)^{s-p} dx \\
&\le (2\|F(\boldsymbol{\varphi})\|_{\boldsymbol{C}(\overline{\Omega})} + 1)^{s-p} \int_\Omega |\mathrm{curl}\,(\boldsymbol{u}_{\boldsymbol{\psi}} - \boldsymbol{u}_{\boldsymbol{\varphi}})|^p dx \\
&= (2\|F(\boldsymbol{\varphi})\|_{\boldsymbol{C}(\overline{\Omega})} + 1)^{s-p} \|\boldsymbol{u}_{\boldsymbol{\psi}} - \boldsymbol{u}_{\boldsymbol{\varphi}}\|^p_{\mathbb{V}^p(\Omega)}.
\end{aligned}$$

Applying Proposition 3.1, there exists a constant C depending only on p, $\|\boldsymbol{f}\|_{\boldsymbol{L}^{q'}(\Omega)}$, $\|\boldsymbol{g}\|_{\boldsymbol{L}^{r'}(\Gamma)}$ and η such that

$$\|\boldsymbol{u}_{\boldsymbol{\psi}} - \boldsymbol{u}_{\boldsymbol{\varphi}}\|^{p\vee 2}_{\mathbb{V}^p(\Omega)} \le C\|F(\boldsymbol{\psi}) - F(\boldsymbol{\varphi})\|_{\boldsymbol{C}(\overline{\Omega})} \le C\varepsilon$$

for all $\boldsymbol{\psi} \in \boldsymbol{C}(\overline{\Omega})$ satisfying $\|\boldsymbol{\psi} - \boldsymbol{\varphi}\|_{\boldsymbol{C}(\overline{\Omega})} < \delta$. Thus the map T is continuous. If we choose s so that $s > 3$, then the inclusion map $i : \mathbb{V}^s(\Omega) \hookrightarrow \boldsymbol{C}(\overline{\Omega})$ is compact. So the map $S = i \circ T : \boldsymbol{C}(\overline{\Omega}) \to \boldsymbol{C}(\overline{\Omega})$ is continuous and compact.

When $p > 3$, we choose $s = p$. Then it follows from Proposition 2.6 that the image of S is relatively compact. Hence from the Schauder fixed point theorem, S has a fixed point in $\boldsymbol{C}(\overline{\Omega})$. Thus the fixed point $\boldsymbol{u}_{\boldsymbol{\varphi}}$ is a solution of (11).

When $p \le 3$, we use the Leray-Schauder fixed point theorem. Since the map $S\colon \boldsymbol{C}(\overline{\Omega}) \to \boldsymbol{C}(\overline{\Omega})$ is continuous and compact, it suffices to prove that there exists some $M > 0$ such that

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{C}(\overline{\Omega})} \le M$$

for any $\boldsymbol{\varphi} \in \boldsymbol{C}(\overline{\Omega})$ and any $\lambda \in [0,1]$ satisfying $\boldsymbol{\varphi} = \lambda S\boldsymbol{\varphi}$ (cf. Gilbarg and Trudinger [9, Theorem 11.3]). Since $s > 3$ and $\mathbb{V}^s(\Omega) \hookrightarrow \boldsymbol{C}(\overline{\Omega})$, it suffices to prove the set

$$\mathcal{A} = \{\boldsymbol{\varphi} \in \boldsymbol{C}(\overline{\Omega});\ \boldsymbol{\varphi} = \lambda T\boldsymbol{\varphi} \ \text{ for some } \lambda \in [0,1]\}$$

is bounded in $\mathbb{V}^s(\Omega)$. In the hypothesis of Theorem 2.5, we can assume that $\alpha > 3/2$. By Lemma 4.1, for the sequence $\{s_n\}$ defined by (19), there exists $n \in \mathbb{N}$ such that $1 < s_n \leq p$. If $\boldsymbol{\varphi} \in \mathcal{A}$, then

$$|\mathrm{curl}\, \boldsymbol{u}_{\boldsymbol{\varphi}}| \leq F(\boldsymbol{\varphi}) \leq c_0 + c_1|\boldsymbol{\varphi}|^\alpha = c_0 + c_1\lambda^\alpha|\boldsymbol{u}_{\boldsymbol{\varphi}}|^\alpha.$$

Let $i < n$. Then

$$\begin{aligned}
\|\boldsymbol{u}_{\boldsymbol{\varphi}}\|_{\mathbb{V}^{s_{i-1}}(\Omega)} &= \left(\int_\Omega |\mathrm{curl}\, \boldsymbol{u}_{\boldsymbol{\varphi}}|^{s_{i-1}} dx\right)^{1/s_{i-1}} \\
&\leq \left(\int_\Omega (c_0 + c_1\lambda^\alpha|\boldsymbol{u}_{\boldsymbol{\varphi}}|^\alpha)^{s_{i-1}} dx\right)^{1/s_{i-1}} \\
&\leq C\left\{c_0|\Omega| + c_1\lambda^\alpha \left(\int_\Omega |\boldsymbol{u}_{\boldsymbol{\varphi}}|^{\alpha s_{i-1}} dx\right)^{1/s_{i-1}}\right\} \\
&= C(c_0|\Omega| + c_1\lambda^\alpha\|\boldsymbol{u}_{\boldsymbol{\varphi}}\|^\alpha_{\boldsymbol{L}^{\alpha s_{i-1}}}).
\end{aligned}$$

Since $s_i = 3\alpha s_{i-1}/(3 - \alpha s_{i-1})$ is equivalent to $\alpha s_{i-1} = 3s_i/(3 - s_i)$, αs_{i-1} is the Sobolev critical exponent for the embedding of $\mathbb{V}^{s_i}(\Omega)$ into $L^q(\Omega)$. Hence

$$\|\boldsymbol{u}_{\boldsymbol{\varphi}}\|_{\mathbb{V}^{s_{i-1}}(\Omega)} \leq C(1 + \|\boldsymbol{u}_{\boldsymbol{\varphi}}\|^\alpha_{\mathbb{V}^{s_i}(\Omega)}).$$

Repeating the process finite times, it follows from Proposition 2.6 that

$$\|\boldsymbol{u}_{\boldsymbol{\varphi}}\|_{\mathbb{V}^s(\Omega)} \leq C(1 + \|\boldsymbol{u}_{\boldsymbol{\varphi}}\|^{\alpha n}_{\mathbb{V}^{s_n}(\Omega)}) \leq C(1 + \|\boldsymbol{u}_{\boldsymbol{\varphi}}\|^{\alpha n}_{\mathbb{V}^p(\Omega)}) \leq C_1.$$

Therefore $\|\boldsymbol{\varphi}\|_{\mathbb{V}^s(\Omega)} \leq C_1$ for all $\boldsymbol{\varphi} \in \mathcal{A}$.

Moreover, since $\mathbb{K}^p(F(\boldsymbol{\varphi})) \hookrightarrow \mathbb{V}^s(\Omega)$ for any $s \geq p$, it follows from the Sobolev embedding theorem that $\boldsymbol{u}_{\boldsymbol{\varphi}}$ belongs to $\boldsymbol{C}^\beta(\overline{\Omega})$ for any $0 < \beta < 1$. This completes the proof of Theorem 2.5.

References

[1] C. Amrouche, N. H. Seloula: *L^p-theory for vector potentials and Sobolev's inequalities for vector fields*, C. R. Acad. Sci. Paris, Ser. 1, 349 (2011) 529–534.

[2] C. Amrouche, N. H. Seloula: *L^p-theory for vector potentials and Sobolev inequalities for vector fields: Application to the Stokes equations with pressure boundary conditions*, Math. Models Methods Appl. Sci. 23(4) (2013) 37–92.

[3] J. Aramaki: *Quasilinear parabolic-type system arising in electromagnetism in a multi-connected domain*, Int. J. Evolution Equation 10(2) (2016) 119–144.

[4] J. Aramaki: *Existence of a solution to a quasivariational inequality with a perturbation term*, Adv. Diff. Equations Control Processes 18(1) (2017) 39–54.

[5] A. Azevedo, F. Miranda, L. Santos: *Stationary quasivariational inequalities with gradient constraint and nonhomogeneous boundary conditions*, in: *From Particle Systems to Partial Differential Equations*, C. Bernardin and G. Conçalves (eds.), Springer Proceedings in Mathematics and Statistics 75, Springer, Berlin (2014) 95–112.

[6] R. Dautray, J. L. Lions: *Mathematical Analysis and Numerical Method for Science and Technology, Vol. 3*, Springer, New York (1990).

[7] E. DiBenedetto: *Degenerate Parabolic Equations*, Springer, New York (1993).

[8] I. Ekeland, R. Témam: *Convex Analysis and Variational Problems*, SIAM, Philadelphia (1999).

[9] D. Gilbarg, N. S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin (2001).

[10] V. Girault, P. A. Raviart: *Finite Element Methods for Navier-Stokes equations*, Springer, Berlin (1979).

[11] Y. B. Kim, C. F. Hempstead, A. R. Strnad: *Critical persistent currents in hard superconductors*, Phys. Rev. Letters 9 (1962) 306–309.

[12] Y. B. Kim, C. F. Hempstead, A. R. Strnad: *Magnetization and critical supercurrents*, Phys. Rev. 129 (1963) 528–535.

[13] F. Miranda, J. F. Rodrigues, L. Santos: *A class of stationary nonlinear Maxwell systems*, Math. Models Methods Appl. Sci. 19(10) (2009) 1883–1905.

[14] F. Miranda, J. F. Rodrigues, L. Santos: *On a p-curl system arising in electromagnetism*, Discrete Cont. Dyn. Systems, Ser. S, 6(3) (2012) 605–629.

[15] L. Prigozhin: *On the Bean critical-state model in superconductivity*, Euro. J. Appl. Math. 7 (1996) 237–247.

[16] S. Sobolev: *Applications of Functional Analysis in Mathematical Physics*, Translation of Mathematical Monographs 7, American Mathematical Society, Providence (1963).